One-Visibility Cops and Robber on Trees

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Tanzina Akter, candidate for the degree of Master of Science in Computer Science, has presented a thesis titled, ***One-Visibility Cops and Robber on Trees***, in an oral examination held on December 14, 2020. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

Graph searching comes into the limelight of research topics since it provides mathematical models for many real-world problems. Cops and robber is one aspect of graph searching that has recently gained much interest. This thesis provides results on a graph searching model, one-visibility cops and robber model.

The one-visibility cops and robber game is a variation of the zero-visibility cops and robber game, where one-visibility means that the robber is seen by a cop when the distance between them is at most one. Suppose that a robber is hiding on vertices of a graph. The focus of this thesis is finding the minimum number of one-visibility cops required to capture the robber. Such a number is called the one-visibility cop number. The literature does not provide enough results on the one-visibility cop number. In this thesis, we consider the one-visibility cops and robber game on trees. We prove a lower bound for the one-visibility cop number of trees. We propose strategies to clear trees according to their structures. We give a linear-time algorithm for computing the labels of all vertices of a tree. The first component in the label of the root is the cop number of that tree. We provide an optimal clearing guide for the cop player to clear a tree. We also present some interesting relations between zero- and one-visibility cop numbers on trees.

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Chapter 1

Introduction

Graph searching is closely related to real-world problems such as eliminating a computer virus in a network, computer games, and even counter-terrorism. Graph searching problems consider two notions *cops* and *robber*. In some research cops are referred to as *searchers* and robber is referred to as *fugitive* or *intruder*. The purpose of cops (searchers) is to capture the robber (fugitive). A natural question here is: What is the fewest number of cops (or searchers) required to capture the robber? This thesis is motivated by this question and also considers a few other constraints like visibility and actions of the cops. In literature, graph searching is referred by many names such as Cops and Robber game or pursuit-evasion game. The broad overview of many graph searching models are given in [3] and some aspects of cops and robber are described in [2]. Here in the first section, we give a brief overview of graph searching games and in the later section, we give a brief overview of cops and robber game.

1.1 Graph Searching Games

In many graph searching problems, there are searchers trying to capture a fugitive (intruder) throughout the game. In searching games, the robber can hide in an edge or vertex and can move at high speed at any time (the movement depends on the model). When the robber moves at a great speed in any direction then the famous moniker searching is used. Graph searching has become an important topic and been used in contexts such as pebbling, VLSI circuit design, assuring privacy while using a bugged channel [3]. Graph searching games are linked with various parameters in literature such as treewidth, pathwidth, cutwidth. They can give deep insight into graph structures and can provide efficient algorithms. In the last three decades, a large number of searching games have been evolved.

The first searching model was introduced by Parsons [14], and used where continuous searching is required. There are many versions of Parsons model and the difference mainly relies on the movement of the robber such as i) the robber is invisible and active, and ii) the robber is visible or lazy. Edge searching, node searching, and mixed searching lie in the first category where there are restrictions on searchers but no restrictions on the robber. The edge searching game was introduced by Megiddo et al. [12]. The three actions performed by searchers in the edge searching are: placing a searcher on a vertex, removing a searcher from a vertex, and sliding a searcher from one vertex to another along an edge. In edge searching, the robber can hide in edges and searchers will traverse the edges to capture the robber. The robber

will be captured when a searcher and the robber both occupy the same vertex. The goal of edge searching is to find the minimum number of searchers to clear a given graph. Yang [20] introduced the fast edge searching model, which uses the same setting as edge searching but the goal is to find the minimum number of steps to clear trees. The node searching game was introduced by Kirousis and Papadimitriou [11]. There are only two actions for searchers: placing a searcher on a vertex and removing a searcher from a vertex. In node searching, the robber is captured either when both searchers and robber occupy the same vertex or the robber is on an edge whose endpoints are both occupied by searchers. The mixed searching game was introduced by Bienstock and Seymour [1]. In mixed searching, the operations performed by the searchers are the same as edge searching but the conditions for capturing the robber are the combination of edge searching and node searching. Fast searching model was introduced by Dyer et al. [9]. The goal of fast searching is different than other searching models. In this model, the main focus is to clear the graph in a minimum number of steps. This model is important in cases where it is affordable to use more searchers instead of allowing a fugitive to be free. For example, it is always important to capture a dangerous hidden fugitive as early as possible regardless of the cost of using more policemen. The fast edge search game has close relation with fast search game. The difference between fast search and fast edge search is that in fast search every edge can be traversed exactly once by a searcher and searchers can not be removed from the graph. Another searching model is graph guarding introduced by Fomin et al. [10]. In this model, a region in the graph is protected by searchers. The

robber will try to enter the protected region and if he can then it will be a *robber win* graph. The searchers have all the information about others and if they can protect the region from robber then it is a *cop win* graph. The goal of all major search problems is to find the minimum number of searchers. A brief survey about searching models is found in [3].

1.2 Cops and Robber Game

In recent years another aspect of graph searching “Cops and Robber” has received much attention. In this thesis, we will focus on the Cops and Robber game. Cops and Robber is a well-known pursuit-evasion game played on graphs. This game is played by two players: the first player controls a collection of agents, known as cops and the second player controls a single robber. The cops move through the graph attempting to capture the robber, while the robber moves to avoid the cops. In the cops and robber game, cops and robber occupy only vertices and they move alternatively to their neighbours. The basic actions performed by a cop player are, i) place: a cop is placed on a vertex, ii) move: a cop moves from one vertex to another along an edge, iii) pass: a cop stays on the same vertex. The game is played over a sequence of rounds on a graph. At round 0, cops are placed on a set of vertices of the graphs, and the robber is placed on a vertex of the graph. Cops move first to an adjacent vertex in round 1 and then the robber moves to an adjacent vertex. The robber is *captured* if a cop and the robber occupy the same vertex after some finite number of rounds. We define a graph as cop-win if the robber

is captured by a cop. If this situation never happens then it is a robber-win graph. A cop win strategy is a set of rules followed by the cops which results in the capture of the robber. A robber win is a strategy where robber can evade the cops indefinitely. If we place a cop on each vertex then the robber will be captured in round 0. So the minimum number of cops to capture a robber is a well defined positive integer and it is defined as *cop number* of the graph G . The cop number of a graph G is denoted by $c(G)$. If $c(G) = k$ then G is called a k -cop win graph. If $k = 1$, then it is a special case and G is simply called *cop win* graph.

An example is given here to understand how to play the game. Let us play the game on a 6-cycle C_6 . The vertices are labeled as 1,2,3,4,5,6 in Figure 1.1. Suppose, a cop is placed on vertex 1. If the robber chooses vertex 1 then it

Figure 1.1: A labeled 6-cycle, C_6

is a suicide and the cops win in round 0. If the robber chooses vertex 2 or 6 then the robber might be captured in the next round. So, by choosing 3,4, or 5 the robber can evade the capture in round 1. So, maintaining distance two from the cop in every step is sufficient for the robber to win. But if we put

two cops in round 0, one cop on vertex 1 can stay there and another cop can visit the vertices one by one. So at some point, cops and robber will occupy the same vertex and then cops win.

The cops and robber game was first introduced by Quilliot [15] and then independently considered by Nowakowski and Winkler [13]. In the classical cops and robber game in [13, 15], both opponents have full information about each other's location as well as the structure of the graph. In those first research works, they focus on cop win graphs; they characterize those graphs where one cop is sufficient to capture the robber. Based on those, Clarke and MacGillivray [5] characterized k -cop win graphs; that is, graphs where k cops are required. They also showed that even $k = 2$ is nontrivial. Many variants of cops and robber have been considered. Those variants evolved from imposing restrictions on cops' movement or limiting cops' knowledge of the robber's position. Zero-visibility cops and robber games is an example of limiting cops' knowledge [6]. The zero-visibility cops and robber game was introduced by Tošić [17], which can be considered as a hybrid of the cops and robber game [13, 15] and the edge searching model [12]. Like the cops and robber game, the cops and robber take turns alternatively and each individual moves from the current vertex to one of its neighbours. Like the edge searching model, the robber is invisible. Tošić characterized the zero-visibility cop-win graphs for paths, cycles, complete graphs, and complete bipartite graphs [17]. Dereniowski et al.[8] established a relationship between the zero-visibility cop number and the pathwidth of a graph. They introduced a monotonic version of the zero-visibility cops and robber model and gave an upper bound and

a lower bound on the monotonic zero-visibility cop number. Tang[16] gave a quadratic time algorithm for computing the zero-visibility cop number of trees. Dereniowski et al.[7] improved this result by presenting a linear-time algorithm. They also proved that the problem of determining the zero-visibility cop number of a graph is NP-complete. Recently, Xue et al.[18] proposed a partition method for proving lower bounds for the zero-visibility cops and robber game. They also investigated the monotonic zero-visibility cop number of graph products. Xue et al. gave lower bounds and upper bounds for the zero-visibility cop number of graph joins, lexicographic products of graphs, complete multipartite graphs and split graphs [19].

Clarke et al.[4] considered a variation of the cops and robber game, called the ℓ -visibility cops and robber game. This game has the same setting as the cops and robber game except that the cops have the information about the location of the robber only when the distance between the cops and the robber is less than or equal to ℓ . There are two sub-tasks for cops: seeing and capturing. In the first phase, the cops move within the distance ℓ of the robber and in the second phase, they capture the robber. They characterize those trees for which k cops are sufficient to guarantee capture of the robber for all $\ell \geq 1$. When $\ell = 1$, the game is called the one-visibility cops and robber game, which was considered by Yang in [23]. A brief survey of cops and robber model is given in [2].

Let us play a 1-visibility cops and robber game on the graph in Figure 1.2. If we place a cop λ_1 on vertex 1 then vertex 1 is occupied and as vertex 2,3 and 4 is seen by λ_1 so robber will not occupy those vertices at round 0. Suppose,

Figure 1.2: A labeled cube Q_3

the robber occupies vertex 6. In round 1, if the cop λ_1 moves to an adjacent vertex, say 4 then vertices 1,5 and 6 are seen by cop λ_1 and the robber is within distance 1 of the cop. But if the robber occupies vertex 7 at round 0 then it is safe as whatever vertex is chosen by λ_1 at round 1 vertex 7 will be at distance 2. To capture the robber we need at least two cops here. We keep one cop λ_1 vibrating between vertex 1 and 4. So, vertices 1, 2, 3, 4, 5, 6 are either occupied or seen by cop λ_1 . Only vertex 7 or 8 could be occupied by a robber. If we place another cop λ_2 on vertex 8 and the robber occupies vertex 7 then it is at distance 1 of cop λ_2 and will be captured in round 1. So here we need two cops to see and capture the robber.

1.3 Main Contribution

In this thesis, we investigate one-visibility cops and robber game on trees. Our first contribution is to develop a theorem to find a lower bound of one-visibility cop number of trees. We find the smallest number of cops that suffices to clear

a tree. This theorem leads us to the key concepts needed for the clearing strategy of a tree, the concepts of “Hub”, “Avenue”, and “Street”. We use the theorem to find the one-visibility cop number of a perfect binary tree and k -ary tree as a function of the height of the tree. This theorem is also used in the rest of the thesis to obtain other results. Our second contribution is to provide clearing strategies for trees containing hub, avenue, and street.

Our third contribution is that we propose a linear time algorithm to calculate the labels of vertices of a given tree. The complexity of our algorithm is $O(n)$. The first component in the label of the root represents the cop number of the tree. The label of each vertex is needed for developing an optimal clearing guide of a tree. We propose an optimal clearing guide to clear a given tree. That is our fourth contribution.

Our fifth contribution is to find the relations between zero-visibility cop number and one-visibility cop number of a tree T . We prove that the ratio of a zero-visibility cop number and one-visibility cop number of a tree T is less than or equal two. We also present some interesting relations between zero-visibility cop number and one-visibility cop number of some special trees such as perfect binary tree and k -ary tree which is our sixth contribution.

1.4 Organization

The structure of the thesis is organized as follows. In Chapter 2, we give definitions and introduce common notations used in this thesis. We also discuss

and give examples of one-visibility cops and robber game on trees. The results of this thesis are divided into three chapters. In Chapter 3, we explain an essential theorem to calculate the lower bound of cop numbers on trees. We also provide search strategies according to the structure of the tree in Chapter 3. An algorithm for calculating the labels of all vertices of an input tree is given in Chapter 4. The correctness and complexity of the algorithm are also provided in this chapter. We also provide an optimal clearing guide for a given tree. In Chapter 5, we examine relations between one- and zero-visibility cop numbers of a tree. We provide the ratio between zero-visibility cop number and one-visibility cop number. We also provide relations between one-visibility cop number and zero-visibility cop number of some special trees. Finally, in Chapter 6, we present a conclusion and future work.

Chapter 2

Preliminaries

In section 2.1 we introduce some notations and definitions used in this thesis and in section 2.2 we discuss about one-visibility cops and robber game.

2.1 Definitions

We consider here a special case of limited visibility cops and robber game which is referred to as one-visibility cops and robber game. In this thesis, we explore this game on trees. Let G be a graph. The vertex set of G is denoted by $V(G)$. Let $u, v \in V(G)$. If u and v are joined by an edge, we say that u and v are *adjacent* or u and v are *neighbours*. We use uv to denote an edge with endpoints u and v . Similarly, we use $u_1 \cdots u_m$ to denote a path with end vertices u_1 and u_m . The *length of a path* is the number of edges on the path. The *distance* between u and v , denoted by $\text{dist}_G(u, v)$, is the length of the

shortest path between u and v in G . Let H be a subgraph of G . The distance between u and H is defined to be $\text{dist}_G(u, H) := \min\{\text{dist}_G(u, v) \mid v \in V(H)\}$. Let $U \subseteq V(G)$. The distance between u and U can be defined similarly. The *neighbourhood* of v is the set $N_G(v) := \{u \in V(G) \mid \text{dist}_G(u, v) = 1\}$. The *closed neighbourhood* of v is the set $N_G[v] := \{u \in V(G) \mid \text{dist}_G(u, v) \leq 1\}$. For $k \geq 0$, we generalize this concept to the k -th *closed neighbourhood* of v , which is the set

$$N_G^k[v] := \{u \in V(G) \mid \text{dist}_G(u, v) \leq k\}.$$

The *closed neighbourhood* of U is defined as the set $N_G[U] = \{u \in V(G) \mid \text{dist}_G(u, U) \leq 1\}$. If there is no ambiguity we omit the subscript G and simply use $\text{dist}(u, v)$, $N^k[v]$, $N[U]$.

The *degree* of v is the number of edges incident on v , denoted $\deg_G(v)$. A *leaf* is a vertex that has degree one. For $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph induced by U , which consists of all vertices of U and all of the edges that connect vertices of U in G . We use $G - U$ to denote the subgraph $G[V(G) - U]$. If U contains a single vertex u , then for simplicity, we use $G - u$ for $G - \{u\}$. Let E' be a subset of edges of G . Similarly, we use $G - E'$ to denote the subgraph obtained from G by deleting all edges of E' from G .

A rooted tree is a tree where a single vertex is marked as the root. Let $T^{[r]}$ denote a rooted tree T with root r . Every vertex $v \neq r$ of the tree is connected with root r by a unique path where the *parent* of v is the sole neighbour of v in the unique path. If u is the parent of v , then v is a *child* of u . For a vertex

$v \in V(T^{[r]})$, if a vertex u is on the unique path from r to v , then we say that v is a *descendant* of u , and u is an *ancestor* of v . For a vertex $v \in V(T^{[r]})$, we will use $T^{[v]}$ to denote the subtree of $T^{[r]}$ induced by v and all its descendants, where v is the root of this subtree. We will extensively use the notation of $T^{[v]} - v$ in Chapter 4 to denote the forest induced by $V(T^{[v]}) - \{v\}$, where $T^{[v]}$ is a rooted subtree of $T^{[r]}$. Note that each component in the forest $T^{[v]} - v$ is rooted at the vertex that is a child of v in $T^{[v]}$. We will also use $T^{[r]} - v$ to denote the forest induced by $V(T^{[r]}) - \{v\}$.

