HEXAGONAL CYLINDRICAL LATTICES: A UNIFIED HELICAL STRUCTURE IN 3D PITCH SPACE FOR MAPPING FLAT MUSICAL ISOMORPHISM

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SUPERVISORY AND EXAMINING COMMITTEE

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Abstract

An isomorphic keyboard layout is an arrangement of notes of a scale such that any musical construct has the same shape regardless of the root note. The mathematics of some specific isomorphisms have been explored since the 1700s, however, only recently has a general theory of isomorphisms been developed such that any set of musical intervals can be used to generate a valid layout. These layouts have been implemented in the design of electronic musical instruments and software applications. This thesis presents a new extension to the theory of isomorphic layouts, taking advantage of the repetition of notes in these layouts to produce a three-dimensional representational mapping onto a cylinder. Isomorphic layouts can be produced using rectangular or hexagonal grids, and the mathematics of Fullerene molecules from organic chemistry is borrowed to regularize the mapping of hexagonal isomorphisms onto cylinders.

This new cylindrical mapping model is also applied to the study of tonal pitch space models, a branch of musicology which seeks to explore the underlying perceptual relationships between harmonically related pitches. Tonal pitch space models are spatial networks of the perceptual "closeness" of pitches, and researchers have experimented with flat networks, cylindrical arrangements, and even torus (or donut) shaped models. Cylindrical models are often helical or spiral in nature, and the new cylindrical isomorphic model developed in this thesis is applied to existing helical tonal pitch space models.

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Last but not least, thanks to my girlfriend Xuan Sun. Without her help I would never have a chance to study in Regina.

Dedication

This thesis is dedicated to my mother Jinghang Zhang and my father Dan Hu who demonstrated their selfless love and unremitting support in the pursuit of this degree.

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Chapter 1

Introduction

1.1 Overview

The human experience of music is universal. Every culture on earth has some form of rhythmic or tonal experience that goes beyond mere communication. Although there is disagreement about the evolutionary basis for music, people have used music for expressing emotion and to unite communities during events such as marriages and funerals. The experience of music has also appeared in religious ceremonies, and as a form of entertainment.

People have tried to find good metaphors for music to describe the complex interactions of harmony. In ancient Greece, Plato described music as a "twinned study" with astronomy, since music pleases the ear and astronomy pleases the eye. Indeed, to Plato, both involve the study of harmony. Astrological maps which describe the position of Sun, Moon and other stars have been used to imagine the harmonic relationships of music [4]. In ancient Asia, early Indian and Chinese philosophers also regarded the study of harmony as a part of Science. The ancient Chinese used the "Yin" and "Yang" to categorize pitches into dullness and brightness [23]. In ancient India, philosophers used algebra to explore the number of tones in a tuning system [10]. These examples show how the study of music has been associated with the study of mathematics since early times.

Mathematics is used today to understand music as well. Musicians and musicologists attempt to share new concepts of composition and perception by using abstract algebra, geometrical metaphors, and set theory. Researchers use computational models of musical knowledge to understand both music and cognition, This emerging

research field, which includes cognitive musicology and music information retrieval, is a part of the wider field of artificial intelligence.

In one specific application of mathematics to music, researchers modelled the perceptual and harmonic relationships between pitches in a scale, representing each pitch as a point in space and finding a way to show that even though some pitches may be far apart in frequency, they are perceived as being close together in harmony. Indeed, the Semitone (the closest interval in 12-tone equal tempered scale) sounds dissonant, and harmonically far away, while the Perfect Fifth (an interval of 7 semitones) sounds consonant and harmonically close. Many researchers have building graphical and mathematical models that attempt to describe these relationships. These models are called *Tonal Pitch Space* models [34]. There have been many tonal pitch space models introduced throughout history. The first of these models likely existed as a one-dimensional representation. Today, new models in higher dimensional spaces are coming to the forefront of research [52].

While musicologists have used mathematics and computer science to explore the way music is constructed, At the same time, composers and performers have used computer science to make and perform new kinds of music, and to perform more traditional music on new kinds of instruments. The study of New Interfaces for Musical Expression (NIME) is popular today and hundreds of thousands of new interfaces have been created for music expression. Many of these new interfaces remain experimental, but recently, a number of new interfaces have emerged as popular tools for creating and performing music to large audiences. Although many new interfaces and new forms of expression have moved away from the idea of "tonal" music, most popular music remains tonal.

When focusing on tonal music, instrument designers tend to focus on a particular tonal pitch space, or arrangement of notes. To do this, designers refer to the perceived harmonic and melodic distance between pitches in order to arrange an instrument's keys [13]. However, because instruments are physical devices, designers must also account for the "reachability" of notes, and the fingerings necessary to play them. As a result, musical devices often represent a tradeoff between the physical distances between pitches and the perceptual (or harmonic) distances between pitches. Based on this concept, some designs such as the traditional piano keyboard may focus more on the physical distance than the perceptual distance. As a consequence, musical

constructs such as chords or melodies may be more or less difficult to play depending on the root key of the music. This assumption has therefore permeated music education: that one must learn different techniques for the same musical construct (for example, a scale) when played in different keys. To overcome this fingering problem, a series of keyboards named *isomorphic keyboards* (see Chapter 4) have been developed since the late 1800s, which arrange the notes in such a way that musical constructs always have the same shape and can be played with the same fingering and technique regardless of the key [56].

To date, experiments in the development of isomorphic keyboard layouts have been restricted to two-dimensional arrangements of notes, which are reminiscent of existing tonal pitch space models. Some researchers have proposed three-dimensional tonal pitch space models, taking into account the perceptual similarity between notes, not just sequentially and harmonically, but also repeating octave-to-octave in a spiral pattern. Researchers have yet to apply these spiral or helical tonal pitch space models to the development of isomorphic keyboards.

This thesis draws upon research in spiral tonal pitch space models and isomorphic keyboards to develop and present a unified cylindrical isomorphic tone space model which can be used to map any isomorphic keyboard layout onto a spiral tonal pitch space model. The structure and features of this unified cylindrical isomorphic model are presented in detail, showing how it applies to existing spiral pitch space models. Moreover, specific instances of this model will be discussed so as to explore the opportunities for building physical musical instruments based on this model in the future.

1.2 Motivation

Interdisciplinary research, which involves two or more academic disciplines grouped into one research activity, has been encouraged in academia. When researchers fail to find existing models that solve a particular problem in their area of study, it is possible that they may find a solution by exploring similar models from other fields. The research in this thesis is an interdisciplinary exploration, drawing from Music Theory, Graph Theory, Musical Instrument Design, Organic Chemistry, and many other fields. The cylindrical hexagonal lattices which will be shown to represent tonality in

music theory, are derived from topological and geometrical models originating from mathematical chemistry.

1.3 Contributions

Considering both the perceptual distance in tonal pitch space and the principle of designing a musical keyboard's physical appearance, this thesis introduces a new cylindrical hexagonal lattice structure model for pitch, built upon the mathematics of a specific class of organic molecules (Fullerenes). This new model is an extension of existing isomorphic layout research, and is explored and validated in the following ways:

- An algorithm is presented for mapping the location of musical notes within this model. The concept of "chiral angle" is presented to regularize and validate new and existing helical models.
- This model is validated for all existing isomorphisms.
- This model is used to map existing popular isomorphisms onto cylindrical structures,
- This model is applied to two prominent spiral / helical tonal pitch space models, and is used to verify the physical validity for applicability of these models.

Finally, by way of a discussion of future work, this thesis explores opportunities for implementing these cylindrical models as musical instruments, focusing on the possibilities of playability, musicality and novelty.

1.4 Organization

The remainder of this thesis is organized as follows:

In Chapter 2, the thesis delivers a background of relevant music fundamentals, including the definitions of Frequency, Pitch, Note, Tone and Key, as well as definitions of Equal Temperament and Scales. To provide a basis for a discussion of pitch space in n-dimensions, the thesis also mentions several historical models of tonal pitch space.

In Chapter 3, the thesis's focus shifts to discussing the physical arrangement of keys on a musical keyboard controller. After reviewing five evolving periods of microtonal keyboard design, this thesis summarizes four principles of key arrangement, invoking the realization of musical isomorphism in preparation of Chapter 4. Afterwards, several modern three-dimensional musical controllers or models are discussed as examples of exploring and extending those principles from two-dimensional to three-dimensional space.

Chapter 4 presents the definition of musical isomorphism and introduces some examples of isomorphic keyboards, along with their respective histories. By analyzing the regular and semi-regular polyhedron in three-dimensional space, the concept of pseudo-isomorphism is discussed.

The thesis then shifts to considering the possibility of hexagonal isomorphisms in a cylindrical arrangement. Chapter 5 begins by exploring the topology and geometry of Fullerene structures, and proceeds to propose a new series of lattices for mapping musical notes in three-dimensional space. As a special example of these lattices, a cylindrical hexagonal lattice, which has been extensively studied in the context of carbon nanotubes, is suggested for building the physical structure of a keyboard. This is the key to the development of the proposed helical three-dimensional isomorphic pitch space model.

Chapter 6 covers the algorithm of wrapping a flat isomorphism into a tube-like lattice. In addition, the implementation of pitch perception helical models using the tube-like lattices with special chiral angles is explored.

In Chapter 7, the diameters of these tube-like lattices are given by their particular chiral angle. While considering the possible construction and physical appearance of an instrument, three issues are discussed: instrument size as it varies by the size of the keys, instrument size as it varies by note duplications, boundary conditions, and note reachability. This chapter then explores the potential for the playability and playing modes of such an instrument.

The content of Chapter 5, 6 and 7 are based on papers published by the author in the International Computer Music Conference [25] and the Sound and Music Computing Conference [24] in 2015.

To end, a summary, a conclusion, and suggestions for future work are presented in Chapter 8.

Chapter 2

Applicable Music Theory Fundamentals and Tonal Pitch Space Models

2.1 Pitch and Scale

"Frequency," "Pitch," "Note," "Tone," and "Key" are five similar terms used in music terminology. It is difficult for non-professionals to understand the difference between them. How different are they?

The modern piano arranges keys in repetitions of the same twelve tones in what is known as a chromatic scale. Is it conceivable that a scale could contain eleven or thirteen tones instead of twelve? Both of these questions are discussed in this Chapter.

2.1.1 Frequency, Pitch, Note, Tone, and Key

In physics, sound is modelled as a vibration that propagates as mechanical wave of pressure and displacement. The number of cycles per unit of time of this vibration is called the *frequency* [9]. Frequency is measured in cycles per second (cps), or Hertz (Hz). According to [28], the range of sound wave frequencies within the spectrum of human hearing is ideally between 20 Hz and 20 kHz.

Pitch is a perceptual property of sounds that allows for their ordering on a frequency-related scale (roughly the logarithm of frequency) from low to high [30]. Generally speaking, a pitch is a particular frequency of sound, such as 440 Hz. However, pitch is not a purely objective physical property, it also has subjective psychoacoustical attributes [40].

A musical *note* is seen as an interchangeable alias for pitch, or as a sign to represent a pitch sound. For example, western music refers to a pitch of 440 Hz as the note A_4.

Under some circumstances, the term *tone* can be interpreted as a synonym for timbre, which represents tonal qualities from psychoacoustics. More often, as in this thesis, a "tone" is used as a synonym for a "note".

Lastly, musical *keys* represent tonic notes and chords which provide a subjective sense of arrival and rest, representing the tonal center of a musical construct. A musical piece may be written in the key of "G major". However, the meaning of the term "key" in musical instrument design may be different from this musical key. A key in musical instrument design usually refers to a single control trigger for activating a specific musical note. A complete set of these keys in a certain arrangement can be called musical keyboard.

2.1.2 Equal Temperament and 12-TET

According to [37], there are two meanings of *tuning*:

1. The act of tuning an instrument or voice, as a tuning practice, and
2. the various systems of pitches (notes) used to tune an instrument, as a tuning system.

Just intonation is a music tuning system which implements pure intervals, where the frequencies of notes are correlated to ratios of small-integers in order to meet the other requirement of the system.

If, however, every pair of adjacent pitches is separated by the *same* interval, the system can be called an *equal temperament* tuning system. In Western music, the most common tuning system is the *twelve-tone equal temperament system* (also known as *12-TET*). This system divides an octave (the interval between one pitch and another with half or double its frequency) into twelve equal logarithmic steps [22].

To avoid ambiguity between equal temperament in the octave or another interval, the term *equal division of the octave* (also known as *EDO*) is used in music theory. In this thesis, the 12-TET and 12-EDO systems are used interchangeably as a tuning system which divides an octave into twelve pieces, all of which are equal on a logarithmic scale of frequency.

2.1.3 Scale

Because the order of notes repeats from one octave to another, it is intuitive to arrange pitches linearly. In other words, each pitch is a sequential partition of the whole structure. This fact can be used to tile a tuning system in a linear way.

Such a linear structure is one-dimensional. The structure is unique in the increasing direction; a tuning system determines both the initial note and the distance between each pitch. In music theory, this structure is called a *scale*, which describes any set of ordered pitches. For example, an increasing C-major scale (described in more detail below) includes the notes C-D-E-F-G-A-B and returns back to C in a higher octave. "C" represents the beginning note, or *tonic*, of the scale. Here, "major" refers to the sequence of intervals between the notes in a scale, consisting of whole tones (two consecutive steps of pitch in 12-TET), and semi-tones (a single step of pitch in 12-TET). The step sequence for this scale is: tone-tone-semitone-tone-tone-tone-semitone. The C-major scale on the music staff is shown in Fig. 2.1.

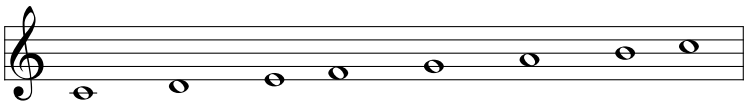


Figure 2.1: C major scale.

2.2 n-Dimensional Pitch Space

For a given pitch scale, there exists exactly one linear pitch arrangement in one-dimensional space. Different pitch scales correspond to different linear pitch arrangements, varied by tonic, direction, and interval. Considering the combination of linear structures, it is possible to create a pitch space with a higher dimension than one. In this section, one-, two-, and three-dimensional pitch spaces are discussed.

Thanks to the repetition of pitches within each octave, it is also possible to develop a circular repetition structure. This section also describes such a circular structure in detail.

Changing the tonic note, intervals between pitches, or the direction of pitches can generate many different linear pitch arrangements. This thesis focuses on four of

these arrangements.

The diatonic scale contains seven distinct pitches in one octave. There are eight steps when traversing from one tonic to the same tonic in an adjacent octave. Since the prefix "oct" means "eight," the musical "octave" is aptly named.

The *chromatic* scale was introduced by adding a subset of five notes into the diatonic scale to balance the arrangement of each note by a half step (also known as semi-tone) above or below another pitch. These intermediate notes, when related to the key of C Major, are presented as black notes on the piano keyboard, and represented by accidentals (♯ or ♭) in music notation. For example, the C chromatic scale is C-C♯-D-D♯-E-F-F♯-G-G♯-A-A♯-B and then back to C, as shown in Fig. 2.2. Note that each accidental can be "spelled" using a ♭ instead of a ♯ depending on the key, for example, C♯ is the same note as D♭ in 12-TET.

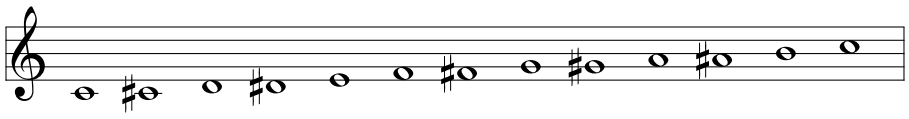


Figure 2.2: C chromatic scale.

