YOUNG'S INEQUALITY IN COMPACT OPERATORS

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Abstract

In 1995, T. Ando [2] proved a matrix-valued version of Young's inequality: let $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$; then for any pair a, b of $n \times n$ complex matrices, there is a unitary matrix u , depending on a and b , such that $u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$.

In this thesis, Young's inequality is extended to compact operators: for any compact operators a and b acting on a complex separable Hilbert space, and for every $p, q \in (1, \infty)$ that satisfy $1/p + 1/q = 1$, there is a partial isometry u such that $u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$. (Here, $y \leq x$ means $x - y$ is a positive operator.) Furthermore, if $|ab^*|$ is injective, then the partial isometry u can be a unitary operator.

For operators acting on an infinite- or finite-dimensional Hilbert space, the case of equality is examined and a complete characterisation of when equality holds is given for certain classes of operators.

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Chapter 1

INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

This dissertation formulates and establishes a version of a generalized arithmetic-geometric mean inequality, known as Young's inequality, in the context of inequalities between positive operators acting on complex Hilbert spaces.

The first section of this chapter discusses the motivation for this work and briefly overviews some of the previous literature relevant to the subject of study. The second section states the main results obtained herein. The third section details the definitions and notations to be used in describing and developing these results, and the final section provides a concrete example, for operators on a 2-dimensional Hilbert space, of the main result.

1.1 Motivation

Operator and matrix versions of classical inequalities are of considerable interest,

and there is an extensive body of literature treating this subject; see, for example, [1-4, 6,7,10,11,15-17]. In one direction, many of the operator inequalities to have come under study are inequalities between the norms of operators. However, a second line of research is concerned with inequalities arising from the partial order on Hermitian operators acting on a Hilbert space. It is in this latter direction that this dissertation aims.

To a large degree, the results of this dissertation are motivated by R.C. Thompson's pioneering approaches [15, 16, 17] to the first fundamental inequality in all of analysis, namely the triangle inequality: $|\lambda + \mu| \leq |\lambda| + |\mu|$, for all complex numbers λ and μ . It is natural to ask if, and how, the triangle inequality might extend to complex matrices, to operators, or to elements in a C^* -algebra. These questions have been addressed by—in addition to Thompson—C.A. Akemann, J. Anderson, and G.K. Pedersen [1], as well as by D. R. Farenick and P. J. Psarrakos [7], resulting in interesting, nontrivial formulations of the triangle inequality for matrices and operators. Certainly a high-point in this direction is the Akemann–Anderson–Pederson triangle inequality: if a and b are elements of a von Neumann algebra M , then there are isometries $u, v \in M$ such that $|a + b| \leq u|a|u^* + v|b|v^*$, where, for any $x \in M$, $|x|$ denotes the operator $(x^*x)^{1/2}$. However, not all issues surrounding the triangle inequality have been resolved in the operator context, even in the case of compact operators; see [1] for further discussion.

Another fundamental inequality between positive real numbers is the arithmetic-geometric mean inequality, which is of particular interest herein, as is its generalisation in the form of Young's inequality.

For the positive real numbers a, b , the arithmetic-geometric mean inequality says that

$$\sqrt{ab} \leq \frac{1}{2}(a + b).$$

Replacing a, b by their squares, this could be written in the form

$$ab \leq \frac{1}{2}(a^2 + b^2).$$

R. Bhatia and F. Kittaneh [4] extended the arithmetic-geometric mean inequality to positive (semi-definite) matrices a, b in the following manner: for any $n \times n$ positive matrices a, b , there is a $n \times n$ unitary matrix u such that

$$u|ab|u^* \leq \frac{1}{2}(a^2 + b^2).$$

(The modulus $|y|$ is defined by

$$|y| = (y^*y)^{\frac{1}{2}},$$

for any $n \times n$ complex matrix y .) We note that the product ab of two positive matrices a and b is not necessarily positive.

Young's inequality is a generalisation of the arithmetic-geometric mean inequality: for any positive real numbers a, b , and any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

T. Ando [2] showed Young's inequality admits a matrix-valued version analogous to the Bhatia–Kittaneh theorem: if $p, q \in (1, \infty)$ satisfy that $1/p + 1/q = 1$, then for any pair a, b of $n \times n$ complex matrices, there is a unitary matrix u such that

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Although finite-rank operators are norm-dense in the set of all compact operators acting on a fixed Hilbert space, the Ando–Bhatia–Kittaneh inequalities, like most matrix inequalities, do not immediately carry over to compact operators via the

usual approximation methods, and consequently only a few of the fundamental matrix inequalities are known to hold in compact operators.

The main purpose of this dissertation, therefore, is to develop a technique through which the Ando–Bhatia–Kittaneh results extend to compact operators. Some limitations of the technique are discussed in the final chapter of this work.

Because the inequalities established herein involve the spectra of positive operators, Young’s inequality persists in passing to unitarily invariant norms, such as those discussed in the paper [11] by F. Hiai and H. Kosaki.

1.2 The Statement of the Main Results

The main result to be established herein is the following version of Young’s inequality.

Theorem 1.1 If a and b are compact operators acting on a complex separable Hilbert space, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then there is a partial isometry u such that the initial space of u is $(\ker(|ab^*|))^\perp$ and

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q. \quad (1.1)$$

Furthermore, if $|ab^*|$ is injective, then the operator u in the inequality above can be taken to be a unitary.

In the corollary below, an attractive rephrasing of Theorem 1.1 is made in an important special case.

Corollary 1.2 If a and b are positive compact operators with trivial kernels, and if $t \in [0, 1]$, then there is a unitary u such that

$$u|a^t b^{1-t}|u^* \leq ta + (1-t)b.$$

Observe, further, that if H is the 1-dimensional Hilbert space of complex numbers, and if $t = 1/2$, then Corollary 1.2 is precisely the arithmetic-geometric mean inequality for positive real numbers.

The operator inequality (1.1) has two necessary twists to it: the first appears in the use of the adjoint of b , the second is by way of conjugation by a partial isometry u . These are forced upon us by the noncommutativity of operator algebra in the first case, and by the infinite dimensionality of the Hilbert space in the second case—but they can be dropped if a and b are commuting normal operators (see Theorem 1.3).

Theorem 1.1 differs from Ando's matricial Young inequality [2] in that the operator u that arises here is typically a partial isometry, whereas in the matrix case u can always be taken to be a unitary. This is explained by the fact that, for operators acting on infinite-dimensional spaces, the dimension of the kernel of a positive operator is not in general determined by the dimension of the operator's range. If $|ab^*|$ has trivial kernel, then this partial isometry u is in fact an isometry; however, as we shall show in Chapter 5, the operator u can, in this case, be taken to be unitary.

For commutative C^* -algebras, we have the following result.

Theorem 1.3 If A is a commutative C^* -algebra with multiplicative identity, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ for all $a, b \in A$. Furthermore,

if the equality holds, then $|b| = |a|^{p-1}$.

We also deal with the cases of equality for the Young's inequality in compact operators and matrices and obtain the following results.

Theorem 1.4 Assume that $a, b \in B(H)$ are commuting compact normal operators, each being injective and having dense range. If there exists unitary u such that $u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$, for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $|b| = |a|^{p-1}$.

Corollary 1.5 If $\dim H < \infty$, then Theorem 1.4 holds even in the case where $|ab^*|$ is not injective.

The case where a and b do not commute is much more delicate to handle.

Theorem 1.6 Assume $a, b \in M_n(\mathbf{C})$ with $a \geq 0, b \geq 0$. If there exists a unitary u such that $u|ab|u^* = \frac{1}{p}a^p + \frac{1}{q}b^q$, for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then b is unitarily equivalent to a^{p-1} .

Theorem 1.7 Assume $n \leq 3$, $a, b \in M_n(\mathbf{C})$ with $a \geq 0, b \geq 0$. If there exists a unitary u such that

$$2u|ab|u^* = a^2 + b^2,$$

then u commutes with a , and $b = a$.

For proving our main result, Theorem 1.1, this thesis follows Ando's model, but makes modifications to it in many places to account for spaces of infinite dimension. Ando's methods are of two sorts. The first is through the use of operator inequalities, which the thesis develops further for non-compact operators in Chapter 4. The second

is by means of spectral inequalities, which is made possible here by the tractable spectral theory of compact Hermitian operators. The two methods are then coupled by way of an approximation of compact operators by non-compact operators.

1.3 Concepts from Operator Theory

Assume that H denotes a complex, separable Hilbert space of finite- or infinite-dimension. If the inner product of vectors $\xi, \eta \in H$ is denoted by $\langle \xi, \eta \rangle$, then the equation

$$\|\xi\| = \sqrt{\langle \xi, \xi \rangle}, \quad \text{for } \xi \in H,$$

defines a norm on H . If $x : H \rightarrow H$ is a linear transformation, then x is called an operator (on H) if x is also continuous with respect to the norm-topology on H . While all linear transformations on finite-dimensional Hilbert spaces are operators, there exist linear transformations on infinite-dimensional Hilbert spaces that are not continuous.

Continuity for linear maps is equivalent to boundedness. A linear transformation $x : H \rightarrow H$ is bounded if $\|x\| < \infty$, where $\|x\|$ is defined to be

$$\|x\| = \sup\{\|x\xi\| : \xi \in H, \|\xi\| \leq 1\}. \quad (1.2)$$

It is a basic fact that a linear transformation is bounded if and only if it is continuous.

The complex algebra of all operators on H is denoted by $B(H)$. Equation (1.2) is a norm on $B(H)$ under which the algebra $B(H)$ is a complex Banach algebra.

The adjoint x^* of $x \in B(H)$ is the unique linear transformation x^* on H for which the equation $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle$ holds for all $\xi, \eta \in H$. It is not difficult to prove that $\|x^*\| = \|x\|$ and that, consequently, $*$ is an isometric involution on $B(H)$. Moreover, $\|x^*x\| = \|x\|^2$, for every $x \in B(H)$; thus, $B(H)$ is a C^* -algebra.

The spectrum of $x \in B(H)$ is the subset $\sigma(x) \subseteq \mathbf{C}$ of all $\lambda \in \mathbf{C}$ for which $x - \lambda 1$

(where 1 is the identity operator $\xi \rightarrow \xi$ on H) is not invertible in $B(H)$. (An operator $z \in B(H)$ is invertible if there is a $y \in B(H)$ such that $yz = zy = 1$.)

An operator x on H is said to be Hermitian if $x^* = x$. Hermitian operators are precisely those operators $x \in B(H)$ for which $\langle x\xi, \xi \rangle \in \mathbf{R}$ for $\xi \in H$. Because, for any operator x , $\sigma(x)$ is a subset of the closure of $\{\langle x\xi, \xi \rangle : \xi \in H, \|\xi\| = 1\}$, one has that $\sigma(x) \subseteq \mathbf{R}$ if x is Hermitian.

A Hermitian operator x is positive if $\sigma(x) \subseteq \mathbf{R}_0^+$, where $\mathbf{R}_0^+ = \{r \in \mathbf{R} : r \geq 0\}$. Equivalently, $x \in B(H)$ is positive if and only if

$$\langle x\xi, \xi \rangle \geq 0, \quad \text{for all } \xi \in H.$$

By this criterion, y^*y is a positive operator for every $y \in B(H)$.

Every positive operator x has a unique positive square root; that is, there is a unique positive $b \in B(H)$ such that $b^2 = x$. This operator b will be denoted by $x^{\frac{1}{2}}$. The modulus of every $y \in B(H)$ is denoted by $|y|$ and is the positive operator defined by the equation

$$|y| = (y^*y)^{\frac{1}{2}}.$$

Now if $a, b \in B(H)$ are Hermitian, then $a \leq b$ shall henceforth denote that $b - a$ is positive. Therefore, the following statements are equivalent for Hermitian operators a and b :

- (i) $a \leq b$;
- (ii) $\sigma(b - a) \subseteq \mathbf{R}_0^+$;
- (iii) $\langle a\xi, \xi \rangle \leq \langle b\xi, \xi \rangle$, for every $\xi \in H$.

Besides Hermitian and positive operators, a few other classes of operators will be of interest here. An operator $u \in B(H)$ is:

- (i) an isometry if $\|u\xi\| = \|\xi\|$, for all $\xi \in H$;
- (ii) a partial isometry if $\|u\xi\| = \|\xi\|$, for all $\xi \in (\ker(u))^\perp$;
- (iii) a unitary if u is an invertible isometry.

In (ii), $\ker(u) = \{\eta \in H : u\eta = 0\}$ (the “kernel” of u), and for any set $S \subseteq H$ the set $S^\perp$ denotes the closed subspace

$$S^\perp = \{\gamma \in H : \langle \gamma, \eta \rangle = 0, \forall \eta \in S\}.$$

The space $(\ker(u))^\perp$, upon which u acts isometrically, is called the initial space of the partial isometry u . An operator u is isometric if and only if $u^*u = 1$, whereas u is unitary if and only if $u^*u = uu^* = 1$.

More generally, an operator $x \in B(H)$ for which $x^*x = xx^*$ is called normal. Thus, Hermitian operators, positive operators, and unitary operators are all examples of normal operators.

An operator $y \in B(H)$ has finite rank if the range of y has finite dimension, and an operator $x \in B(H)$ is compact if for every $\varepsilon > 0$ there is a finite-rank operator y_ε such that $\|x - y_\varepsilon\| < \varepsilon$. Thus, if $K(H)$ denotes all compact operators acting on H , then $K(H)$ is a closed ideal of $B(H)$. Moreover, $x^* \in K(H)$ for all $x \in K(H)$. Hence, $K(H)$ is a C^* -subalgebra of $B(H)$, and $B(H) = K(H)$ if and only if H has finite dimension.