2.2 One-Visibility Cops and Robber

The one-visibility cops and robber game is played on a graph by two players: cop player and robber player. The cop player controls a set of cops and the robber player controls a single robber. The robber has full information about the locations of all cops, but the cops have the information about the location of the robber only when there is a cop whose distance to the robber is at most one. The game is played over a sequence of rounds. Each *round* consists of a cops' turn followed by a robber's turn. At round 0, the cops are placed on a set of vertices and then the robber is placed on a vertex. At each of the following rounds, the cops move first and the robber move next. At round i , $i \geq 1$, each cop either moves from the current vertex to a neighbouring vertex or stays still, then the robber does the same. The cops *see* the robber if the closed neighbourhood of the cops contains the robber. The cops *capture* the robber if one of them occupies the same vertex as the robber. If this happens

in a finite number of rounds, then the *cops win*; otherwise, the *robber wins*. The *one-visibility cop number* of a graph G , denoted by $c_1(G)$, is the minimum number of cops required to capture the robber on G .

If G is not connected, from the above definition, we know that $c_1(G)$ is the sum of the one-visibility cop number of each component of G . We define $c_1^*(G)$ to be the largest possible $c_1(G')$, where G' is a component in G .

The zero-visibility cops and robber game has the same setting as the one-visibility cops and robber game except that the cops have no information about the location of the robber at any time, i.e., the robber is invisible to the cops. However, if a cop occupies the same vertex as the robber at some moment, then the robber is captured. The *zero-visibility cop number* of a graph G , denoted by $c_0(G)$, is the minimum number of cops required to capture the robber on G .

Throughout the thesis, we assume that both cops and the robber play optimally, which means the cops want to capture the robber as soon as possible while the robber wants to make the number of rounds of the game as large as possible. We sometimes use 1-visibility for “one-visibility” and 0-visibility for “zero-visibility”.

Note that in a tree, once the robber has been seen by a cop, this cop can capture the robber in a finite number of rounds by moving to the vertex occupied by the robber when he was seen and eventually forcing the robber to a leaf. The next observation is from [4].

Proposition 2.1. *Suppose that the 1-visibility cops and robber game is played*

on a tree. If the robber is seen by a cop at some moment, then the robber can be captured by this cop in a finite number of rounds.

From Proposition 2.1, we know that seeing implies capturing on trees.

A *cops' strategy* is a sequence of cops' actions to capture the robber. A *clearing guide* is a sequence of cops' actions and by following those actions the cop player can clear a tree. We say that a cop *vibrates* between two adjacent vertices x and y for a consecutive sequence of rounds if in these rounds, the cop alternates two actions: "sliding from x to y " and "sliding from y to x ". We define a cop as *vibrating cop* if the cop vibrates between two adjacent vertices x and y for more than two consecutive rounds.

We say a subgraph is *cleared* at some moment if it is certain that this subgraph does not contain the robber at the moment; otherwise, the subgraph is *contaminated*. Another important concept is *recontamination* of a vertex. If a vertex is defined clean by cops but later the robber can occupy this cleaned vertex then the vertex is recontaminated and this situation is called recontamination.

In Figure 2.1, let T be a tree formed by the solid edges and the dotted edge, and let T' be a subtree of T formed by only the solid edges. Lets play one visibility cops and robber game on tree T given in Figure 2.1. First, place one cop λ_1 on a_1 . The cop does not need to visit the leaf as the visibility is one and the cop can see the robber if the robber is on the leaf. Then λ_1 moves towards a and clear all the vertices on the path $a_1 \dots a$. Then λ_1 moves towards a_2 and when the cop is at a_2 then vertex a is unobserved for two subsequent

Figure 2.1: $c_1(T) = 3$ and $c_1(T') = 2$

cops terms and the robber can pass through a to the cleared vertices. So, we need to guard the cleared vertices and for this, we need another cop λ_2 on a . So, λ_2 stays on a and λ_1 clears the descendants of a . After clearing all the descendants of a both λ_1 and λ_2 move towards r along the path $a \dots r$. Now, the cop λ_2 can vibrate between d' and d'' to protect d . Then vertex r is unobserved for two subsequent cops' terms. So, the robber can recontaminate the cleared vertices by passing through r . So, we need another cop λ_3 to stay on r . The cop λ_2 can stay on d or can vibrate between d' and d'' and λ_1 can clear the descendants of d . After clearing all the descendants of d both λ_1 and λ_2 return to r through $d \dots r$. The rest of the dirty vertices can be cleared similarly. So, we need three cops to clear tree T .

Now we show that computing the one-visibility cop number of a given tree T is a tricky algorithmic problem with an example. It is easy to see that $c_1(T) = 3$, but it is probably not immediately obvious that $c_1(T') = 2$. Now we show how we can clear the tree T' with two cops. As the subtree of T' rooted at r'_1 is symmetrical to the subtree of T rooted at r'_1 , two cops can clear the subtree

of T' rooted at r'_1 . After that both λ_1 and λ_2 move to r . The cop λ_2 vibrates between r'_3 and d' to guard the cleared vertices and λ_1 moves to d . At first λ_1 moves to d_1 . Then λ_1 returns to d through $d_1 \dots d$. After that, λ_1 moves to d'_3 and then to d''_3 and clear all the rightmost descendants of d . In the meantime if the robber moves to d then the cop can see the robber in the next round and can capture the robber in a finite number of rounds. After that, λ_1 clears the remaining dirty vertices and returns to d . As all the descendants of r'_3 are now cleared both λ_1 and λ_2 return to r . Now both λ_1 and λ_2 move towards b through $r \dots b$. As all the descendants of both r'_1 and r'_3 are cleared, two cops can clear the descendants of b by utilizing the same strategy applied for clearing the descendants of a . So two cops can clear T' .

Chapter 3

Lower Bound on $c_1(T)$

In this Chapter, we show a structural theorem to provide a lower bound of cop number of a given tree T . We introduce different structures of a tree and provide strategies to clear trees according to their structures. We begin by proving some useful and essential lemmas for lower bound theorem.

3.1 Lower Bound Theorem

The notion of subtree plays an important role in constructing our structure theorem. Let $T = (V, E)$ be a tree. We here prove that for any subtree H of T , $c_1(H) \leq c_1(T)$. The idea of the proof of this result is similar to Lemma 4.2 in [8].

Lemma 3.1. *For a tree T and a subtree H of T , $c_1(H) \leq c_1(T)$.*

Proof. Suppose that $c_1(T) = k$. We show that $c_1(H) \leq k$. We consider a pair

of 1-visibility games played in parallel on T and H . Suppose the strategy to play the game on T with k cops is S_T . Now we produce a strategy S_H to play the game on H . The k cops play according to a winning strategy on T . The cops move in S_H depends on the cops move in S_T . For each cop λ_j , $1 \leq j \leq k$, if the cop λ_j is initially placed in H , in S_H we place the cop λ_j in H . We do the following for each cop's move in S_T ,

- In S_T , whenever a cop moves within H , in S_H the corresponding cop also moves in H ;
- In S_T , whenever a cop enters H , in S_H we place the corresponding cop on H ;
- In S_T , whenever a cop exits H , in S_H the corresponding cop stays on H .

Since S_T is a successful strategy on T , the strategy described above also successfully cleans H . Therefore, $c_1(H) \leq c_1(T)$. ■

Note that this Lemma does not hold for general graphs. For example, for any integer $k \geq 2$, we can apply Theorem 3.5 in this section to construct a tree T with $c_1(T) \geq k$, but for any complete graph K_n (with $n \geq |V(T)|$), we have $c_1(K_n) = 1$.

Lemma 3.2. *Let T be a tree and $v \in V(T)$. Let H be a component in the forest $T - v$. If $c_1(H) = k$, then in any cops' strategy to clear T , there is a moment at which at least k cops are on $N_T[V(H)]$.*

Proof. Let u be the unique neighbour of v in H , and let H_v be the subtree obtained by adding the vertex v and edge uv to H . Note that $N_T[V(H)] = V(H_v)$. Assume, for the sake of contradiction, that there is a cops' strategy S_T for clearing T such that at any moment there are at most $k - 1$ cops on H_v . We construct a pseudo strategy S to clear H_v as follows: In S_T , whenever a cop is initially placed in H_v , in S we place the corresponding cop on the same vertex. In S_T , whenever a cop moves within H_v , in S the corresponding cop has the same action in H_v . In S_T , whenever a cop enters H_v , in S we place the corresponding cop on v . In S_T , whenever a cop leaves H_v , in S we remove the corresponding cop from v . We can easily modify the pseudo strategy S to a cops' strategy S' for H_v such that the number of cops used in S' is the same as that in S . From the assumption, at any moment in S there are at most $k - 1$ cops on H_v . So H_v can be cleared by S' using at most $k - 1$ cops. It follows from Lemma 3.1 that H is cleared by at most $k - 1$ cops. Thus $c_1(H) \leq k - 1$, which is a contradiction. ■

Lemma 3.3. *Let T be a tree and $v \in V(T)$. Let H be a component in the forest $T - v$. If $c_1(H) = k$, then there is a strategy to clear T such that at any moment there are at most k cops on $N_T[V(H)]$.*

Proof. Let u be the unique neighbour of v in H , and let H_v be the subtree obtained by adding the vertex v and edge uv to H . Note that $N_T[V(H)] = V(H_v)$. Assume that in any cops' strategy S_T for clearing T there is a moment such that at least $k + 1$ cops are on H_v . But from Lemma 3.2 we know if $c_1(H) = k$, then in any cops' strategy to clear T , there is a moment at which at least k cops are on H_v . So we can say in this case $c_1(H) = k + 1$, which is

a contradiction. ■

Note that the condition in Lemma 3.3 does not hold for general subtree H of T . For example consider the tree T in Figure 3.1, where H is a subtree of T consisting of vertices in the dotted region. We can see in any cops strategy to clear T there is a moment at which 2 cops are on $N_T[V(H)]$.

Figure 3.1: A tree T with $c_1(T) = 2$ and $c_1(H) = 1$.

Now consider the Figure 3.2, where the subtree H of T is indicated by the dotted region and $c_1(T) = 3$ and $c_1(H) = 1$. In that figure in any cops strategy to clear T there is a moment at which 2 cops are on $V(H)$.

Figure 3.2: A tree T with $c_1(T) = 3$ and $c_1(H) = 1$.

Let T be a tree and let S be a cops' strategy that clears T in m rounds. Recall that a vertex v is *contaminated* at some moment if v is occupied by the robber or the cops do not know if the robber is on v at the moment. Let V_b^i , $1 \leq i \leq m$, be the set of contaminated vertices of T before the cops' move in round i , let V_a^i , $1 \leq i \leq m$, be the set of contaminated vertices of T just after the cops' move in round i , and let V_c^i , $0 \leq i \leq m$, be the set of vertices of T occupied by cops after the cops' move in round i . In the next lemma we show that the robber's territory is not far away from the set of vertices occupied by cops.

Lemma 3.4. *Let T be a tree and let S be a cops' strategy that clears T in m rounds. Then*

- (i) *for every component H of $T[V_b^i]$, $1 \leq i \leq m$, there is a vertex $v \in V_c^{i-1}$ such that $\text{dist}_T(v, H) \leq 2$; and*
- (ii) *for every component H of $T[V_a^i]$, $1 \leq i \leq m-1$, there is a vertex $v \in V_c^i$ such that $\text{dist}_T(v, H) \leq 3$.*

Proof. Let V_r^i , $0 \leq i \leq m-1$, be the set containing a single vertex that is occupied by the robber after the robber's move in round i . At round 0, cops are placed on vertices of V_c^0 , and then the robber is placed on the vertex in V_r^0 . So at round 1, $V_b^1 = (V(T) - N_T[V_c^0]) \cup V_r^0$, and thus, statement (i) holds. Since every cop can move to a neighbouring vertex or stays still, it is easy to see that statement (ii) also holds. Suppose both statements hold at round i . We will show that both of them are true at round $i+1$.

At round $i + 1$, we have $V_b^{i+1} = (N_T[V_a^i] - N_T[V_c^i]) \cup V_r^i$. Let H be a component of $T[V_b^{i+1}]$. Since $V_a^i \subseteq V_b^{i+1}$, there exists a component H' in $T[V_a^i]$ such that H' is a subgraph of H . By the assumption, there is $v \in V_c^i$ such that $\text{dist}_T(v, H') \leq 3$. If $\text{dist}_T(v, H') \leq 2$, then $\text{dist}_T(v, H) \leq 2$. If $\text{dist}_T(v, H') = 3$, then there is a vertex $x \in N_T[V_a^i]$ such that $\text{dist}_T(x, v) = 2$, and thus $x \in V_b^{i+1}$. Hence $\text{dist}_T(v, H) \leq 2$ and so statement (i) is true. Just after the cops' move at round $i + 1$, we have $V_a^{i+1} = (V_b^{i+1} - N_T[V_c^{i+1}]) \cup V_r^i$. Let H be a component in $T[V_a^{i+1}]$. Since $V_a^{i+1} \subseteq V_b^{i+1}$, there is a component H'' in $T[V_b^{i+1}]$ such that H is a subgraph of H'' . By the above, there is $v \in V_c^i$ such that $\text{dist}_T(v, H'') \leq 2$. Then for any vertex $x \in N(v) \cap V_c^{i+1}$, $\text{dist}_T(x, v) \leq 3$. Thus statement (ii) is true. ■

From the above two lemmas, we can show an essential theorem of this section.

Theorem 3.5. *Let T be a tree and let k be a positive integer. If there is a vertex $v \in V(T)$ such that the forest $T - N_T^3[v]$ contains three components with cop number at least k and the path in T connecting any pair of these three components contains v , then $\mathbf{c}_1(T) \geq k + 1$.*

Proof. Let X_1, X_2, X_3 be three components in $T - N_T^3[v]$ with $\mathbf{c}_1(X_i) \geq k$, $1 \leq i \leq 3$. For $1 \leq i \leq 3$, let $p_i \in V(X_i)$ with $\text{dist}_T(p_i, v) = 4$. Let T_1 be a subtree of T which is formed from the disjoint union of X_i , $1 \leq i \leq 3$, together with the paths of length four from p_i to v (see Figure 3.3). For the sake of contradiction, assume T_1 can be cleared by k cops. Then there is a round t_1 such that just after the cops' move in this round, only one of the three

components, say X_1 , is cleared and it remains cleared through the following rounds. There is also a round t_2 such that just after the cops' move, one of the other two components, say X_2 , is cleared. Since $c_1(X_2) \geq k$, from Lemma 3.2, there is a round t_3 between t_1 and t_2 such that at the moment t^* just after the cops' move in round t_3 , all k cops are simultaneously present in $N_T[V(X_2)]$. As X_3 is contaminated at t^* and $\text{dist}_{T_1}(v, N_T[V(X_2)]) = 3$, by Lemma 3.4, v is contaminated at t^* . So the robber will recontaminate X_1 , which derives a contradiction. Therefore $c_1(T_1) \geq k + 1$. It follows from Lemma 3.1 that $c_1(T) \geq k + 1$. ■

Figure 3.3: T_1 consists of X_1, X_2, X_3 and the paths to v .

3.2 Cops' Strategies for Trees

Definition 3.6. (hub, street, and avenue vertices) Suppose that $c_1(T) = k$. A vertex v is called a *hub* of T if each component X in the forest $T - N_T^3[v]$ has $c_1(X) < k$. A vertex v is called an *avenue vertex* of T if $T - N_T^3[v]$ has two components with cop number k satisfying that the path in T connecting

Figure 3.4: A tree T with $c_1(T) = k$, where v is a hub.

them contains v (see Figure 3.5). A vertex v is called a *street vertex* of T if T does not have a hub or an avenue vertex, and $T - N_T^2[v]$ has two components with cop number k such that the path in T connecting them contains v (see Figure 3.6).

Figure 3.5: A tree T with $c_1(T) = k$, where v is an avenue vertex.

Theorem 3.7. *Every tree contains a hub, an avenue vertex, or a street vertex.*

Proof. Let T be a tree with $c_1(T) = k \geq 1$. If there is a vertex v in T such that $T - N_T^3[v]$ is empty, then $c_1(T) = 1$ and v is a hub of T . Suppose that there is no vertex v in T such that $T - N_T^3[v]$ is empty.

Figure 3.6: A tree T with $c_1(T) = k$, where u and v are street vertices.