A chromatic scale can also be represented by a circular pitch space (Fig. 2.3a) satisfying the following two conditions:

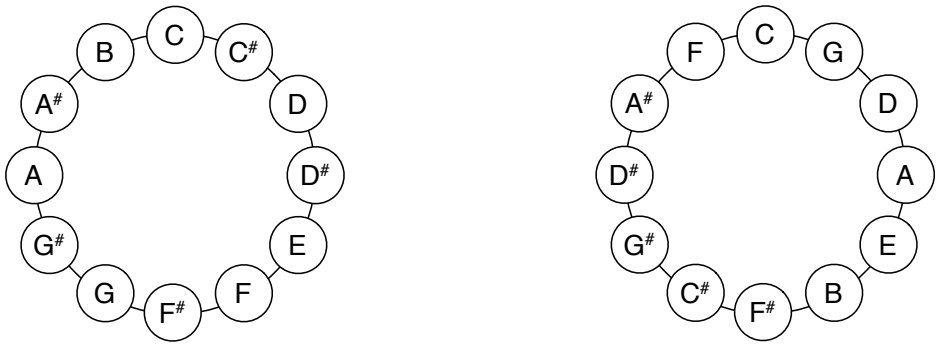
1. The intervals between adjacent notes in this scale are all identical, and

2. The tonics are the same regardless of the octave.

The modern piano was designed based on the diatonic and chromatic scales. In an octave, there are seven white keys corresponding to the seven notes within the diatonic scale in C-major. There are also an additional five black keys within an octave; when combined with the white keys, there is a total of twelve keys, which correspond to the twelve notes within the chromatic scale.

In addition to the diatonic and chromatic scales, there are two other related one-dimensional pitch space representations that are worth discussing.

One of these is the *mel* scale. In 1873, physicist Gustav Fechner explored the problem of quantifying and measuring pitch perception [15]. By using the concept of measuring perceived quantity in vision research, he proved that the perceived quantity



(a) Chromatic Circle. (b) Circle of Fifths

Figure 2.3: Circular Pitch Scales

in pitch is approximately a logarithmic transformation of its physical quantity (that is, its frequency). Steven *et al* [54] constructed the *mel* scale that reflected the psychological reality of how people hear musical tones.

The other one-dimensional pitch space worth mentioning is *Heinichen's regional circle*, which is a one-dimensional circular structure proposed by Heinichen in 1728 [27] and shown in Fig. 2.4. The figure displays how the major circle of fifths alternates with its relative minor counterpart. The word "relative" in this case refers to two scales having the same key signature. This circle progresses through all the members of the chromatic collection by perfect fifth intervals before returning to its starting point. This structure handles the major-minor relationship, but as it is a single circle, it is still a one-dimensional pitch space representation. The modern circle of fifths is presented in Fig. 2.3b.

2.2.1 Two-Dimensional Pitch Space

Expanding further on Heinichen's regional circle, Kellner (1737) developed a similar structure by using double circles [34] instead of one. These circles distinguish between major and minor keys by using different circles rather than different characters. This model elevated Heinichen's pitch space into two dimensions.

Later, G. Weber (1821-1824) introduced *Weber's regional chart* [57], which presents the circle of fifths with both a vertical axis and a horizontal axis, having alternating major-minor relationships. The chart is shown in Fig. 2.5.

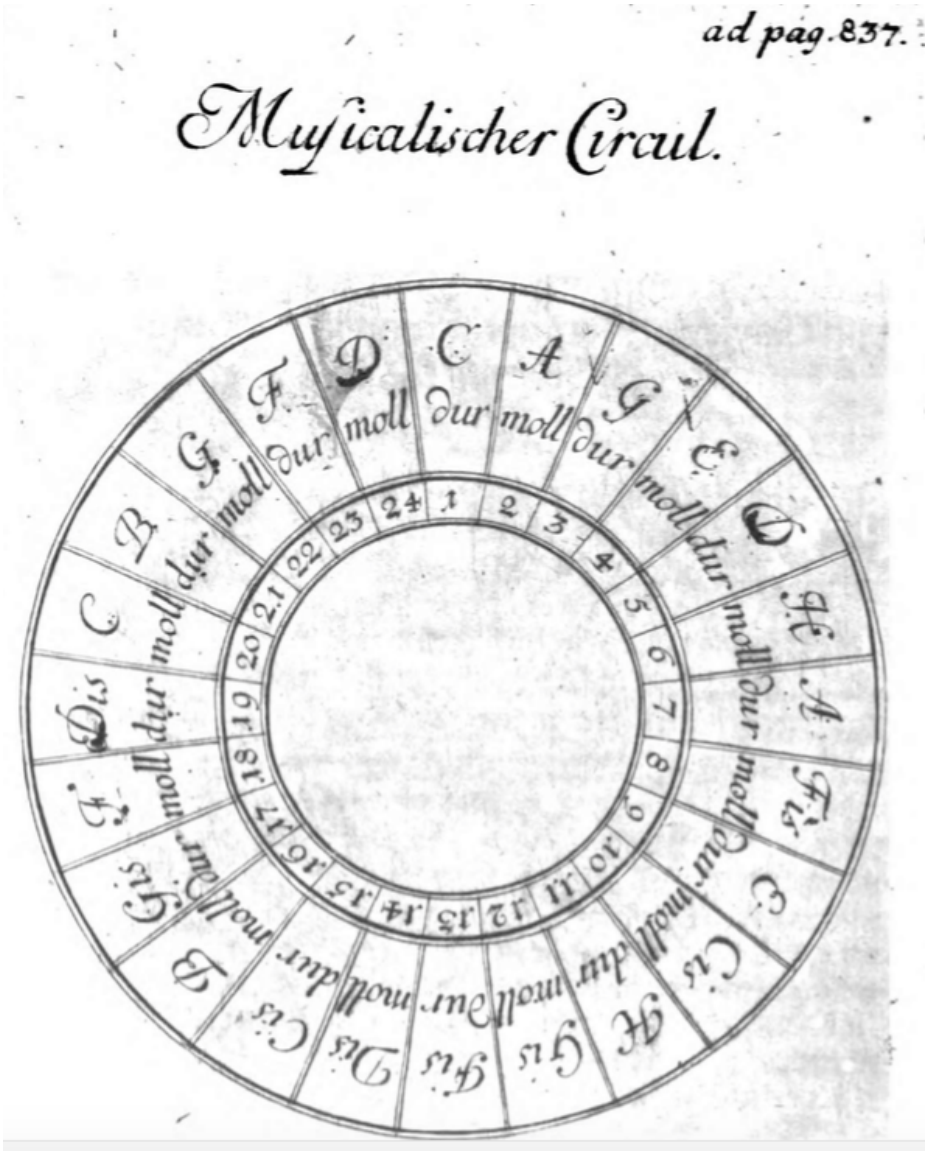


Figure 2.4: Heinichen's circle of fifths (German: musicalischer circul) (1728) [20]

In the field of music theory research, an early example of a musical lattice structure was developed by the mathematician Euler (1739) who introduced this chart as a way of representing just intonation [14]. The chart displays perfect fifths to the left and major thirds to the right of each key (Fig. 2.6) showing how the keys interrelate. Influenced by Euler, Riemann (1902) created a more readable chart, as seen in Fig. 2.7, calling it the *Triangular Tonnetz* [48]. In this case, Perfect Fifths are horizontal, Major Thirds are vertical toward the right, and Minor Thirds are vertical

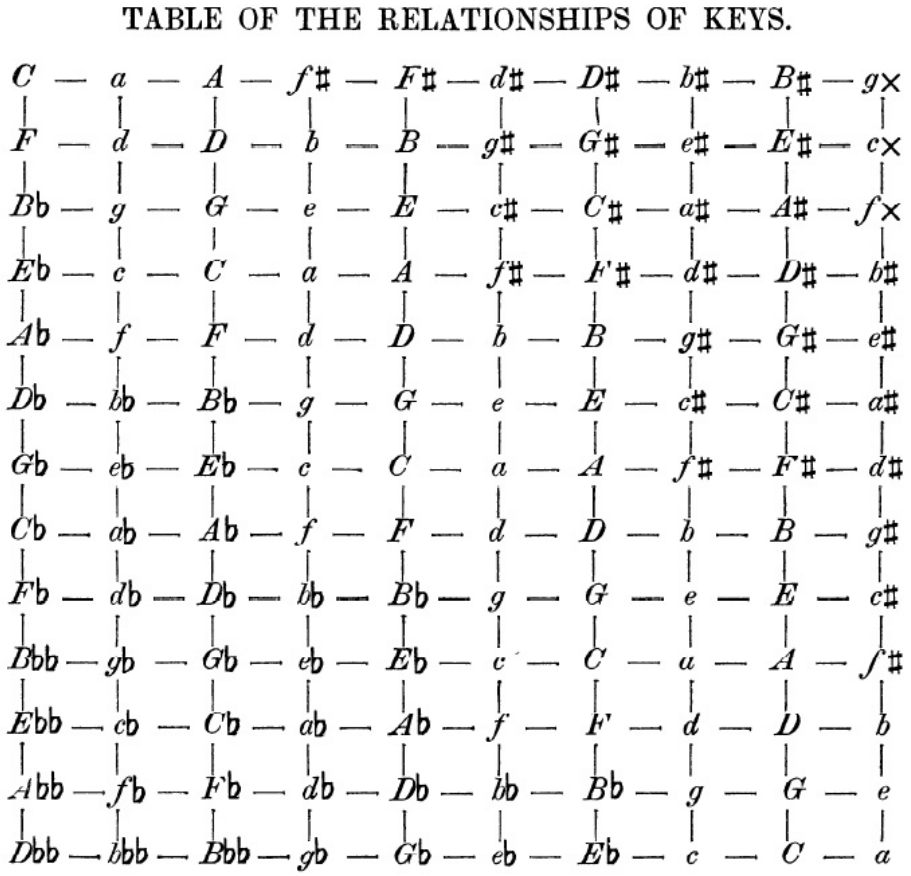


Figure 2.5: Weber's regional chart [57]

toward the left.

Shepard's "Harmonic map" (1982) and Cohn's work (1997) tried to use neo-Riemannian theory to adjust Tonnetz (table) to equal temperament [34].

2.2.2 Three-Dimensional Pitch Space

Music theory has a long history of encapsulating the concept of the circle of fifths as it shows close perceptual relationships between notes. Researchers have also explored three-dimensional structures of pitch, which can help to show the close perceptual relationships between specific pairs notes.

Drobisch (1855) was the first to propose the notion that pitch height can be represented by a helix [34].

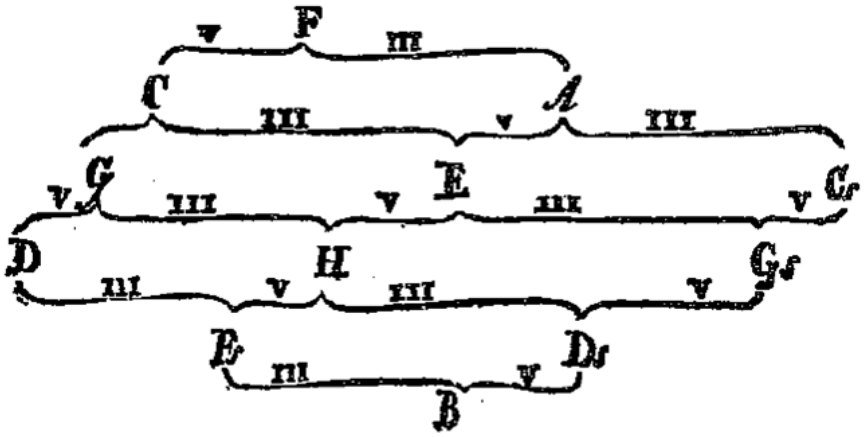


Figure 2.6: Euler's Tonnetz [14]

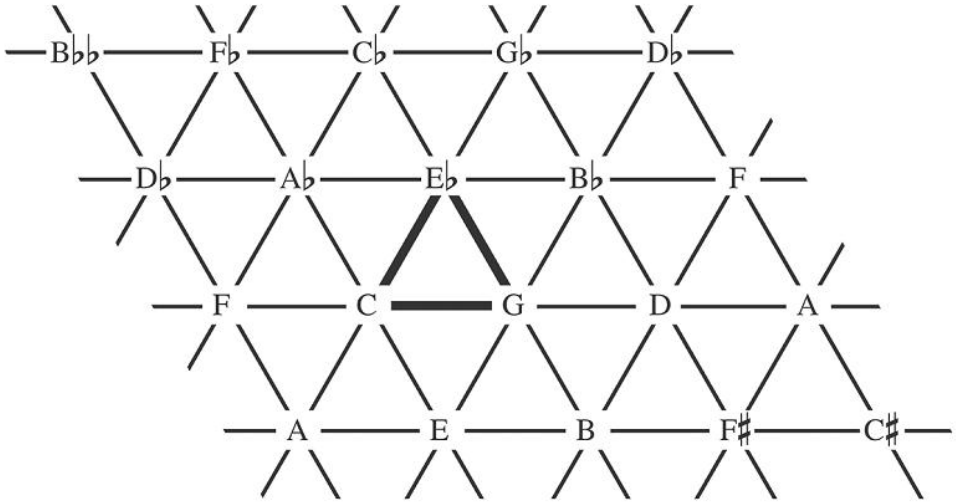


Figure 2.7: Tonnetz as regularized and extended by Riemann and others [55]

Based on Drobisch's model, Shepard (1982) introduced an equal-spaced helical model which arranges twelve chromatic pitches over a regular, symmetrical, transformation-invariant surface [33], as shown in Fig. 2.8. Shepard also extends this approach to combine the semitone and fifth cycles to yield a double helix called the *melodic map* [52].

Krumhansl (1979) used empirical data to unveil the relationships of pitch in tonality. She proposed a conical structure of pitch intervals, which corroborates the perceptual neo-Riemannian transformation and does not contradict Shepard's model [32].

After considering Shepard's and Krumhansl's models, Chew (2002) suggested an

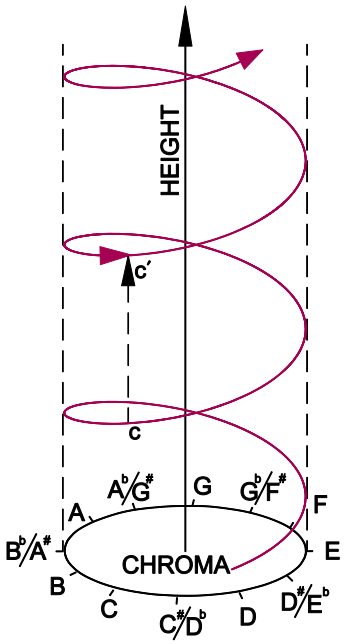


Figure 2.8: Shepard's Helical Model [52]

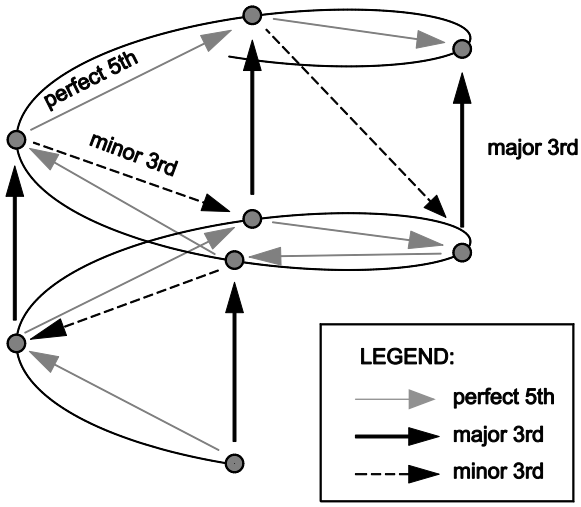


Figure 2.9: Chew's Spiral Array Model [5]

abstracted spiral array model, for mapping Tonnetz-based representations to helixes, providing an identical distance between each perfect fifth interval, each major third interval, and each minor third interval [5]. Chew's spiral array is shown in Fig. 2.9.

Both Shepard and Krumhansl's models are based on the psychological perception of pitch. Their models are abstracted structures that do not indicate the exact distance between octaves. Shepard mentions that the distance between C and C$'$ (the same note in different octave), pictured in Fig. 2.8, can either be shrunk or stretched along the vertical axis, meaning that the size and shape of the area representing each note may change.

In all three models, including Chew's model, the position of a pitch is defined by height h and radius r. The angle of the helix itself from the plane can be calculated as the ratio of these two values, h/r, and is useful when finding a musical key.

However, Shepard and Chew admit the distance between each pitch in the model does not correspond to a physical distance in terms of the appearance of a musical instrument. It must therefore be proven if the distance between pitch and the angles of each spiral shown in those models can be used to calculate the position of each key when designing the physical appearance of an instrument using these models.

These three helical models begin first with the idea of using a flat circular structure. Each model then adds an additional linear arrangement perpendicular to the circular structure to turn the model into one that exists in three-dimensional pitch space.

These theoretical representations of musical pitch can be realized into real musical instruments, and although many attempts have been made to create alternative arrangements of notes, most have not been accepted into the mainstream, and the linear arrangement of the piano remains the most popular and most familiar arrangement of notes in the 12-TET system.