1.4 An Example

We here give an example to illustrate that, in general, we do not have

$$|ab^*| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

But, for this example, there exists a unitary u such that

$$u|ab^*|u^* \leq \frac{1}{2}(|a|^2 + |b|^2).$$

Example 1.1 If

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$\frac{1}{2}(a^2 + b^2) = \begin{pmatrix} 3 & 1 \\ 1 & \frac{3}{2} \end{pmatrix},$$

and

$$|ab^*| = |ab| = \begin{pmatrix} \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \end{pmatrix}.$$

However,

$$c = \left(\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2\right) - |ab| = \begin{pmatrix} 3 - \frac{\sqrt{10}}{2} & 1 - \frac{\sqrt{10}}{2} \\ 1 - \frac{\sqrt{10}}{2} & \frac{3}{2} - \frac{\sqrt{10}}{2} \end{pmatrix}$$

is not a positive matrix, i.e., $c = (\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2) - |ab| \not\geq 0$. (In fact, the determinant of c satisfies that $\det(c) < 0$.) So, we do not have

$$|ab^*| \leq \frac{1}{2}(|a|^2 + |b|^2).$$

But the spectrum of $|ab|$ is

$$\sigma(|ab|) = \{\sqrt{10}, 0\},$$

and the spectrum of $\frac{1}{2}(|a|^2 + |b|^2)$:

$$\sigma\left(\frac{1}{2}(|a|^2 + |b|^2)\right) = \left\{\frac{7}{2}, 1\right\}.$$

Therefore, there exists a unitary matrix u such that

$$u|ab^*|u^* \leq \frac{1}{2}(|a|^2 + |b|^2).$$

We compute the unitary matrix u as follows.

Taking unitary matrices

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$$

and

$$u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we then have

$$u_1 \left(\frac{1}{2}(a^2 + b^2) \right) u_1^* = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$u_2 |ab| u_2^* = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$u_2 |ab| u_2^* \leq u_1 \left(\frac{1}{2}(a^2 + b^2) \right) u_1^*.$$

By taking a unitary matrix

$$u = u_1^* u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 & -1 \\ -1 & 3 \end{pmatrix},$$

we get

$$u |ab| u^* \leq \frac{1}{2}(a^2 + b^2).$$

Chapter 2

SPECTRAL THEORY

This chapter reviews the aspects of spectral theory that arise in the proof of Young's inequality throughout Chapters 4, 5, and 6.

Recall that the spectrum $\sigma(x)$ of an operator $x \in B(H)$ is the set of all $\lambda \in \mathbf{C}$ for which $x - \lambda 1$ is not invertible in $B(H)$. The spectrum is typically divided into various subsets, each of which pertains to a particular way in which $x - \lambda 1$ can fail to be invertible. Our interest is almost entirely with the simplest part of the spectrum: namely, the set of eigenvalues.

Definition 2.1 If $x \in B(H)$, then $\lambda \in \mathbf{C}$ is an eigenvalue of x if $\ker(x - \lambda 1) \neq \{0\}$. Every nonzero vector of $\ker(x - \lambda 1)$ is called an eigenvector of x .

Observe that $\eta \in \ker(x - \lambda 1)$ if and only if $x\eta = \lambda\eta$; this is the eigenvalue equation that is familiar in various parts of analysis and linear algebra. While it is true that if λ is an eigenvalue of $x \in B(H)$, then $\lambda \in \sigma(x)$, it is also the case that $\sigma(x)$ can have elements that are not eigenvalues, and there exist operators x that have no eigenvalues at all.

2.1 Spectra of Compact Normal Operators

Recall that $x \in B(H)$ is a compact operator if for each $\varepsilon > 0$ there is a finite-rank operator y_ε such that $\|x - y_\varepsilon\| < \varepsilon$.

The basic facts about the spectrum $\sigma(x)$ of a compact operator $x \in B(H)$ are:

- (i) $0 \in \sigma(x)$ if H has infinite dimension;
- (ii) $\sigma(x)$ is finite or countably infinite, and 0 is the only possible cluster point of $\sigma(x)$;
- (iii) $\lambda \in \sigma(x) \setminus \{0\}$ if and only if λ is an eigenvalue of x ; and
- (iv) $\dim(\ker(x - \lambda 1)) < \infty$, $\forall \lambda \neq 0$.

Compact normal operators are decomposed by their nonzero spectra. (Recall that x is normal if $x^*x = xx^*$.) The following notational convention will be employed in the remainder of this thesis.

Notation 2.1 The letter ν shall denote an element from $\{1, 2, 3, \dots\} \cup \{\infty\}$, and so $\{\omega_k\}_{k=1}^\nu$ denotes a finite or infinite sequence.

Theorem 2.1 (Spectral Theorem) If x is a nonzero compact normal operator, then there exist orthonormal vectors ϕ_k ($k = 1, \dots, \nu$) and nonzero complex numbers λ_k ($k = 1, \dots, \nu$) such that

- (i) $x\phi_k = \lambda_k\phi_k$, for all k ,
- (ii) $\lim_{k \rightarrow \infty} \lambda_k = 0$ if $\nu = \infty$, and
- (iii) $x\xi = \sum_{k=1}^\nu \lambda_k \langle \xi, \phi_k \rangle \phi_k$, for all $\xi \in H$.

Theorem 2.1 is the diagonalisation theorem from linear algebra if $\dim(H) < \infty$. If H has infinite dimension, the sequence $\{\phi_k\}_{k=1}^\nu$ need not form an orthonormal basis

for H , although it will do so if $\ker(x) = \{0\}$.

The case in which x is positive is of greatest interest. In this case, the nonzero eigenvalues λ_k of x can be arranged in descending order: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots > 0$. This will be the conventional notation henceforth.

Notation 2.2 If $x \in B(H)$ is a positive compact operator, then $\lambda_k(x)$ denotes the k^{th} nonzero eigenvalue of x , where $\{\lambda_j(x)\}_{j=1}^\nu$ is an arrangement of the nonzero eigenvalues of x in descending order.

A variational principle recovers $\lambda_k(x)$ for every k and every positive compact operator x .

Theorem 2.2 (The Minimum-Maximum Principle) If $x \in B(H)$ is a positive compact operator, then

$$\lambda_k(x) = \min_{\substack{M \subseteq H \\ \dim(M)=k-1}} \max_{\substack{\xi \in M^\perp \\ \|\xi\|=1}} \langle x\xi, \xi \rangle,$$

for every $1 \leq k \leq \nu$.

2.2 Functional Calculus

A C^* -algebra is a complex normed algebra A with an involution $*$ such that

- (i) A is complete with respect to the norm,
- (ii) $\|xy\| \leq \|x\|\|y\|$, $\forall x, y \in A$,
- (iii) $\|x^*x\| = \|x\|^2$, $\forall x \in A$.

Two examples of C^* -algebra are:

(1) Any norm-closed subalgebra A of $B(H)$ such that x^* (the adjoint of x) belongs to A if x does.

(2) The algebra $C(\Omega)$ of continuous complex-valued functions on a compact Hausdorff space, where for each $f \in C(\Omega)$,

- (i) $\|f\| = \max\{|f(\omega)| : \omega \in \Omega\}$, and
- (ii) $f^*(\omega) = \overline{f(\omega)}$.

An isometric $*$ -isomorphism between two C^* -algebras A and B is a bijective homomorphism $\phi : A \rightarrow B$ such that, for all $a \in A$, $\phi(a^*) = \phi(a)^*$ and $\|\phi(a)\| = \|a\|$.

Theorem 2.3 (Gelfand's Theorem) If A is a commutative C^* -algebra with multiplicative identity, then there is a compact Hausdorff space Ω such that A and $C(\Omega)$ are isometrically $*$ -isomorphic.

For a commutative C^* -algebra A that has the multiplicative identity 1, let

$$\Omega = \{\varphi \in A^* : \varphi \text{ is a homomorphism, } \varphi(1) = 1\}.$$

Give Ω the relative weak* topology that it has as a subset of A^* (the Banach space dual of A). Ω with this topology is called the maximal ideal space of A . It is a compact Hausdorff space. In fact, the space Ω that appears in the statement of Theorem 2.3 is the maximal ideal space of A .

The smallest C^* -algebra $C^*(1, x)$ that contains the identity operator 1 and a normal operator $x \in B(H)$ is commutative. In this case, Gelfand's theorem is as follows.

Theorem 2.4 (Functional Calculus) If $x \in B(H)$ is normal, then there is an isometric $*$ -isomorphism $\phi : C(\sigma(x)) \rightarrow C^*(1, x)$ under which the function $f(t) = t$ is mapped

by ϕ to x .

We shall require the square root function throughout this thesis. That is, if $x \in B(H)$ is positive, then, from functional calculus, the function $f(t) = \sqrt{t}$ leads to an operator $f(x) = x^{\frac{1}{2}}$ for which $(x^{\frac{1}{2}})^2 = x$, $(x^{\frac{1}{2}})^* = x^{\frac{1}{2}}$, and $\sigma(x) = \{\sqrt{\lambda} : \lambda \in \sigma(x)\}$.

The operator $x^{\frac{1}{2}}$ can be understood analytically as follows by way of the stone-Weierstrass Theorem. If $p(t) = \sum_{j=0}^n \alpha_j t^j$ is a polynomial, then $p(x) = \sum_{j=0}^n \alpha_j x^j$. The Stone-Weierstrass Theorem states that there is a sequence of polynomials p_k such that

$$\lim_{k \rightarrow \infty} (\max_{t \in \sigma(x)} |\sqrt{t} - p_k(t)|) = 0.$$

Hence, by functional calculus, $\lim_{k \rightarrow \infty} \|x^{\frac{1}{2}} - p_k(x)\| = 0$.

2.3 Projection-Valued Measure

Let β be the σ -algebra of Borel sets in $\mathbf{R}$, and let H be a Hilbert space. A resolution of the identity (on β) is a mapping

$$e : \beta \rightarrow B(H)$$

with the following properties:

- (a) $e(\phi) = 0, e(\mathbf{R}) = 1$;
- (b) Each $e(\omega)$ is a projection, $\omega \in \beta$;
- (c) $e(\omega_1 \cap \omega_2) = e(\omega_1) \cdot e(\omega_2), \forall \omega_1, \omega_2 \in \beta$;
- (d) If $\omega_1 \cap \omega_2 = \phi$, then $e(\omega_1 \cup \omega_2) = e(\omega_1) + e(\omega_2), \forall \omega_1, \omega_2 \in \beta$;
- (e) $\forall \xi \in H$ and $\forall \eta \in H$, the function $e_{\xi, \eta}$ defined by

$$e_{\xi, \eta}(\omega) = \langle e(\omega)\xi, \eta \rangle$$

is a complex regular measure on β .

Let f be a complex β -measurable function on $\mathbf{R}$. There is a countable collection $\{D_i\}$ of open discs which forms a base for the topology of $\mathbf{C}$. Let V be the union of those D_i for which $e(f^{-1}(D_i)) = 0$. We know that $e(f^{-1}(V)) = 0$. Also, V is the largest open subset of $\mathbf{C}$ with this property.

The essential range of f is the complement of V . It is the smallest closed subset of $\mathbf{C}$ that contains $f(t)$ for e -almost all $t \in \mathbf{R}$, that is for all $t \in \mathbf{R}$ except those that lie in some set $\omega \in \beta$ with $e(\omega) = 0$.

We say that f is essentially bounded if its essential range is bounded. In this case, the supremum of $|\lambda|$, as λ runs through the essential range of f , is called the essential supremum $\|f\|_\infty$ of f .

Let B be the algebra of all bounded complex β -measurable functions on $\mathbf{R}$; with the norm

$$\|f\| = \sup\{|f(t)| : t \in \mathbf{R}\}.$$

We can easily see that B is a Banach algebra and

$$N = \{f \in B : \|f\|_\infty = 0\}$$

is an ideal of B which is closed. Hence the quotient algebra B/N is a Banach algebra, and we denote it by $L^\infty(\mathbf{R}, e)$.

The norm of any coset $[f] = f + N$ of $L^\infty(\mathbf{R}, e)$ is then equal to $\|f\|_\infty$, and its spectrum $\sigma([f])$ is the essential range of f . As is usually done in measure theory, the distinction between f and its equivalence class $[f]$ will be ignored.

The following result is from [13, p.303].

Theorem 2.5 If e is a resolution of the identity, then the formula

$$\langle \psi(f)\xi, \eta \rangle = \int_{\mathbf{R}} f de_{\xi, \eta}, \quad \forall \xi, \eta \in H \quad (2.3)$$

defines an isometric isomorphism ψ of the Banach algebra $L^\infty(\mathbf{R}, e)$ onto a closed normed subalgebra A of $B(H)$, and this isomorphism satisfies

$$\psi(\bar{f}) = \psi(f)^*, \quad \forall f \in L^\infty(\mathbf{R}, e).$$

Moreover, an operator $b \in B(H)$ commutes with every $e(\omega)$ if and only if b commutes with every $\psi(f)$.

We abbreviate equation (2.3) as

$$\psi(f) = \int_{\mathbf{R}} f de.$$

Theorem 2.6 If $a \in B(H)$ is Hermitian, then there is a resolution of the identity e such that the isometric $*$ -isomorphism $\psi : L^\infty(\mathbf{R}, e) \rightarrow B(H)$ sends $f(t) = t$ to a ; i.e., if $f(t) = t$, then $\psi(f) = a$. Therefore, in the notation established above,

$$a = \int_{\mathbf{R}} \lambda de.$$

2.4 Polar Decomposition

Recall that an operator $u \in B(H)$ is a partial isometry if and only if there exist subspaces H_1, H_2 of H for which $H = H_1 \oplus H_2$ and

- (i) $\|u\xi\| = \|\xi\|, \forall \xi \in H_1$, and
- (ii) $\ker(u) = H_2$.