If there is a vertex v such that each component in $T - N_T^3[v]$ has cop number less than k , then by Definition 3.6 v is a hub.

If T does not have a hub, then for every vertex v in T , $T - N_T^3[v]$ must contain at least one component of cop number k . If there is a vertex v such that $T - N_T^3[v]$ has three components with cop number k and the path in T connecting any pair of these three components contains v , then, by Theorem 3.5, $c_1(T) \geq k + 1$. This is a contradiction as $c_1(T) = k$.

If there is a vertex u such that $T - N_T^3[u]$ has two components of cop number k and the path between them contains u , then from Definition 3.6, u is an avenue vertex.

Suppose that T does not have a hub or an avenue vertex. Let v be an arbitrary vertex in T . Then $T - N_T^3[v]$ has at least one component of cop number k , but it does not have two components with cop number k such that the path between them contains v . So all components with cop number k in

$T - N_T^3[v]$ must be subgraphs of the same component in the forest $T - v$. Let v' be a neighbour of v such that all components with cop number k in $T - N_T^3[v]$ be subgraphs of the component in $T - v$ which contains v' . This component is denoted by $T_{vv'}$ which is associated with the edge vv' . Since the number of vertices in T is one more than that of edges, by the pigeonhole principle there must exist at least one edge xy that is associated with two components T_{xy} and T_{yx} . Thus $T - N_T^3[x]$ contains a set F_1 of components with cop number k such that the path in T between any pair of components from F_1 does not contain x ; similarly, $T - N_T^3[y]$ contains a set F_2 of components with cop number k such that the path in T between any pair of components from F_2 does not contain y . Note that the components in F_1 and F_2 are subgraphs of components in $T - N_T^2[x]$. Hence $T - N_T^2[x]$ has two components with cop number k , where one is a supergraph of a component in F_1 and the other is a supergraph of a component in F_2 , such that the path in T connecting them contains x . From Definition 3.6, x is a street vertex. Similarly, y is also a street vertex. ■

From Definition 3.6 and the proof of Theorem 3.7, every tree has the following property.

Corollary 3.8. *Every tree contains exactly one of the three kinds of vertices:*

(i) *hubs*, (ii) *avenue vertices*, and (iii) *street vertices*.

Proof. Let T be a tree with $c_1(T) = k$. Suppose we have an avenue vertex v in T . As T contains an avenue vertex by definition T can not have a street vertex. Since a vertex u of the form hub can not be on the avenue, and, if it were off the avenue, the component in $T - N^3[u]$ that included the avenue

would have cop number k , contradicting the requirement for a hub that all component in $T - N^3[u]$ have cop number less than k .

Assume that T contains two street vertices uv . By definition, T can not have a hub or an avenue vertex.

Suppose that T contains a hub v . If there exists an avenue vertex u in T , at least one component in $T - N^3[v]$ must include that avenue, which contradicts with the requirement for a hub. The same applies to a street vertex. So if T contains a hub then T does not contain a street or an avenue vertex. ■

Note that a tree may have some vertices which are none of the three kinds in Corollary 3.8 (see Figure 3.7).

Figure 3.7: A tree T with $c_1(T) = 2$, where x_1 and x_2 are avenue vertices of T and all others are not hubs, street vertices or avenue vertices of T .

A graph could have many hubs and the distance between hubs could be arbitrarily large. Figure 3.8 shows an example where every vertex in the tree is a hub.

Lemma 3.9. *If a tree contains hubs, then the subgraph induced by all hubs of the tree is connected.*

Figure 3.8: A tree T with $c_1(T) = 2$, in which every vertex is a hub.

Proof. Let T be tree with $c_1(T) = k$ and let V_h be the set of all hubs of T . Suppose $T[V_h]$ is not connected and v_1 and v_2 are two vertices of V_h in two different components of $T[V_h]$. Then there is a vertex $u \notin V_h$ on the path in T connecting v_1 and v_2 . Since each component in $T - N_T^3[v_i]$, $1 \leq i \leq 2$, has cop number less than k , we know that $T - N_T^3[u]$ cannot contain a component with cop number k . So u is a hub of T . This is a contradiction. Thus $T[V_h]$ is connected. ■

We now describe a cops' strategy to clear a tree that contains a hub.

function HUB-STRATEGY(T, v)

Let T be a tree rooted at v such that $c_1(T) = k$ and v is a hub of T . Let $v_1, \dots, v_d$ be all children of v . For each v_i , $1 \leq i \leq d$, let v_{ij} , $1 \leq j \leq m_i$, be all children of v_i .

- 1: Place all k cops on v .
 - 2: For each v_i , $1 \leq i \leq d$, for each v_{ij} , $1 \leq j \leq m_i$, let one cop vibrate between v_i and v_{ij} , in the meantime, slide other cops into each subtree rooted at a grandchild of v_{ij} until the subtree rooted at v_{ij} is cleared.
 - 3: Slide k cops back to v .
-

Lemma 3.10. *Let T be a tree which does not have a hub or an avenue vertex. Then T has exactly two street vertices, and they are adjacent to each other.*

Proof. Let $c_1(T) = k$. Since T does not have a hub or an avenue vertex of T ,

from the proof of Theorem 3.7, there is at least one edge whose endpoints are street vertices of T . Following that proof, suppose there exists two edge x_1x_2 and x_3x_4 such that each x_i , $1 \leq i \leq 4$, is a street vertex of T and the path between x_1 and x_4 in T contains x_2 and x_3 . From Definition 3.6, $T - N_T^2[x_1]$ has two components with cop number k such that the path in T connecting them contains x_1 ; similarly, $T - N_T^2[x_4]$ has two components with cop number k such that the path in T connecting them contains x_4 . Thus $T - N_T^3[x_2]$ has two components with cop number k such that the path in T connecting them contains x_2 . Hence x_2 is an avenue vertex of T , which is a contradiction. So there is only one edge whose endpoints are street vertices of T . ■

The edge whose endpoints are street vertices of T is called a *street* of T .

Lemma 3.11. *Suppose that T is a tree with $c_1(T) = k$ and uv is a street of T . Let T_{uv} be a component in the forest $T - u$ which contains a component of $T - N_T^3[u]$ with cop number k . Then T_{uv} contains all components of $T - N_T^3[u]$ with cop number k , u is a hub of $T - V(T_{uv})$, and v is a hub of T_{uv} .*

Proof. Since v is a street vertex of T , from Definition 3.6, $T - N_T^2[v]$ has two components with cop number k such that the path in T connecting them contains v . Thus $T - N_T^3[u]$ has two components with cop number k such that the path in T connecting them contains v . Since u is not an avenue vertex of T , all components with cop number k in $T - N_T^3[u]$ must be contained in one component in the forest $T - u$, which must be T_{uv} . Therefore, the forest $T - V(T_{uv}) - N_T^3[u]$ does not have any component with cop number k .

Since u is a street vertex of T , $T - N_T^2[u]$ has at least two components

with cop number k such that the path in T connecting them contains u . So $T - V(T_{uv}) - N_T^2[u]$ has at least one component with cop number k , and thus $c_1(T - V(T_{uv})) = k$. Hence u is a hub of $T - V(T_{uv})$. Similarly, we can define T_{vu} , and symmetrically, v is a hub of $T - V(T_{vu})$. Note that $T - V(T_{vu}) = T_{uv}$. Therefore, v is a hub of T_{uv} . ■

From Lemma 3.11, we have the following cops' strategy to clear a tree that contains a street.

function STREET-STRATEGY(T, uv)

Let T be a tree with $c_1(T) = k$, and let uv be the street of T . Let T_{uv} be the component in $T - u$ which contains all components of $T - N_T^3[u]$ with cop number k .

- 1: Place all k cops on u ; then HUB-STRATEGY($T - V(T_{uv}), u$).
 - 2: Slide all cops from u to v ; then HUB-STRATEGY(T_{uv}, v).
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In the next lemma, we show that the subgraph induced by avenue vertices is a path.

Lemma 3.12. *Let A be the set of all avenue vertices of a tree T . Then $T[A]$ is a path.*

Proof. Let $k = c_1(T)$. Suppose that $T[A]$ is not connected. Let v_1 and v_2 be two vertices of A in two different components of $T[A]$. Let $u \notin A$ be a vertex on the path in T connecting v_1 and v_2 . From Definition 3.6, $T - N_T^3[v_i]$, $1 \leq i \leq 2$, has two components with cop number k such that the path in T connecting them contains v_i . So $T - N_T^3[u]$ has two components with cop number k satisfying that the path in T connecting them contains u . Thus, u is an avenue vertex, which is a contradiction. Hence $T[A]$ is connected.

Suppose there is a vertex $v \in A$ that has at least three neighbours in A . Then $T - N_T^3[v]$ has at least three components with cop number k such that the path in T connecting any two of them contains v . It follows from Theorem 3.5 that $c_1(T) \geq k + 1$. This is a contradiction. Thus $T[A]$ does have a vertex of degree more than two. Therefore $T[A]$ is a path. ■

From Lemma 3.12, it is easy to see that, the path $T[A]$ induced by the avenue vertices is unique. The path $T[A]$ in Lemma 3.12 is called the *avenue* of T .

Corollary 3.13. *Let T be a tree with $c_1(T) = k$ and let $x_1 \cdots x_m$ be the avenue of T . Then*

- (i) x_1 has a unique neighbour $v \neq x_2$ such that $T - N_T^2[v]$ has two components with cop number k such that the path in T connecting them contains v .
- (ii) v is a hub of $T - V(T^*)$, where T^* is a component in $T - v$ that contains the avenue.

Proof. (i) As x_1 is an avenue vertex, by Definition 3.6, $T - N_T^3[x_1]$ has two components T_1 and T_2 with cop number k such that the path in T connecting them contains x_1 . Without loss of generality, suppose that x_2 is on the path connecting T_2 to x_1 . Let v be a neighbour of x_1 on the path connecting T_1 to x_1 . So T_1 is also a component in $T - N_T^2[v]$. Note that $T - N_T^2[v]$ has a component that contains T_2 as a subtree. Let this component be denoted by T'_2 . So $T - N_T^2[v]$ has two components, i.e., T_1 and T'_2 , with cop number k such that the path in T connecting them contains v . If x_1 has another neighbour

$v' \notin \{v, x_2\}$ such that $T - N_T^2[v']$ has two components with cop number k such that the path in T connecting them contains v' , then from Theorem 3.5, $c_1(T) \geq k + 1$. This is a contradiction. Thus v is unique to have the property.

(ii) Note that v is not an avenue vertex. So for any two components with cop number k in $T - N_T^3[v]$, the path in T connecting them does not contain v . So every component with cop number k in $T - N_T^3[v]$ is a subtree of T^* . Hence $T - V(T^*) - N_T^3[v]$ does not contain any component with cop number k . Since $T - V(T^*) - N_T^2[v]$ contains the component T_1 with cop number k , we have $c_1(T - V(T^*)) = k$. Therefore v is a hub of $T - V(T^*)$. ■

The vertex v in Corollary 3.13 is called a *pre-avenue vertex* of T . If a tree has an avenue, then it has exactly two pre-avenue vertices, each of which is adjacent to an end vertex of the avenue.

From Lemma 3.12 and Corollary 3.13, we have the following cops' strategy to clear a tree that contains an avenue.

function AVENUE-STRATEGY(T, A)

Let T be a tree with $c_1(T) = k$, and let $x_1 \cdots x_m$ be the avenue of T , where $A = \{x_1, \dots, x_m\}$ is the set of all avenue vertices of T . Let x_0 (resp. x_{m+1}) be the pre-avenue vertex adjacent to x_1 (resp. x_m). Let Y_0 (resp. Y_{m+1}) be the component in $T - A$ which contains x_0 (resp. x_{m+1}). Let Y_i , $1 \leq i \leq m$, be the component containing x_i in the forest $T - \{x_{i-1}x_i, x_i x_{i+1}\}$.

- 1: Place all k cops on x_0 .
 - 2: **for** $i := 0$ to m **do**
 - 3: HUB-STRATEGY(Y_i, x_i); then slide all cops from x_i to x_{i+1} .
 - 4: HUB-STRATEGY(Y_{m+1}, x_{m+1}).
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Chapter 4

One-Visibility Cop Number of a Tree

The main work of this section is to propose an $O(n)$ time algorithm to compute the cop number of a tree. We also propose an optimal clearing guide of a tree. First of all, we introduce some notations and definitions.

4.1 Notations and Definitions

From Section 2.2, we know $c_1^*(G) = \max\{c_1(G') \mid G' \text{ is a component in } G\}$. It is easy to see that if G is connected, $c_1^*(G) = c_1(G)$.

For simplicity, we will use $T^{[v]} - N^3[v]$ for $T^{[v]} - N_{T^{[v]}}^3[v]$, which is the forest obtained from the rooted tree $T^{[v]}$ by deleting the vertices of $N_{T^{[v]}}^3[v]$. Similarly, if there is no ambiguity we will simply use $\text{dist}(u, v)$, $N[v]$, $N^2[v]$

and $N^3[v]$ without subscripts.

Definition 4.1. (k -pre-branching, k -weakly-branching, k -branching)

Let $T^{[v]}$ be a rooted tree with $c_1(T^{[v]}) = k \geq 1$. We call v a *k -pre-branching vertex* if $c_1^*(T^{[v]} - N^2[v]) = k$ and $c_1(T_{2v}^{[u]}) = k$, where $T_{2v}^{[u]}$ is a tree obtained from two copies of $T^{[v]}$ by connecting each root v to a new root u .

We call v a *k -weakly-branching vertex* if one of the three forests, $T^{[v]} - v$, or $T^{[v]} - N[v]$, or $T^{[v]} - N^2[v]$, has exactly two components whose root is a k -pre-branching vertex in the component.

We call v a *k -branching vertex* if $c_1^*(T^{[v]} - N^2[v]) = k$, the forest $T^{[v]} - N^2[v]$ has exactly one component whose root is a k -weakly-branching vertex in the component, and the forest $T^{[v]} - N^3[v]$ has no component whose root is a k -weakly-branching vertex.

Let u be a child of v in $T^{[v]}$. If u is a k -pre-branching vertex (resp. k -weakly-branching vertex, k -branching vertex) in $T^{[u]}$, then we say that u is a *k -pre-branching child* (resp. *k -weakly-branching child*, *k -branching child*) of v . Similarly, we can define the *k -pre-branching descendant*, *k -weakly-branching descendant*, and *k -branching descendant* of v .

Definition 4.2. Let $T^{[v]}$ be a rooted tree with $c_1(T^{[v]}) = k \geq 1$. The *k -pre-branching indicator* $I_{\text{pb}}^k(v)$ and the *k -weakly-branching indicator* $I_{\text{wb}}^k(v)$ are defined to be:

$$I_{\text{pb}}^k(v) = \begin{cases} 1, & \text{if } v \text{ is a } k\text{-pre-branching vertex in } T^{[v]}; \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\text{wb}}^k(v) = \begin{cases} 1, & \text{if } v \text{ is a } k\text{-weakly-branching vertex in } T^{[v]}; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.3. Let $T^{[v]}$ be a rooted tree with $c_1(T^{[v]}) = k \geq 1$. The k -initial-counter $J^k(v)$ and the k -weakly-counter $J_{\text{w}}^k(v)$ are defined as follows:

$$J^k(v) = \begin{cases} 0, & I_{\text{pb}}^k(v) = 0 \text{ and } c_1^*(T^{[v]} - v) = k - 1; \\ 1, & I_{\text{pb}}^k(v) = 0, c_1^*(T^{[v]} - N[v]) = k - 1 \text{ and } c_1^*(T^{[v]} - v) = k; \\ 2, & I_{\text{pb}}^k(v) = 0, c_1^*(T^{[v]} - N^2[v]) = k - 1 \text{ and } c_1^*(T^{[v]} - N[v]) = k; \\ 0, & \text{otherwise.} \end{cases}$$

$$J_{\text{w}}^k(v) = \begin{cases} 0, & \text{if } I_{\text{wb}}^k(v) = 1 \text{ and } v \text{ has exactly two } k\text{-pre-branching children} \\ & \text{and no } k\text{-weakly-branching child;} \\ 1, & \text{if } I_{\text{wb}}^k(v) = 1 \text{ and } v \text{ has exactly one } k\text{-weakly-branching child} \\ & \text{and this child } u \text{ has } J_{\text{w}}^k(u) = 0; \\ 2, & \text{if } I_{\text{wb}}^k(v) = 1 \text{ and } v \text{ has exactly one } k\text{-weakly-branching child} \\ & \text{and this child } u \text{ has } J_{\text{w}}^k(u) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Remark. We know that the k -initial-counter $J^k(v) = 0$ both for $I_{\text{pb}}^k(v) = 0$ and $I_{\text{pb}}^k(v) = 1$. But in the context of this thesis, it will not create any ambiguity.