Chapter 3

Mapping Musical Notes into Grids and Lattices

Musical instruments are designed to provide varied access to all pitches. The ordinary method is to provide direct access to each note in a grid or a lattice. This raises the question: is there a particular grid or lattice most suitable for performing?

Chomsky (1965) noted that many errors in grammar are obvious upon recordings of actual speech. This means the grammar that is correct may be different from what people actually use in speaking [6]. Similarly in representing musical intervals, any grid, lattice or scale can be modelled well to calculate perceptive distance, but that model may not be most suitable for performing. Although arranging keys on a musical keyboard based on a controller is a dynamic research field, designers should consider the most appropriate finger or hand position for performers to use (also known as *fingering*). In this chapter, several existing physical layouts will be discussed and explored as well as the design principles behind them.

3.1 One-Dimensional Linear Tessellation

As presented in Section 2.1.3, each key can be designed with an identical size on a musical keyboard corresponding to the chromatic scale. However, the designers of the acoustic piano did not choose to use an identical key size. Prior to discussing linear tessellation in one-dimension, the design of the acoustic piano should be reviewed.

The word "piano" is an abbreviation of *pianoforte* [13], where "piano" means

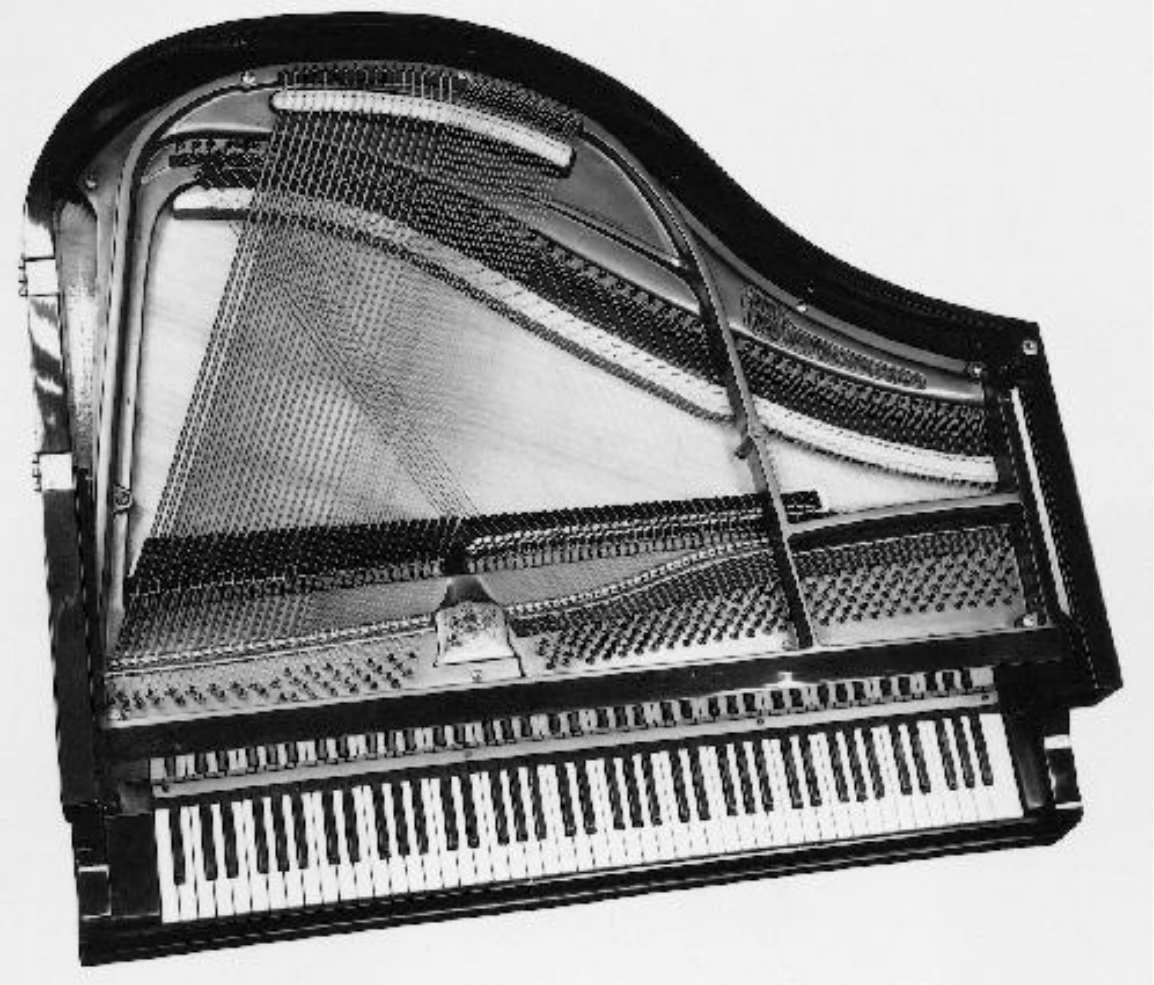


Figure 3.1: Baby grand piano (the 'Elfin') by Broadwood, London, manufactured in 1924—30 (private collection) [49]

"soft" and "forte" means "loud," referring to the variation in volume in response to a pianist's touch on the keys. A traditional piano (Fig. 3.1) has 88 keys (including 52 white and 36 black keys). It is protected by a wooden case surrounding the soundboard and metal strings. Once the key is pressed down, it will trigger the pad of the hammer inside the soundboard to strike the strings. The vibration of the string will cause frequency energy to transmit through the air. The sounds with different pitches are generated by using strings with different lengths, thicknesses, and tensions, corresponding to their resonance frequency.

The pitch arrangement of the keyboard is on a scale from low to high, tiled from left to right. Designers consider performance a priority: the identical size of the

keys(white keys are all the same width to each other and the black keys are the same width to each other) is good for fingering, despite causing a gap between perceptive distance representation and performing. Furthermore, it is easy to remember the pitch arrangement, from low to high octaves, in one direction.

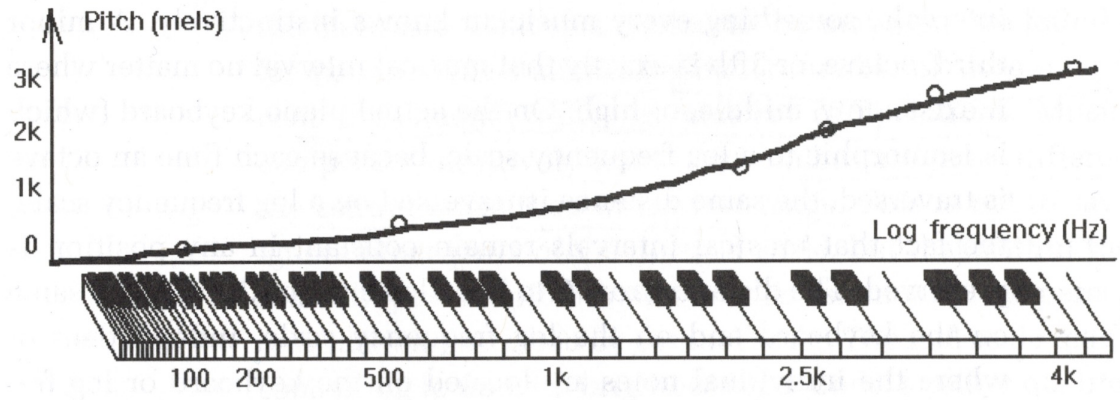


Figure 3.2: The Mel Scale and a warped keyboard depicting the scale [7]

The results of distributing the size of each key by using log-frequency rather than pitch are shown in Fig. 3.2. Once mapping a piano onto the mel scale, which shows the psychological reality of how people hear musical tones, it is obvious how the size of each key continuously increases from low to high pitch. This design also arranges pitches from low to high and left to right. However, the size of the key increasing logarithmically makes it very difficult for fingers to reach some particular keys either in the low end of the keyboard on the left or the high end of the keyboard on the right.

Based on those two examples, the principle of designing a musical keyboard is clear: consider performance where fingering is the priority. After considering fingering, designers use perceptive distance to determine the size of each key. However, researchers never give up on finding a model which can balance fingering (performance) and key size (perceptive distance). They tried to find the answer from microtonality.

3.2 Two and Three-Dimensional Tessellation

There are many designs for physical appearance using *microtonality*, which involves music having intervals smaller than a semitone. These designs map notes

from one dimension to two dimensions. Suggested by [29], the history of microtonal keyboard design can be roughly separated into four periods: before the nineteenth century, the nineteenth century, the early twentieth century, and the twentieth century and later.

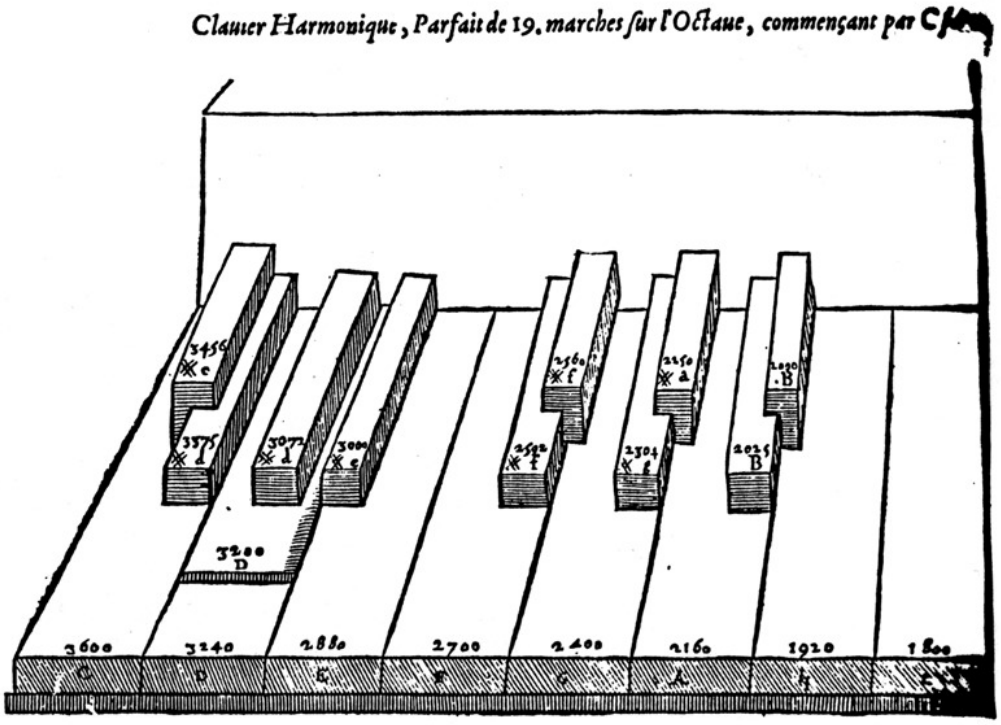


Figure 3.3: Harpsichoard built by Joan Albert Ban (1639) [29]

In the period before fourteenth century, evidence from musical and literary writings suggests that keyboards had seven white keys (coming from a diatonic scale) and five black keys (which represent a part of the keys in the chromatic scale) per octave. By the end of seventeenth century, the remaining keys in the chromatic scale were added into the design [43]. In order to avoid violating the spatial pattern of the existing keys, those newcomer keys were added in a row above the super row, as shown in Fig. 3.3. From that point up to the the end of eighteenth century, the following designs of microtonal keyboards used the same idea of adding an upper row such as Fig. 3.4. The process is suggested by [29] and named *accretion*. The process of accretion marks the tiling of the music keys in a keyboard from one dimension to two dimensions.

Figure 3.4: Archicembalo Harpsichord built by Vito Trasuntino (1606) [11]

In the nineteenth century, other two main ideas had been explored: [29] *transposition invariance* and *Bosanquet*.

Robert Bosanquet's keyboard was designed using regular cyclic temperaments in representing a microtonal scale. This idea was similar to the idea introduced in Section 2.2.1 as an instance of one-dimensional pitch space. The difference here is that Bosanquet suggests more than one cycle of fifths in a microtonal representation. The combination of cycles creates higher dimensional tessellation. Therefore, Bosanquet can also be seen as a three-dimensional keyboard design.

Transposition invariance allows moving the constructs (single note, intervals, chords, etc.) to any pitch level (where the pitch starts on) while maintaining the same spatial

Figure 3.5: Keyboard of R. Bosanquet's enharmonic harmonium (1876) [2]

relationships or fingerings between the keys [37]. In Fig. 3.5, the design complicates the appearance of the instrument, but simplifies playing using logical tiling; the entire tone rows rise upward to make sure all major scales can be fingered like a standard keyboard with the same consistency for all constructs.

In the early twentieth century, researchers focused on *quarter-tone* keyboard design. The quarter-tone keyboard uses a 24-tone equal temperament tonal system which features a microtonal scale as a subset of the twelve-tone equal temperament system [12]. One possible way is to duplicate the standard twelve-tone equal temperament system by spreading apart two music notes for making a quarter-tone. Aimed

Figure 3.6: The Quarter-tone piano designed by August Forste. Original photo taken by Bob L. Sturm. Used with permission (10 Nov 2015)

at simplifying fingering, most of the quarter-tone keyboard designs chose parallel duplicates with certain shifting degrees, even to add the third duplicate in the back as shown in Fig. 3.6.

It can be learned from this period that the idea of duplicating had been suggested for musical keyboard design realization. During the prototyping of the quarter-tone keyboard, researchers brought *just intonation matrices* into the mapping process [29]. For example the *Tonality Diamond*, as shown in Fig. 3.7, provides the frequency ratio to the reference pitch for each key [16].

From the middle of the twentieth century, there are many commercial productions based on the idea of extending the principle of accretion, transposition invariance, duplicating, and the just intonation matrix. These commercial productions emphasize the shape of the keys on the keyboard, such as Bosanquet's long, narrow, overlapping keys [29], Archifoon's shorter, rectangular keys [29], as shown in Fig. 3.9, Scalatron's oval keys [29], as shown in Fig. 3.8, and Wilson's hexagonal keys [59].

In addition to the shape of the key, another feature of design in the late twentieth century is the implementation of a programmable or re-mappable keyboard pattern or layout, providing the maximum flexibility on key configuration.

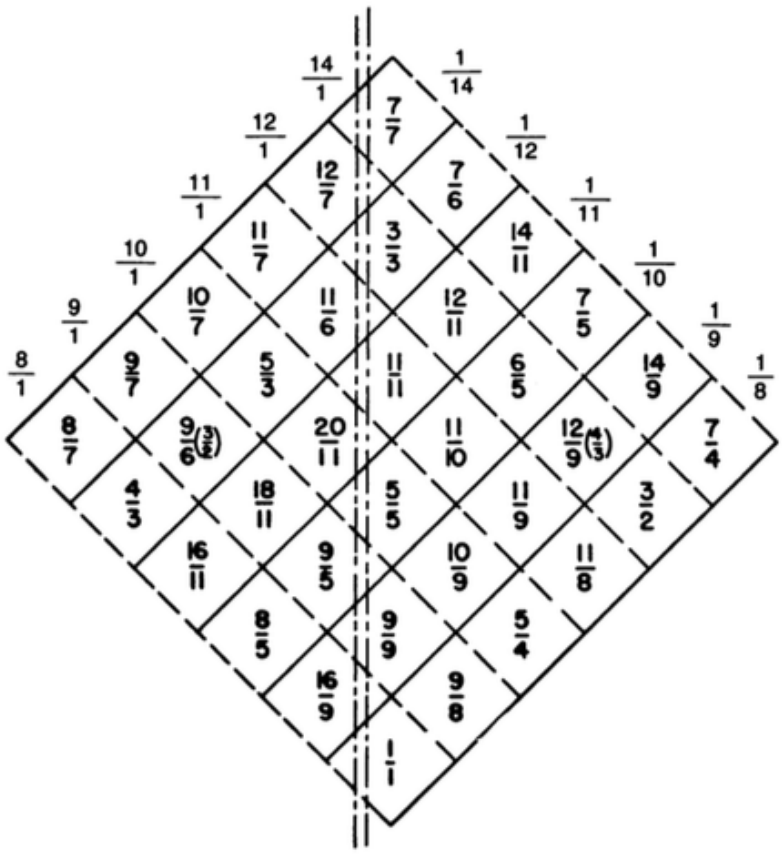


Figure 3.7: Tonality Diamond [29]

Although the number of keys in the chromatic scale is less than that in the microtonal scale, some ideas for the design referred to chromatic keyboards such as 12-TET isomorphic keyboards, which are introduced later in this thesis. Researchers in the late twentieth century predicted that in the future, three-dimensional sensors with the function of creating arbitrary key layouts will probably enable the realization of a physical appearance configured by software. Park and Gerhard released a pattern-configurable keyboard named Rainboard in 2013 [42] which use a two-dimensional matrix grid to present musical isomorphism.