H_1 is the initial space of u , and $u(H_1)$ is the final space of u . Actually, the initial space is $(\ker(u))^\perp$, and the final space is $\text{ran}(u)$.

Theorem 2.7 (Polar Decomposition) Let a be an operator on a Hilbert space H . Then there exists a unique partial isometry $u \in B(H)$ such that

$$a = u|a|,$$

and

$$\ker(u) = \ker(a).$$

Moreover, $(\ker(a))^\perp$ is the initial space, and $(\text{ran}(a))^\perp$ is the final space. Lastly, u^*u is the projection onto the initial space, uu^* is the projection onto the final space.

Chapter 3

FUNDAMENTAL OPERATOR INEQUALITIES

Our purpose in this expository chapter is to discuss operator monotone and operator convex functions, and to introduce and prove Young's inequality in its most basic form.

3.1 Operator Monotone and Operator Convex Functions

In this section we deal with an important and useful class of functions called operator monotone functions. These are real functions whose extensions to positive operators preserve order. They are closely related to properties of operator convex functions. We are going to study both of them here.

If $I \subseteq \mathbf{R}$ is an interval, and if $f : I \rightarrow \mathbf{R}$ is a continuous function, then f is said to be operator monotone on I if $f(a) \leq f(b)$, for all Hermitian operators a and b for

which $\sigma(a) \cup \sigma(b) \subseteq I$ and $a \leq b$.

If $I \subseteq \mathbf{R}$ is an interval, and if $f : I \rightarrow \mathbf{R}$ is a continuous function, then f is said to be operator convex on I if, for all Hermitian operators a and b for which $\sigma(a) \cup \sigma(b) \subseteq I$ and for all real numbers $0 \leq \lambda \leq 1$,

$$f((1 - \lambda)a + \lambda b) \leq (1 - \lambda)f(a) + \lambda f(b).$$

A continuous function f is operator concave if $-f$ is operator convex.

It is obvious that the set of operator monotone functions and the set of operator convex functions are both closed under positive linear combinations and also under pointwise limits. The same is true for operator concave functions.

We remark that a monotone function is not necessarily an operator monotone function. The same is true for an operator convex function.

For example, $f(t) = t^2$ is a monotone function but it is not an operator monotone function on $[0, \infty)$. To see this, take

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$b = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$b - a = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

is positive but

$$b^2 - a^2 = \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix}$$

is not.

On the other hand, the function $f(t) = t^3$ is convex but is not operator convex. To see this, take

$$c = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$\frac{1}{2}(a^3 + c^3) - \left(\frac{1}{2}(a + c)\right)^3 = \begin{pmatrix} -21 & -17 \\ -17 & -\frac{27}{2} \end{pmatrix}.$$

This is not positive.

Lemma 3.1 Let λ_1, λ_2 be positive real numbers and c be an operator in $B(H)$ such that $\lambda_1 \lambda_2 1 \geq c^* c$. Then

$$a = \begin{pmatrix} \lambda_1 1 & c^* \\ c & \lambda_2 1 \end{pmatrix}$$

is a positive operator in $B(H \oplus H)$.

Proof. If $\xi = (\xi_1, \xi_2) \in H \oplus H$, then

$$\left\langle \begin{pmatrix} \lambda_1 1 & c^* \\ c & \lambda_2 1 \end{pmatrix} \xi, \xi \right\rangle = \lambda_1 \langle \xi_1, \xi_1 \rangle + \langle c^* \xi_2, \xi_1 \rangle + \langle c \xi_1, \xi_2 \rangle + \lambda_2 \langle \xi_2, \xi_2 \rangle \in \mathbf{R}.$$

Noting that

$$\begin{aligned} & \lambda_1 \langle \xi_1, \xi_1 \rangle + \lambda_2 \langle \xi_2, \xi_2 \rangle \\ &= \lambda_1 \|\xi_1\|^2 + \lambda_2 \|\xi_2\|^2 \\ &\geq 2(\lambda_1 \lambda_2)^{\frac{1}{2}} \|\xi_1\| \cdot \|\xi_2\| \\ &\geq 2\|cc^*\|^{\frac{1}{2}} \|\xi_1\| \cdot \|\xi_2\| \\ &= 2\|c\| \cdot \|\xi_1\| \cdot \|\xi_2\|, \end{aligned}$$

and

$$\begin{aligned} & |\langle c^* \xi_2, \xi_1 \rangle + \langle c \xi_1, \xi_2 \rangle| \\ &= |\langle \xi_2, c \xi_1 \rangle + \langle c \xi_1, \xi_2 \rangle| \\ &= |\overline{\langle c \xi_1, \xi_2 \rangle} + \langle c \xi_1, \xi_2 \rangle| \end{aligned}$$

$$\begin{aligned}
&= |2\operatorname{Re}\langle c\xi_1, \xi_2 \rangle| \\
&\leq 2\|c\xi_1\| \cdot \|\xi_2\| \\
&\leq 2\|c\| \cdot \|\xi_1\| \cdot \|\xi_2\|,
\end{aligned}$$

we get

$$\left\langle \begin{pmatrix} \lambda_1 1 & c^* \\ c & \lambda_2 1 \end{pmatrix} \xi, \xi \right\rangle \geq 0, \forall \xi \in H \oplus H.$$

This implies that

$$\begin{pmatrix} \lambda_1 1 & c^* \\ c & \lambda_2 1 \end{pmatrix}$$

is a positive operator.

Lemma 3.2. Let a, b , and $c \in B(H)$ and

$$\begin{pmatrix} a & c^* \\ c & b \end{pmatrix}$$

be a Hermitian operator on $H \oplus H$. Then for every $\varepsilon > 0$, there is a $\lambda > 0$ such that

$$\begin{pmatrix} a & c^* \\ c & b \end{pmatrix} \leq \begin{pmatrix} a + \varepsilon 1 & 0 \\ 0 & \lambda 1 \end{pmatrix}.$$

Proof. Take a positive number $\delta > 0$ such that

$$(\varepsilon\delta)1 \geq c^*c.$$

(We note that $\langle c^*c\xi, \xi \rangle \leq \|c^*c\| \cdot \|\xi\|^2 = \|c\|^2 \cdot \|\xi\|^2$, and $\langle \varepsilon\delta\xi, \xi \rangle = \varepsilon\delta\|\xi\|^2$. If $\delta > 0$ such that $\varepsilon\delta \geq \|c\|^2$, then $\varepsilon\delta 1 \geq c^*c$.)

Choose $\lambda > 0$ for which $(\lambda - \delta)1 \geq b$, i.e., $\lambda 1 \geq b + \delta 1$. Then

$$\begin{pmatrix} a + \varepsilon 1 & 0 \\ 0 & \lambda 1 \end{pmatrix} - \begin{pmatrix} a & c^* \\ c & b \end{pmatrix} \geq \begin{pmatrix} \varepsilon 1 & -c^* \\ -c & \delta 1 \end{pmatrix}.$$

By Lemma 3.1 we have

$$\begin{pmatrix} a + \varepsilon 1 & 0 \\ 0 & \lambda 1 \end{pmatrix} \geq \begin{pmatrix} a & c^* \\ c & b \end{pmatrix}.$$

The following important theorem is fundamental in the proof of Young's inequality in compact operators.

Proposition 3.3 The function $f(t) = t^r$ is operator monotone and operator concave on $[0, \infty)$ for $0 \leq r \leq 1$.

Proof. We first prove that $f(t) = t^r$ is operator monotone.

If $b \geq a \geq 0$, and if b is invertible, then $b^{-\frac{1}{2}}bb^{-\frac{1}{2}} \geq b^{-\frac{1}{2}}ab^{-\frac{1}{2}}$ implies that

$$1 \geq b^{-\frac{1}{2}}ab^{-\frac{1}{2}} = (a^{\frac{1}{2}}b^{-\frac{1}{2}})^*(a^{\frac{1}{2}}b^{-\frac{1}{2}}).$$

This implies $\|(a^{\frac{1}{2}}b^{-\frac{1}{2}})^*(a^{\frac{1}{2}}b^{-\frac{1}{2}})\| \leq 1$. Therefore

$$\|a^{\frac{1}{2}}b^{-\frac{1}{2}}\| \leq 1.$$

Let $\text{spr}(c)$ denote the spectral radius of an operator c . From $\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}$ and the fact that $\text{spr}(c) \leq \|c\|$, we have

$$\text{spr}(b^{-\frac{1}{4}}a^{\frac{1}{2}}b^{-\frac{1}{4}}) = \text{spr}(a^{\frac{1}{2}}b^{-\frac{1}{2}}) \leq \|a^{\frac{1}{2}}b^{-\frac{1}{2}}\| \leq 1 = \text{spr}(1).$$

Note that because $b^{-\frac{1}{4}}a^{\frac{1}{2}}b^{-\frac{1}{4}}$ is positive, and from $x \leq 1$ if and only if $\|x\| \leq 1$ for x positive, we get $b^{-\frac{1}{4}}a^{\frac{1}{2}}b^{-\frac{1}{4}} \leq 1$. And so

$$b^{\frac{1}{2}} \geq a^{\frac{1}{2}}.$$

If $b \geq a \geq 0$ and if b is not invertible, then $\forall \varepsilon > 0, b + \varepsilon 1$ is invertible. According to the discussion above, $(b + \varepsilon 1)^{\frac{1}{2}} \geq a^{\frac{1}{2}}$. Letting $\varepsilon \rightarrow 0_+$, by the continuity of the inner product one has

$$b^{\frac{1}{2}} \geq a^{\frac{1}{2}} \quad \forall a, b \geq 0. \quad (3.4)$$

Let r be a number of the form $r = \frac{m}{2^n}$, where n is any positive integer and $1 \leq m < 2^n$. We will prove that $b^r \geq a^r$, assuming $b \geq a \geq 0$, using mathematical induction on n .

Equation (3.4) shows that $b^r \geq a^r$ when $n = 1$ (and hence $m = 1$). Suppose it is true that $b^{\frac{m}{2^j}} \geq a^{\frac{m}{2^j}}$ for all $j = 1, \dots, n-1$.

By assumption, $1 \leq m < 2^n$. Thus, either $m \leq 2^{n-1}$ or $2^{n-1} < m < 2^n$. Assume first that $m \leq 2^{n-1}$. By the induction hypothesis, $b^{\frac{m}{2^{n-1}}} \geq a^{\frac{m}{2^{n-1}}}$. Hence, By equation (3.4), $(b^{\frac{m}{2^{n-1}}})^{\frac{1}{2}} \geq (a^{\frac{m}{2^{n-1}}})^{\frac{1}{2}}$; that is, $b^r \geq a^r$. Assume the second possibility: $2^{n-1} < m < 2^n$. Set $k = m - 2^{n-1}$ and note that $k < 2^n - 2^{n-1} = 2^{n-1}$. Assume further that a is invertible. Then $b \geq a > 0$ implies $a^{-1} \geq b^{-1}$ and so $b^r a^{-1} b^r \geq b^r b^{-1} b^r = b^{\frac{k}{2^{n-1}}} \geq a^{\frac{k}{2^{n-1}}}$. Thus $a^{-\frac{1}{2}} b^r a^{-1} b^r a^{-\frac{1}{2}} \geq a^{-\frac{1}{2}} a^{\frac{k}{2^{n-1}}} a^{-\frac{1}{2}} = a^{\frac{m}{2^{n-1}} - 2}$ (induction-hypothesis). This can also be written as

$$(a^{-\frac{1}{2}} b^{\frac{m}{2^n}} a^{-\frac{1}{2}})^2 \geq a^{\frac{m}{2^{n-1}} - 2}.$$

By (3.4),

$$a^{-\frac{1}{2}} b^{\frac{m}{2^n}} a^{-\frac{1}{2}} \geq (a^{\frac{m}{2^{n-1}} - 2})^{\frac{1}{2}} \geq a^{\frac{m}{2^n} - 1}.$$

Hence, $b^{\frac{m}{2^n}} \geq a^{\frac{m}{2^n}}$ if a is invertible. If a is not invertible, then replace a and b with $a + \varepsilon 1$ and $b + \varepsilon 1$ and let $\varepsilon \rightarrow 0_+$ to obtain $b^r \geq a^r$. This completes the proof of operator monotonicity of f by induction.

Secondly, we aim to show $f(t) = t^r$ is operator concave.

Let z be any projection in $B(H)$, $a \geq 0$,

$$w_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$u = \begin{pmatrix} z & 1-z \\ 1-z & -z \end{pmatrix}.$$

Then u is a unitary and

$$u^*w_1u = \begin{pmatrix} zaz & za(1-z) \\ (1-z)az & (1-z)a(1-z) \end{pmatrix}.$$

Hence by Lemma 3.2, for each $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$u^*w_1u \leq \begin{pmatrix} zaz + \varepsilon & 0 \\ 0 & \lambda 1 \end{pmatrix}.$$

Replacing w_1 by $f(w_1)$ we have

$$\begin{aligned} u^*f(w_1)u &= \begin{pmatrix} zf(a)z & zf(a)(1-z) \\ (1-z)f(a)z & (1-z)f(a)(1-z) \end{pmatrix} \\ &= f(u^*w_1u) \\ &\leq \begin{pmatrix} f(zaz + \varepsilon) & 0 \\ 0 & f(\lambda)1 \end{pmatrix}, \end{aligned}$$

the second equality is obtained by computing directly and the last inequality is by the operator monotonicity of f .

Therefore $zf(a)z \leq f(zaz + \varepsilon)$, for each $\varepsilon > 0$. Then

$$zf(a)z \leq f(zaz), \tag{3.5}$$

for any projection z .