Definition 4.4. (utmost- k -pre-branching vertex) Suppose $T^{[v]}$ is a rooted

tree where the root v is a k -pre-branching vertex. If v does not have any k -pre-branching child then v is the utmost- k -pre-branching vertex in $T^{[v]}$. If v has only one k -pre-branching child v_1 then there exists a y^{th} neighbour z_1 of v such that for $y \geq 1$, $\#_{pb}^k(T^{[v]} - N^{y-1}[v]) = 1$, and $\#_{pb}^k(T^{[v]} - N^y[v]) = 0$. If $y = 1$ then v_1 is the utmost- k -pre-branching vertex in $T^{[v]}$; otherwise z_1 is the utmost- k -pre-branching vertex in $T^{[v]}$. Note that in this case z_1 is also a descendant of v_1 . All vertices on the path $z_1 \dots v$ are k -pre-branching vertices. If v is a k -weakly-branching vertex in $T^{[v]}$ then v has two k -pre-branching children say v_1 and v_2 . Then there exists an utmost- k -pre-branching vertex z_1 in $T^{[v_1]}$ and an utmost- k -pre-branching vertex z_2 in $T^{[v_2]}$ and z_1, z_2 are two utmost- k -pre-branching vertices in $T^{[v]}$. If v is a k -branching vertex in $T^{[v]}$ then v has a k -weakly-branching descendant say w and the two utmost- k -pre-branching vertices z_1, z_2 in $T^{[w]}$ are two utmost- k -pre-branching vertices in $T^{[v]}$.

Definition 4.5. (road) If there exists an avenue in $T^{[v]}$ then from Corollary 3.13 we know there exist two pre-avenue vertices in $T^{[v]}$. Any path connecting two pre-avenue vertices is the road in $T^{[v]}$, where avenue vertices are internal road vertices; or any path connecting the utmost- k -pre-branching vertex and the root is a road in $T^{[v]}$, where the utmost- k -pre-branching vertex is a pre-avenue vertex and another pre-avenue vertex is on the road. If there exists a street in $T^{[v]}$ then any path connecting two street vertices is the road in $T^{[v]}$, or any path connecting the utmost- k -pre-branching vertex and the root is a road in $T^{[v]}$, where the utmost- k -pre-branching vertex and the parent of the utmost- k -pre-branching vertex are two street vertices. If there is no avenue or

street vertex then the root v is a hub as well as the road in $T^{[v]}$; or any path connecting the utmost- k -pre-branching vertex and the root is a road in $T^{[v]}$ where the utmost- k -pre-branching vertex is a hub in $T^{[v]}$. We denote a road in $T^{[v]}$ by $P_{T^{[v]}}$.

Remark. *If there exists an utmost- k -pre-branching vertex z_1 in $T^{[v]}$ then one of the three cases can happen i) z_1 is a hub in $T^{[v]}$, ii) z_1 is one street vertex and the parent of z_1 is another street vertex in $T^{[v]}$ iii) z_1 is a pre-avenue vertex and one vertex on the path $v \sim z_1$ is another pre-avenue vertex in $T^{[v]}$.*

Definition 4.6. (label $L_{T^{[v]}}(v)$, value $|L_{T^{[v]}}(v)|$) Let $T^{[v]}$ be a rooted tree. The label of v in $T^{[v]}$, denoted by $L_{T^{[v]}}(v)$, is a sequence

$$(s_1, v_1, P_1; s_2, v_2, P_2; \dots; s_m, v_m, P_m; I_{\text{wb}}^{s_m}(v), J_{\text{w}}^{s_m}(v); I_{\text{pb}}^{s_m}(v), J^{s_m}(v)),$$

where s_i , v_i and P_i are defined in the following procedure:

1. If $T^{[v]}$ contains only one vertex, then $s_1 = 1$, $v_1 = \perp$, $P_1 = v$, $I_{\text{wb}}^{s_1}(v) = J_{\text{w}}^{s_1}(v) = I_{\text{pb}}^{s_1}(v) = J^{s_1}(v) = 0$, and return $L_{T^{[v]}}(v) = (1, \perp, v; 0, 0; 0, 0)$; otherwise, set $i \leftarrow 1$ and $T_1^{[v]} \leftarrow T^{[v]}$.
2. Set $s_i \leftarrow c_1(T_1^{[v]})$. Then we have one of the following cases:
 - (a) If v is an s_i -branching vertex in $T_1^{[v]}$, then $I_{\text{wb}}^{s_i}(v) = J_{\text{w}}^{s_i}(v) = I_{\text{pb}}^{s_i}(v) = J^{s_i}(v) = 0$, and P_i contains two utmost- k -pre-branching vertices z_1 and z_2 in $T_1^{[v]}$; and return $L_{T^{[v]}}(v) = (s_1, v_1, P_1; \dots; s_i, v, z_1 \sim z_2; 0, 0; 0, 0)$.

- (b) If v has an s_i -branching descendant in $T_1^{[v]}$, let v_i be this vertex. Set $T_1^{[v]} \leftarrow T_1^{[v]} - V(T_1^{[v_i]})$, $i \leftarrow i + 1$, and go back to Step 2.
- (c) If v is an s_i -weakly-branching vertex in $T_1^{[v]}$, then $I_{\text{pb}}^{s_i}(v) = J^{s_i}(v) = 0$, $I_{\text{wb}}^{s_i}(v) = 1$, and $J_{\text{w}}^{s_i}(v)$ can be determined by Definition 4.3, and P_i contains two utmost- k -pre-branching vertices z_1 and z_2 in $T_1^{[v]}$; return $L_{T^{[v]}}(v) = (s_1, v_1, P_1; \dots; s_i, \perp, z_1 \sim z_2; 1, J_{\text{w}}^{s_i}(v); 0, 0)$.
- (d) If v is an s_i -pre-branching vertex in $T_1^{[v]}$, then $I_{\text{wb}}^{s_i}(v) = J_{\text{w}}^{s_i}(v) = J^{s_i}(v) = 0$, $I_{\text{pb}}^{s_i}(v) = 1$, and either v is the utmost- k -pre-branching vertex then return $L_{T^{[v]}}(v) = (s_1, v_1, P_1; \dots; s_i, \perp, v; 0, 0; 1, 0)$; or P_i contains an utmost- k -pre-branching descendant z_1 ; then return $L_{T^{[v]}}(v) = (s_1, v_1, P_1; \dots; s_i, \perp, v \sim z_1; 0, 0; 1, 0)$.
- (e) $I_{\text{wb}}^{s_i}(v) = J_{\text{w}}^{s_i}(v) = I_{\text{pb}}^{s_i}(v) = 0$, and $J^{s_i}(v)$ can be determined by Definition 4.3, and P_i contains the root v ; return $L_{T^{[v]}}(v) = (s_1, v_1, P_1; \dots; s_i, \perp, v; 0, 0; 0, J^{s_i}(v))$.

The *value* of $L_{T^{[v]}}(v)$, denoted by $|L_{T^{[v]}}(v)|$, is equal to s_1 . The cop number of $T^{[v]}$ is the first component s_1 . The road in $T^{[v]}$ denoted by $P_{T^{[u]}}$ is equal to P_1 .

If there is no ambiguity, we simply use $L(v)$ for $L_{T^{[v]}}(v)$. In Algorithm 1, we will compute the cop number of subtrees in the reverse order $s_m, s_{m-1}, \dots, s_1$. For convenience, in the rest of the thesis we let $t_i = s_{m-i+1}$, $x_i = v_{m-i+1}$, and $P_i = P_{m-i+1}$; i.e.,

$$L(v) = (t_m^v, x_m^v, P_m^v; \dots; t_1^v, x_1^v, P_1^v; I_{\text{wb}}^{t_1^v}(v), J_{\text{w}}^{t_1^v}(v); I_{\text{pb}}^{t_1^v}(v), J^{t_1^v}(v)), \quad (4.1)$$

where the superscript v in t_i^v , x_i^v and P_i^v are used to refer to the vertex v . Here P_i^v is the road in tree $T^{[v]} - \bigcup_{j=i+1}^m V(T^{[x_j^v]})$. So, $|L(v)| = t_m^v$. Note that only x_1^v can be a “ $\perp$ ” sign, which means that neither v is an t_1^v -branching vertex in $T^{[v]} - \bigcup_{i=2}^m V(T^{[x_i^v]})$ nor it has an t_1^v -branching descendant in $T^{[v]} - \bigcup_{i=2}^m V(T^{[x_i^v]})$. We call the first three components (t_m^v, x_m^v, P_m^v) an *item* associated with $T^{[v]}$, and call each triple (t_i^v, x_i^v, P_i^v) , $1 \leq i < m$, an *item* associated with subtree $T^{[v]} - \bigcup_{j=i+1}^m V(T^{[x_j^v]})$, where t_i^v is called the *key* of the item and x_i^v and P_i^v are *attributes*.

Figure 4.1: Labels of vertices

We explain labels with the following example.

Example 4.7. Consider the tree in Figure 4.1(a). Suppose that v is the root and $L(s_1) = (k, \perp, s_1; 0, 0; 0, 0)$ and $L(s_2) = (k, \perp, s_2; 0, 0; 0, 0)$. Then $J^k(p_1) = J^k(p_2) = 1$ and $J^k(q_1) = J^k(q_2) = 2$. So $I_{pb}^k(z_1) = I_{pb}^k(z_2) = 1$ and both z_1 and z_2 are k -pre-branching vertices with label $(k, \perp, z_1; 0, 0; 1, 0)$ and $(k, \perp, z_2; 0, 0; 1, 0)$ respectively. Note that y_2 is also a k -pre-branching vertex with label $(k, \perp, y_2 \sim z_2; 0, 0; 1, 0)$. The vertex w is a k -weakly-branching vertex since there are exactly two components in the forest $T^{[w]} - w$ such that

each of the two components is rooted at a k -pre-branching vertex. The vertex v is a k -branching vertex as it has a k -weakly-branching child y and $J_w^k(y) = 2$.

In Figure 4.1(b), suppose that v is the root and $L(z_1) = (k, \perp, s_1 \sim s_2; 1, 0; 0, 0)$ and $L(z_2) = (k, \perp, s_3 \sim s_4; 1, 0; 0, 0)$. Then x_1 and x_2 are k -branching vertices with labels $(k, x_1, s_1 \sim s_2; 0, 0; 0, 0)$ and $(k, x_2, s_3 \sim s_4; 0, 0; 0, 0)$ respectively. As w has two k -branching children x_1 and x_2 , the label of w is $(k + 1, \perp, w; 0, 0; 0, 0)$. Hence, v is a $(k + 1)$ -pre-branching vertex with label $(k + 1, \perp, v; 0, 0; 1, 0)$.

Lemma 4.8. *Let $T^{[u]}$ be a tree with $c_1(T^{[u]}) = k$. Then $T^{[u]}$ has at most two utmost- k -pre-branching vertices. Moreover if u is a k -pre-branching vertex then $T^{[u]}$ has a unique utmost- k -pre-branching vertex, if u is a k -branching or k -weakly-branching vertex then $T^{[u]}$ has two utmost- k -pre-branching vertices.*

Proof. Suppose that $T^{[u]}$ has at least three utmost- k -pre-branching descendants of u . We know an utmost- k -pre-branching descendant is a k -pre-branching vertex. So in the forest $T^{[u]} - N^3[u]$, there are at least three components with cop number k . From Theorem 3.5, we have $c_1(T^{[u]}) \geq k + 1$, which is a contradiction. So $T^{[u]}$ can not have three or more utmost- k -pre-branching descendants.

Suppose that u is a k -pre-branching vertex and $T^{[u]}$ does not have any utmost- k -pre-branching descendant. As u is a k -pre-branching vertex by definition u is the utmost- k -pre-branching vertex and its unique.

Suppose that u is a k -pre-branching vertex and $T^{[u]}$ has a k -pre-branching descendant. Then in the forest $T^{[u]} - u$, $\#_{pb}^k(T^{[u]} - u) = 1$. So there exist

a y^{th} neighbour z of u such that for $y \geq 1$, $\#_{pb}^k(T^{[u]} - N^{y-1}[u]) = 1$, and $\#_{pb}^k(T^{[u]} - N^y[u]) = 0$. z is the utmost- k -pre-branching vertex and its unique.

Let u is a k -weakly-branching vertex in $T^{[u]}$. From definition 4.1 we know $T^{[u]}$ has exactly two k -pre-branching vertices. We can see $T^{[u]}$ has exactly two utmost- k -pre-branching vertices.

Let u is a k -branching vertex in $T^{[u]}$. From Definition 4.1 we know $T^{[u]} - N^2[u]$ has exactly one component containing a k -weakly-branching vertex as the root. Similar to above we can say $T^{[u]}$ has exactly two utmost- k -pre-branching vertices. ■

Lemma 4.9. *Let $T^{[v]}$ be a tree where*

$$L(v) = (t_m^v, x_m^v, P_m^v; \dots; t_1^v, \perp, P_1^v; I_{wb}^{t_1^v}(v), J_w^{t_1^v}(v); I_{pb}^{t_1^v}(v), J^{t_1^v}(v)).$$

Then $T_m^{[v]} = T^{[v]}$ and for $m > i \geq 2$, $T_i^{[v]} = T^{[v]} - \bigcup_{j=i+1}^m V(T^{[x_j^v]})$ each subtree contains an avenue.

Proof. For $2 \leq i \leq m$, x_i^v is a t_i^v -branching vertex in $T_i^{[v]}$. From the Definition 4.1 we know the forest $T_i^{[v]} - N^2[x_i^v]$ has exactly one component whose root say w_i is a t_i^v -weakly-branching vertex in that component. So, $T_i^{[v]} - N^3[w_i]$ has two components with cop number t_i^v such that the path between them contains w_i . According to Definition 3.6, w_i is an avenue vertex in $T_i^{[v]}$. So, $T_i^{[v]}$ contains an avenue. ■

From Lemma 4.9 we have the following corollary,

Corollary 4.10. *Let $T^{[v]}$ be a tree where*

$$L(v) = (t_m^v, x_m^v, P_m^v; \dots; t_1^v, \perp, P_1^v; I_{\text{wb}}^{t_1^v}(v), J_{\text{w}}^{t_1^v}(v); I_{\text{pb}}^{t_1^v}(v), J_{\text{p}}^{t_1^v}(v)).$$

Then the two utmost- k -pre-branching vertices in $T_m^{[v]} = T^{[v]}$ and for $2 \leq i < m$, the two utmost- k -pre-branching vertices in each subtree $T_i^{[v]} = T^{[v]} - \bigcup_{j=i+1}^m V(T^{[x_j^v]})$ are two pre-avenue vertices in that subtree.

Definition 4.11. Let $T^{[u]}$ be a tree with root u whose children are $v_1, \dots, v_d$. Suppose $c_1^*(T^{[u]} - u) = k \geq 1$. The counters $\#_{\text{pb}}^k(T^{[u]} - u)$, $\#_{\text{wb}}^k(T^{[u]} - u)$, $\#_{\text{c}}^k(T^{[u]} - u)$, $h^k(T^{[u]} - u)$ and $h_{\text{w}}^k(T^{[u]} - u)$ are defined as follows:

$$\#_{\text{pb}}^k(T^{[u]} - u) = \sum_{j=1}^d I_{\text{pb}}^k(v_j),$$

$$\#_{\text{wb}}^k(T^{[u]} - u) = \sum_{j=1}^d I_{\text{wb}}^k(v_j),$$

$$\#_{\text{c}}^k(T^{[u]} - u) = \left| \left\{ j \mid c_1(T^{[v_j]}) = k \text{ for } j \in \{1, \dots, d\} \right\} \right|,$$

$$h^k(T^{[u]} - u) = \max \left\{ J^k(v_j) \mid j \in \{1, \dots, d\} \right\},$$

$$h_{\text{w}}^k(T^{[u]} - u) = \max \left\{ J_{\text{w}}^k(v_j) \mid j \in \{1, \dots, d\} \right\}.$$

4.2 Algorithm for Computing $c_1(T)$

The labels in Definition 4.6 have the following properties.