3.3 Musical Mappings to Regular Polyhedra and Sphere

Theoretically, it is possible to map any number of notes onto a three dimensional structure. But an unordered or geometrically asymmetrical structure will not be

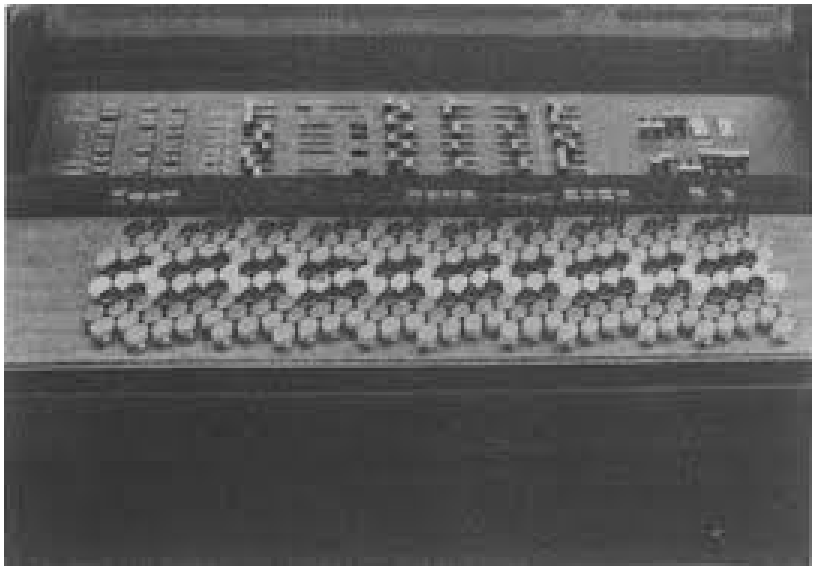


Figure 3.8: Motorola Scalatron Keyboard (1975) [29]

Figure 3.9: Archifoon's Shorter Rectangular Keys (1970) [1]

considered. Following this paragraph, both Dodecaudion [47] and Skoog [53], which are regular polyhedron based musical controllers, are introduced.

Dodecaudion, as shown in Fig. 3.10, was designed by a group from Poland. By installing sensors into a dodecahedron and cutting a hole on each face of the dodecahedron, twelve note sensors can be set correspondingly to each face. The performer can play music on the chromatic scale when his or her hands move over each hole, and can control the volume by changing the distance between the hand and the face of the dodecahedron. Which note maps to each face depends on computer configuration. The edges and vertices of the dodecahedra have the potential of playing

intervals and chord sounds.

Skoog, as shown in Fig. 3.11, was designed more as a tool rather than a professional musical instrument. However, there is some evidence showing it is possible to use this physical appearance for music education [39]. Skoog was designed based on a cube structure. It allows a player to tap, shake, squeeze, and stroke the instrument to make sound. Since it needs one face as the base, there are only five facets that can be used to map musical notes. Similar to the Dodecaudion, the edges and vertices have the potential for playing musical constructs.

From these two examples, it can be learned that by using regular polyhedron, the physical representation of each note can be shaped identically so that performers will not be confused when they play with the controller. However, from an aesthetics view, spheres may be preferred to polyhedra. The following are ways that notes can be mapped onto a sphere:

1. Leave a gap between each tiling piece so that the shape of each piece is identical, then map each note to a piece,

2. Distribute each piece equally from a pole to the other like a hesahedra, and

3. Allow a different shape or size for each piece.

Option 1 will be discussed in Section 4.4 as to which tessellation can make a full angle in a vertex. option 2 will be discussed in Section 4.4.2 as mapping musical notes onto prisms and hosohedron. For option 3, the alphasphere [44], is an example.

The Alphasphere, as shown in Fig. 3.12, uses pressure sensitive pads arranged on a spherical surface. Each round pad represents a note. The size of the pads is decreasing from the equator to the poles. The eight pads found at the equator present the diatonic scale. A chromatic scale can be played with a combination of the adjacent two lines, and the beginning pitch can be configured by a computer. Although the different sizes of pads can confuse a performer, this design provides an idea of a tiling in three-dimensional space with a single type of polygon of differing sizes.

Figure 3.10: Dodecaudion (2011) [47]

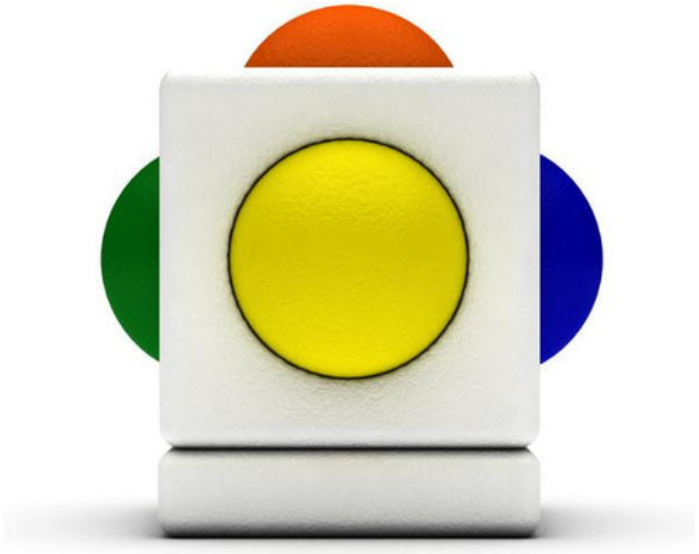


Figure 3.11: Skoog (2008) [53]

Figure 3.12: AlphaSphere (2013) [44]

Chapter 4

Isomorphism in Music

Recently, research on musical isomorphism is becoming more popular as it can be used to create interactive applications for mobile devices [19]. This chapter will answer a number of questions: What is musical isomorphism? Why do research on musical isomorphism? Where pursue the different isomorphic layouts come from? Is it possible to map musical isomorphism into three-dimensional space?

4.1 The Tiling Problem in Music Theory

The main idea of the tiling problem is comparing different mathematical partitions (sometimes called mosaics, such as cyclic, dihedral, or affine symmetry groups) on a musical construct (such as melodies, intervals, chords, etc.) to determine if a symmetry exists motivated by the resulting arrangement of musical notes. Isomorphic tessellation is a subset of the tiling problem. The tiling problem in music theory is detailed in [17], and the interested reader is referred to this paper. In this thesis, we focus only on the challenge of isomorphism in music notation.

4.2 Musical Isomorphism in Notations

The word "isomorphism" has the prefix "iso," which means "equal," and an affix term "morph," which means "shape." Isomorphism, then, refers to the property of having an identical shape or form. The concept of isomorphism applied to music notations is that for an isomorphic arrangement of notes, any musical construct (such as an interval, chord, or melody) has the same shape regardless of the root pitch of

the construct. The pattern of constructs should be consistent in the relationship of its representation, both in position and tuning. Corresponding to transposition invariance, *tuning invariance*[1] is another requirement of musical isomorphism. Most modern musical instruments (like the piano and guitar) are *not* isomorphic. The guitar in standard tuning uses Perfect Fourth intervals between strings, except for the B string which is a Major Third from the G string below it. Because of this different interval for one pair of strings, the guitar is not isomorphic.

For the chromatic scale, the number of different patterns is decided by the number of factors of twelve. Except for 1 and 12, which cannot make an appropriate pattern, the numbers 2, 3, 4, and 6 correspond to cycles of each 2 (binary), 3 (tertiary), 4 (quaternary), and 6 (senary) semitones, which make the four basic types of musical isomorphism in notation as shown in Table 4.1.

Types	A	A♯	B	C	C♯	D	D♯	E	F	F♯	G	G♯
Binary	1	2	1	2	1	2	1	2	1	2	1	2
Tertiary	1	2	3	1	2	3	1	2	3	1	2	3
Quaternary	1	2	3	4	1	2	3	4	1	2	3	4
Senary	1	2	3	4	5	6	1	2	3	4	5	6

Table 4.1: Four basic types of musical isomorphism in notations (musical note "A" used as the root).

Table 4.1 shows that each pattern type divides the chroma into several subgroups. The intervals (the distances from the beginning number to the end number) in each type of pattern are consistent beyond these subgroups. In other words, the isomorphic notation system is a pitch-proportional system which repeats patterns regularly. For example, the "binary" lies a major second apart and "tertiary" lies a minor third apart.

[1] *Tuning invariance: where all constructs must have identical geometric shape of the continuum*

4.3 Musical Note Arrangement with Isomorphism

Isomorphic instruments are musical hardware which can play the same musical patterns regardless of the starting pitch. Isomorphic arrangements of musical notes introduce a number of benefits to performers [19]. The most notable of these is that fingerings are identical in all musical keys, making learning and performing easier. Modern instruments which display isomorphism include stringed instruments such as violin, viola, cello, and string bass [42]. It should be noted in this case that although the relative position of intervals is the same for every note on the fingerboard of a violin, the relative size of each note zone may change, with the notes being smaller as you move closer to the bridge of the instrument. The traditional piano keyboard is not isomorphic since it includes seven major notes and five minor notes as a 7-5 pattern, mentioned in section 3.1.

Thanks to this 7-5 pattern design, performers can easily distinguish in-scale and out-of-scale notes by binary colours, but the performer must remember which white notes and which black notes are in the scale in which they are performing. Because the piano is not isomorphic, different fingerings and patterns are required when performers play intervals and chords in different keys. This is one of the reasons that the piano is difficult to learn: each musical construct (*e.g.* the Major scale) must be learned separately for each musical key (*e.g.* C Major, G Major, F Major *etc.*)

Aiming to find an easier way to learn fingerings on a keyboard, musicians, mathematicians, and computer scientists have used the mathematics of tonal pitch space models (Chapter 2) to explore new isomorphic tessellations of musical notes on a keyboard. In 1863, Helmholtz suggested several patterns in his book [21]. His purpose was to show the relationship between major chords and minor chords in some sequence within a particular mode.

A brief history of the microtonal keyboard is discussed in Chapter 3.2. The general principles of building a microtonal keyboard such as accretion, transposition invariance, duplicating, and the just intonation matrix are the same for building an isomorphism in two-dimensional space, while the exploration of appropriate shape and size of a physical appearance of an isomorphic keyboard is ongoing.

The first physical appearance of an isomorphic layout was decided by Hungarian pianist Paul von Jankó in 1882 [56]. The *Jankó keyboard* shown in Fig. 4.1 was

originally designed for pianists who have small hands that can cause fingering difficulties when stretching to reach the ninth interval, or even the octave, on a traditional keyboard. By setting every second key into the upper row and shaping all keys identically, the size of the keyboard in the horizontal direction shrinks by about half within one octave. After making three duplicates, the performer can play intervals or chords by putting the fingers up or down to reach the desired notes. Each vertical column of keys to the adjacent column are a semitone away, and the horizontal row of keys to the adjacent row is a whole step away. This design never became popular since performers are not convinced of the benefits of this keyboard and they would instead have to spend more time learning a new system [38].

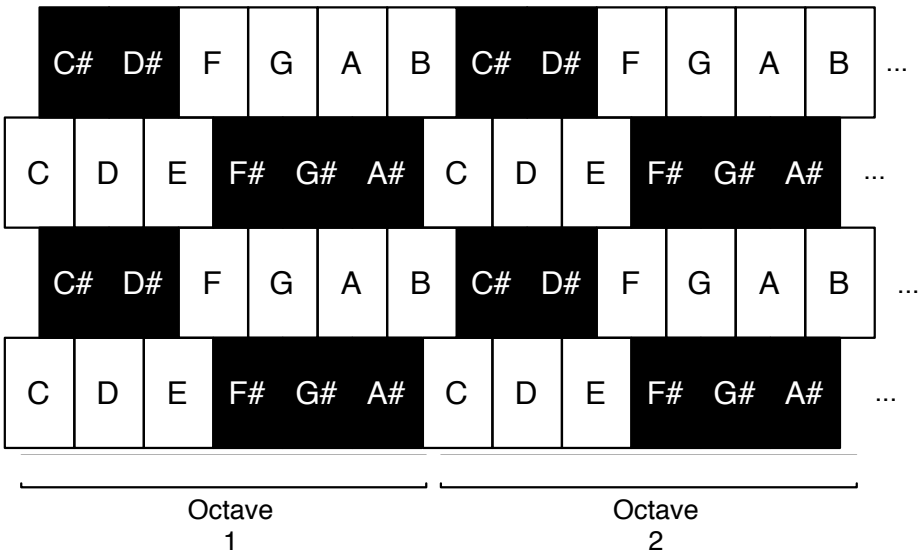


Figure 4.1: Jankó Keyboard tessellation

This arrangement of notes on the Jankó keyboard is isomorphic because a musical construct has the same shape regardless of key. Consider a Major triad (Fig. 4.2). the C-Major triad has the notes C-E-G, while the D-Major triad has the notes D-F♯-A. On the piano keyboard, these triads have different shapes, but on the Janko keyboard (and on any isomorphic keyboard) these triads have the same shape. In fact, every major triad has the same shape on an isomorphic keyboard.

In 1896, a Swiss inventor named Kaspar Wicki applied for a patent for a new layout which adds more keys in the keyboard part of a bandoneon. This layout was independently refined and patented again in 1986 by a concertina player named

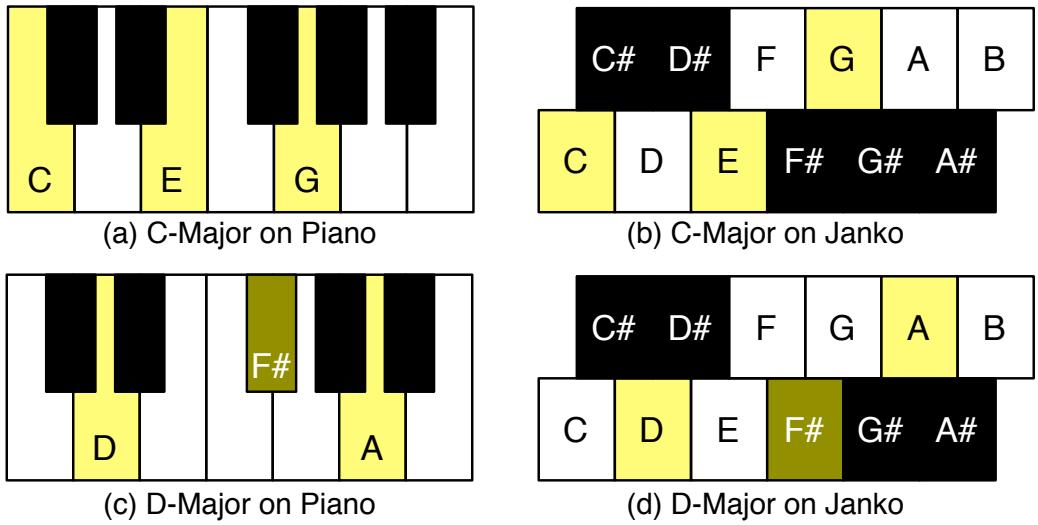


Figure 4.2: Isomorphism in the Jankó keyboard as compared to polymorphism in the piano keyboard.

Hayden in his *Hayden System Duet Concertina* [18].

This layout is the first isomorphic layout which represents each note by using a regular hexagonal grid and is given a name as a combination of Wicki's and Hayden's patent: the *Wicki-Hayden* layout, as shown in Fig. 4.3. The main feature of this layout is that, without losing musical isomorphism, the notes of the major scale are grouped together, and it is easy to switch to a different key by changing to a different column of notes. The disadvantage of this layout is that the order of the keys in this layout is not chromatic, and semitones are far apart, which brings difficulty in reading and remembering for a new learner. However, since it places all the notes of the diatonic scale under the finger and does not require moving the hand to reach notes which are musically far away, it has become popular with small accordions [58].

Many other researchers have tried to make instruments that obey some form of isomorphism. In 1974, after reviewing Euler's Tonnetz, Wilson tried to find an appropriate lattice to map this harmonic table onto a microtonal matrix [59]. Wesley introduced the array Mbira in 2001 [26]. The Harpejji [35] was introduced as an isomorphic string system, which was inspired by the Jankó keyboard, in 2007.