Let a and b be positive,

$$w_2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$w_3 = \begin{pmatrix} \lambda^{\frac{1}{2}}1 & -(1-\lambda)^{\frac{1}{2}}1 \\ (1-\lambda)^{\frac{1}{2}}1 & \lambda^{\frac{1}{2}}1 \end{pmatrix},$$

where $0 \leq \lambda \leq 1$ is a real number.

Then w_3 is unitary and

$$pw_3^*w_2w_3p = \begin{pmatrix} \lambda a + (1 - \lambda)b & 0 \\ 0 & 0 \end{pmatrix},$$

So

$$\begin{aligned} & \begin{pmatrix} f(\lambda a + (1 - \lambda)b) & 0 \\ 0 & f(0) \end{pmatrix} \\ &= f(pw_3^*w_2w_3p) \\ &\geq pf(w_3^*w_2w_3)p \\ &= pw_3^*f(w_2)w_3p \\ &= \begin{pmatrix} \lambda f(a) + (1 - \lambda)f(b) & 0 \\ 0 & f(0) \end{pmatrix}. \end{aligned}$$

The inequality is by equation (3.5).

Therefore

$$f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b).$$

Hence f is operator concave.

Proposition 3.4 Let $f(t) = -t^r$, $r \in (0, 1]$, then $f(z^*az) \leq z^*f(a)z$ for all $a, z \in B(H)$ such that $\|z\| \leq 1$ and a is positive.

Proof. For $\forall \xi \in H$, $\langle z^*z\xi, \xi \rangle = \langle z\xi, z\xi \rangle = \|z\xi\|^2 \leq \|\xi\|^2 = \langle \xi, \xi \rangle$. Hence $1 - z^*z \geq 0$, and thus $1 - z^*z$ has a unique positive square root $(1 - z^*z)^{\frac{1}{2}}$. Because z^*z and zz^* have the same nonzero spectra, $1 - zz^*$ is also positive and has a positive square root $(1 - zz^*)^{\frac{1}{2}}$.

Let

$$w = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = a \oplus 0 \in B(H \oplus H),$$

$$u = \begin{pmatrix} z & (1 - zz^*)^{\frac{1}{2}} \\ (1 - z^*z)^{\frac{1}{2}} & -z^* \end{pmatrix},$$

and

$$v = \begin{pmatrix} z & -(1 - zz^*)^{\frac{1}{2}} \\ (1 - z^*z)^{\frac{1}{2}} & z^* \end{pmatrix}.$$

We note that u, v are unitary operators on $H \oplus H$ [9, p.121], and that

$$\frac{1}{2}(u^*wu + v^*wv) = \begin{pmatrix} z^*az & 0 \\ 0 & (1 - zz^*)^{\frac{1}{2}}a(1 - zz^*)^{\frac{1}{2}} \end{pmatrix}.$$

Therefore, by the fact that f is operator convex,

$$\begin{aligned} & \begin{pmatrix} f(z^*az) & 0 \\ 0 & f((1 - zz^*)^{\frac{1}{2}}a(1 - zz^*)^{\frac{1}{2}}) \end{pmatrix} \\ & \leq \frac{1}{2}(f(u^*wu) + f(v^*wv)) \\ & = \frac{1}{2}(u^*f(w)u + v^*f(w)v) \\ & = \frac{1}{2} \left(u^* \begin{pmatrix} f(a) & 0 \\ 0 & f(0) \end{pmatrix} u + v^* \begin{pmatrix} f(a) & 0 \\ 0 & f(0) \end{pmatrix} v \right) \\ & = \begin{pmatrix} z^*f(a)z & 0 \\ 0 & (1 - zz^*)^{\frac{1}{2}}f(a)(1 - zz^*)^{\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

(Note that $f(0) = 0$.) Consequently, we get $f(z^*az) \leq z^*f(a)z$.

Proposition 3.5 Suppose that $r \in (0, 1]$, $s \in [1, 2]$, and that x, y , and z are Hermitian operators such that $0 \leq x \leq y$ and $\|z\| \leq 1$. Then

- (a) $x^r \leq y^r$,
- (b) $z^*x^rz \leq (z^*xz)^r$, and
- (c) $(z^*xz)^s \leq z^*x^sz$.

Proof. If $r \in (0, 1]$, then the function $t \mapsto t^r$, defined on $[0, \infty)$, is operator monotone and operator concave (by Proposition 3.3); hence, $0 \leq x \leq y$ implies that $x^r \leq y^r$.

Let $f(t) = -t^r$, which is operator convex on $[0, \infty)$ and satisfies $f(0) = 0$. By Proposition 3.4, $f(z^*xz) \leq z^*f(x)z$, and so $z^*x^rz \leq (z^*xz)^r$, proving (b).

Next, let $s \in (1, 2]$, $g(t) = t^s$, and $h(t) = \frac{1}{t}g(t) = t^{s-1} = t^r$, where $r = s-1 \in (0, 1]$. As h is operator monotone on $(0, \infty)$, g is necessarily operator convex on $[0, \infty)$, implying that $(z^*xz)^s = g(z^*xz) \leq z^*g(x)z = z^*x^sz$. The case $s = 1$ is trivial.

3.2 Poincaré's Inequality

In this section, we state and prove Poincaré's inequality for compact operators.

Theorem 3.6 (Poincaré's Inequality) If $h \in B(H)$ is positive and compact, and if $p \in B(H)$ is a projection, then the k^{th} nonzero eigenvalue of php is no larger than the k^{th} (nonzero) eigenvalue of h ; that is,

$$\lambda_k(php) \leq \lambda_k(h)$$

for each k .

The proof requires a preliminary lemma.

Lemma 3.7 If $x \in B(H)$ is compact, and if $\lambda \in \mathbb{C} \setminus \{0\}$, then

$$\dim(\ker(x^*x - \lambda 1)) = \dim(\ker(xx^* - \lambda 1)).$$

Proof. Assume that λ is a nonzero eigenvalue of x^*x , and suppose that $\phi \in H$ is any unit eigenvector corresponding to λ . Write x in its polar decomposition: $x = v|x|$. Thus $x^*x = |x|^2$ and $xx^* = v|x|^2v^*$. Because ϕ is an element in the closed range of $|x|$, and because v maps $(\text{ran}(|x|))^\perp$ isometrically (into the closed range of x), $v^*v\phi = \phi$. Hence $v\phi \neq 0$ and

$$xx^*(v\phi) = v|x|^2v^*v\phi = v|x|^2\phi = v(x^*x)\phi = v(\lambda\phi) = \lambda(v\phi).$$

Thus, $v\phi \in \ker(xx^* - \lambda 1)$. This shows that v is an isometric linear map from $\ker(x^*x - \lambda 1)$ to $\ker(xx^* - \lambda 1)$, and hence $\dim(\ker(x^*x - \lambda 1)) \leq \dim(\ker(xx^* - \lambda 1))$. By interchanging the roles of x and x^* in the arguments above, equality is achieved. This proves that $\dim(\ker(x^*x - \lambda 1)) = \dim(\ker(xx^* - \lambda 1))$ for all nonzero λ .

Proof of Poincaré's Inequality. Fix k and consider the k^{th} nonzero eigenvalue $\lambda_k(php)$ of php . From $php = ph^{\frac{1}{2}}h^{\frac{1}{2}}p = (ph^{\frac{1}{2}})(ph^{\frac{1}{2}})^*$, and from Lemma 3.7, we may rewrite $\lambda_k(php)$ as follows:

$$\lambda_k(php) = \lambda_k((ph^{\frac{1}{2}})^*(ph^{\frac{1}{2}})).$$

Pick any subspace $\mu \subset H$ of dimension $k - 1$. If $\xi \in \mu^\perp$ is a unit vector, then

$$\langle (ph^{\frac{1}{2}})^*(ph^{\frac{1}{2}})\xi, \xi \rangle = \|ph^{\frac{1}{2}}\xi\|^2 \leq \|p\|^2 \cdot \|h^{\frac{1}{2}}\xi\|^2 = \langle h\xi, \xi \rangle.$$

Thus,

$$\max_{\substack{\xi \in \mu^\perp \\ \|\xi\|=1}} \langle (ph^{\frac{1}{2}})^*(ph^{\frac{1}{2}})\xi, \xi \rangle \leq \max_{\substack{\xi \in \mu^\perp \\ \|\xi\|=1}} \langle h\xi, \xi \rangle.$$

Now, if one minimizes each maximum over all possible $(k - 1)$ -dimensional subspaces μ , the Minimum-Maximum Principle (Theorem 2.2) yields

$$\lambda_k(php) = \min_{\dim(\mu)=k-1} \max_{\substack{\xi \in \mu^\perp \\ \|\xi\|=1}} \langle (ph^{\frac{1}{2}})^*(ph^{\frac{1}{2}})\xi, \xi \rangle \leq \min_{\dim(\mu)=k-1} \max_{\substack{\xi \in \mu^\perp \\ \|\xi\|=1}} \langle h\xi, \xi \rangle = \lambda_k(h).$$

This completes the proof.

3.3 Young's Inequality

We now consider the logarithm function $f(t) = \log t$, $t \in (0, \infty)$. We know that it is a strictly concave function, i.e., for any $\tau \in [0, 1]$ we have

$$f((1 - \tau)t_1 + \tau t_2) \geq (1 - \tau)f(t_1) + \tau f(t_2),$$

for every $t_1, t_2 \in (0, \infty)$; and if for some $t_1, t_2 \in (0, \infty)$,

$$f((1 - \tau)t_1 + \tau t_2) = (1 - \tau)f(t_1) + \tau f(t_2),$$

then we must have $t_1 = t_2$.

Therefore for every $p, q \geq 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\frac{1}{p}f(t_1) + \frac{1}{q}f(t_2) \leq f\left(\frac{1}{p}t_1 + \frac{1}{q}t_2\right),$$

which implies that

$$e^{\frac{1}{p}f(t_1) + \frac{1}{q}f(t_2)} \leq e^{f\left(\frac{1}{p}t_1 + \frac{1}{q}t_2\right)},$$

i.e.,

$$e^{\log t_1^{\frac{1}{p}}} e^{\log t_2^{\frac{1}{q}}} \leq e^{\log\left(\frac{1}{p}t_1 + \frac{1}{q}t_2\right)}.$$

This shows that

$$t_1^{\frac{1}{p}} t_2^{\frac{1}{q}} \leq \frac{1}{p}t_1 + \frac{1}{q}t_2.$$

Write $\lambda = t_1^{\frac{1}{p}}$, $\mu = t_2^{\frac{1}{q}}$ to obtain

$$\lambda\mu \leq \frac{1}{p}\lambda^p + \frac{1}{q}\mu^q.$$

This is the classical version of Young's inequality.

Furthermore, by the strict concavity of the logarithm, if

$$\lambda\mu = \frac{1}{p}\lambda^p + \frac{1}{q}\mu^q,$$

Then $\mu = \lambda^{p-1}$.

In taking $p = q = 2$ and if $\lambda = \alpha^{\frac{1}{2}}, \mu = \beta^{\frac{1}{2}}$, then Young's inequality implies the arithmetic-geometric mean inequality

$$\sqrt{\alpha\beta} \leq \frac{1}{2}(\alpha + \beta),$$

where α, β are any positive numbers.

The following theorem is the Young's inequality for the case of a commutative C^* -algebra.

Theorem 3.8 If A is a commutative C^* -algebra with the multiplicative identity, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ for all $a, b \in A$. If the equality holds, then $|b| = |a|^{p-1}$.

Proof. Since a, b are commuting, so are a^*a and b^*b . Therefore $(a^*a)^{\frac{1}{2}}$ and $(b^*b)^{\frac{1}{2}}$ are also commuting. Hence

$$(b^*b)^{\frac{1}{2}}(a^*a)^{\frac{1}{2}} \cdot (b^*b)^{\frac{1}{2}}(a^*a)^{\frac{1}{2}} = b^*b \cdot a^*a = b^*a^* \cdot ab,$$

which means that

$$(b^*ba^*a)^{\frac{1}{2}} = (a^*ab^*b)^{\frac{1}{2}} = (b^*b)^{\frac{1}{2}}(a^*a)^{\frac{1}{2}} = (a^*a)^{\frac{1}{2}}(b^*b)^{\frac{1}{2}}.$$

That is to say

$$|ab| = |a||b| = |b||a|.$$

(So $|a||b| = |b||a|$ is a positive operator.)

Let Ω be the maximal ideal space of A . By Theorem 2.3, there is a $*$ -isomorphism $\phi : A \rightarrow C(\Omega)$, where $C(\Omega)$ is the C^* -algebra of all continuous complex-valued functions on Ω . Given arbitrary $a, b \in A$, let $f = \phi(|a|)$ and $g = \phi(|b|)$. Then for every

$w \in \Omega$, we have Young's inequality in real numbers:

$$0 \leq f(w)g(w) \leq \frac{1}{p}f(w)^p + \frac{1}{q}g(w)^q.$$

Thus, as elements of $C(\Omega)$, $0 \leq fg \leq \frac{1}{p}f^p + \frac{1}{q}g^q$, which, via ϕ , implies that $0 \leq |a||b| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$. According to the discussion above, we get

$$0 \leq |ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Similarly, if equality holds, i.e., if

$$0 \leq |ab| = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,$$

then

$$0 \leq f(w)g(w) = \frac{1}{p}f(w)^p + \frac{1}{q}g(w)^q.$$

According to Young's inequality in scalars, we get $g(\omega) = f(\omega)^{p-1}$. Therefore

$$|b| = |a|^{p-1}.$$

Chapter 4

YOUNG'S INEQUALITY IN NONZERO EIGENVALUES

Young's inequality in compact operators is proved through a comparison of the eigenvalues of $|ab^*|$ with those of $\frac{1}{p}|a|^p + \frac{1}{q}|b|^q$. Thus, in this chapter, Young's inequality is established at the level of eigenvalues. The following chapter passes from eigenvalues to operators.