Theorem 4.12. *Let $T^{[u]}$ be a tree with root u whose children are $v_1, \dots, v_d$.*

Suppose that $c_1^*(T^{[u]} - u) = k \geq 1$ and for $1 \leq j \leq d$,

$$L_{T^{[v_j]}}(v_j) = (t^{v_j}, \perp, P^{v_j}; I_{\text{wb}}^{t^{v_j}}(v_j), J_{\text{w}}^{t^{v_j}}(v_j); I_{\text{pb}}^{t^{v_j}}(v_j), J^{t^{v_j}}(v_j)). \quad (4.2)$$

Then the label $L_{T^{[u]}}(u)$ must be of the form $(t^u, x^u, P^u, I_{\text{wb}}^{t^u}(u), J_{\text{w}}^{t^u}(u); I_{\text{pb}}^{t^u}(u), J^{t^u}(u))$

which can be determined as follows:

- (1) If $\#_{\text{wb}}^k(T^{[u]} - u) > 1$, then $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.
- (2) If $\#_{\text{wb}}^k(T^{[u]} - u) = 1$ and $\#_{\text{pb}}^k(T^{[u]} - u) \geq 1$, then $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.
- (3) If $\#_{\text{wb}}^k(T^{[u]} - u) = 1$, $\#_{\text{pb}}^k(T^{[u]} - u) = 0$ and $\#_{\text{c}}^k(T^{[u]} - u) \geq 2$, then $/^*$

Suppose v_1 is the k -weakly-branching child of u . $^*/$

- (3.1) if $h_{\text{w}}^k(T^{[u]} - u) = 2$, then $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.
- (3.2) if $h_{\text{w}}^k(T^{[u]} - u) = 1$ and $h^k(T^{[u]} - u) \geq 1$, then $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.
- (3.3) if $h_{\text{w}}^k(T^{[u]} - u) = 1$ and $h^k(T^{[u]} - u) = 0$, then $L_{T^{[u]}}(u) = (k, \perp, P^{v_1}; 1, 2; 0, 0)$.
- (3.4) if $h_{\text{w}}^k(T^{[u]} - u) = 0$ and $h^k(T^{[u]} - u) = 2$, then $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.
- (3.5) if $h_{\text{w}}^k(T^{[u]} - u) = 0$ and $h^k(T^{[u]} - u) \leq 1$, then $L_{T^{[u]}}(u) = (k, \perp, P^{v_1}; 1, 1; 0, 0)$.

- (4) If $\#_{\text{wb}}^k(T^{[u]} - u) = 1$ and $\#_{\text{c}}^k(T^{[u]} - u) = 1$, then

$/^*$ Suppose v_1 is the k -weakly-branching child of u . $^*/$

(4.1) if $h_{\text{w}}^k(T^{[u]} - u) = 2$, then $L_{T^{[u]}}(u) = (k, u, P^{v_1}; 0, 0; 0, 0)$.

(4.2) if $h_{\text{w}}^k(T^{[u]} - u) = 1$, then $L_{T^{[u]}}(u) = (k, \perp, P^{v_1}; 1, 2; 0, 0)$.

(4.3) if $h_{\text{w}}^k(T^{[u]} - u) = 0$, then $L_{T^{[u]}}(u) = (k, \perp, P^{v_1}; 1, 1; 0, 0)$.

(5) If $\#_{\text{wb}}^k(T^{[u]} - u) = 0$, then

(5.1) if $\#_{\text{pb}}^k(T^{[u]} - u) \geq 3$, then $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

(5.2) if $\#_{\text{pb}}^k(T^{[u]} - u) = 2$, then $L_{T^{[u]}}(u) = (k, \perp, z_1 \sim z_2; 1, 0; 0, 0)$.

/ z_1 and z_2 are two utmost- k -pre-branching vertices in $T^{[u]}$. */*

(5.3) if $\#_{\text{pb}}^k(T^{[u]} - u) = 1$, then $L_{T^{[u]}}(u) = (k, \perp, z_1 \sim u; 0, 0; 1, 0)$.

/ z_1 is the utmost- k -pre-branching vertex in $T^{[u]}$ */*

(5.4) if $\#_{\text{pb}}^k(T^{[u]} - u) = 0$, then

(5.4.1) if $h^k(T^{[u]} - u) = 2$, then $L_{T^{[u]}}(u) = (k, \perp, u; 0, 0; 1, 0)$.

(5.4.2) if $h^k(T^{[u]} - u) = 1$, then $L_{T^{[u]}}(u) = (k, \perp, u; 0, 0; 0, 2)$.

(5.4.3) if $h^k(T^{[u]} - u) = 0$, then $L_{T^{[u]}}(u) = (k, \perp, u; 0, 0; 0, 1)$.

Proof. (1) Since $\#_{\text{wb}}^k(T^{[u]} - u) = \sum_{j=1}^d I_{\text{wb}}^k(v_j) \geq 2$, we can suppose that $I_{\text{wb}}^k(v_1) = I_{\text{wb}}^k(v_2) = 1$. For each $i = 1, 2$, from Definition 4.1, $T^{[v_i]} - v_i$, $T^{[v_i]} - N[v_i]$, or $T^{[v_i]} - N^2[v_i]$ has exactly two components whose root is a k -pre-branching vertex in the component. Since each $L_{T^{[v_j]}}(v_j)$ contains $\perp$ (see Equ. (4.2)), we know that u does not have a descendant that is a k -branching vertex. Thus, in the forest $T^{[u]} - N^3[u]$, there are at least four components with cop number k . It follows from Theorem 3.5 that $c_1(T^{[u]}) \geq k + 1$. Since $c_1^*(T^{[u]} - u) = k \geq 1$, we can clear $T^{[u]}$ by stationing one cop on u and using

the other k cops to clear each component with cop number at most k . Thus $t^u = c_1(T^{[u]}) = k + 1$. Since u and all its descendants are not k -branching, we have $x^u = \perp$. As $c_1(T^{[u]}) = k + 1$ and all components in $T^{[u]} - N^3[u]$ has cop number at most k , u is a hub in $T^{[u]}$, and by Definition 4.5 the hub u is a road in $T^{[u]}$. Note that neither u is $(k + 1)$ -weakly-branching nor it has a $(k + 1)$ -weakly-branching descendant. So $I_{wb}^{k+1}(u) = 0$ and $J_w^{k+1}(u) = 0$. Similarly, we have $I_{pb}^{k+1}(u) = 0$ and $J^{k+1}(u) = 0$. Therefore $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

(2) Since $\#_{wb}^k(T^{[u]} - u) = 1$ and $\#_{pb}^k(T^{[u]} - u) \geq 1$, in the forest $T^{[u]} - u$, there is exactly one component whose root is a k -weakly-branching vertex and at least one component whose root is a k -pre-branching vertex. W.l.o.g., suppose that v_1 is the k -weakly-branching vertex in $T^{[v_1]}$ and v_2 is a k -pre-branching child of u . Thus, in the forest $T^{[u]} - N^3[u]$, there are at least three components with cop number k . According to Theorem 3.5, $c_1(T^{[u]}) \geq k + 1$. We can clear $T^{[u]}$ with $k + 1$ cops by utilizing HUB-STRATEGY. Similar to case (1) we can prove $x^u = \perp$, $P^u = u$, $I_{wb}^{k+1}(u) = 0$, $J_w^{k+1}(u) = 0$, $I_{pb}^{k+1}(u) = 0$ and $J^{k+1}(u) = 0$. Therefore $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

(3) Since $\#_{wb}^k(T^{[u]} - u) = 1$ and $\#_c^k(T^{[u]} - u) \geq 2$, in the forest $T^{[u]} - u$, there is exactly one component whose root is a k -weakly-branching vertex. W.l.o.g., suppose that v_1 is the k -weakly-branching vertex in $T^{[v_1]}$ and v_2 is a child of u with $c_1(T^{[v_2]}) = k$.

(3.1) If $h_w^k(T^{[u]} - u) = 2$, from Definitions 4.3 and 4.11, we know $J_w^k(v_1) = 2$ and $c_1^*(T^{[v_1]} - N[v_1]) = k$, and further, there is exactly one component in the forest $T^{[v_1]} - N[v_1]$ such that the root of this component is a k -weakly-

branching vertex in the component. Let w be this k -weakly-branching vertex. From Definition 4.1, we know $T^{[w]} - N^3[w]$ has at least two components with cop number k . Furthermore, since $c_1(T^{[v_2]}) = k$, $T^{[u]} - N^3[w]$ has at least three components with cop number k . From Theorem 3.5, we have $c_1(T^{[u]}) \geq k + 1$. Note that we can use $k + 1$ cops to clear $T^{[u]}$ by HUB-STRATEGY. Thus $t^u = c_1(T^{[u]}) = k + 1$. Similar to Case (1), we have $x^u = \perp$, $P^u = u$, and $I_{wb}^{k+1}(u) = J_w^{k+1}(u) = I_{pb}^{k+1}(u) = J^{k+1}(u) = 0$. So, $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

(3.2) If $h_w^k(T^{[u]} - u) = 1$ and $h^k(T^{[u]} - u) \geq 1$, from Definitions 4.3 and 4.11, we know $J_w^k(v_1) = 1$ and $c_1^*(T^{[v_1]} - v_1) = k$. So there is exactly one component in the forest $T^{[v_1]} - v_1$ such that the root of this component is a k -weakly-branching vertex in the component. Let w be this k -weakly-branching vertex (see Figure 4.2). From Definition 4.1, we know $T^{[w]} - N^3[w]$ has at least two components with cop number k . Since $c_1(T^{[v_2]}) = k$ and $h^k(T^{[u]} - u) \geq 1$, we know that $T^{[u]} - N^3[w]$ has at least three components with cop number k . Thus $c_1(T^{[u]}) \geq k + 1$. Similar to Case (3.1), we have $L_{T^{[u]}}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

(3.3) If $h_w^k(T^{[u]} - u) = 1$ and $h^k(T^{[u]} - u) = 0$, similar to case (3.2), there is exactly one component in the forest $T^{[v_1]} - v_1$ such that the root w of this component is a k -weakly-branching vertex in $T^{[w]}$ (see Figure 4.3). So $T^{[w]} - N^3[w]$ has two components with cop number k . Since $c_1(T^{[v_2]}) = k$ and $h^k(T^{[u]} - u) = 0$, we have $\#_c^k(T^{[v_2]} - v_2) = k - 1$. Note that $\text{dist}(w, v_2) = 3$, which implies that $T^{[u]} - N^3[w]$ has exactly two components with cop number k . So there is an avenue in $T^{[u]}$ that contains w . Note that $L_{T^{[u]}}(w) = (k, \perp, z_1 \sim z_2; 1, 0; 0, 0)$. By AVENUE-STRATEGY, we can use k cops to clear $T^{[u]}$. So $t^u = c_1(T^{[u]}) = k$. Since there is no k -branching vertex in $T^{[u]}$, it is easy

Figure 4.2: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k + 1$, where $L_{T^{[v_1]}}(v_1) = (k, \perp, z_1 \sim z_2; 1, 1; 0, 0)$ and $L_{T^{[v_2]}}(v_2) = (k, \perp, v_2; 0, 0; 0, 1)$.

to see that $x^u = \perp$. Since all components in $T^{[u]} - N^3[u]$ with cop number k are the components in $T^{[v_1]} - N^3[v_1]$, the road in $T^{[v_1]}$ is also the road in $T^{[u]}$. Hence $P^u = P^{v_1}$. Since $\#_{wb}^k(T^{[u]} - u) = 1$ and $h_w^k(T^{[u]} - u) = 1$, we have $I_{wb}^k(u) = 1$, $J_w^k(u) = 2$, $I_{pb}^k(u) = 0$ and $J^k(u) = 0$. Therefore $L_{T^{[u]}}(u) = (k, \perp, P^{v_1}; 1, 2; 0, 0)$.

Figure 4.3: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$, where $L_{T^{[v_1]}}(v_1) = (k, \perp, z_1 \sim z_2; 1, 1; 0, 0)$ and $L_{T^{[v_2]}}(v_2) = (k, \perp, v_2; 0, 0; 0, 0)$.

(3.4) Similar to Case (3.2), we have $L_{T[u]}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

(3.5) Similar to Case (3.3), we can show $L_{T[u]}(u) = (k, \perp, P^{v_1}; 1, 1; 0, 0)$.

(4.1) Since $\#_{\text{wb}}^k(T^{[u]} - u) = 1$, $\#_c^k(T^{[u]} - u) = 1$ and $h_w^k(T^{[u]} - u) = 2$, we can suppose that $I_{\text{wb}}^k(v_1) = 1$. Then $J_w^k(v_1) = 2$, $c_1^*(T^{[v_1]} - N[v_1]) = k$ and there is exactly one component in the forest $T^{[v_1]} - N[v_1]$ such that the root say w , is a k -weakly-branching vertex in that component. So $T^{[w]} - N^3[w]$ has exactly two components with cop number k . As $\#_c^k(T^{[u]} - u) = 1$, the forest $T^{[u]} - N^3[u]$ has only two components with cop number k . Similar to Case (3.3), we can clear $T^{[u]}$ with k cops. Thus $t^u = k$. Since $c_1^*(T^{[u]} - N^2[u]) = k$ and there is exactly one component in the forest $T^{[u]} - N^2[u]$ whose root w is a k -weakly-branching vertex, we have $x^u = u$. Similar to Case (3.3), $P^u = P^{v_1}$. From the structure of $T^{[u]}$, we have $I_{\text{wb}}^k(u) = J_w^k(u) = I_{\text{pb}}^k(u) = J^k(u) = 0$. Thus $L_{T[u]}(u) = (k, u, P^{v_1}; 0, 0; 0, 0)$.

(4.2) Similar to Case (3.3), we have $L_{T[u]}(u) = (k, \perp, P^{v_1}; 1, 2; 0, 0)$.

(4.3) Similar to Case (3.3), we can show $L_{T[u]}(u) = (k, \perp, P^{v_1}; 1, 1; 0, 0)$.

(5.1) Suppose that $\#_{\text{wb}}^k(T^{[u]} - u) = 0$ and $\#_{\text{pb}}^k(T^{[u]} - u) \geq 3$. W.l.o.g., suppose that v_1, v_2 and v_3 are k -pre-branching vertices in $T^{[v_1]}$, $T^{[v_2]}$ and $T^{[v_3]}$, respectively (see Figure 4.4). For each $i = 1, 2, 3$, from Definition 4.1, we know $c_1^*(T^{[v_i]} - N^2[v_i]) = k$. So in the forest $T^{[u]} - N^3[u]$, there are at least three components with cop number k . It follows from Theorem 3.5 that $c_1(T^{[u]}) \geq k + 1$. We can clear $T^{[u]}$ by using HUB-STRATEGY. So $t^u = c_1(T^{[u]}) = k + 1$. It is easy to see that $x^u = \perp$, $P^u = u$, and $I_{\text{wb}}^k(u) = J_w^k(u) = I_{\text{pb}}^k(u) = J^k(u) = 0$. Thus $L_{T[u]}(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.

Figure 4.4: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$, where v_1, v_2 and v_3 are three k -pre-branching children of u .

(5.2) Suppose that $\#_{\text{wb}}^k(T^{[u]} - u) = 0$ and $\#_{\text{pb}}^k(T^{[u]} - u) = 2$. W.l.o.g., suppose that v_1 and v_2 are k -pre-branching vertices in $T^{[v_1]}$ and $T^{[v_2]}$, respectively (see Figure 4.5). So $T^{[u]} - N^3[u]$ has exactly two components with cop number k . Furthermore, there is an avenue in $T^{[u]}$ that contains u . By AVENUE-STRATEGY, we can use k cops to clear $T^{[u]}$. So $t^u = c_1(T^{[u]}) = k$. Since there is no k -branching vertex in $T^{[u]}$, it is easy to see that $x^u = \perp$. As u is a k -weakly-branching vertex by Lemma 4.8, $T^{[u]}$ has two utmost- k -pre-branching vertices. Let z_1 and z_2 be the two utmost- k -pre-branching vertices. Then $z_1 \sim z_2$ is the road in $T^{[u]}$. It follows from Definitions 4.1 and 4.3 that $I_{\text{wb}}^k(u) = 1$ and $J_{\text{w}}^k(u) = I_{\text{pb}}^{t_1}(u) = J^{t_1}(u) = 0$. Hence $L_{T^{[u]}}(u) = (k, \perp, z_1 \sim z_2; 1, 0; 0, 0)$.