Maupin, Park, and Gerhard introduced a configurable mobile app for presenting all possible isomorphic layouts in 2009 [36]. They consider both rectangular and regular hexagonal grids. Since they proved it is possible to use the note in a 45°

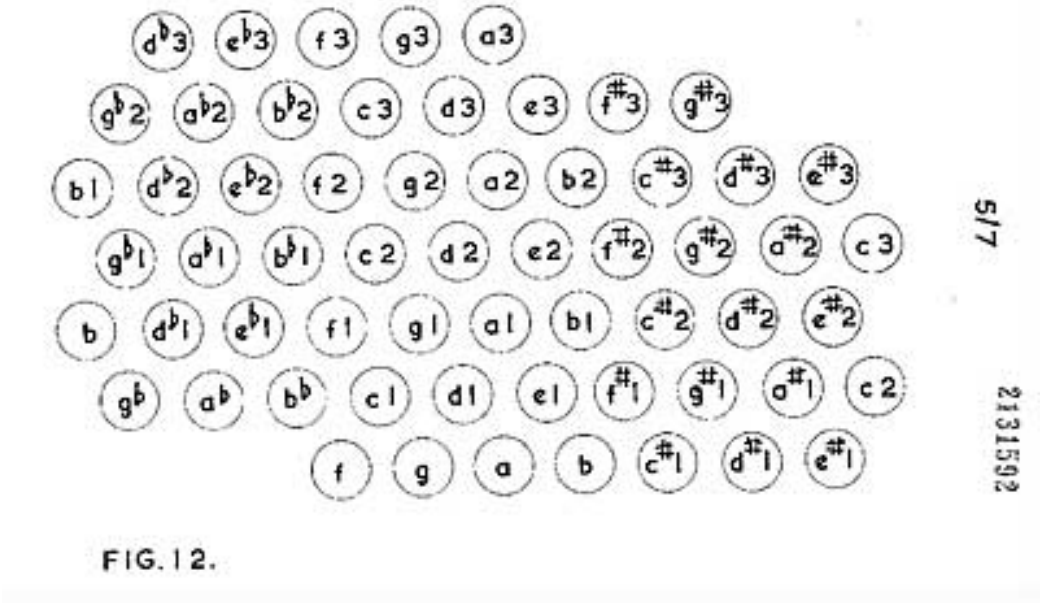


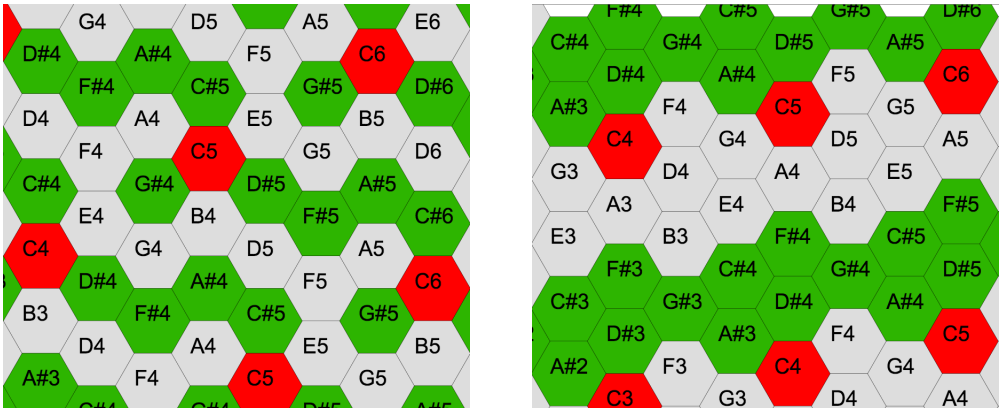
Figure 4.3: Fig-12 from Hayden's Patent(1986) [18]

position for presenting the half step between the adjacent, vertical, and horizontal notes, they suggested representing all isomorphic layouts by using a hexagonal grid rather than a rectangular grid.

4.3.1 Notation for Defining Isomorphisms

An isomorphism can be arranged in a hexagonal grid or a rectangular grid. For the remainder of this discussion, a hexagonal grid will be considered, but the presentation could be equally considered with a rectangular grid.

Considering the examples presented above, there are many different isomorphisms that can be created. Each different isomorphism is defined by the intervals that are associated with adjacent notes. In the Jankó keyboard layout, adjacent intervals are a whole tone in the horizontal direction, and a semitone above and to the left or right. In the Wicki-Hayden layout, the adjacent intervals are whole tones in the horizontal direction, Perfect Fifths up and to the right, and Perfect Fourths up and to the left. These three intervals form a triangle, and if you follow the intervals around the triangle, you arrive back at the original note, for example, a Perfect Fifth minus a Perfect Fourth gives a whole tone.



(a) Gerhard Layout (b) Park Layout

Figure 4.4: Typical isomorphic layouts. Root note (C) is marked in red, and notes that would normally be black on a piano keyboard are marked in green.

Any isomorphism can be created using any set of three intervals, as long as one of the three intervals can be produced as a difference of the other two. Hayden developed a theory for describing any isomorphism using three intervals. By considering Hayden's original notation on the hexagonal grid, Park and Gerhard suggested a "unified isomorphic layout *UIL*" notation to distinguish different isomorphic layouts, in which "G" represents the greatest positive interval value of the three, "L" is the least positive interval value, and "D" is the difference between "G" and "L." These three are the same as in Hayden's GLD notation, but Park and Gerhard extended the theory to include "R", representing a clockwise rotation of the layout, "M" indicating if the layout is mirrored or not, "S" indicating the amount of shear, and "T" representing the number of tones in the scale which allows the theory to be extended to microtonal applications [41]. Based on this structure, some typical isomorphic layouts are marked with this notation in Table 4.2.

In 2011, by using this UIL format as shown in table 4.3, they re-introduced two new interesting musical isomorphic layouts named the *Gerhard layout*, as shown in Fig. 4.4a, and the *Park layout*, as shown in Fig. 4.4b, respectively, which are considered useful for performance. The main advantage of the Gerhard layout is the arrangement lends itself very well to major and minor triads, but the disadvantage is that adding octaves to the chord is difficult. For the Park layout, the major and minor triads are easy to play as well, but the dominant seventh is hard to reach.

Layout Name	L	G	D	R	M
Wicki-Hayden	2	7	5	30	True
Harmonic Table	3	7	4	0	False
Janko	1	2	1	90	False
Bajan	1	3	2	90	True
B-System	1	3	2	270	False
C-System	1	3	2	270	True

Table 4.2: Typical isomorphic layouts by using UIL notation

Layout Name	UIL Format	L	G	D	R	M
Gerhard	1,4,3;R60	1	4	3	60	False
Park	2,5,3;R90M	2	5	3	90	True

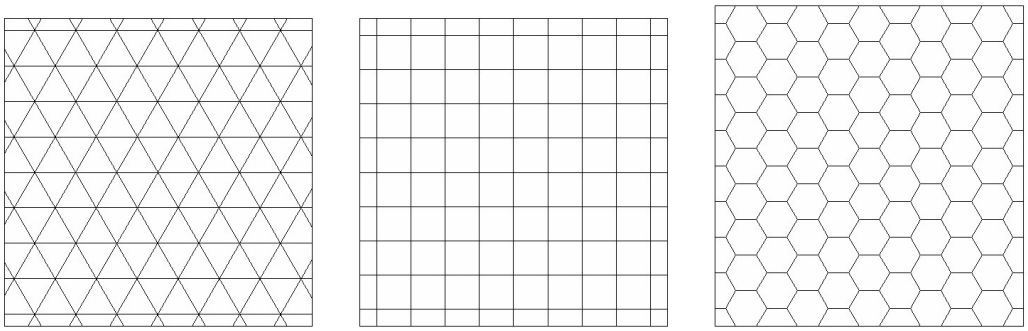
Table 4.3: Gerhard layout and Park layout by using UIL notation

4.4 Alternative Lattices and Three-Dimensional Arrangements

Recently, Ragzpole, a cylindrical isomorphic musical controller, was introduced as a system mapping musical isomorphism into a higher dimension than two-dimensions. As mentioned in Section 3.3, mapping tones into higher dimensions is not new research. However, at first, analysis seems impossible to map tones into higher dimensions while keeping strict musical isomorphism as defined in [41].

There are three regular tessellation on the plane: equilateral triangular grid, the square grid, and the hexagonal grid, as in Fig. 4.5a, Fig. 4.5b, and Fig. 4.5c. At each crossing point of the grid lines, the tiles make a full 360° angle, which means there is no way to make strict three-dimensional isomorphism by using the flat isomorphic grids directly except by folding. However, this does not affect mapping notes to three-dimensional space so as to make general music controllers for performance like the examples in Section 3.3.

The next section explores the benefits of regular and semi-regular polyhedra, including prisms and their transformation, and the Hosohedron for mapping notes. The



(a) Triangular grid (b) Square grid (c) Hexagonal grid

Figure 4.5: Three regular tessellation on the plane

following section then suggests the possibilities of pseudo-isomorphism and quasi-isomorphism.

4.4.1 Regular Polyhedron and Non-Isomorphism

There are exactly five regular polyhedra [8]. They are shown in Table 4.4.

Polyhedron	Vertices	Edges	Faces	Edges per face	Edges per vertex
Tetrahedron	4	6	4	3	3
Cube	8	12	6	4	3
Octahedron	6	12	8	3	4
Dodecahedron	20	30	12	5	3
Icosahedron	12	30	20	3	5

Table 4.4: Five regular polyhedron

From the Table 4.4, it is clear that four out of the five regular polyhedron have the number "12" in either vertices, edges, or faces except for the tetrahedron. In fact, there are two pairs of duals: the Cube with Octahedron, and the Dodecahedron with Icosahedron. Starting from either proposition of a pair, such as the position of vertices or position of faces, others can be inferred by interchanging the corresponding parts such as the position of faces or the position of vertices.

It is also easy to see that, including the tetrahedron, the regular polyhedra all have the number "3" in either "edges per face" or "edges per vertex. " The special

number "12" allows particular regular polyhedra to map the chromatic scale on it, which corresponds to 12-TET. The number "3" allows either face or vertex to play triads. Since an edge always joins two faces of polyhedron, the edge can be used for interval representation. This is one of the motivations for the development of the Dodecaudion and Skoog mentioned in Section 3.3.

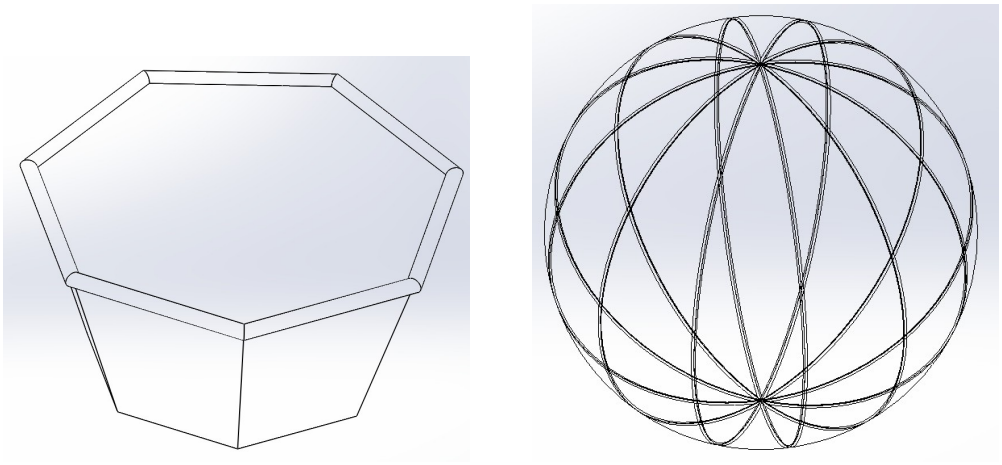
However, a regular polyhedron can be split along the edge to obtain a two-dimensional tessellation. It is easy to find the tessellations of regular polyhedron that are not isomorphic. Therefore, on either Dudecaudion or Skoog, it is possible to map each note of the chromatic scale, but it is impossible to use these notes to play all triads in the same shape. For example, in the tessellation of Dodecahedron, if the note C is assigned into the middle pentagon, with the other four adjacent pentagons tiled notes being E, G, F, A, regardless of what note the fifth adjacent pentagon tiles, the player can play C-E-G or F-A-C, using a single vertex, but cannot play G♯-C-D♯. This arrangement is therefore not isomorphic.

4.4.2 Prisms and Hosohedron

For semi-regular polyhedron, the prism as shown in Fig. 4.6a is a very special one. Since all the side faces of a prism are identical, it is possible to map notes on each side face, without considering the top and bottom faces. Then, the number of side faces corresponding to the number of notes to map onto can be configured. The joint edge of two side faces can be used to represent musical intervals. A prism is the simplest model of a pseudo three-dimensional concept for mapping music notes. If, however, each side of the prism is allowed to represent multiple notes, or even a portion of an existing isomorphic layout, then the entire structure could be extended to become isomorphic. This concept is extended in the design presented in Section 6.1.

The Hosohedron, as shown in Fig. 4.6b, is a sphere and also a variant of the prism. By extending the joint edges of the side faces, the top and bottom faces of the prism shrink to two poles, so that a prism transforms into a sphere. As a prism, each piece of the side faces has to perceptually-balance[2] because of the identical shape. Hosohedron provides a basic idea of the potential for a sphere-like three-dimensional musical controller design.

[2]Perceptually-balance: a unit step anywhere in the shape scale produces a perceptually-uniform difference in shape



(a) Prism (with 7 side faces) (b) Hosohedron (with 12 pieces)

Figure 4.6

These different 3-dimensional pseudo-isomorphic arrangements of control surfaces provide motivation for discovering a true isomorphic 3-dimensional representation. Considering that popular isomorphisms rely on a hexagonal grid, are there three-dimensional structures built from hexagonal grids. Might it be possible to apply isomorphisms to these 3-dimensional hexagonal structures? In the next chapter, we explore the mathematical foundations of cylindrical hexagonal lattices in preparation for combining these structures with existing isomorphisms.

Chapter 5

Cylindrical Hexagonal Lattices

A cylindrical hexagonal lattice structure is introduced in this chapter which has been extensively studied in the context of carbon nanotubes. The related mathematical background will be given for presenting the tube-like structure and calculating the chiral angle. In the following chapter, this model will be applied to the tonal pitch model in three-dimensional space and the principle of isomorphic keyboard design.

It may seem disjunct to switch from considering music theory to considering organic chemistry, but the mathematical foundations of cylindrical hexagonal lattices are primarily found in the study of Fullerene and carbon nanotubes. We will borrow this mathematical foundation and apply it to the construction of new helical tonal pitch models and musical isomorphisms.

5.1 Fullerene Structure

Carbon Nanotubes and Fullerene are both members of a family of organic molecules consisting of carbon atoms arranged in a regular hexagonal grid. The first Fullerene was discovered in 1985 by Smalley *et al.* at Rice University. This spherical molecule was nicknamed "buckminsterfullerene" after the architect Buckminster Fuller because it resembled the geodesic domes Fuller was known for. This form of Fullerene has 60 carbon molecules in a spherical lattice [3]. The research on Fullerene structure typically focuses on its physical stability, and materials made with carbon nanotubes (a related Fullerene) have been shown to have very high tensile strength.

A Fullerene is a molecule of carbon which has a spherical, ellipsoid, tube-like, or any other shaped form [50]. After cylindrical Fullerene, carbon nanotubes (CNTs)

were discovered from observations of formations of Fullerenes, the mathematical topology of carbon nanotubes became a subject of scrutiny in mathematical chemistry research [51] and [46].

5.2 Carbon Nanotube Structure and Cylindrical Hexagonal Lattices

In its simplest form, the carbon nanotube structure is a cylindrical lattice of hexagons. This structure can be generated by curling a flat hexagonal sheet until the edges of the sheet join together. Because of the different symmetries of a hexagonal lattice, there are three different ways to curl a flat sheet of hexagons into a tube, as shown in Fig. 5.1. Either the zig-zag edges are cut and curled, in which case the end of the tube looks like an "armchair" pattern; or the armchair edges are cut and curled, in which case the end of the tube looks like a "zig-zag", or the sheet is cut along an irregular edge, in which case the tube is called "chiral" and there are many variants of this type of tube depending on the angle of the irregular edge.