4.1 A Compression Inequality

A key technique in Ando's paper [2] appears here as the following proposition. Its formulation below is more technical than Ando's original because of a significant new complicating factor: compact operators on infinite-dimensional spaces do not have inverses [8]. Furthermore, the result below is (in infinite dimensions) slightly weaker than Ando's result for matrices, for what is lost is finer information about the geometric multiplicities of individual eigenvalues.

Proposition 4.1 Suppose that a and b are positive operators acting on a separable Hilbert space H , and assume that b is invertible. Let e denote the spectral resolution of the identity for the positive operator $(ba^2b)^{1/2}$. For each $t \in [0, \infty)$, set $e_t = e([t, \infty))$ and let h_t denote the projection onto the (closed) subspace $H_t = b^{-1}e_t(H)$. Then, given $p \in (1, 2]$ and $q = \frac{p}{(p-1)}$,

$$th_t \leq h_t \left(\frac{1}{p}a^p + \frac{1}{q}b^q \right) h_t, \text{ for every } t \in [0, \infty).$$

Proof. Fix $t \in [0, \infty)$, and choose any $\eta \in H$. Then $h_t\eta \in H_t$ and so $h_t\eta = b^{-1}e_t\xi$ for some $\xi \in H$. Therefore, $bh_t\eta = e_t\xi = e_t^2\xi = e_tbh_t\eta$, which implies that $e_tbh_t = bh_t$. Passing to adjoints yields $h_tb = h_tbe_t$. By the definition of h_t , $b^{-1}e_t = h_tb^{-1}e_t$ and $e_tb^{-1} = e_tb^{-1}h_t$. Thus,

$$(h_tb^2h_t)(b^{-1}e_tb^{-1}) = h_t \text{ and } (b^{-1}e_tb^{-1})(h_tb^2h_t) = h_t.$$

From this we make the following conclusions:

- (i) that $b^{-1}e_tb^{-1}$ and $h_tb^2h_t$ map H_t bijectively onto H_t ;
- (ii) that, as operators on H_t , $b^{-1}e_tb^{-1}$ and $h_tb^2h_t$ are inverse to each other; and
- (iii) $b^{-1}e_tb^{-1}$ and $h_tb^2h_t$ vanish on $H_t^\perp$.

Using e , express $(ba^2b)^{1/2}$ as $\int_0^\infty \lambda de$; then, by the spectral theorem,

$$ba^2b = \int_0^\infty \lambda^2 de \geq \int_t^\infty \lambda^2 de \geq t^2 \int_t^\infty de = t^2 e_t.$$

Therefore, $t^2(b^{-1}e_tb^{-1}) \leq a^2$. Now set $r = \frac{p}{2}$; because $r \in (0, 1]$ by the hypothesis on p , Proposition 3.5 implies that

$$t^p(b^{-1}e_tb^{-1})^{p/2} = [t^2(b^{-1}e_tb^{-1})]^r \leq (a^2)^r = a^p.$$

From $h_t(b^{-1}e_tb^{-1})h_t = (h_tb^{-1}e_t)b^{-1}h_t = (b^{-1}e_t)b^{-1}h_t = b^{-1}(e_tb^{-1}h_t) = b^{-1}e_tb^{-1}$, the inequality above yields

$$t^p(b^{-1}e_tb^{-1})^{p/2} = t^2h_t(b^{-1}e_tb^{-1})^{p/2}h_t \leq h_t a^p h_t.$$

Let x and y be operators on H_t ; we write $x \leq_t y$ to denote the inequalities $\langle x\eta, \eta \rangle \leq \langle y\eta, \eta \rangle$ for every $\eta \in H_t$. Because as an operator on H_t the inverse of $b^{-1}e_tb^{-1}$ is $h_t b^2 h_t$ [by (ii)], the inequality above becomes

$$t^p(h_t b^2 h_t)^{-p/2} \leq_t h_t a^p h_t. \quad (4.6)$$

The remainder of this proof splits into two cases.

Case # 1: $q \in [2, 4]$. Set $s = \frac{q}{2}$. Because $s \in [1, 2]$ and $\|h_t\| = 1$, Proposition 3.5(c) implies that $(h_t b^2 h_t)^{q/2} = (h_t b^2 h_t)^s \leq h_t (b^2)^s h_t = h_t b^q h_t$. This inequality, coupled with (4.6), results in the following inequality for operators on H_t :

$$\frac{1}{p} \left[t(h_t b^2 h_t)^{-1/2} \right]^p + \frac{1}{q} \left[(h_t b^2 h_t)^{1/2} \right]^q \leq_t h_t \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) h_t. \quad (4.7)$$

As the positive operators $(h_t b^2 h_t)^{-1/2}$ and $(h_t b^2 h_t)^{1/2}$ commute when acting on H_t , Theorem 3.8 gives the first of the two inequalities that follow:

$$\begin{aligned} t(h_t b^2 h_t)^{-1/2} (h_t b^2 h_t)^{1/2} &\leq_t \frac{1}{p} \left[t(h_t b^2 h_t)^{-1/2} \right]^p + \frac{1}{q} \left[(h_t b^2 h_t)^{1/2} \right]^q \\ &\leq_t h_t \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) h_t. \end{aligned}$$

The second inequality above follows from (4.7). On both sides of the inequality above, the operators in question vanish on $H_t^\perp$, implying that the inequality extends from H_t to H :

$$t h_t \leq h_t \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) h_t,$$

as desired.

Case # 2: $q \in (4, \infty)$. Set $u = \frac{q}{2}$ so that $\frac{2}{u} \in (0, 1)$ and $\frac{q}{u} = 2$. By Proposition 3.5(c), using $s = \frac{q}{u}$, we have

$$(h_t b^u h_t)^2 = (h_t b^u h_t)^{q/u} \leq h_t (b^u)^{q/u} h_t = h_t b^q h_t. \quad (4.8)$$

Using $r = \frac{2}{u}$, Proposition 3.5(b) yields

$$h_t b^2 h_t = h_t (b^u)^r h_t \leq (h_t b^u h_t)^r = (h_t b^u h_t)^{2/u}.$$

Apply Proposition 3.5(a), using $r = \frac{p}{2}$, to the inequality above to obtain

$$(h_t b^2 h_t)^{p/2} \leq (h_t b^u h_t)^{(2/u)(p/2)} = (h_t b^u h_t)^{p/u}.$$

Because $h_t b^2 h_t$ is invertible as an operator on H_t , the inequality above implies that $h_t b^u h_t$ is also invertible as an operator on H_t . Thus,

$$(h_t b^u h_t)^{-p/u} \leq_t (h_t b^2 h_t)^{-p/2}. \quad (4.9)$$

Now $(h_t b^u h_t)^{-1/u}$ and $(h_t b^u h_t)^{1/u}$ are positive operators commuting on H_t , and so Theorem 3.8 can be invoked to obtain the first inequality below:

$$\begin{aligned} t(h_t b^u h_t)^{-1/u} (h_t b^u h_t)^{1/u} &\leq_t \frac{1}{p} \left[t(h_t b^u h_t)^{-1/u} \right]^p + \frac{1}{q} \left[(h_t b^u h_t)^{1/u} \right]^q \\ &\leq_t \frac{1}{p} \left[t^p (h_t b^2 h_t)^{-p/2} \right] + \frac{1}{q} \left[(h_t b^u h_t)^2 \right] \\ &\leq_t h_t \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) h_t. \end{aligned}$$

The middle inequality above follows from (4.9) and the fact that $\frac{q}{u} = 2$; the third inequality above follows from (4.6) and (4.8). The left-hand side of the inequality above is the operator th_t on H_t , and therefore the entire inequality extends to H (because both sides vanish on $H_t^\perp$) to get

$$th_t \leq h_t \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) h_t,$$

as desired.

4.2 Young's Inequality in Nonzero Eigenvalues

In this section we use Proposition 4.1 to establish an inequality between the nonzero eigenvalues of $(ba^2b)^{1/2}$ and those of $\frac{1}{p}a^p + \frac{1}{q}b^q$, where a and b are positive compact operators. Because compact operators are not invertible, we first need to consider perturbations of b by $\varepsilon 1$ (for $\varepsilon > 0$) in order to invoke Proposition 4.1.

Lemma 4.2 If A is a unital C^* -algebra, if $b \in A$ is positive, and if $q \in (1, \infty)$, then there is a function $\gamma : (0, \infty) \rightarrow [0, \infty)$ such that

- (a) $\lim_{\varepsilon \rightarrow 0^+} \gamma(\varepsilon) = 0$, and
- (b) for each $\varepsilon > 0$ there is a positive $y_\varepsilon \in A$ with $(b + \varepsilon 1)^q = b^q + y_\varepsilon$ and $\|y_\varepsilon\| \leq \gamma(\varepsilon)$.

Proof. Let $\beta > 0$ be such that $\sigma(b) \subseteq [0, \beta]$, and for each $\varepsilon > 0$ let g_ε be the function $g_\varepsilon(t) = t + \varepsilon$. Then $g_\varepsilon(t) \rightarrow t$ uniformly on $[0, \beta]$ as $\varepsilon \rightarrow 0^+$; hence, $(t + \varepsilon)^q \rightarrow t^q$ uniformly on $[0, \beta]$. Set $f_\varepsilon(t) = (t + \varepsilon)^q - t^q$, so that $f_\varepsilon \rightarrow 0$ uniformly on $[0, \beta]$ as $\varepsilon \rightarrow 0^+$. Define $\gamma(\varepsilon) = \max_{t \in [0, \beta]} f_\varepsilon(t)$ and, using functional calculus, let $y_\varepsilon = f_\varepsilon(b)$. Then y_ε is positive, $\|y_\varepsilon\| \leq \gamma(\varepsilon)$, and $\gamma(\varepsilon) \rightarrow 0$. As $y_\varepsilon = (b + \varepsilon 1)^q - b^q$, we have $(b + \varepsilon 1)^q = b^q + y_\varepsilon$.

Lemma 4.3 If a and b are positive compact operators, and if $0 < \varepsilon < 1$ and $b_\varepsilon = b + \varepsilon 1$, then, for any k ,

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_k((b_\varepsilon a^2 b_\varepsilon)^{1/2}) = \lambda_k((ba^2b)^{1/2}).$$

Proof. Fix k . For every $\varepsilon > 0$, $b_\varepsilon a^2 b_\varepsilon = ba^2b + \varepsilon(ba^2 + a^2b + \varepsilon a^2) = ba^2b + z_\varepsilon$, where z_ε is compact and $\|z_\varepsilon\| \leq \varepsilon M$, given $M = \|ba^2\| + \|a^2b\| + \|a^2\|$. Thus

$$\|b_\varepsilon a^2 b_\varepsilon - ba^2b\| \leq \varepsilon M$$

which implies that $|\langle b_\varepsilon a^2 b_\varepsilon \xi, \xi \rangle - \langle ba^2 b \xi, \xi \rangle| \leq \varepsilon M$, for every unit vector $\xi \in H$. Therefore, by the minimum-maximum principle, $|\lambda_k(b_\varepsilon a^2 b_\varepsilon) - \lambda_k(ba^2 b)| \leq \varepsilon M$. Thus,

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_k(b_\varepsilon a^2 b_\varepsilon) = \lambda_k(ba^2 b).$$

On passing to the positive square roots of the positive compact operators $b_\varepsilon a^2 b_\varepsilon$ and $ba^2 b$, the geometric multiplicities of the nonzero eigenvalues are unchanged. Hence,

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_k((b_\varepsilon a^2 b_\varepsilon)^{1/2}) = \lambda_k((ba^2 b)^{1/2}).$$

We now arrive at the crucial spectral inequality between nonzero eigenvalues that leads to Theorem 1.1.

Proposition 4.4 If a and b are positive compact operators, and if $p \in (1, 2]$ and $q = \frac{p}{p-1}$, then, for every k ,

$$\lambda_k((ba^2 b)^{1/2}) \leq \lambda_k\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right).$$

Proof. Let $\varepsilon > 0$ and consider $b_\varepsilon = b + \varepsilon 1$. By Lemma 4.2, there is a positive operator y_ε such that $(b_\varepsilon)^q = b^q + y_\varepsilon$ and $\|y_\varepsilon\| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Because the operator $(b_\varepsilon a^2 b_\varepsilon)^{1/2}$ is positive and compact, there is an orthonormal sequence $\{\phi_j^{(\varepsilon)}\}_j$ in H consisting of eigenvectors:

$$(b_\varepsilon a^2 b_\varepsilon)^{1/2} \phi_j^{(\varepsilon)} = \lambda_j((b_\varepsilon a^2 b_\varepsilon)^{1/2}) \phi_j^{(\varepsilon)}.$$

Fix k . Let e_ε denote the spectral resolution of the identity for $(b_\varepsilon a^2 b_\varepsilon)^{1/2}$, and set $t_\varepsilon = \lambda_k((b_\varepsilon a^2 b_\varepsilon)^{1/2})$. Let e'_ε be the projection onto the span of $\{\phi_1^{(\varepsilon)}, \dots, \phi_k^{(\varepsilon)}\}$ and let p_ε denote the projection $p_\varepsilon = e_\varepsilon([t_\varepsilon, \infty))$. Then

$$e'_\varepsilon(H) \subseteq \sum_{\mu \geq t_\varepsilon}^\oplus \ker((b_\varepsilon a^2 b_\varepsilon)^{1/2} - \mu 1) = e_\varepsilon([t_\varepsilon, \infty))(H) = p_\varepsilon(H).$$

Therefore, $b_\varepsilon^{-1}e'_\varepsilon(H) \subseteq b_\varepsilon^{-1}p_\varepsilon(H)$. Let h'_{t_ε} and h_{t_ε} be the projections onto $b_\varepsilon^{-1}e'_\varepsilon(H)$ and $b_\varepsilon^{-1}p_\varepsilon(H)$ respectively. Then,

$$t_\varepsilon h'_{t_\varepsilon} \leq t_\varepsilon h_{t_\varepsilon} \leq h_{t_\varepsilon} \left(\frac{1}{p}a^p + \frac{1}{q}b_\varepsilon^q \right) h_{t_\varepsilon} \leq h_{t_\varepsilon} \left(\frac{1}{p}a^p + \frac{1}{q}b^q \right) h_{t_\varepsilon} + \frac{1}{q}h_{t_\varepsilon}y_\varepsilon h_{t_\varepsilon}.$$

The second inequality above is from Proposition 4.1.