(5.3) Suppose that $\#_{\text{wb}}^k(T^{[u]} - u) = 0$, $\#_{\text{pb}}^k(T^{[u]} - u) = 1$, then there is only one child of u , say v_1 , which is a k -pre-branching vertex. So $T^{[u]} - N^3[u]$ has exactly one component with cop number k . It is not hard to see that $T^{[u]}$ can be cleared by k cops. Thus $t^u = c_1(T^{[u]}) = k$ and $x^u = \perp$. According to Lemma 4.8, $T^{[u]}$ has one utmost- k -pre-branching descendant of u . Suppose z_1 is the utmost- k -pre-branching vertex. Since v_1 is the only k -pre-branching

Figure 4.5: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$, where $L_{T[v_1]}(v_1) = (k, \perp, v_1; 0, 0; 1, 0)$ and $L_{T[v_2]}(v_2) = (k, \perp, v_2 \sim z_1; 0, 0; 1, 0)$ and $L_{T[v_3]}(v_3) = (k, \perp, v_3; 0, 0; 0, 2)$.

child of u , if $T^{[u]}$ contains an avenue then by Definition 4.5 z_1 is an pre-avenue vertex and the path connecting z_1 and u is the road containing the avenue. Figure 4.6 shows an example where the road $u \sim z_1$ in $T^{[u]}$ contains the only avenue vertex y_1 . Similarly if $T^{[u]}$ contains a street then by Definition 4.5

Figure 4.6: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$, where $u \sim z_1$ is the road containing an avenue vertex y_1 .

$z_1 \sim u$ is the road containing the street vertices z_1 and the parent of z_1 . See figure 4.7.

If there is no avenue or street then by Definition 4.5 $z_1 \sim u$ is the road containing a hub z_1 in $T^{[u]}$. See figure 4.8. Since there is no k -weakly-branching

Figure 4.7: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$, where $u \sim z_1$ is the road containing street vertices z_1 and p_1 .

Figure 4.8: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$ and with no street or avenue vertex, the road $u \sim z_1$ is the road containing a hub z_1 .

vertex, we have $I_{\text{wb}}^k(u) = J_{\text{w}}^k(u) = 0$. It follows from Definition 4.1 that $I_{\text{pb}}^k(u) = 1$. From Definition 4.3, $J^k(u) = 0$. Therefore $L_{T^{[u]}}(u) = (k, \perp, u \sim z_1; 0,0; 1,0)$.

(5.4.1) Suppose that $\#_{\text{wb}}^k(T^{[u]} - u) = 0$, $\#_{\text{pb}}^k(T^{[u]} - u) = 0$ and $h^k(T^{[u]} - u) = 2$. From Definitions 4.11 and 4.3, we have $t^u = k$, $x^u = \perp$, and $I_{\text{wb}}^k(u) = J_{\text{w}}^k(u) = 0$. Similar to Case (5.3), we can show that $I_{\text{pb}}^k(u) = 1$ and $J^k(u) = 0$. Since $T^{[u]} - N^3[u]$ has no components with cop number k , u is a hub in $T^{[u]}$. So u is a road in $T^{[u]}$. Hence $L_{T^{[u]}}(u) = (k, \perp, u; 0,0; 1,0)$.

Figure 4.9: A tree $T^{[u]}$ with $c_1(T^{[u]}) = k$ and a hub u in $T^{[u]}$.

(5.4.2) Suppose that $\#_{\text{wb}}^k(T^{[u]} - u) = 0$, $\#_{\text{pb}}^k(T^{[u]} - u) = 0$ and $h^k(T^{[u]} - u) = 1$. Similar to Case (5.4.1), we have $t^u = k$, $x^u = \perp$, $P^u = u$ and $I_{\text{wb}}^k(u) = J_{\text{w}}^k(u) = 0$. From Definitions 4.1 and 4.3 we know that $I_{\text{pb}}^k(u) = 0$ and $J^k(u) = 2$. So $L_{T^{[u]}}(u) = (k, \perp, u; 0, 0; 0, 2)$.

(5.4.3) Similar to Case (5.4.2), we have $L_{T^{[u]}}(u) = (k, \perp, u; 0, 0; 0, 1)$. ■

Algorithm 1 is a bottom-up approach for computing the labels of a tree. In this algorithm, we first assign labels to each vertex that has no children. Then for each vertex whose children have been labeled, we compute the label of this vertex using the rules proved in Theorems 4.12, 4.13 and 4.14. Finally, the first component in the label of the root is the 1-visibility cop number of the tree and third component is the road. Recall that the label of v in $T^{[v]}$ has the form below:

$$L_{T^{[v]}}(v) = (t_m^v, x_m^v, P_m^v; \dots; t_1^v, x_1^v, P_1^v; I_{\text{wb}}^{t_1^v}(v), J_{\text{w}}^{t_1^v}(v); I_{\text{pb}}^{t_1^v}(v), J^{t_1^v}(v)), \quad (4.3)$$

where the first pair (t_m^v, x_m^v, P_m^v) is an *item* associated with $T^{[v]}$, and each pair (t_i^v, x_i^v, P_i^v) , $1 \leq i < m$, is an *item* associated with subtree $T^{[v]} - \bigcup_{j=i+1}^m V(T^{[x_j^v]})$.

Note that t_i^v is called the *key* of the item and x_i^v and P_i^v are its *attributes*.

Let u be the root of the tree in Figure 4.10 which has five children v_1, v_2, v_3, v_4 , and v_5 . If we consider in Step 4 of Algorithm 1 the labels of v_1, v_2, v_3, v_4 , and v_5 are those given in the Figure 4.10 then in Step 5, T_1 should be the subtree inside dotted region. In Step 6, we get $L_{T_1[u]}(u) = (4, \perp, v_1 \sim v_2; 1, 0; 0, 0)$. After executing Steps 7-11 and then going back to Step 3, the algorithm returns $L_{T[u]}(u) = (8, x_2, s_3 \sim s_4; 7, \perp, u; 0, 0; 0, 0)$.

Figure 4.10: Computing the label of u in $T[u]$.

Algorithm 1 Computing the labels of a tree with road information

Input: A tree T with $n \geq 3$ vertices.

Output: The label of the root of T .

- 1: Pick a vertex of T as its root.
 - 2: Sort the vertices of T to a list $u_1, \dots, u_n$ such that every vertex is before its parent in the list. For each vertex u that has no child, set its label as $(1, \perp, u; 0, 0; 0, 0)$.
 - 3: If the root u_n has obtained a label $L_{T[u_n]}(u_n)$, then output this label; otherwise, let u be the first unlabeled vertex in the list currently. Run Steps 4 to 11 to compute $L_{T[u]}(u)$.
 - 4: Let $v_j, 1 \leq j \leq d$, be all children of u with labels $L_{T[v_j]}(v_j)$ in the form of Equ. (4.3). Let $I_\perp$ be the subset of children whose label contains $\perp$ and let I_b be the subset of children whose label does not contain $\perp$.
 - 5: Construct $T_1^{[u]}$, where $T_1^{[u]} = T^{[u]} - \bigcup_{y \in I_b} V(T^{[x_1^y]}) - \bigcup_{y \in I_\perp} V(T^{[x_2^y]})$. For $v \in V(T_1) - \{u\}$, $L_{T_1^{[u]}}(v)$ is obtained by deleting all components except the last seven components from $L_{T[v]}(v)$.
 - 6: If $T_1^{[u]}$ contains only one vertex u then $L_{T_1^{[u]}}(u) = (1, \perp, u; 0, 0; 0, 0)$, otherwise, compute $L_{T_1^{[u]}}(u)$ by Theorem 4.12. Let $k = |L_{T_1^{[u]}}(u)|$.
 - 7: For $1 \leq j \leq d$, if $v_j \in I_\perp$, let L_j be a list obtained from $L_{T[v_j]}(v_j)$ by deleting the last seven components and items whose key is less than k ; if $v_j \in I_b$, let L_j be a list obtained from $L_{T[v_j]}(v_j)$ by deleting the last four components and items whose key is less than k . Let L_{d+1} be a list containing only the first item of $L_{T_1^{[u]}}(u)$.
 - 8: If no key in $L_1, \dots, L_d, L_{d+1}$ is repeated, then $L_{T^{[u]}}(u) \leftarrow L_{T_1^{[u]}}(u)$ and insert the items of $L_1, \dots, L_d$ into $L_{T^{[u]}}(u)$. Go to Step 3.
 - 9: Find the largest repeated key k^* in the lists $L_1, \dots, L_d, L_{d+1}$.
 - 10: Let $K = (k_1, \dots, k_\ell)$ be a list containing the distinct keys from $L_1, \dots, L_{d+1}$ satisfying that the keys in K are decreasing and are greater than or equal to k^* .
 - 11: Find the smallest index h in K , where $1 \leq h \leq \ell$, such that $k_h = k_{h+1} + 1 = \dots = k_\ell + (\ell - h)$. Update $K \leftarrow (k_1, \dots, k_{h-1}, k'_h)$ where $k'_h = k_h + 1$. Create a list $X = (Q_1, \dots, Q_{h-1}, Q_h)$, where $Q_i = (k_i, x_i, P_i)$, $1 \leq i \leq h-1$, is an item with key k_i and attributes x_i and P_i (note that $x_i, 1 \leq i \leq h-1$, is a k_i -branching vertex in some subtree and P_i is the road in that subtree) and $Q_h = (k'_h, \perp; u)$. Insert $(0, 0; 0, 0)$ at the end of X . Set $L_{T^{[u]}}(u) \leftarrow X$. Go to Step 3.
-

4.3 Correctness and Time Complexity

Theorem 4.13. *Let $T^{[u]}$ be a rooted tree and $v_1, \dots, v_d$ be the children of the root u . Let $c_1(T^{[v_j]}) \leq k$, for $1 \leq j \leq d$. Suppose that $L_{T^{[v_j]}}(v_j) =$*

$$\begin{cases} (t_1^{v_j}, \perp, P_1^{v_j}; I_{\text{wb}}^{t_1^{v_j}}(v_j), J_{\text{w}}^{t_1^{v_j}}(v_j); I_{\text{pb}}^{t_1^{v_j}}(v_j), J_{\text{p}}^{t_1^{v_j}}(v_j)), & \text{if } 1 \leq j \leq d_1, \\ (t_{m_j}^{v_j}, x_{m_j}^{v_j}, P_{m_j}^{v_j}; \dots; t_1^{v_j}, \perp, P_1^{v_j}; I_{\text{wb}}^{t_1^{v_j}}(v_j), J_{\text{w}}^{t_1^{v_j}}(v_j); I_{\text{pb}}^{t_1^{v_j}}(v_j), J_{\text{p}}^{t_1^{v_j}}(v_j)), & \text{if } d_1 < j \leq d_2, \\ (t_{m_j}^{v_j}, x_{m_j}^{v_j}, P_{m_j}^{v_j}; \dots; t_1^{v_j}, x_1^{v_j}, P_1^{v_j}; 0, 0; 0, 0), & \text{if } d_2 < j \leq d, \end{cases}$$

where $x_1^{v_j}$, $d_2 < j \leq d$, are $t_1^{v_j}$ -branching vertices of $T^{[u]}$. Let $T_1^{[u]} = T^{[u]} - \bigcup_{j=d_1+1}^{d_2} V(T^{[x_2^{v_j}]}) - \bigcup_{j=d_2+1}^d V(T^{[x_1^{v_j}]})$ and $c_1(T_1^{[u]}) = k$. If $t_{m_j}^{v_j} < k$ for each $j \in \{d_1+1, \dots, d\}$, then $c_1(T^{[u]}) = k$ and $P_{T^{[u]}} = P_{T_1^{[u]}}$. If $t_{m_j}^{v_j} = k$ for at least one $j \in \{d_1+1, \dots, d\}$, then $c_1(T^{[u]}) = k+1$ and $P_{T^{[u]}} = u$.

Proof. For $j \in \{d_1+1, \dots, d\}$, since $t_{m_j}^{v_j} < k$, we can use $k-1$ cops to clear $T^{[v_j]}$. Let $T_1^{[u]} = T^{[u]} - \bigcup_{j=d_1+1}^{d_2} V(T^{[x_2^{v_j}]}) - \bigcup_{j=d_2+1}^d V(T^{[x_1^{v_j}]})$. Since $c_1(T_1^{[u]}) = k$, we can clear tree $T_1^{[u]}$ with k cops. We now give a cops' strategy to clear the whole tree $T^{[u]}$ with k cops.

CASE 1. Let u be a hub in $T_1^{[u]}$. So the road in $T_1^{[u]}$ is u . As $t_{m_j}^{v_j} < k$, $d_1+1 \leq j \leq d$, $T^{[u]} - N^3[u]$ does not have any components with cop number k . So u is also a hub in $T^{[u]}$ as well as a road in $T^{[u]}$. To clear $T^{[u]}$, we first use k cops to clear $T_1^{[u]}$, and then let one cop stay on u and use $k-1$ cops to clear the subtrees $T^{[v_j]}$, $d_1+1 \leq j \leq d$.

CASE 2. Let u be a street vertex in $T_1^{[u]}$ and $u \sim v$ be the street where v is

a child of u . As $t_{m_j}^{v_j} < k$, $d_1 + 1 \leq j \leq d$, all components in $T^{[u]} - N^3[u]$ (resp. $T^{[v]} - N^3[v]$) with cop number k are the components in $T_1^{[u]} - N^3[u]$ (resp. $T_1^{[v]} - N^3[v]$). So u and v are also street vertices in $T^{[u]}$. Hence $u \sim v$ is the street in $T^{[u]}$. For clearing $T^{[u]}$, k cops first clear the subtree $T^{[v_1]}$ rooted at v_1 using HUB-STRATEGY. After that all k cops move to u to clear the subtree $T^{[u]} - V(T^{[v_1]})$ rooted at u using HUB-STRATEGY.

CASE 3. Let u is an avenue vertex and $z_1 \sim z_2$ is the road in $T_1^{[u]}$ where z_1 and z_2 are two utmost- k -pre-branching vertices in $T_1^{[u]}$. As $t_{m_j}^{v_j} < k$, $d_1 + 1 \leq j \leq d$, components in $T^{[u]} - N^3[u]$ containing z_1 and z_2 are the components in $T_1^{[u]} - N^3[u]$. So z_1 and z_2 are also two pre-avenue vertices in $T^{[u]}$. Hence $z_1 \sim z_2$ is also the road in $T^{[u]}$. The k cops can clear $T^{[u]}$ using AVENUE-STRATEGY, during which when all cops move to the vertex u on the avenue, $k - 1$ cops can clear each subtree $T^{[v_j]}$, $d_1 + 1 \leq j \leq d$, attached to u , as $t_{m_j}^{v_j} < k$.