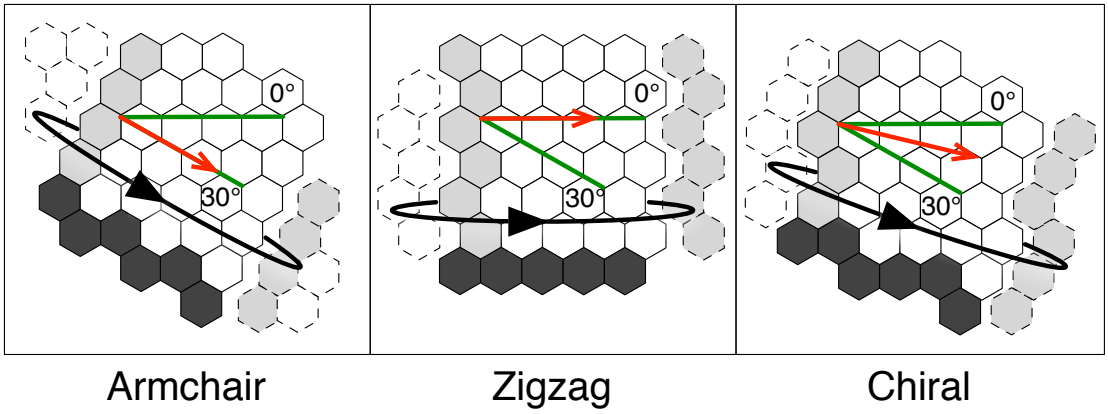


Figure 5.1: The three types of hexagon lattice cuttings. Dark grey indicates the "end" of the resulting tube, and light grey indicates the "seam" of the tube. Two green rays and a red arrow indicates the chiral angle.

Once cut and curled, these three types of chiral angles produce three types of cylindrical hexagonal tubes, shown in the side view in Fig. 5.2. A cylindrical hexagonal lattice (n, m) can be defined, where $n \geq m$. Each such lattice is associated with

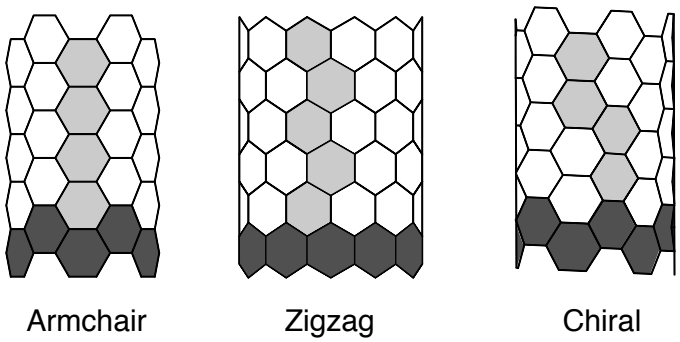


Figure 5.2: Three types of cylindrical hexagonal tubes, generated by cutting the planar hexagonal lattice as in Fig. 5.1

a chiral vector, showing the direction and length of repetition of the cutting of the hexagonal sheet. The definition of the chiral vector is:

$$\overrightarrow{Ch} = n\overrightarrow{a1} + m\overrightarrow{a2}, \tag{5.1}$$

where $\overrightarrow{a1}$ and $\overrightarrow{a2}$ are two vectors within 60° on the grid.

Figure 5.3, shows that $\overrightarrow{a1}$ and $\overrightarrow{a2}$ can be expressed in Cartesian coordinates (x, y) as

$$\overrightarrow{a1} = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) a \tag{5.2}$$

and

$$\overrightarrow{a2} = \left(\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) a, \tag{5.3}$$

where a is a the length between two vertices in a hexagon.

Each intersection point on a two-dimensional hexagonal grid can be represented by using these two vectors ($\overrightarrow{a1}$ and $\overrightarrow{a2}$). When choosing an origin, the other points are labelled with hexagonal coordinates (n, m). In Fig. 5.3, these points are vertices of the lattice, but this vector representation is not limited to such vertices; it can be any point inside the hexagon or on the boundary.

It is also possible to group these into these three types by distinguishing the chiral angle, Θ, as the angle between the chiral vector and the zigzag direction, as shown in Fig. 5.3:

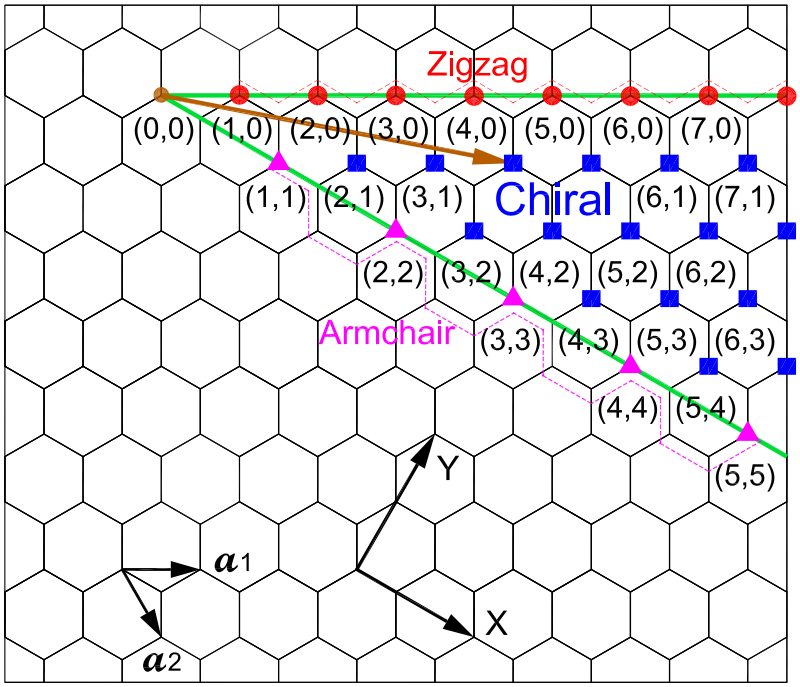


Figure 5.3: Three types of chiral angle given by hexagonal coordinates

$$\Theta = \tan^{-1}\left[\frac{\sqrt{3}m}{m+2n}\right] \tag{5.4}$$

By using different values for n and m in Equation (5.4), we can again see the three types of tubes:

Armchair $(m = n)$: $\Theta = \tan^{-1}\left[\frac{1}{\sqrt{3}}\right] = 30°$, the trace shown by the purple dash line with purple triangles in Fig. 5.3.

Zigzag $(m = 0)$: $\Theta = \tan^{-1}[0] = 0°$, the trace shown by the red dash line with red dots in Fig. 5.3.

Other chiral tubes (called "Chiral"): $0° < \Theta < 30°$, the area between the zigzag and armchair angles in Fig. 5.3.

In the next chapter, we again consider hexagonal isomorphisms and find similarities between the concept of the chiral angle and the direction of increasing pitch in an isomorphism. Using this similarity, and adhering to certain constraints, it is possible

to map an isomorphism onto a hexagonal cylindrical lattice and maintain the harmonic relationships between pitches. This will be the first and primary contribution of the thesis.

Chapter 6

Mapping Isomorphic Layouts onto Cylindrical Hexagonal Lattices and the Implementation of Helix Models

The previous two chapters presented two very different subjects - that of musical isomorphisms and fullerene structures. This chapter combines these ideas into the presentation of the main contribution of this thesis: the mathematical mapping of a musical isomorphism onto a fullerene structure. This chapter will present the mapping model, and a series of examples, and will conclude by applying the model to both the Shepard helical tone model and the Chew spiral tone model.

6.1 Isomorphic Layouts and Cylindrical Hexagonal Lattices

To successfully wrap a flat isomorphic cutting sample into a tube, the intervals must be preserved such as in Fig. 6.1. This puts a strict constraint on the way that isomorphisms can be wrapped: *the circumference of the tube must be in a direction in which the repeated notes are found on the original flat layout.* If we proceed around the circumference of the tube, we must eventually arrive back where we started. On an isomorphic layout, this means that the relationships between notes must be arranged such that if we proceed from note to note around the cylinder, we must be able to arrive back at the original note without losing the isomorphic characteristics of the layout.

Conveniently, the *GLD* notation of isomorphisms provides such a direction. In [41], the *isotone axis* is defined as a line which contains all the instances of a particular

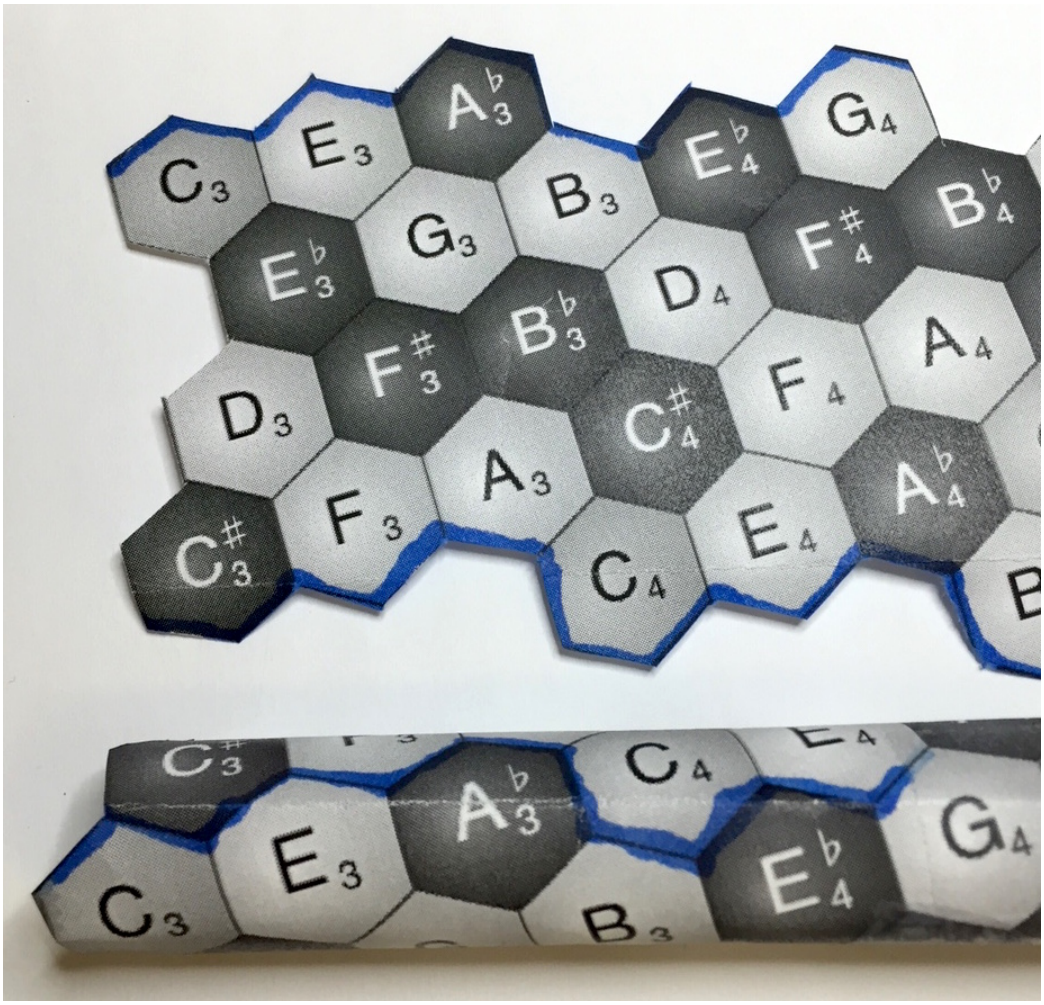


Figure 6.1: By curling a planar hexagonal lattice in a specific direction along the edges of the hexagons, the resulting sheet becomes a cylinder

note in an isomorphic layout. The *pitch axis* is a line orthogonal to the isotone axis, and is the direction in which pitch increases in the smallest degree (by semitones, for 12-TET). Fig. 6.2 through 6.7 show examples of some of the more common hexagonal isomorphisms, with their pitch axis indicated by a green arrow and their isotone axis indicated by a dashed green line. The "zigzag" direction of the hexagonal grid is shown as a blue line.

By choosing a chiral vector in hexagonal coordinates (n, m) to be equal to the isotone axis in the *GLD* notation of musical isomorphism, with an appropriately chosen chiral vector length, any isomorphic layout can be mapped from a two-dimensional

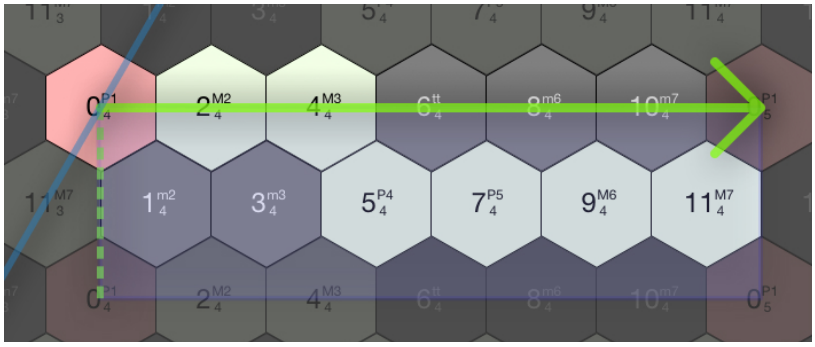


Figure 6.2: Jankó (2,1)

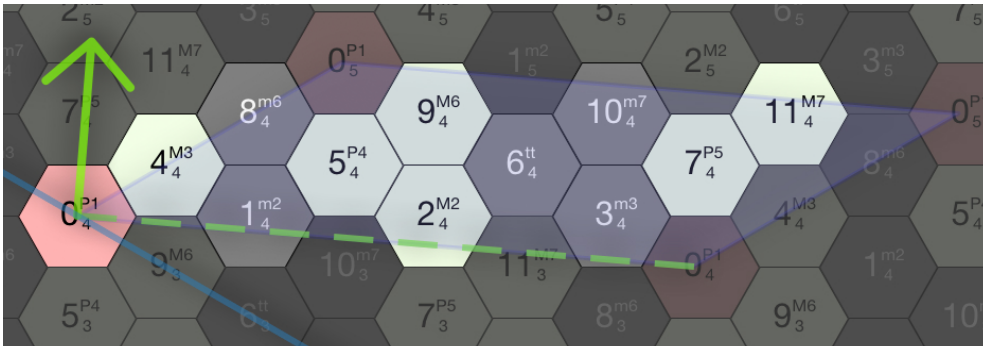


Figure 6.3: Harmonic (4,3)

grid into a three-dimensional cylindrical hexagonal lattice. This result means that if an isomorphism exists on a two-dimensional plane, a corresponding cylindrical isomorphism can be found, *for any such isomorphism*.

6.1.1 Mapping Isotone Axis Into Chiral Vector Direction

In *GLD* notation, either the isotone axis range or pitch axis range can be transposed by using rotation and reflection. However, since the hexagon is a member of the dihedral group (a mathematically defined set of symmetries of a regular polyhedron, which include reflection and rotation), it is possible to focus on the area in hexagonal coordinates (n, m) with Θ (the chiral angle) as $0° \leq \Theta \leq 30°$. Besides, either D, $-L$ directions or $-D$, L directions, there has to be a 60 degree opening which is the same as the $\overrightarrow{a1}, \overrightarrow{a2}$ vectors. Therefore, the isotone axis can be set in each isomorphic layout equal to a chiral vector direction by mapping D and -L into $\overrightarrow{a1}$ and $\overrightarrow{a2}$ directions, respectively, after having applied an appropriate mirroring or rotation.