Recall (from linear algebra) that if $M, N \subseteq H$ are subspaces such that $\dim M < \dim N < \infty$, then $M^\perp \cap N \neq \{0\}$. Therefore, because the projection h'_{t_ε} has rank k , $L^\perp \cap (h'_{t_\varepsilon}(H)) \neq \{0\}$ for every $(k-1)$ -dimensional subspace $L \subset H$. Thus,

$$\min_{\dim L=k-1} \max_{\substack{\xi \in L^\perp \\ \|\xi\|=1}} \langle h'_{t_\varepsilon} \xi, \xi \rangle = 1.$$

What this implies further is:

$$\begin{aligned} \lambda_k((b_\varepsilon a^2 b_\varepsilon)^{1/2}) &= t_\varepsilon = \min_{\dim L=k-1} \max_{\substack{\xi \in L^\perp \\ \|\xi\|=1}} \langle t_\varepsilon h'_{t_\varepsilon} \xi, \xi \rangle \\ &\leq \min_{\dim L=k-1} \max_{\substack{\xi \in L^\perp \\ \|\xi\|=1}} \left\langle \left(\frac{1}{p}a^p + \frac{1}{q}b_\varepsilon^q \right) \xi, \xi \right\rangle \\ &\leq \left(\min_{\dim L=k-1} \max_{\substack{\xi \in L^\perp \\ \|\xi\|=1}} \left\langle \left(\frac{1}{p}a^p + \frac{1}{q}b^q \right) \xi, \xi \right\rangle \right) + \frac{1}{q} \|y_\varepsilon\| \\ &= \lambda_k \left(\frac{1}{p}a^p + \frac{1}{q}b^q \right) + \frac{1}{q} \|y_\varepsilon\|. \end{aligned}$$

Hence, by the inequalities above and Lemmas 4.2 and 4.3,

$$\begin{aligned} \lambda_k((ba^2b)^{1/2}) &= \lim_{\varepsilon \rightarrow 0^+} \lambda_k((b_\varepsilon a^2 b_\varepsilon)^{1/2}) \\ &\leq \lambda_k \left(\frac{1}{p}a^p + \frac{1}{q}b^q \right) + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{q} \|y_\varepsilon\| \\ &= \lambda_k \left(\frac{1}{p}a^p + \frac{1}{q}b^q \right), \end{aligned}$$

which is precisely the sought after inequality.

Chapter 5

YOUNG'S INEQUALITY IN COMPACT OPERATORS

In this chapter, we are going to prove our main result, Young's inequality in compact operators.

5.1 Transfer of Point Spectra

To this stage we have worked only with positive operators. The present section facilitates the transition from positive operators of the type $(ba^2b)^{1/2}$ to positive operators of the form $|xy^*|$.

Proposition 5.1 For any operators a and b acting on a Hilbert space,

$$\dim(\ker(|ab^*| - \lambda 1)) = \dim\left(\ker\left(\left||a||b|\right| - \lambda 1\right)\right), \text{ for every nonzero } \lambda.$$

Proof. Let $b = w|b|$ denote the polar decomposition of b . Observe that $\left||a||b|\right| = (|b||a|^2|b|)^{1/2}$. Thus, because the closures of the ranges of a positive operator and

its square root coincide, the closures of the ranges of $|b||a|^2|b|$ and $|a||b|$ are equal.

Moreover, as $w^*w|a||b| = |a||b|$, we have that

$$f(w(|b||a|^2|b|)w^*) = wf(|b||a|^2|b|)w^*, \quad (5.10)$$

for all polynomials f . Choose $\delta > 0$ so that $\sigma(|b||a|^2|b|) \subseteq [0, \delta]$. By the Weierstrass approximation theorem, there is a sequence of polynomials f_n such that $f_n(t) \rightarrow \sqrt{t}$ uniformly on $[0, \delta]$. Thus, from (5.10) and functional calculus,

$$(w(|b||a|^2|b|)w^*)^{1/2} = w(|b||a|^2|b|)^{1/2}w^* = w|a||b|w^*.$$

Let $a = v|a|$ be the polar decomposition of a . Then the left-hand term in the equalities above expands as follows:

$$\begin{aligned} (w(|b||a|^2|b|)w^*)^{1/2} &= (w|b||a|v^*v|a||b|w^*)^{1/2} \\ &= (ba^*ab^*)^{1/2} \\ &= |ab^*|. \end{aligned}$$

Thus, $|ab^*| = w|a||b|w^*$.

Next, let λ be a nonzero eigenvalue of $|a||b|$ and let $\{\phi_j\}_j$ be a sequence of orthonormal basis vectors for $\ker(|a||b| - \lambda 1)$. Fix j . Then $|a||b|\phi_j = \lambda\phi_j$ implies that $\lambda^2\phi_j = |a||b|^2\phi_j \in (|b||a|^2|b|)(H) \subseteq |b|(H) \subseteq [b^*(H)]^-$. Because ϕ_j belongs to the closure of the range of $b^*(H)$ and because w^*w is the projection onto this closed range, $\phi_j = w^*w\phi_j$. Thus, $\|w\phi_j\| = 1 \neq 0$. We aim to show that $w\phi_j$ is an eigenvector of $|ab^*|$ and that λ is its corresponding eigenvalue. This is a straightforward verification:

$$|ab^*|w\phi_j = w|a||b|w^*(w\phi_j) = w|a||b|\phi_j = w(\lambda\phi_j) = \lambda(w\phi_j).$$

The above relation is true for every j . If a finite sum $\sum_j \alpha_j(w\phi_j) = 0$, then

$$0 = w^* \left(\sum_j \alpha_j(w\phi_j) \right) = \sum_j \alpha_j(w^*w\phi_j) = \sum_j \alpha_j\phi_j,$$

and so $\alpha_j = 0$ for all j . This proves that

$$\dim \left(\ker \left(|a| \cdot |b| - \lambda 1 \right) \right) \leq \dim(\ker(|ab^*| - \lambda 1)).$$

Conversely, let λ be a nonzero eigenvalue of $|ab^*|$, and let $\{\psi_i\}_i$ be an orthonormal basis of $\ker(|ab^*| - \lambda 1)$. Fix i . Then $|ab^*|\psi_i = \lambda\psi_i$ implies that

$$\lambda^2\psi_i = (ba^*ab^*)\psi_i \in b(H).$$

Thus, $ww^*\psi_i = \psi_i$, because ww^* is the projection onto the closure of $b(H)$; in particular, $\|w^*\psi_i\| = 1 \neq 0$. Now $\lambda\psi_i = |ab^*|\psi_i = w \left(|a||b| \right) w^*\psi_i$ implies that

$$\lambda(w^*\psi_i) = w^*w \left(|a||b| \right) w^*\psi_i = \left(|a||b| \right) w^*\psi_i,$$

and hence $w^*\psi_i \in \ker \left(\left(|a||b| \right) - \lambda 1 \right)$. As we argued earlier, the vectors $\{w^*\psi_i\}_i$ are linearly independent and so

$$\dim(\ker(|ab^*| - \lambda 1)) \leq \dim \left(\ker \left(\left(|a||b| \right) - \lambda 1 \right) \right).$$

These arguments also show that a nonzero complex number λ is an eigenvalue of $|ab^*|$ if and only if λ is an eigenvalue of $\left(|a||b| \right)$, which completes the proof.

Lemma 5.2 For any two operators x and y acting on a Banach space,

$$\dim(\ker(xy^2x - \lambda 1)) = \dim(\ker(yx^2y - \lambda 1)), \text{ for all nonzero } \lambda \neq 0.$$

Proof. If ξ is a nonzero vector, $\lambda \neq 0$, and $xy^2x\xi = \lambda\xi$, then $yx\xi \neq 0$. Thus $yx^2y(yx\xi) = yx(xy^2x)\xi = \lambda(yx\xi)$, implying that λ is an eigenvalue of yx^2y . Assume that $\xi_1, \dots, \xi_k$ are linearly independent eigenvectors of xy^2x corresponding to the eigenvalue λ , and assume that $\alpha_1, \dots, \alpha_k$ are scalars such that $0 = \sum_j \alpha_j(yx\xi_j)$. Then

$$0 = xy \left(\sum_j \alpha_j yx\xi_j \right) = \sum_j \alpha_j (xy^2x\xi_j) = \lambda \sum_j \alpha_j \xi_j.$$

Because $\xi_1, \dots, \xi_k$ are linearly independent and $\lambda \neq 0$, we must have $\alpha_1 = \dots = \alpha_k = 0$. Thus $yx\xi_1, \dots, yx\xi_k$ are linearly independent. This proves that

$$\dim(\ker(xy^2x - \lambda 1)) \leq \dim(\ker(yx^2y - \lambda 1)) \quad \forall \lambda \neq 0.$$

The reverse inequality is proved similarly.

For the proposition that follows, recall that $\lambda_k(x)$ refers only to the *nonzero* eigenvalues of a positive compact operator x .

Proposition 5.3 If a and b are compact operators, then $\lambda_k(|ab^*|) = \lambda_k(|ba^*|)$.

Proof

$$\begin{aligned} \lambda_k(|ab^*|)^2 &= \lambda_k\left(\left||a||b|\right|\right)^2 && \text{[by Proposition 5.1]} \\ &= \lambda_k\left(\left||a||b|\right|^2\right) && \text{[by the Spectral Mapping Theorem]} \\ &= \lambda_k(|b||a|^2|b|) \\ &= \lambda_k(|a||b|^2|a|) && \text{[by Lemma 5.2]} \\ &= \lambda_k\left(\left||b||a|\right|\right)^2 \\ &= \lambda_k(|ba^*|)^2 && \text{[by Proposition 5.1].} \end{aligned}$$

Thus, $\lambda_k(|ab^*|) = \lambda_k(|ba^*|)$.

5.2 Young's Inequality in Compact operators

This section brings together the results that have been thus far established to prove the main theorem (Theorem 1.1) concerning Young's inequality.

Theorem 5.4 If a and b are compact, then there is a partial isometry u such that the initial space of u is $(\ker |ab^*|)^\perp$ and

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad (5.11)$$

for any $p, q \in (1, \infty)$ that satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Assume that a and b are arbitrary compact operators acting on H , and fix any pair $p, q \in (1, \infty)$ for which $\frac{1}{p} + \frac{1}{q} = 1$. Enumerating the nonzero eigenvalues of positive compact operators in the usual descending order, we have by Propositions 5.1 and 5.3 that

$$\lambda_k(|ab^*|) = \lambda_k(|ba^*|) = \lambda_k(|a||b|) = \lambda_k(|b||a|), \quad \forall k.$$

First assume that $p \in (1, 2]$; then Proposition 4.4 states that

$$\lambda_k(|a||b|) \leq \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right).$$

Second, if $p > 2$, then $q < 2$; so, Proposition 4.4 states that

$$\lambda_k(|b||a|) \leq \lambda_k\left(\frac{1}{q}|b|^q + \frac{1}{p}|a|^p\right).$$

In either case, the end result, for each k , is:

$$\lambda_k(|ab^*|) \leq \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right).$$

This means that $\frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ has at least as many nonzero eigenvalues as $|ab^*|$ has. Therefore, there are orthonormal sequences $\{\phi_k\}_k$ and $\{\psi_k\}_k$ in H such that, for every $\xi \in H$,

$$\begin{aligned} |ab^*|\xi &= \sum_k \lambda_k(|ab^*|) \langle \xi, \phi_k \rangle \phi_k, \\ \left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)\xi &= \sum_k \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) \langle \xi, \psi_k \rangle \psi_k. \end{aligned}$$

Let $u : H \rightarrow H$ be the operator for which $u\phi_k = \psi_k$ for every k , and for which $u\xi = 0$ for every $\xi \in \{\phi_k : k = 1, 2, \dots\}^\perp$. Then u is a partial isometry and, for every $\xi \in H$,

$$\begin{aligned} \langle u|ab^*|u^*\xi, \xi \rangle &= \sum_k \lambda_k(|ab^*|) |\langle \xi, u\phi_k \rangle|^2 \\ &= \sum_k \lambda_k(|ab^*|) |\langle \xi, \psi_k \rangle|^2 \\ &\leq \sum_k \lambda_k \left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q \right) |\langle \xi, \psi_k \rangle|^2 \\ &= \left\langle \left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q \right) \xi, \xi \right\rangle. \end{aligned}$$

Hence, $u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$.

To establish Young's inequality via unitary conjugation in the case where $|ab^*|$ has trivial kernel, the following variant of Lemma 2.2 of [7] is necessary.

Lemma 5.5 Suppose that z is a contraction and a is a positive compact operator. If $\ker(a) = \ker(z) = \{0\}$, then there is a unitary operator u such that $z^*az \leq u^*au$.