CASE 4. Suppose that u is not a hub, a street vertex, or an avenue vertex in $T_1^{[u]}$. Assume that $T_1^{[u]}$ has an avenue $x_1 \cdots x_m$. Let x_0 be the pre-avenue vertex adjacent to x_1 and let x_{m+1} be the pre-avenue vertex adjacent to x_m . Since u is not an avenue vertex in $T_1^{[u]}$, the avenue is contained in only one component of $T_1^{[u]} - u$. W.l.o.g., suppose that $T^{[v_1]}$ contains all x_i , $0 \leq i \leq m + 1$. So the road in $T^{[v_1]}$ is the road in $T_1^{[u]}$. Since $c_1(T_1^{[u]}) = k$ and $T^{[v_1]}$ contains the avenue of $T_1^{[u]}$, we know that $c_1(T^{[v_1]}) = k$. Among the vertices in $\{x_0, x_1, \dots, x_m, x_{m+1}\}$, let x_j be the closest vertex to u in $T^{[u]}$. We now modify the AVENUE-STRATEGY for clearing $T_1^{[u]}$ to a strategy for clearing $T^{[u]}$. When we use the AVENUE-STRATEGY to clear $T_1^{[u]}$, there is a moment at which all

k cops are located at x_j . While one cop protects x_j by staying on it or its neighbour x'_j between x_j and u , or visiting x'_j every other term, the other $k-1$ cops can clear all components $T_1^{[v_j]}$, $2 \leq j \leq d_1$. Since $T_1^{[v_j]}$ is the same as components $T^{[v_j]}$, $2 \leq j \leq d_1$, and $t_{m_j}^{v_j} < k$, $d_1 + 1 \leq j \leq d$, these $k-1$ cops can also clear components $T^{[v_j]}$, $d_1 + 1 \leq j \leq d$. Thus k cops can clear the tree $T^{[u]}$. As $t_{m_j}^{v_j} < k$, $d_1 + 1 \leq j \leq d$, components in $T^{[u]} - N^3[u]$ containing the avenue are the components in $T_1^{[u]} - N^3[u]$. So $P_{T^{[u]}} = P_{T_1^{[u]}}$.

Similarly, we can show that if $T_1^{[u]}$ has a hub x or a street vertex x and u is a vertex in some component of $T_1^{[u]} - x$, we can still use k cops to clear $T^{[u]}$ and the path in $T_1^{[u]}$ is the path in $T^{[u]}$.

Now consider $t_{m_j}^{v_j} = k$ for at least one $j \in \{d_1 + 1, \dots, d\}$. Then there exist a k -branching child of u in $T^{[u]}$, and $T^{[u]} - N^3[u]$ has a k -weakly-branching vertex say w_1 in $T^{[u]}$. From Definitions 4.1, there are exactly two components in the forest $T^{[w_1]} - w_1$ such that the root of this component is a k -pre-branching vertex. Thus in the forest $T^{[u]} - N^3[w_1]$, there exist at least three components with cop number k . It follows from Theorem 3.5 that $c_1(T^{[u]}) \geq k + 1$. We can clear $T^{[u]}$ by stationing one cop on u and using the other k cops to clear each component with cop number at most k . Thus $c_1(T^{[u]}) = k + 1$. As $T - N^3[u]$ does not have any component with cop number $k + 1$, u is a hub in $T^{[u]}$. Thus u is a road in $T^{[u]}$. ■

From the above Theorem we know if $c_1(T^{[u]}) = k$ then $P_{T^{[u]}} = P_{T_1^{[u]}}$. So if $P_{T_1^{[u]}}$ contains a k -branching vertex then $x_1^u = u$, otherwise $x_1^u = \perp$.

The following theorem is an extension of Theorem 3.5.

Theorem 4.14. *Let $T^{[u]}$ be a rooted tree and $v_1, \dots, v_d$ be the children of the root u . Suppose that for $1 \leq j \leq d$,*

$$L_{T^{[v_j]}}(v_j) = (t_{m_j}^{v_j}, x_{m_j}^{v_j}, P_{m_j}^{v_j}; \dots; t_1^{v_j}, x_1^{v_j}, P_1^{v_j}; 0, 0; 0, 0),$$

where $t_{m_1}^{v_1} \geq t_{m_2}^{v_2} \geq \dots \geq t_{m_d}^{v_d}$ and $x_1^{v_j} \neq \perp$. Let $k = c_1^(T^{[u]} - u)$. If $\mathbf{c}_1(T^{[v_1]}) = \mathbf{c}_1(T^{[v_2]}) = k$, then $L(u) = (k + 1, \perp, u; 0, 0; 0, 0)$.*

Proof. Since v_1 and v_2 are two k -branching children of u in $T^{[u]}$, each $T^{[v_i]}$, $i = 1, 2$, has a k -weakly-branching vertex w_i , $i = 1, 2$. From Definitions 4.1, there are exactly two components in the forest $T^{[w_1]} - w_1$ such that the root of this component is a k -pre-branching vertex. Thus in the forest $T^{[u]} - N^3[w_1]$, there exist at least three components with cop number k . It follows from Theorem 3.5 that $\mathbf{c}_1(T^{[u]}) \geq k + 1$. We can clear $T^{[u]}$ with $k + 1$ cops in the following way: First one cop protects w_1 and the other k cops clear all components in $T^{[w_1]} - w_1$; then all cops move along the path from w_1 to w_2 to clear all branches attached on the internal vertices of the path; finally, one cop stays on w_2 and the other k cops clear all components in $T^{[w_2]} - w_2$. Therefore $\mathbf{c}_1(T^{[u]}) = k + 1$.

As $T - N^3[u]$ does not have any component with cop number $k + 1$, u is a hub in $T^{[u]}$. According to Definition 4.5, u is a road in $T^{[u]}$. Since u and all its descendants are not $(k + 1)$ -branching, we have $x^u = \perp$. Note that neither u is $(k + 1)$ -weakly-branching nor it has a $(k + 1)$ -weakly-branching descendant. So $I_{\text{wb}}^{k+1}(u) = 0$ and $J_{\text{w}}^{k+1}(u) = 0$. Similarly, we have $I_{\text{pb}}^{k+1}(u) = 0$ and $J^{k+1}(u) = 0$. Hence $L(u) = (k + 1, \perp, u; 0, 0; 0, 0)$. ■

We next show the correctness of Algorithm 1.

Theorem 4.15. *For a tree T with at least three vertices, Algorithm 1 computes the label of the root of T .*

Proof. First, arbitrarily select a vertex as the root of T , which induces the parent-child relation in the tree. Then the vertices of T can be ordered such that each vertex is listed before its parent. Let $u_1, \dots, u_n$ be such an ordered list of vertices. So u_n is the root of T . If a vertex has no child, then from Definition 4.6, its label is $(1, \perp, u; 0, 0; 0, 0)$. Let u be a vertex in the list whose children's labels have been computed. Let $v_1, \dots, v_d$ be all children of u with labels in the form of Eq. (4.3). Let $T_1^{[u]}$ be the tree defined in Step 5 of Algorithm 1. For any $v \in V(T_1) - \{u\}$ we compute the label $L_{T_1^{[v]}}(v)$ by simply deleting all components except the last seven components from $L_{T[v]}(v)$.

From Theorem 4.12, we can compute $L_{T_1^{[u]}}(u)$ in Step 6. The road in $L_{T_1^{[u]}}(u)$ is calculated at that time.

Note that the cop number of $T_1^{[u]}$ is the largest key k in $L_{T_1^{[u]}}(u)$. From the proof of Theorem 4.13, we know by utilizing the road in $T_1^{[u]}$, we can use this cop number k to clear the subtrees whose corresponding items have keys less than k , which results in the lists $L_1, \dots, L_d, L_{d+1}$ created in Step 7. If no key in $L_1, \dots, L_d, L_{d+1}$ is contained in two different lists, from Theorem 4.13, $L_{T^{[u]}}(u)$ consists of the items of $L_1, \dots, L_d$ and $L_{T_1^{[u]}}(u)$, and thus, it is indeed the label of u in the subtree $T^{[u]}$; otherwise, let k^* be the largest key appears in at least two lists in $L_1, \dots, L_d, L_{d+1}$, which is found in Step 9. Let $K = (k_1, \dots, k_\ell)$ be a decreasing sequence created in Step 10, and let the corresponding items

be $(Q_1, \dots, Q_\ell)$, where $Q_i = (k_i, x_i, P_i)$, $1 \leq i \leq \ell$, is an item with key k_i and attributes x_i and P_i . Note that $k_\ell = k^*$. So either u has at least two k_ℓ -branching descendants or u has one k_ℓ -branching descendant and $c_1(T_1^{[u]}) = k$. Similar to the proof of Theorem 4.14 and Theorem 4.13, we have $c_1(T^{[u]} - \bigcup_{j=1}^{\ell-1} V(T^{[x_j]})) \geq k_\ell + 1$. If $k_{\ell-1} > k_\ell + 1$, then in Step 11, we have $h = \ell$ and we update K by $(k_1, \dots, k_{\ell-1}, k'_\ell)$, where $k'_\ell = k_\ell + 1$. If $k_{\ell-1} = k_\ell + 1$, then we have $c_1(T^{[u]} - \bigcup_{j=1}^{\ell-2} V(T^{[x_j]})) \geq k_\ell + 2$. Keep going in this way, we can find the smallest index h in K , where $1 \leq h \leq \ell$, such that $k_h = k_{h+1} + 1 = \dots = k_\ell + (\ell - h)$. Thus we can update K by $(k_1, \dots, k_{h-1}, k'_h)$ where $k'_h = k_h + 1$ in Step 11. Let $T_2^{[u]} = T^{[u]} - \bigcup_{j=1}^{h-1} V(T^{[x_j]})$. As $T_2^{[u]} - N^3[u]$ does not have any components with cop number k'_h then u is a hub in $T_2^{[u]}$. According to Definition 4.5, u is a road in $T_2^{[u]}$. From the structure of $T_2^{[u]}$, we have $x_h = \perp$, and $I_{\text{wb}}^{k'_h}(u) = J_{\text{w}}^{k'_h}(u) = I_{\text{pb}}^{k'_h}(u) = J^{k'_h}(u) = 0$. From Lemma 4.9 we know each $T^{[x_i]}$, $1 \leq i \leq h-1$ contains a road containing two utmost- k -pre-branching vertices. Therefore, the list X generated in Step 11 is indeed the label of u in the subtree $T^{[u]}$. We repeat Steps 4–11 until the root u_n of the tree has a label $L_{T^{[u_n]}}(u_n)$. ■

In the remainder of this section, we analyze the running time of Algorithm 1.

Lemma 4.16. *Suppose that each $S_i = (t_{m_i}^i, t_{m_i-1}^i, \dots, t_1^i)$, $1 \leq i \leq d$, is a list of strictly decreasing positive integers. Let s be the largest number in $S_1, \dots, S_d$ which occurs in at least two lists if one exists; otherwise, let $s = 0$. Then s can be determined in $O(\max\{t_{m_1}^1, \dots, t_{m_d}^d\} + d)$ time.*

Proof. It takes $O(d)$ time to sort the first components $t_{m_i}^i$, $1 \leq i \leq d$, in the

lists in descending order using the bucket sort. W.l.o.g., suppose after sorting we have $t_{m_1}^1 \geq t_{m_2}^2 \geq \dots \geq t_{m_d}^d$. Then s can be determined by the following procedure.

Initially set $s \leftarrow 0$ and $S \leftarrow S_1$.

For $i := 2$ to d , do

For $t \in \{t_{m_i}^i, \dots, t_1^i\} \cup \{s+1\}$ from $t_{m_i}^i$ down to $\max\{t_1^i, s+1\}$, do

If t is found in S , then update $s \leftarrow t$ and

if $s = t_{m_i}^i$ or $i = d$, then return s , else, break the inner loop;

else, insert t into S .

In this procedure, we scan S from the left (the largest number) to the right to find the target number t or insert it into S . Since S contains at most $t_{m_1}^1$ numbers, the number of insertions is at most $t_{m_1}^1$. So the runtime of the procedure is $O(t_{m_1}^1 + d)$, which is $O(\max\{t_{m_1}^1, \dots, t_{m_d}^d\} + d)$. ■

Lemma 4.17. *Let $f(n)$ be a function defined on the positive integers by the recurrence equation*

$$f(n) = \begin{cases} c, & n = 1 \\ f(n_1) + c, & d = 1, n \geq 2, \\ \max_M \{ \sum_{i=1}^d f(n_i) + c(\lceil \log n_1 \rceil + d) \}, & d \geq 2, n \geq 3, \end{cases}$$

where $M = \{(n_1, \dots, n_d) \mid n_1 \geq \dots \geq n_d \geq 1, \text{ and } \sum_{i=1}^d n_i = n - 1\}$, and $c \geq 1$ is a constant. Then $f(n)$ is $O(n)$.

Lemma 4.17 can be proved similarly to Lemma 3.13 in [22].

Theorem 4.18. *Algorithm 1 can be implemented in linear time.*

Proof. Let T be a tree with $n \geq 3$ vertices. In Algorithm 1, Step 2 requires $O(n)$ time by the topological sort. In each iteration of the loop from Steps 4 to 11, we compute the label of u in the subtree $T^{[u]}$. Let $v_1, \dots, v_d$ be the children of u whose labels have been calculated. When we calculate those labels, we use a linked list for each label. We also use a flag associated with every label to indicate whether the label contains $\perp$. So Step 4 requires $O(d)$ time to find $I_\perp$ and I_b .

In Step 5, we construct the subtree $T_1^{[u]}$. In order to save time for scanning the labels, we compress the label representation. For a sub-list of items with consecutive keys in a label, we use an interval to represent them. For any $v \in V(T_1) - \{u\}$ we compute the label $L_{T_1^{[v]}}(v)$ by simply deleting all components except the last seven components from $L_{T^{[v]}}(v)$. As we are using linked list to store the labels Step 5 requires $O(d)$ time.

In Step 6 we compute $L_{T_1^{[u]}}(u)$ from $L_{T_1^{[v_1]}}(v_1), \dots, L_{T_1^{[v_d]}}(v_d)$ by using Theorem 4.12. Its easy to see that all cases except Case (5.2) and Case (5.3) in Theorem 4.12 takes $O(d)$ time.

In (5.3) to find the utmost- k -pre-branching vertex in $T_1^{[u]}$ we check the sixth component of each label of d children of u and find the child v_1 with $I_{\text{pb}}^k(v_1) = 1$. Now we check the third component of v_1 . If P^{v_1} contains v_1 then v_1 itself is the utmost- k -pre-branching vertex in $T_1^{[v_1]}$. Then uv_1 is the road in $T_1^{[u]}$. If P^{v_1} contains v_1 and the utmost- k -pre-branching vertex z_1 in $T_1^{[v_1]}$, then $u \sim z_1$ is the road. So to find the road we check the label of each d children

of u . Similarly we can find the utmost- k -pre-branching vertices in Case (5.2). So all cases in Theorem 4.12, take $O(d)$ time to compute the road in $L_{T_1^{[u]}}(u)$. So the complexity after executing Step 6 for each vertex is bounded by the number of incident edges $O(d)$.

In Step 7, it takes $O(d)$ time to modify the labels to obtain the lists $L_1, \dots, L_{d+1}$. Let $t_{m_i}^i$, $1 \leq i \leq d+1$, be the largest key in L_i . In Steps 8 and 9, from Lemma 4.16, it takes $O(\max\{t_{m_1}^1, \dots, t_{m_{d+1}}^{d+1}\} + d)$ time to determine if all keys in $L_1, \dots, L_{d+1}$ are different or find the largest repeated key k^* in the lists. Using the compressed representation of labels, the runtime of Steps 10 and 11 is $O(\max\{t_{m_1}^1, \dots, t_{m_{d+1}}^{d+1}\} + d)$.

From Lemma 4.17, the total runtime of the loop from Steps 4 to 11 is $O(n)$. Thus Algorithm 1 can be implemented in linear time. ■

4.4 An Optimal Clearing Guide of a Tree

Algorithm CLEAR($T^{[u]}$) is an optimal clearing guide of a tree $T^{[u]}$.

Figure 4.11: Tree $T_i^{[x_i^u]}$, $i \in \{2, 3, \dots, m\}$.¹

¹In Figure 4.11, $T_i^{[x_i^u]}$ is a subtree of $T_i^{[u]}$.

Algorithm 2 Clear($T^{[u]}$)

Input: A rooted tree $T^{[u]}$ with $n \geq 3$ vertices and the labels of all vertices in $T^{[u]}$. The label of root u in $T^{[u]}$ is $L(u) = (t_m^u, x_m^u, P_m^u, \dots; t_1^u, x_1^u, P_1^u; I_{wb}^{t_1^u}(u), J_w^{t_1^u}(u); I_{pb}^{t_1^u}(u), J_p^{t_1^u}(u))$.

Output: A guide to clear $T^{[u]}$ with $c_1(T^{[u]})$ cops.

If the robber is seen by a cop at any moment then the robber will be captured by the cop in at most b rounds, where b is the diameter of the tree.