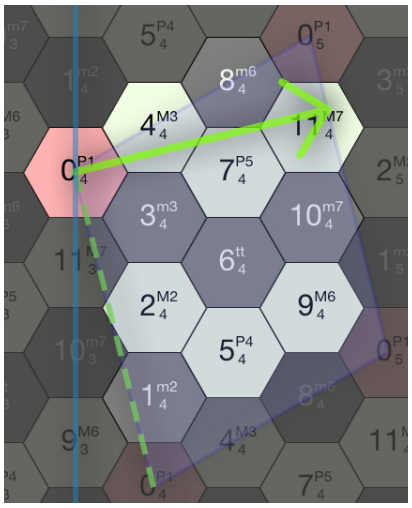


Figure 6.4: Gerhard (3,1)

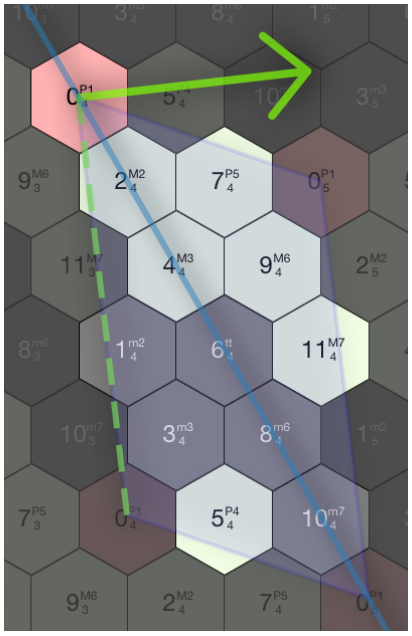


Figure 6.5: Park (3,2)

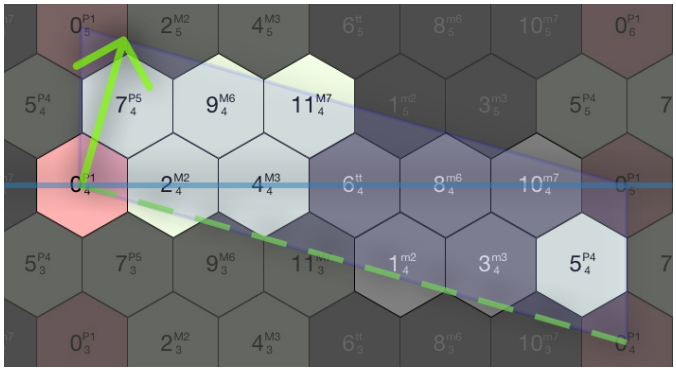


Figure 6.6: Wicki-Hayden (5,2)

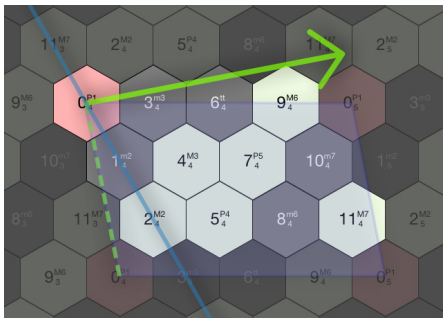


Figure 6.7: Bajan (2,1)

A new notation (D, L) can now be defined to fully represent the isomorphic cylinder corresponding to the isotone axis range in the LGD notation. Correspondingly, the vector perpendicular to the chiral vector, which is called the *translation vector*, goes in the same direction as the pitch axis and represents the direction of the axis of the resulting cylinder.

A subset of an isomorphic layout consisting of a single copy of each note from a single octave (12 notes for 12-TET, but this could be extended to microtonal systems) can be considered. This sample "patch" of notes represents the smallest unit that can be considered when curling such an isomorph into a tube. Along the isotone axis, these patches repeat identically, and represent a further constraint – each tube must have around its circumference a whole number of copies of this patch.

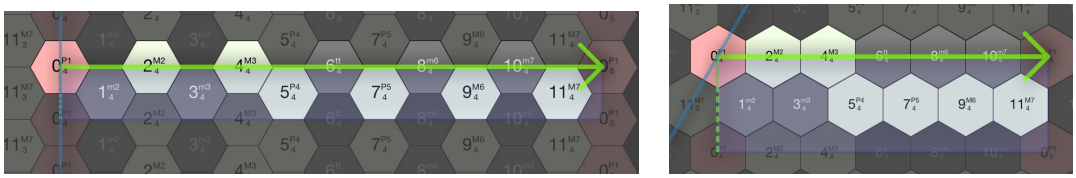
Considering Figs. 6.2–6.7 again, the blue line representing the zigzag direction serves as a reference for determining the chiral vector, which is the angle between the zigzag direction (blue line) and the isotone axis (dashed green line). This means that each layout maps to a cylindrical hexagonal lattice structure with a specific chiral vector. The resulting chiral angles of these common isomorphic layouts (in degrees to two significant digits) is calculated using equation (6.6) and are shown in Table 6.1.

Layout	(D, L)	Chiral angle
Jankó	(1,1)	30.00°
Harmonic Table	(4,3)	25.29°
Gerhard	(3,1)	13.90°
Park	(3,2)	23.41°
Wicki-Hayden	(5,2)	16.10°
Bajan	(2,1)	19.10°

Table 6.1: Chiral vectors for typical isomorphic layouts

6.1.2 Special Edge Cases

There are two special cases of isomorphic layouts mentioned in [41]. The first one is where L= 0, which only happens for intervals 0, 1, 1 in GLD notation. This case results in a **Zigzag** type lattice (1,0). The second case is where D=L, which happens



(a) Zigzag (1,0) (b) Armchair (1,1)

Figure 6.8: Two special cases exist in the lattices.

for intervals of 1, 2, 1 in *GLD* notation. This case makes the **Armchair** type lattice (1,1). The samples of those two special cases are shown in Fig. 6.8.

6.2 Implementing Spiral Tonal Pitch Space Models

The tonal pitch space models presented in Section 2.2.2 represent attempts to show how pitch is related not just linearly and harmonically, but in repeated cycles octave by octave. The prevaling 3-dimesnional representation of a tonal pitch space is a spiral or helix model, and the two prominent models in the literature are Shepard's tone spiral and Chew's tone spiral. In this section, the hexagonal lattice representation of isomorphisms will be shown to also represent both of these models (inasmuch as they are self-consistent). Further, the cylindrical hexagonal model can be used to represent any other helical tonal pitch space model.

6.2.1 Shepard's Model

For Shepard's model, the pitch increases by semitones around the spiral, with a complete turn around the cylinder corresponding to an octave increase in pitch. This means that starting at any note, the next note along the axis of the cylinder is an octave different in pitch, and that note can also be reached by moving to adjacent hexagons laterally around the cylinder. In order to advance an octave in the $\overrightarrow{a1}$ direction, twelve semitones (hexagons) around the tube in $\overrightarrow{a2}$ direction must be passed. The chiral angle in this case is:

$$\Theta = \tan^{-1}\left[\frac{\sqrt{3m}}{m+2n}\right] = \tan^{-1}\left[\frac{\sqrt{3\cdot 12}}{12+2}\right] = 23.2^{\circ} \tag{6.5}$$

The hexagon lattice cutting for implementation of Shepard's model and the resulting chiral tube are shown in Fig. 6.9. Shepard's original model also allows for a differential stretching or shrinking of the vertical extent of an octave of the helix relative to its diameter. In the cylindrical hexagonal implementation of this model, this differential may be accomplished by allowing duplicates of the cutting, resulting in a larger-diameter tube.

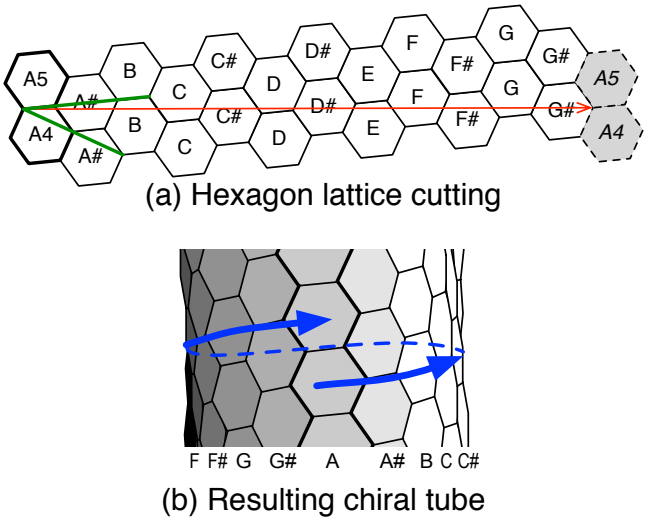
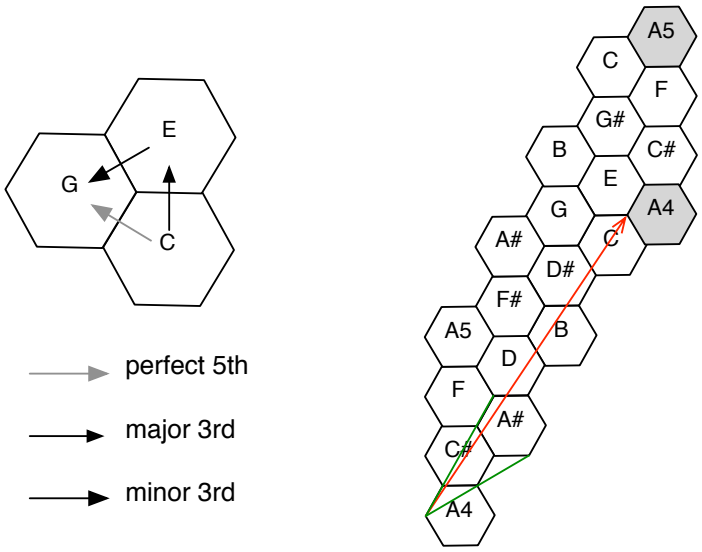


Figure 6.9: Chiral tube version of Shepard's helix model.

6.2.2 Chew's Model

Chew's spiral array model proposed using a Major Third in the vertical direction and a Perfect Fifth around the spiral direction, as shown in Fig. 2.9. This model can be mapped onto the hexagonal cylindrical model as shown in Fig. 6.10. This mapping, however, is evidence that Chew's model is not internally consistent. Using the intervals as proposed, the isotone vector (indicated as a red arrow) is not horizontal. A horizontal isotone axis is required in order to produce a cylindrical hexagonal isomorph, because proceeding note to note around the circumference of the cylinder must bring you back to the original note. If we allow the model to have different space between the notes, then a physically implementable arrangement may be produced, but this modified arrangement would not be isomorphic, nor would it be guaranteed to be self-consistent.

There are four points arranged in Chew's model which are around a circle, each 90° apart, providing the same exact distance between all notes. However, this model fails to deliver on note adjacency within a musical scale. For example, C_1 is set as the beginning note of the spiral, then the spiral will traverse G_1 - D_2 - A_2 and E_3 to complete the circle. This means Chew's model passes over note D_1, and it will never be reached. Although Chew indicates that in the vertical direction the pitch distance is a major 3rd (4 steps in the chromatic scale), it is actually at the distance of two octaves and a major 3rd (27 steps). Therefore, it can be concluded that without using duplicates, Chew's model is not internally consistent.



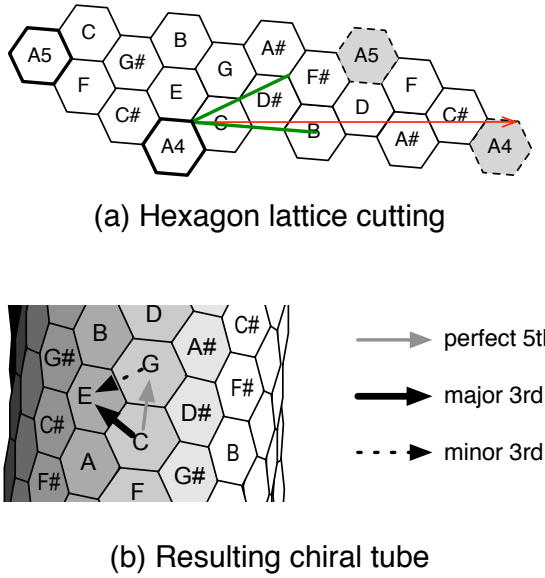
(a) Chew's model on a hexgonal lattice

(b) Cutting required to implement Chew's model

Figure 6.10: Chew's original model cannot be implemented with fixed note size. The chiral angle (isotone axis) is not horizontal, and therefore the cutting cannot be made into a self-consistent tube.

An alternative or modified version of Chew's model is hereby proposed to implement similar pitch relationships in a self-consistent isomorphic model. This modified Chew model is shown in Fig. 6.11. After rotating and mirroring the original model, the modified version places Major Thirds are along the spiral, with Perfect Fifths in the vertical direction. Recall that in the original Chew model, Perfect Fifths were along the spiral, and Major Thirds were in the vertical direction. In this way, the

horizontal chiral angle is satisfied, and a self-consistent model can be produced.



(a) Hexagon lattice cutting

(b) Resulting chiral tube

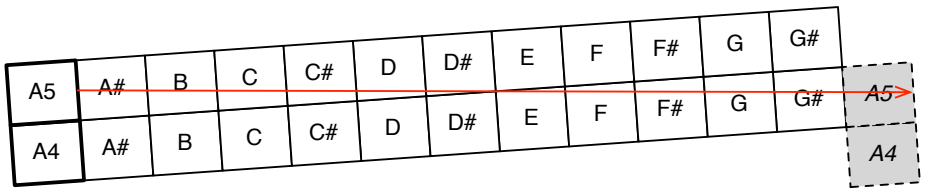
Figure 6.11: Modified Chew tone spiral, and the resulting chiral tube

6.3 Spiral and Helical Pitch Models Using Rectangular Lattices

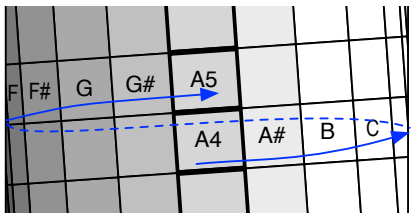
Although hexagonal isomorphisms are the most common, square and rectangular lattices can also be used to form isomorphisms, and these square models are realized in stringed instruments like violin and bass. These rectangular isomorphisms can also be wrapped into cylinders creating tone spirals using similar models.

The chiral angle of cylindrical rectangular lattices can be calculated by using the angle between the isotone and the lattice direction. Although many rectangular isomorphic layouts are degenerate (do not contain all of the notes in the scale) [36], the rectangular isomorphism corresponding to an octave in one direction and a semitone in the other direction is not degenerate. This means that a cylindrical rectangular lattice can also be used to implement Shepard's model. The chiral angle for Shepard's model in a rectangular isomorphism is determined by the angle between twelve steps in the horizontal direction and one step in the vertical direction:

$$\Theta = \tan^{-1}\left[\frac{1}{12}\right] = 4.8^{\circ} \tag{6.6}$$



(a) Rectangle lattice cutting



(b) Resulting tube

Figure 6.12: Rectangular tube version of Shepard's model.

And the resulting model is shown in Fig. 6.12. Chew's spiral model cannot be implemented using adjacent cylindrical rectangular lattices since the corresponding flat isomorphic layout for major third, perfect fifth (vertical: +4, horizontal +7) is a degenerate one. This result further reinforces the result in Section 1.1 about how Chew's spiral model is not self-consistent. However, Chew's model can be implemented with a cylindrical square lattice by skipping adjacent notes between octaves, by adding more duplicates, or by leaving space between notes.

Chapter 7

Toward the Construction of Isomorphic Cylinders

The previous chapters have presented the main contribution of this thesis, which is the generalized theory of cylindrical isomorphisms. Building upon previous work in regularizing isomorphisms in the plane, we can now take any isomorphic arrangement of notes and construct a self-consistent cylindrical representation. An obvious extension of this would be to actually construct such a cylinder and study how musicians, composers, students, and music theorists may be able to make use of such a device. This chapter explores some practical details of discusses some practical features of variations of these lattices, leading up to how one might construct building a tube-like musical instrument based on these theories.

7.1 Diameters of Cylindrical Hexagonal Lattices

Before considering the making of a physical device based on cylindrical hexagonal lattices, it is helpful to calculate the diameter of a tube in various implementations to help choose which layout and in what configuration would be appropriate. Based on the previous discussion of hexagonal lattice tubes, and considering equations 5.1–5.3, the diameter of a tube is based on the length of the chiral vector corresponding to the lattice cutting that produces that tube. The length of the chiral vector is:

$$\|\overrightarrow{Ch}\| = \sqrt{3}a\sqrt{n^2 + nm + m^2}, \tag{7.7}$$

where a is the length of an edge between two vertices in a hexagon. If the circumference C of the tube is equal to the length of the chiral vector, then the diameter of

the tube $D = C/\pi$ is:

$$D = \frac{\|\overrightarrow{Ch}\|}{\pi} = \frac{\sqrt{3}a\sqrt{n^2 + nm + m^2}}{\pi} \tag{7.8}$$

Based on the calculation of chiral vectors in the previous chapters, Table 7.1 shows the diameter of each tube for every layout in 6.1, calculated using equation 7.8. For the Armchair ($m = n$) and Zigzag ($m = 0$) cases, Table 7.2 shows the corresponding tube parameters.