Proof. Because z is a contraction, the operator matrix w below is a unitary operator on $H \oplus H$ [9, p.121]:

$$w = \begin{pmatrix} z & (1 - zz^*)^{\frac{1}{2}} \\ (1 - z^*z)^{\frac{1}{2}} & -z^* \end{pmatrix}.$$

The operator z^*az appears as the (1,1)-entry of the (compact) operator

$$w^*(a \oplus 0)w = \begin{pmatrix} z^*az & z^*a(1 - zz^*)^{\frac{1}{2}} \\ (1 - z^*z)^{\frac{1}{2}}az & (1 - zz^*)^{\frac{1}{2}}a(1 - zz^*)^{\frac{1}{2}} \end{pmatrix}.$$

Because $\ker(a) = \ker(z) = \{0\}$ and a is positive, the compact positive operator z^*az has trivial kernel, and so its eigenvalues may be enumerated in decreasing order $\lambda_1(z^*az) \geq \lambda_2(z^*az) \geq \dots > 0$, with eigenvalues repeated in this list in accordance

with their geometric multiplicity. The nonzero eigenvalues of $w^*(a \oplus 0)w$ are precisely the eigenvalues of a (as $\ker(a) = \{0\}$) and, moreover, the multiplicity of any nonzero eigenvalue of $w^*(a \oplus 0)w$ is the same as the multiplicity of that eigenvalue of a . Thus, in listing the nonzero eigenvalues of $w^*(a \oplus 0)w$ and the eigenvalues of a in descending order, we have that $\lambda_k(w^*(a \oplus 0)w) = \lambda_k(a)$, for all positive integers k . Now invoke the Poincaré's inequality [3, p.59] to obtain

$$\lambda_k(a) = \lambda_k(w^*(a \oplus 0)w) \geq \lambda_k(z^*az)$$

for all k . Because a and z^*az are injective, there exist orthonormal bases $\{\phi_k\}_k$ and $\{\psi_k\}_k$ of H comprising eigenvectors of a and z^*az respectively. That is, for all k , $a\phi_k = \lambda_k(a)\phi_k$ and $z^*az\psi_k = \lambda_k(z^*az)\psi_k$. Let u be the unique unitary operator on H for which $u\psi_k = \phi_k$, for every k . Then the inequalities $\lambda_k(z^*az) \leq \lambda_k(a)$ imply, by standard arguments, that $z^*az \leq u^*au$.

Corollary 5.6 If a and b are compact operators, if $|ab^*|$ is injective, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then there is a unitary u_0 such that

$$u_0|ab^*|u_0^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Proof. Because $|ab^*|$ has trivial kernel, the partial isometry u that occurs in Young's inequality (5.11) is an isometry. Thus, $u^*u = 1$ and $\ker(u) = \{0\}$. From

$$u^*(u|ab^*|u^*)u \leq u^*\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)u$$

we obtain

$$|ab^*| \leq u^*\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)u.$$

This inequality implies that $u^*\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)u$ is an injective compact operator. Therefore, by Lemma 5.5, there is a unitary operator u_0 such that

$$u^*\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)u \leq u_0^*\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)u_0.$$

Hence,

$$u_0|ab^*|u_0^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,$$

thereby completing the proof.

Chapter 6

CHARACTERISING CASES OF EQUALITY

In this chapter, we discuss the cases of equality in Young's inequality.

6.1 The Commuting Case

At first, here, we deal with the case of commuting normal compact operators.

Lemma 6.1 If $a, b \in B(H)$ are normal and commuting, then

$$|a||b| = |b||a|,$$

and $|a||b|$ is positive.

Proof. We obviously have

$$a^*b^* = b^*a^*.$$

And by Fuglede theorem [9] we get

$$a^*b = ba^*, ab^* = b^*a.$$

On the other hand, if $c, d \in B(H)$ with c, d positive and commuting, then

$$c^{\frac{1}{2}}d^{\frac{1}{2}} \cdot c^{\frac{1}{2}}d^{\frac{1}{2}} = c^{\frac{1}{2}}c^{\frac{1}{2}} \cdot d^{\frac{1}{2}}d^{\frac{1}{2}} = cd.$$

Hence

$$(cd)^{\frac{1}{2}} = c^{\frac{1}{2}}d^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} |a||b| &= (a^*a)^{\frac{1}{2}}(b^*b)^{\frac{1}{2}} \\ &= (a^*ab^*b)^{\frac{1}{2}} \\ &= (b^*b)^{\frac{1}{2}}(a^*a)^{\frac{1}{2}} \\ &= |b||a|. \end{aligned}$$

Which implies that $|a||b|$ is positive and $|a||b| = (|a||b|)^* = |b||a|$. (In fact, $|a||b|$ is the positive square root of the positive operator a^*ab^*b .)

Lemma 6.2 If $a, b \in B(H)$ be normal operators such that $ab = ba$, then the following statements are equivalent:

- (i) $\ker(|ab^*|) = \{0\}$;
- (ii) a and b are injective and have dense range.

Proof. (i) $\Rightarrow$ (ii). Let $b = w|b|$ be the polar decomposition of b . The proof of Proposition 5.1 shows that $|ab^*| = w | |a||b| | w^*$. Because a and b are commuting normal, from Lemma 6.1 $|a||b| = |b||a|$ and $|a||b|$ is positive. This implies that

$$|ab^*| = w|a||b|w^*.$$

If $\xi \in \ker(w^*)$, then $\xi \in \ker(|ab^*|)$. Hence $\ker(w^*) = \{0\}$, which means that $\text{ran}(w) = H$. Hence, w is unitary. By the theorem on polar decomposition [9, p.75], b is injective and has dense range.

Let $a = v|a|$ be the polar decomposition of a . We know that $ab = ba$ implies that $ab^* = b^*a$ (again, by Fuglede theorem). Therefore, we can interchange the role of a and b in the previous paragraph to obtain: a^* is injective and has dense range. Thus, a is injective and has dense range.

(ii) $\Rightarrow$ (i). From the hypothesis we have polar decompositions $a = v|a|$, $b = w|b|$, where v and w are unitary [9, p.75]. Therefore, $\ker(|a|) = \ker(|b|) = \{0\}$. Because $|ab^*| = w|a||b|w^*$ and w is unitary, we have

$$\ker(|ab^*|) = \{0\}.$$

Lemma 6.3 If $x \in B(H)$ is positive, compact, and injective, and if $x \leq u^*xu$ for some unitary u , then u is diagonalisable and commutes with x .

Proof. Because x is injective, the Hilbert space H is the direct sum of the eigenspaces of x :

$$H = \sum_{\lambda \in \sigma_p(x)}^{\oplus} \ker(x - \lambda 1).$$

Let $\sigma_p(x) = \{\lambda_1, \lambda_2, \dots\}$, where $\lambda_1 > \lambda_2 > \dots > 0$ are the (distinct) eigenvalues of x , listed in descending order. Our first goal is to prove that $\ker(x - \lambda_j 1)$ is invariant under u and u^* for every $j \in \mathbf{Z}^+$; we shall do so by induction.

Start with λ_1 ; note that $\lambda_1 = \|x\|$. If $\xi \in \ker(x - \lambda_1 1)$ is a unit vector, then

$$\lambda_1 = \lambda_1 \langle \xi, \xi \rangle = \langle \lambda_1 \xi, \xi \rangle = \langle x\xi, \xi \rangle \leq \langle u^*xu\xi, \xi \rangle = \langle xu\xi, u\xi \rangle \leq \|x\| \cdot \|u\xi\|^2 = \lambda_1.$$

Thus,

$$\langle xu\xi, u\xi \rangle = \lambda_1 = \max\{\langle x\eta, \eta \rangle : \|\eta\| = 1\}.$$

Which means that $u\xi$ is an eigenvector of x corresponding to the eigenvalue λ_1 . Then, $u\xi \in \ker(x - \lambda_1 1)$. Because $\ker(x - \lambda_1 1)$ is finite-dimensional and u is unitary, we have that $u : \ker(x - \lambda_1 1) \rightarrow \ker(x - \lambda_1 1)$ is an isomorphism. Furthermore, $u|_{\ker(x - \lambda_1 1)}$

is diagonalisable because $\dim(\ker(x - \lambda_1 1)) < \infty$. Hence, $\ker(x - \lambda_1 1)$ is invariant under u^* (because $\ker(x - \lambda_1 1)$ has a finite orthonormal basis of eigenvectors of u), which means that if $\eta \perp \ker(x - \lambda_1 1)$, then $u\eta \perp \ker(x - \lambda_1 1)$.

Now choose λ_2 , and pick up a unit vector $\xi \in \ker(x - \lambda_2 1)$. Note that

$$\lambda_2 = \max\{\langle x\eta, \eta \rangle : \|\eta\| = 1, \eta \in \ker(x - \lambda_1 1)^\perp\}.$$

Using the arguments of the previous paragraph,

$$\lambda_2 = \langle x\xi, \xi \rangle \leq \langle xu\xi, u\xi \rangle \leq \lambda_2$$

(Because $u\xi$ is a unit vector orthogonal to $\ker(x - \lambda_1 1)$). Hence, by the minimum-maximum principle, $u\xi \in \ker(x - \lambda_2 1)$. So $u : \ker(x - \lambda_2 1) \rightarrow \ker(x - \lambda_2 1)$ is an isomorphism, $\ker(x - \lambda_2 1)$ has an orthonormal basis of eigenvectors of u . And if $\eta \perp \ker(x - \lambda_1 1) \oplus \ker(x - \lambda_2 1)$, then $u\eta \perp \ker(x - \lambda_1 1) \oplus \ker(x - \lambda_2 1)$.

Inductively, assume that u leaves $\ker(x - \lambda_j 1)$ invariant for all $1 \leq j \leq k$, and look at λ_{k+1} . By the arguments above, $(\sum_{1 \leq j \leq k} \ker(x - \lambda_j 1))^\perp$ is also invariant under u . Hence, if $\xi \in \ker(x - \lambda_{k+1} 1)$ is a unit vector, then $\lambda_{k+1} = \langle x\xi, \xi \rangle \leq \langle xu\xi, u\xi \rangle \leq \max\{\langle x\eta, \eta \rangle : \|\eta\| = 1, \eta \in (\sum_{1 \leq j \leq k} \ker(x - \lambda_j 1))^\perp\} = \lambda_{k+1}$. By the minimum-maximum principle, $u\xi$ is an eigenvector of x corresponding to λ_{k+1} . Hence, $\ker(x - \lambda_{k+1} 1)$ is invariant under u and u^* . This completes the induction process.

What these arguments show is that H has an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of eigenvectors of both x and u ; hence

$$xu\phi_j = ux\phi_j, \quad \forall j \in \mathbf{Z}^+.$$

Consequently,

$$xu\xi = ux\xi, \quad \forall \xi \in H,$$

meaning that

$$xu = ux.$$

Below is a major result of this chapter, stated earlier as Theorem 1.4.

Theorem 6.4 Assume that $a, b \in B(H)$ are commuting compact normal operators, each being injective and having dense range. If there exists a unitary u such that

$$u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,$$

for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|b| = |a|^{p-1}.$$

Proof. By the hypothesis, if $b = w|b|$ is the polar decomposition of b , then w is unitary and $\ker(|ab^*|) = \{0\}$ (Lemma 6.2). Moreover, $|ab^*| = w|a||b|w^*$, as a and b are commuting normals (noting that $|a||b|$ is positive from Lemma 6.1). Thus $u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ becomes

$$uw|a||b|w^*u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q. \quad (6.12)$$

By Theorem 3.8, and because $|a||b| = |b||a|$ (Lemma 6.2), we get $|a||b| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$. Hence from (6.12)

$$|a||b| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q = uw|a||b|w^*u^*. \quad (6.13)$$

Because uw is unitary (since w is unitary from the proof of Lemma 6.2), and because $|a||b|$ is positive, Lemma 6.3 and the equation (6.13) yield that

$$|a||b| = uw|a||b|w^*u^*.$$

Hence, (6.13) becomes

$$|a||b| = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q. \quad (6.14)$$

Let $\lambda_1(|a|) \geq \lambda_2(|a|) \geq \cdots > 0$ and $\lambda_1(|b|) \geq \lambda_2(|b|) \geq \cdots > 0$ be the eigenvalues of $|a|$ and $|b|$. Because $|a|$ and $|b|$ belong to a commutative C^* -algebra, the spectra of $|a||b|$ and $\frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ are determined from the spectra of $|a|$ and $|b|$, i.e., $\forall k \in \mathbf{Z}^+$

$$\lambda_k(|a||b|) = \lambda_k(|a|)\lambda_k(|b|),$$

$$\lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) = \frac{1}{p}\lambda_k(|a|)^p + \frac{1}{q}\lambda_k(|b|)^q.$$

Therefore, the equation (6.14) implies that

$$\lambda_k(|a|)\lambda_k(|b|) = \frac{1}{p}\lambda_k(|a|)^p + \frac{1}{q}\lambda_k(|b|)^q, \forall k \in \mathbf{Z}^+.$$

This is equality in the (scalar) Young's inequality, and hence $\lambda_k(|b|) = \lambda_k(|a|)^{p-1}$, $\forall k \in \mathbf{Z}^+$, which yields $|b| = |a|^{p-1}$.

Corollary 6.5 If $\dim H < \infty$, then Theorem 6.4 holds even in the case where $|ab^*|$ is not injective.

Proof. Similar to the proof of Theorem 6.4 we can get the equation (6.13). From Theorem 3.6 and the assumption we obtain

$$|a||b| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q = u^*|a||b|u. \quad (6.15)$$

Therefore

$$\begin{aligned} 0 &\leq \operatorname{tr}(u^*|a||b|u - |a||b|) \\ &= \operatorname{tr}(u^*|a||b|u) - \operatorname{tr}(|a||b|) \\ &= 0. \end{aligned}$$

That is to say $\operatorname{tr}(u^*|a||b|u - |a||b|) = 0$. However, the trace on $M_n(\mathbf{C})$ is faithful (i.e., the only positive element of trace zero is the zero matrix), and so $u^*|a||b|u - |a||b| = 0$. From this and the equation (6.15) we attain the equation (6.14) again. The rest of the proof is the same as Theorem 6.4.

From Corollary 6.5 we immediately get

Corollary 6.6 If a and b are positive commuting compact operators such that $|ab|$ is injective, and if there is an isometry $v \in B(H)$ for which

$$v|a^t b^{1-t}|v^* = ta + (1-t)b$$

for some $t \in [0, 1]$, then

$$b = a^{t-1}.$$

6.2 The Noncommuting Cases: 2×2 and 3×3 Matrices

This section proves Theorem 1.7.

We are here to continue our discussion for the cases of equality for Young's inequality. We are going to deal with the noncommuting situations. At first, we prove that if the equality holds in Young's inequality for two $n \times n$ matrices a and b , then b and a are unitarily equivalent. Secondly, we show that, for two 2×2 or 3×3 positive matrices a and b , if the equality holds in Young's inequality, then $b = a$.

The singular values of a complex $n \times n$ matrix are the nonnegative real numbers $s_1(a) \geq s_2(a) \geq \cdots \geq s_n(a) \geq 0$ defined to be $s_k(a) = \sqrt{\lambda_k(a^*a)}$, for each $k = 1, 2, \dots, n$. Note that $s_k(|x|) = s_k(x) = \lambda_k(|x|)$, for all k . Moreover, the singular values of a product are weakly majorised by the products of the singular values. That is:

Lemma 6.7 ([3, Theorem IV.2.5]) If a and b are complex $n \times n$ matrices, then

$$\sum_{j=1}^k s_j(ab) \leq \sum_{j=1}^k s_j(a)s_j(b),$$

for all $k = 1, 2, \dots, n$.

Theorem 6.8 Assume $a, b \in M_n(\mathbb{C})$ with $a \geq 0, b \geq 0$. If there exists a unitary u such that $u|ab|u^* = \frac{1}{p}a^p + \frac{1}{q}b^q$, for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then there is a unitary w such that $b = wa^{p-1}w^*$.

Proof. From the assumption we have

$$\text{tr}(|ab|) = \frac{1}{p}\text{tr}(|a|^p) + \frac{1}{q}\text{tr}(|b|^q).$$

So from Lemma 6.7 and because $\lambda_j(|ab|) = s_j(ab)$ we get

$$\begin{aligned} & \frac{1}{p} \sum_{j=1}^n \lambda_j(a)^p + \frac{1}{q} \sum_{j=1}^n \lambda_j(b)^q \\ &= \text{tr}(|ab|) = \sum_{j=1}^n s_j(ab) \leq \sum_{j=1}^n \lambda_j(a)\lambda_j(b), \end{aligned}$$

therefore

$$\sum_{j=1}^n \left(\frac{1}{p} \lambda_j(a)^p + \frac{1}{q} \lambda_j(b)^q \right) \leq \sum_{j=1}^n \lambda_j(a)\lambda_j(b).$$

Then

$$\sum_{j=1}^n \left(\frac{1}{p} \lambda_j(a)^p + \frac{1}{q} \lambda_j(b)^q \right) = \sum_{j=1}^n \lambda_j(a)\lambda_j(b).$$

Since $\frac{1}{p}\lambda_j(a)^p + \frac{1}{q}\lambda_j(b)^q \geq \lambda_j(a)\lambda_j(b)$, ($j = 1, 2, \dots, n$) we get

$$\frac{1}{p}\lambda_j(a)^p + \frac{1}{q}\lambda_j(b)^q = \lambda_j(a)\lambda_j(b), \quad (j = 1, 2, \dots, n).$$

Hence, according to the Young's inequality in scalars we conclude

$$\lambda_j(b) = \lambda_j(a)^{p-1} \quad (j = 1, 2, \dots, n).$$

Thus, b and a^{p-1} are unitarily equivalent.

Next Proposition 6.9 plays an important role in the proof of our main results in this section.

Proposition 6.9 Assume $a, b \in M_n(\mathbf{C})$ with $a \geq 0, b \geq 0$. If there exists a unitary u such that $2u|ab|u^* = a^2 + b^2$, and if there exists a unitary u_1 such that $a^2 = u_1|ab|u_1^*$, then $b = a$.

Proof. By Theorem 6.8 there exists a unitary w such that $b = waw^*$. Thus

$$a^2 = u_1|ab|u_1^* = u_1(u^*(\frac{1}{2}a^2 + \frac{1}{2}wa^2w^*)u)u_1^*.$$

If $x = u_1u^*$, then

$$a^2 = \frac{1}{2}(xa^2x^*) + \frac{1}{2}(xw)a^2(xw)^*,$$

which represents a^2 as a convex combination of elements in the unitary orbit of a^2 . But this can happen only if $a^2 = xa^2x^*$ and $a^2 = (xw)a^2(xw)^*$ [14]. Hence $xa^2 = a^2x$ and $(xw)a^2 = a^2(xw)$. So $x(wa^2) = a^2xw = xa^2w$, which yields that $x(wa^2 - a^2w) = 0$. Thus $wa^2 = a^2w$, which shows that $wa = aw$. Consequently, $b = a$.

The next lemma is quoted from [5, p.334].

Lemma 6.10 (Newton's identity) Assume that $a \in M_n(\mathbf{C})$ and that α_k is the coefficient on λ^{k-1} in the characteristic polynomial

$$\gamma(\lambda) = \det(\lambda 1 - a).$$

Then

$$(-1)^k \alpha_k = \sum_{i=1}^k (-1)^{i-1} \alpha_{i-1} \text{tr}(a^i), \quad 2 \leq k \leq n.$$

Where $\det(a)$ denotes the determinant of a .

We note that for the characteristic polynomial $\gamma(\lambda) = \det(\lambda 1 - a) = \sum_{k=1}^{n+1} \alpha_k \lambda^{k-1}$, we have $\alpha_{n+1} = 1$ and $\alpha_1 = (-1)^n \det(a)$.

Lemma 6.11 Assume $a, b \in M_n(\mathbf{C})$ with $a \geq 0, b \geq 0$. If there exists a unitary u such that

$$2u|ab|u^* = a^2 + b^2,$$

then $\text{tr}(a^2) = \text{tr}(|ab|)$, and $\text{tr}(a^4) = \text{tr}(ab^2a)$. Where $\text{tr}(c)$ denotes the trace of the matrix c .

Proof. From Theorem 6.8 we have $\text{tr}(a^2) = \text{tr}(b^2)$. And $2u|ab|u^* = a^2 + b^2$, implies $2\text{tr}(|ab|) = \text{tr}(a^2) + \text{tr}(b^2)$; therefore $\text{tr}(a^2) = \text{tr}(|ab|)$.

On the other hand, since $2u|ab|u^* = a^2 + b^2$, we have

$$a^4 + a^2b^2 + b^2a^2 + b^4 = 4uba^2bu^*.$$

Hence

$$\text{tr}(a^4 + a^2b^2 + b^2a^2 + b^4) = 4\text{tr}(uba^2bu^*) = 4\text{tr}(ba^2b).$$

Note that

$$\text{tr}(a^2b^2) = \text{tr}(b^2a^2) = \text{tr}(ab^2a)$$

and

$$\text{tr}(a^4) = \text{tr}(b^4).$$

Hence,

$$\text{tr}(a^4) = \text{tr}(ab^2a).$$

Lemma 6.12 Assume $a, b \in M_n(\mathbf{C})$, with $a \geq 0, b \geq 0$. If there exists a unitary u such that

$$2u|ab|u^* = a^2 + b^2,$$

then a^2 is unitary equivalent $|ab|$.

Proof. By Lemma 6.11 we have

$$(i) \operatorname{tr}(a^2) = \operatorname{tr}(|ab|),$$

$$(ii) \operatorname{tr}(a^4) = \operatorname{tr}(ab^2a).$$

By Theorem 6.8, $b = waw^*$ for some unitary w . So

$$\begin{aligned} \det(|ab|) &= \det(ba^2b)^{\frac{1}{2}} \\ &= \det((waw^*a^2waw^*)^{\frac{1}{2}}) \\ &= (\det(waw^*a^2waw^*))^{\frac{1}{2}} \\ &= (\det(a^4))^{\frac{1}{2}} = \det(a^2). \end{aligned}$$

i.e., we have

$$(iii) \det(a^2) = \det(|ab|).$$

Combining (i), (ii) and (iii) and according to Newton's lemma (Lemma 6.10) we conclude that $\det(\lambda 1 - a^2) = \det(\lambda 1 - |ab|)$, which means that the characteristic polynomials of a^2 and $|ab|$ are the same. Therefore, a^2 is unitary equivalent $|ab|$.

The final step in the proof of Theorem 1.7 is below.

Theorem 6.13 Assume $n \leq 3$, $a, b \in M_n(\mathbf{C})$ with $a \geq 0, b \geq 0$. If there exists a unitary u such that

$$2u|ab|u^* = a^2 + b^2,$$

then u commutes with a , and

$$b = a.$$

Proof. By Lemma 6.12 and the Newton's identity (Lemma 6.10) a^2 and $|ab|$ have the same characteristic polynomials. Hence from Proposition 6.9 we conclude that $b = a$.

6.3 Remark [added December 7, 2001]

In November 2001 I learned of the paper [18] of O. Hirzallah and F. Kittaneh that proves Theorem 1.7 for all n .

Chapter 7

CONCLUSION

Operator inequalities that arise from operator-convex or operator-monotone functions, or from purely algebraic considerations, are usually valid in a wide range of settings, ranging from finite matrices to operators (not only compact operators) and elements in C^* -algebras.

Certain matrix inequalities, however, are known to hold solely on the basis of inequalities between the spectra of positive matrices. Ando's version of Young's inequality for complex matrices is of this type. Such matrix inequalities, therefore, require substantial further work if they are to be extended to arbitrary—or just compact—operators.

The main achievements of this dissertation, in summary, are:

- (1) the development of methods whereby the eigenvalue inequalities observed by Ando in the case of matrices are now proven to hold in compact operators as well; and
- (2) the first analysis of cases in which equality occurs in either the matrix or operator Young's inequality.

There are limitations to both contributions (1) and (2) above.

First of all, to achieve (1), the fact that the operators under consideration were

compact was crucial. For example, the spectral theorem, the Minimum–Maximum Principle, and Poincaré’s inequality each had a pivotal role. Although a useful version of the spectral theorem holds for arbitrary Hermitian operators, neither the Minimum–Maximum Principle nor Poincaré’s inequality have been adequately adapted to deal with non-compact operators. Indeed, it is an open question as to whether the results of this dissertation hold among operators of the form $\alpha 1 + x$, where x is compact and α is a complex number, and it appears that an entirely new approach would be required to handle this slightly more general situation.

Second, while it was shown here that the contribution (2) is complete if the operators under consideration are commuting normal operators, the results obtained in Chapter 6 fall short of characterising the case of equality in Young’s inequality for noncommuting positive operators acting on Hilbert spaces of dimension 4 or higher. However, the results in Chapter 6 do yield a new, interesting characterisation of cases of equality for noncommuting positive operators acting on 2- or 3-dimensional Hilbert space; this result also tells us how the conjecture for higher dimensions should be formulated. This conjecture is in fact true; see [18].

While it may seem that Chapter 6 has an excessive amount of theoretical development for results that are valid only in low dimensions, it is worth commenting that I initially approached the problem in 2 dimensions by explicit calculation at the level of individual matrix entries. Despite the amount of effort I put into this approach, it proved to be unsuccessful and intractable because of the high number of nonlinear multivariable equations that required simultaneous solution. Whether I had succeeded or not, the “brute-force calculation” approach also has the further disadvantage that it is wholly unsuited as an approach to solve the problem 3 dimensions or more.

Notwithstanding the independent result of [18], the method I employed herein — specifically the technique of Theorem 6.8 — is useful for developing majorization results for compact operators. In this regard, see [19].

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List of Symbols

$$|a| = (a^*a)^{\frac{1}{2}}$$

$\bar{\alpha}$: the conjugate of the complex number α

$\langle \cdot, \cdot \rangle$: the inner product of the Hilbert space H

$ball(H)$: the unit ball of the Hilbert space H

$B(H)$: the set of operators on the Hilbert space H

$\mathbf{C}$: the set of complex numbers

$\mathbf{C}^n$: the set of $(c_1, c_2, \dots, c_n)$, $\forall c_k \in \mathbf{C}$

$C(\Omega)$: the set of continuous functions on a compact or local-compact set Ω

$C^*(1, a)$: the smallest C^* -algebra containing 1 and a

$\det(a)$: the determinant of the matrix a

$\dim M$: the dimension of the (sub)space M

D^- : the topological closure of the set D

$\ker(a)$: the kernel of the operator a

$K(H)$: the set of compact operators on the Hilbert space H

$\lambda_j(a)$: the j -th eigenvalue of the matrix or operator a , listed in descending order

$M^\perp$: the orthogonal complement of the subset M

$M_n(\mathbf{C})$: the set of $n \times n$ complex matrices

$\oplus, \sum_{i=1}^{\oplus k}$: the direct sum

$\mathbf{R}$: the set of real numbers

$\text{ran}(a)$: the range of the operator a

$\mathbf{R}_0^+$: the set of non-negative real numbers

$\mathbf{R}^n$: the n -Euclidean space

$s_j(a)$: the j -th singular value of the operator a , i.e., the j -th eigenvalue of the operator

$|a| = (a^*a)^{\frac{1}{2}}$, listed in descending order

$sp\{\xi_1, \xi_2, \dots\}$: the subspace spanned by the vectors $\xi_1, \xi_2, \dots$

$spr(a)$: the spectral radius of the matrix or the operator a

$\sigma(a)$: the spectrum of the matrix or the operator a

$tr(a)$: the trace of the matrix a

$u|_M$: the restriction of the operator u to the subspace M

$\mathbf{Z}^+$: the set of nonnegative integers