Layout	Diameter
Jankó	$3a/\pi$
Harmonic Table	$\sqrt{111}a/\pi$
Gerhard	$\sqrt{39}a/\pi$
Park	$\sqrt{57}a/\pi$
Wicki-Hayden	$\sqrt{117}a/\pi$
Bajan	$\sqrt{27}a/\pi$

Table 7.1: Tube diameters for eight typical isomorphic layouts, where a is the length of one side of a hexagon.

Tube	Chiral Length	Tube Diameter
Armchair	$3na$	$3na/\pi$
Zigzag	$\sqrt{3}na$	$\sqrt{3}na/\pi$

Table 7.2: Chiral vector length and tube diameter for armchair and zigzag cases

7.2 Size of Instrument is Varied by Size of Hexagons

When considering the construction of a physical instrument, given that there are different tube sizes required, there are two options: allow the size of the instrument to change, or allow the size of the hexagons to change. To map a specific isomorphic layout onto a tube with a given diameter, the length of the side of the hexagonal

tiles (a) must be changed. As an example, consider the situation where two different isomorphisms are to be mapped onto a tube of a given size. The ratio of the size of two hexagonal buttons can then be calculated from Table 7.1. To map the *Gerhard* and *Wicki-Hayden* layouts on the same tube, the following must be set:

$$\frac{\sqrt{39}a1}{\pi} = \frac{\sqrt{117}a2}{\pi}$$

which also assumes that both cylinders are using the same number of copies of the base set of notes around the circumference. Simplifying, the result is:

$$\frac{a1}{a2} = \frac{\sqrt{3}}{1}$$

which means the size of buttons in the *Gerhard* layout is $\sqrt{3}$ times bigger than that in the *Wicki-Hayden* layout, given the same tube diameter.

7.3 Size of Instrument is Varied by Note Duplications

The size of the tube for any given isomorphism will depend on the number of copies of the base parallelogram that are included around the circumference of the tube. This choice is aesthetic and can be used to influence playability, interaction, note availability, button size, and other factors.

Figure 7.1 presents a set of possible tubes from the same isomorphic layout, in this case, the Gerhard layout. The only difference between the tubes is the number of duplicates that go around the circumference of the tube. If a single copy is used, the tube is quite narrow and each note appears exactly once on the entire structure. Adding more duplicates makes the tube larger, but does not change the shape of any musical constructs on the tube. This point should be emphasized: Changing the number of duplicates only changes the size of the tube: the shape of musical constructs (scales, chords *etc.*) is identical no matter how many duplicates are used. In the limiting case, if infinite duplicates are used, the tube degenerates into a flat isomorphism. Practically speaking, the diameter of the tube can have influence in the playability of certain shapes, with a larger tube being closer to a flat isomorphism, and a smaller tube requiring the fingers to curl around the tube. The consequences of these different tube sizes is a topic for further study.

Figure 7.1: Tube size varied by the number of duplicates; from the left: 4 copies, 3 copies, 2 copies, and 1 copy

7.4 Boundary Conditions and Note Reachability

One of the primary features of any arrangement of note actuators on a musical instrument is to make notes reachable. Adding additional manuals to an organ or additional strings to a bass guitar, for example, serve two purposes: to extend the range of the instrument, and also to make more notes available with less hand travel. On a traditional piano keyboard, only a little more than an octave of notes is available in any one hand position (depending on the size of the player's hand), and the ability to quickly and accurately move your hand to a new position while keeping your eyes on the music is a critical stage in learning how to play the piano.

Isomorphic layouts have the potential to be more compact than those of existing instruments, making more notes available in a single hand position and making all notes a smaller distance from the centre of the layout. However, any attempt to construct a reconfigurable hexagonal instrument that can present different isomorphisms

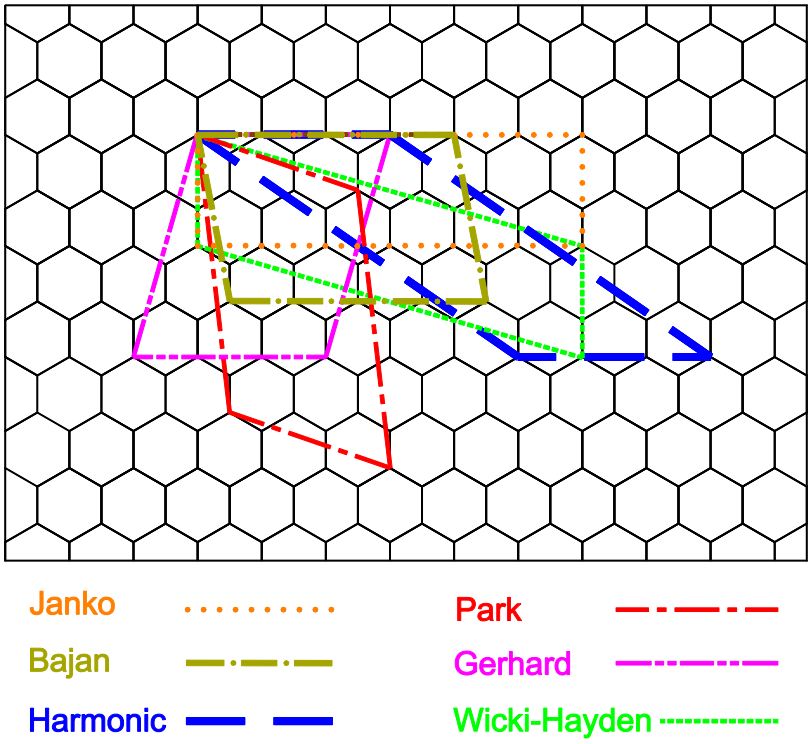


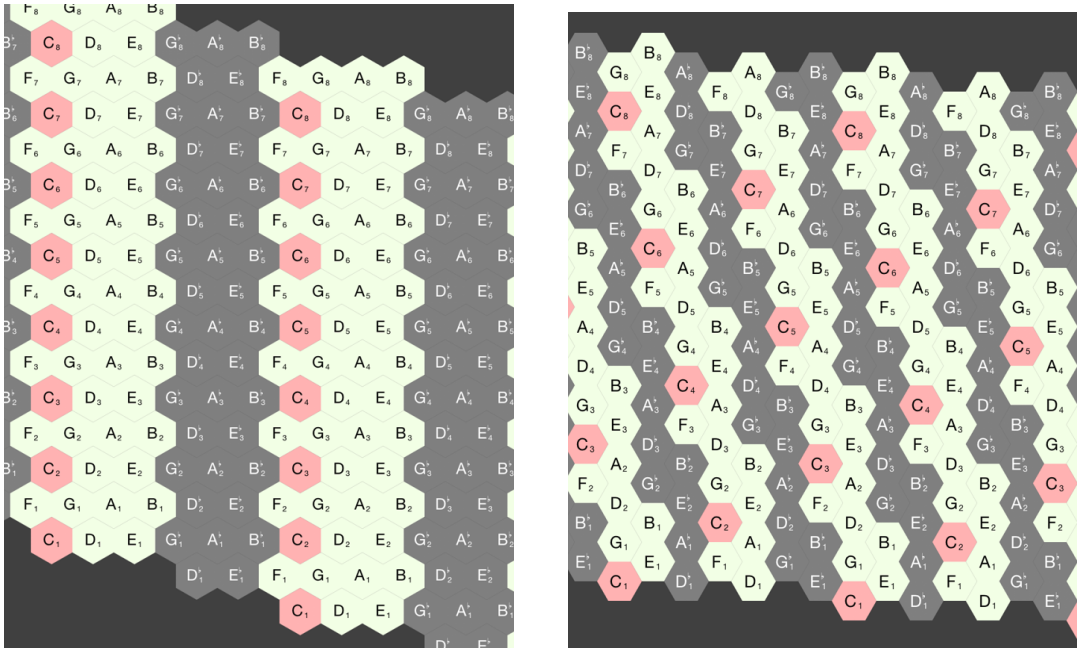
Figure 7.2: Parallelograms of isomorphic layouts. For reference, see Figs. 6.2–6.7

is a challenge: each isomorphism potentially has a different *boundary*, which is the overall shape of the entire layout showing all notes. Fig.7.3a and Fig.7.3b shows the boundaries of two isomorphic layouts.

It would be difficult to create a reconfigurable musical instrument that could represent both of these layouts to their top and bottom boundaries for two reasons. First, the angle of the boundaries is different, and second, the orientation of the hexagons is different. Wicki-Hayden uses a "horizontal" layout, where adjacent hexagons share a vertical face, while the Harmonic Table layout uses a "vertical" layout. Indeed, both layouts represent infinite duplications of notes to the left and right, at different angles, which would add to the challenge of manufacturing such an instrument.

Considering the parallelograms shown in Figs.6.2 through 6.7, and extending these by repeating along the isotone axis and extending along the pitch axis, it is clear that each of the popular layouts will have a very different boundary shape. These boundary shapes are compared in Fig.7.2. This is also related to the *shear*, a characteristic of an isomorphic layout, described in [45].

Considering again the boundary shape of each isomorphism, it should be clear that



(a) Wicki-Hayden (b) Harmonic Table

Figure 7.3: The boundaries of two isomorphic layouts with 8 octaves. Note how the shape of the boundaries is different between layouts

the previous discussion on nanotube mapping and the chiral angle can be simplified by considering an infinite sheet of repetitions of notes, and rolling that sheet in such a way that the repetitions coincide around the circumference of a tube. It should also be clear that the diameter of these tubes will be constrained to a whole number multiple of the distance between identical notes in the same octave. Table 7.1 shows the diameter of the tube corresponding to each of the layouts under discussion, calculated using equation (7.8).

7.5 Playability Exploration

Fingering on a curved keyboard can be a solution for some particular isomorphic layouts which are considered as having "fingering difficulties" on a two-dimensional planar keyboard, but this will require further study to conclusively prove. One can imagine a controller constructed with the ability to "roll" across a table or surface (Fig.7.4), allowing different notes to become available at different times. With the

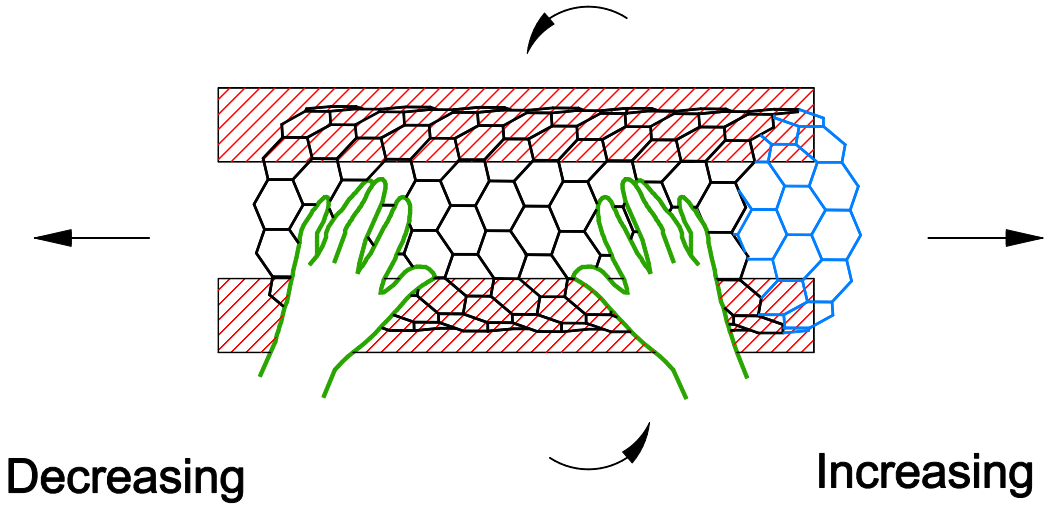


Figure 7.4: An appropriate area along either the decreasing or increasing octave direction

appropriate layout, this could be an additional compositional or performance function, modulating key or tonality or adjusting other musical parameters.

It is also possible to imagine a larger cylinder with keys tiled on the inside of the surface instead of the outside. This could produce a compelling stage presence with players performing inside the lattice, and playing on the inner surface. The inside and outside tilings are shown in Fig.7.5.

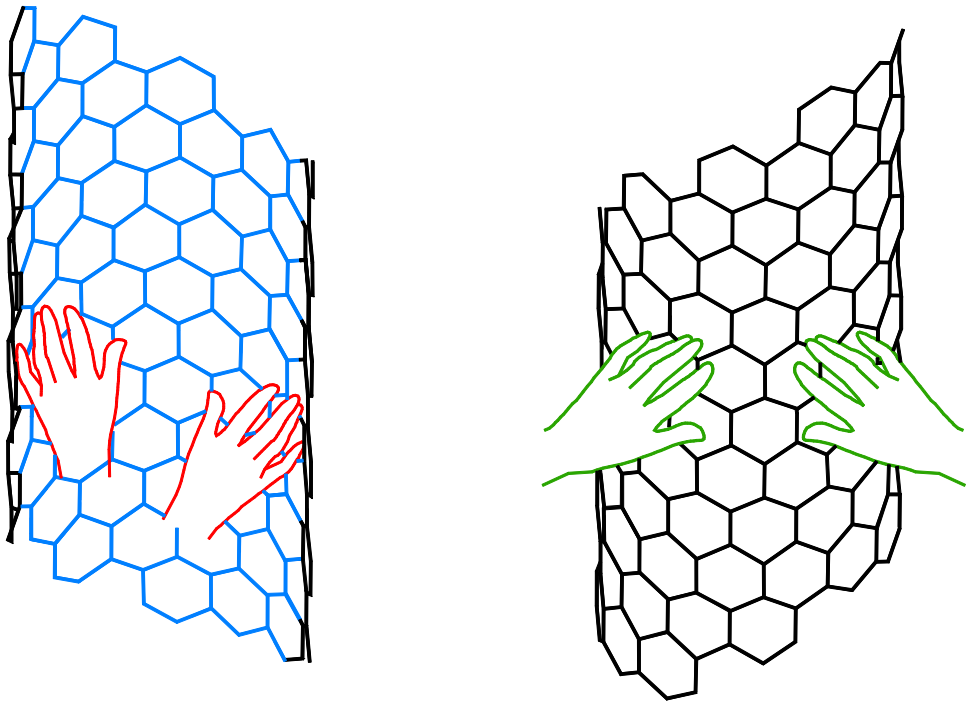


Figure 7.5: Playing on the inner (left) or the outer (right) surface

Chapter 8

Conclusion and Future Research

8.1 Conclusion

In this thesis, a series of new unified helical models in three-dimensional pitch space is introduced. By using this lattice, musical isomorphism can be mapped onto three-dimensional space by using the chiral angle. Moreover, the chiral angle and the unified grid are supplements of the existing helical pitch space model. The lattices can implement the helical pitch space model and provide the exact distance between two notes, as well as the size of the tube based on the size of each hexagon tile. Furthermore, this model provides sufficient details to consider the construction of a physical instrument in the future.

8.2 Future Work

Future work on this topic will begin with brute-force generation of a set of lattices for all possible isomorphisms, either degenerate or non-degenerate, based on the brute-force work of isomorphic layout completeness in [41]. By choosing the intervals on the isomorphic axes, and by changing the number of duplicates and the size of buttons on each cylindrical hexagonal lattice, it is possible to create a wide variety of tube-like lattices of different sizes and structures. Each of these can maintain the strong constraints of isomorphic note arrangements while offering the possibility of new playing interfaces, compositional structures.

The next step will be to construct a cylindrical hexagonal lattice and test the musical playability of the device. Work has begun on the first version of such a device,

which will be called the "Buckytone". This project is now in the prototype stage, as shown in Fig.8.1. The name "Buckytone" is inspired from the name given to spherical fullerenes: "buckminsterfullerene," to commemorate the architect Buckminster Fuller [31]. Buckminsterfullerene is also given the nickname "buckyball," and so the Buckytone is a play on this nickname.

Figure 8.1: Prototype of the Buckytone

Although a physical device with buttons will only be able to implement a single isomorphic layout depending on the chiral angle, a reconfigurable cylindrical isomorphism would be another future goal. Making a cylinder reconfigurable would require

the ability to change the position of the keys or buttons depending on the chiral angle of the desired layout, but if a touch-sensitive cylinder could be constructed, the note locations could be changed in software, allowing any isomorphism to be made available.

After building such a physical appearance, its playability and interactivity will be explored. The hope is that this novel controller and new way of thinking about the arraignment of musical notes will inspire new musical ideas and new techniques, and encourage more people to become musicians.

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