At the beginning of the game place all $c_1(T^{[u]})$ cops on u . All cops are free at this moment. Assign grey color to all vertices of $T^{[u]}$ to indicate them as dirty.

if $m = 1$ **then**,

 BASE-STRATEGY($T^{[u]}$).

return

else

 For $1 \leq i < m$, $T_i^{[u]} = T^{[u]} - \bigcup_{j=i+1}^m V(T_j^{[u]})$ and $T_m^{[u]} = T^{[u]}$. For $2 \leq$

$i \leq m$, P_i^u contains two terminal vertices p_0^i and $p_{q_i+1}^i$ of a road in $T_i^{[u]}$. Let $A^i = \{p_1^i, \dots, p_{q_i}^i\}$ be the set of internal road vertices. Suppose that $p_{a_i}^i \in A^i$ be the great grand child of x_i^u . Let $y_{a_i}^i$ and $z_{a_i}^i$ be two vertices on the path $x_i^u p_{a_i}^i$ where $z_{a_i}^i$ is the parent of $p_{a_i}^i$ and $y_{a_i}^i$ is the child of x_i^u . Let w_i be the parent of x_i^u . Let $Y_{p_0}^i$ (resp. $Y_{p_{q_i+1}}^i$) be the component in $T_i^{[u]} - A^i$ which contains p_0^i (resp. $p_{q_i+1}^i$). Let $Y_{p_j}^i$, $1 \leq j \leq a_i - 1$, (resp. $Y_{p_j}^i$, $a_i + 1 \leq j \leq q_i$) be the component containing p_j^i in the forest $T_i^{[u]} - \{p_{j-1}^i p_j^i, p_j^i p_{j+1}^i\}$. Let $Y_{x_i}^i$ (resp. $Y_{y_{a_i}}^i$, $Y_{z_{a_i}}^i$ and $Y_{p_{a_i}}^i$) be the component containing x_i^u (resp. $y_{a_i}^i$, $z_{a_i}^i$ and $p_{a_i}^i$) in the forest $T_i^{[u]} - \{x_i^u y_{a_i}^i, x_i^u w_i\}$ (resp. $T_i^{[u]} - \{x_i^u y_{a_i}^i, y_{a_i}^i z_{a_i}^i\}$, $T_i^{[u]} - \{y_{a_i}^i z_{a_i}^i, z_{a_i}^i p_{a_i}^i\}$ and $T_i^{[u]} - \{p_{a_i}^i p_{a_i-1}^i, z_{a_i}^i p_{a_i}^i, p_{a_i}^i p_{a_i+1}^i\}$) (see Figure 4.11).

for $i := m$ **to** 2 **do**

 Slide all $c_1(T_i^{[u]}) = t_i^u$ free cops to p_0^i .

for $j := 0$ **to** $a_i - 1$ **do**

 HUB-STRATEGY($Y_{p_j}^i, p_j^i$); slide all free t_i^u cops from p_j^i to p_{j+1}^i .

 One cop vibrates between $z_{a_i}^i$ and $y_{a_i}^i$. Indicate the cop as busy.

 Slide all t_1^u free cops to u ; then BASE-STRATEGY($T_1^{[u]}$).

for $i := 2$ **to** m **do**

 Slide all $t_i^u - 1$ free cops to x_i^u ; then HUB-STRATEGY($Y_{x_i}^i, x_i^u$)

 Slide all $t_i^u - 1$ free cops to $y_{a_i}^i$; then HUB-STRATEGY($Y_{y_{a_i}}^i, y_{a_i}^i$). Indicate t_i^u -th cop as free.

 Slide all t_i^u free cops to $z_{a_i}^i$; then HUB-STRATEGY($Y_{z_{a_i}}^i, z_{a_i}^i$)

Slide all t_i^u free cops to $p_{a_i}^i$; then HUB-STRATEGY($Y_{p_{a_i}}^i, p_{a_i}^i$)

Slide all t_i^u free cops to $p_{a_i+1}^i$;

for $j := a_i + 1$ to q_i **do**

HUB-STRATEGY($Y_{p_j}^i, p_j^i$); then slide all t_i^u free cops from p_j^i to p_{j+1}^i .

HUB-STRATEGY($Y_{p_{q_i+1}}^i, p_{q_i+1}^i$);

Slide all t_m^u free cops back to u .

return

function HUB-STRATEGY($T^{[v]}, v$)

Let $v_1, \dots, v_d$ be all children of v . For each v_i , $1 \leq i \leq d$, let v_{ij} , $1 \leq j \leq m_i$, be all children of v_i . For each v_{ij} , $1 \leq j \leq m_i$, let v_{ijp} , $1 \leq p \leq a_{ij}$, be all children of v_{ij} . For each v_{ijp} , $1 \leq p \leq a_{ij}$, let v_{ijpq} , $1 \leq q \leq b_{ijp}$, be all children of v_{ijp} .

For each v_i , $1 \leq i \leq d$, for each v_{ij} , $1 \leq j \leq m_i$, for each v_{ijp} , $1 \leq p \leq a_{ij}$, for each v_{ijpq} , $1 \leq q \leq b_{ijp}$, if v (resp. v_i) does not have any child, then v (resp. v_i) is cleared by the cop sitting at v , or if v_{ij} (resp. v_{ijp}) does not have any child then sliding a cop from v_i to each v_{ij} clears the subtree rooted at v_{ij} , otherwise, let one cop λ vibrates between v_i and v_{ij} and indicate the cop as busy, in the meantime, slide other free cops to v_{ijpq} and CLEAR($T^{[v_{ijpq}]}$) until the subtree rooted at v_{ij} is cleared.

Indicate λ as free.

Assign green color to grey vertices of $T^{[v]}$ to indicate $T^{[v]}$ as a cleared zone.

return

function BASE-STRATEGY($T^{[u]}$)

if $P_1^u = u$ **then**

HUB-STRATEGY($T^{[u]}, u$)

else

P_1^u contains two terminal vertices $p_0^1 p_{q_1+1}^1$ of a road. Let $A^1 = \{p_1^1, \dots, p_{q_1}^1\}$ be the set of internal road vertices. Let $Y_{p_0}^1$ (resp. $Y_{p_{q_1+1}}^1$) be the component in $T^{[u]} - A^1$ containing p_0^1 (resp. $p_{q_1+1}^1$). Let $Y_{p_j}^1$, $1 \leq j \leq q_1$, be the component containing p_j^1 in the forest $T^{[u]} - \{p_{j-1}^1 p_j^1, p_j^1 p_{j+1}^1\}$.

Slide all t_1^u free cops to p_0^1 .

for $j := 0$ to q_1 **do**

HUB-STRATEGY($Y_{p_j}^1, p_j^1$); then slide all t_1^u free cops from p_j^1 to p_{j+1}^1 .

HUB-STRATEGY($Y_{p_{q_1+1}}^1, p_{q_1+1}^1$)

Slide all t_1^u free cops back to u .

return

Lemma 4.19. *Let $T^{[u]}$ be a tree. If the labels of all vertices are known, then we can construct a clearing guide for $T^{[u]}$ in $O(n \log n)$ time.*

Proof. The correctness of Algorithm 2 follows from our discussion in Sections 4.1, 4.2 and 4.3. We now analyze the running time of Algorithm 2. At the beginning of the game placing $c_1(T^{[u]}) = t$ cops on u takes $O(t)$ time, setting the indicator of t cops as free takes $O(t)$ time, and assigning grey color to all vertices of $T^{[u]}$ takes $O(n)$ time since n is the number of vertices in $T^{[u]}$. To keep track of free cops and their associated location we maintain a queue. The step of sliding t free cops to a vertex v consists of s rounds where s is the distance of the farthest cop from v . Since input tree $T^{[u]}$ has n vertices, the sliding actions of cops at each step take at most $O(n)$ time. The core function of Algorithm 2 is HUB-STRATEGY. We can implement HUB-STRATEGY using depth first search algorithm which takes $O(n)$, and assigning green color to the grey vertices at the end of HUB-STRATEGY also takes $O(n)$ time. Hence the complexity of implementing HUB-STRATEGY is $O(n)$. If $m > 1$ tree $T^{[u]}$ is partitioned into $T_1^{[u]}, T_2^{[u]}, \dots, T_m^{[u]}$ trees. Each $T_i^{[u]}$, $1 < i \leq m$ contains an avenue $p_1^i, \dots, p_q^i$. The t_i -weakly-branching vertex $p_{a_i}^i$ is the great grand child of x_i and is on the avenue. The clearing guide clears the subtree rooted at p_0^i with t_i cops by using HUB-STRATEGY, then all free t_i cops slide from p_0^i to p_1^i , then clear the subtree rooted at p_1^i with t_i cops by using HUB-STRATEGY, and keep doing same until $p_{a_i}^i$. Then the t_i -th cop keeps vibrating on the parent of $p_{a_i}^i$ and the child of x_i until $T_j^{[u]}$, $1 \leq j < i$ is cleared. This vibrating actions take $O(\log n)$ time. We set the indicator of the t_i -th cop as busy which takes $O(1)$ time. We do the above mentioned steps for each $T_i^{[u]}$, $m \geq i \geq 2$. To

accomplish this we need $O(n \log n)$ time, since we divide the avenue of each $T_i^{[u]}$ into half and sliding actions and HUB-STRATEGY both require $O(n)$ time. Tree T_1 is cleared using BASE-STRATEGY which takes $O(n)$ time. Similarly clearing the rest of the dirty vertices in $T_i^{[u]}$, $m \geq i \geq 2$ take $O(n \log n)$ time. If $m = 1$ then $T^{[u]}$ is cleared using BASE-STRATEGY which takes $O(n)$ time. Therefore, the total running time of Algorithm 2 is $O(n \log n)$. ■

Remark. *The clearing guide in Algorithm 2 contains some sliding actions and by simply following those sliding actions of cops the cop player can clear an input tree $T^{[u]}$. A step of sliding t free cops to a vertex v is actually consists of s rounds, where free cops refer to those cops not vibrating on any edge and s is the distance of farthest cop from v . Within s rounds if some cops reach v then those cops will sit on v until the farthest cop reaches to v . So, in a round $t_1 \leq t$ cops sit on v , $t - t_1$ cops move towards v and the rest of the cops in $T^{[u]}$ keep vibrating. If the robber is seen by a cop at some moment, then according to proposition 2.1 the robber can be captured by this cop in a finite number of rounds. If the robber ever enters the cleared zone then the robber will be seen by a cop in the next round and captured by the cop in a finite number of rounds, otherwise the cleared zone will be kept cleared until the end of the game.*

Lemma 4.20. *Let $T^{[u]}$ be a tree. In an optimal cop-win strategy computed by Algorithm 2 let α be the maximum number of vibrating cops in a round. Then $c_1(T^{[u]}) = \alpha + 1$.*

Proof. Let $L(u) = (t_m^u, x_m^u, P_m^u, \dots; t_1^u, x_1^u, P_1^u; I_{wb}^{t_1^u}(u), J_w^{t_1^u}(u); I_{pb}^{t_1^u}(u), J_p^{t_1^u}(u))$ and

$c_1(T^{[u]}) = t_m^u = k > 1$. If $m > 1$, Algorithm 2 calls $\text{HUB-STRATEGY}(T^{[v]}, v)$ to clear a subtree of $T^{[u]}$ rooted at a pre-avenue vertex v . If $m = 1$, then if $P_m^u = u$, Algorithm 2 calls $\text{HUB-STRATEGY}(T^{[u]}, u)$, otherwise Algorithm 2 calls $\text{HUB-STRATEGY}(T^{[v]}, v)$ to clear a subtree of $T^{[u]}$ rooted at a pre-avenue vertex or a street vertex v . As u (resp. v) is a hub in $T^{[u]}$ (resp. $T^{[v]}$) and $c_1(T^{[u]}) = k$ (resp. $c_1(T^{[v]}) = k$), then $T^{[u]} - N^3[u]$ (resp. $T^{[v]} - N^3[v]$) has at least three components H_1, H_2, H_3 with cop number $k - 1$ such that the path between any two of them contains u (resp. v). Algorithm 2 keeps one cop vibrating on a child say u_1 (resp. v_1) and a grandchild say u_2 (resp. v_2) of u (resp. v) and calls itself to clear H_i , $i \in \{1, 2, 3\}$. The vibrating cop keeps vibrating until the subtree rooted at u_2 (resp. v_2) is cleared. As Algorithm 2 calls itself to clear H_i , $i \in \{1, 2, 3\}$, if the road in H_i is equal to the root of H_i say q_i then $\text{HUB-STRATEGY}(H_i^{[q_i]}, q_i)$ is called to clear H_i , otherwise $\text{HUB-STRATEGY}(H_i^{[p_i]}, p_i)$ is called to clear a subtree of H_i rooted at a pre-avenue vertex or a street vertex p_i . As q_i (resp. p_i) is a hub in $H_i^{[q_i]}$ (resp. $H_i^{[p_i]}$) then $H_i^{[q_i]} - N^3[q_i]$ (resp. $H_i^{[p_i]} - N^3[p_i]$) has at least three components S_1, S_2, S_3 with cop number $k - 2$ such that the path between any two of them contains q_i (resp. p_i). If $c_1(H_i^{[q_i]}) > 1$ (resp. $c_1(H_i^{[p_i]}) > 1$), Algorithm 2 keeps one cop vibrating on a child say q_i^1 (resp. p_i^1) and a grandchild say q_i^2 (resp. p_i^2) of q_i (resp. p_i) and calls itself to clear S_i , $i \in \{1, 2, 3\}$. The vibrating cop keeps vibrating until the subtree rooted at q_i^2 (resp. p_i^2) is cleared. So now in a round there are two vibrating cops and $k - 2$ cops move to clear S_i , $i \in \{1, 2, 3\}$. As Algorithm 2 calls itself to clear S_i the previous mentioned steps are followed for S_i , $i \in \{1, 2, 3\}$. So there is a round when three cops are vibrating and

$k - 3$ cops move to clear a subtree of S_i , $i \in \{1, 2, 3\}$ with cop number $k - 3$. From the above discussion we can see there is a round when $k - 1$ cops are vibrating and one cop moves to clear a subtree with cop number one. Let W be the subtree with cop number one. If the root of W is equal to the root of T say w then $\text{HUB-STRATEGY}(W^{[w]}, w)$ is called to clear W , otherwise $\text{HUB-STRATEGY}(W^{[y]}, y)$ is called to clear a subtree of W rooted at a pre-avenue vertex or a street vertex y . As w (resp. y) is a hub in $W^{[w]}$ (resp. $W^{[y]}$) and $c_1(W^{[w]}) = 1$ (resp. $c_1(W^{[y]}) = 1$) then $W^{[w]} - N^3[w]$ (resp. $W^{[y]} - N^3[y]$) is empty. If w (resp. y) does not have any child or any grandchild then the cop will sit on w (resp. y) and the subtree is cleared, otherwise, let w_i (resp. y_i), $1 \leq i \leq d$ be the children and w_{ij} (resp. y_{ij}), $1 \leq j \leq m_d$, be the grandchildren of w (resp. y) and sliding the cop from each w_i (resp. y_i) to each w_{ij} (resp. y_{ij}) will clear the subtree $W^{[w]}$ (resp. $W^{[y]}$). So the cop does not need to vibrate between two vertices for more than two consecutive rounds. Then there is no vibrating cop in that subtree. So according to Algorithm 2 in this case the number of vibrating cops in a round in $T^{[u]}$ is $k - 1$. If the cop does not follow Algorithm 2 then the cop can vibrate between w_i (resp. y_i) and w_{ij} (resp. y_{ij}) for four consecutive rounds and clear $W^{[w]}$ (resp. $W^{[y]}$). In that case the number of vibrating cops in a round in $T^{[u]}$ is k . ■

Chapter 5

Relations between $c_1(T)$ and $c_0(T)$

In this Chapter, we explore relations between zero- and one-visibility cop numbers on trees. We divide our results into two sections here. In the first section, we give a relation between $c_1(T)$ and $c_0(T)$ for a general tree T . The second section contains relations between zero-visibility cop number and one-visibility cop number of some special trees.

5.1 Comparing $c_1(T)$ and $c_0(T)$

The following Theorem proves a relation between $c_1(T)$ and $c_0(T)$ for a general tree T .

Theorem 5.1. *For any tree T , $c_1(T) \leq c_0(T) \leq 2c_1(T)$.*

Proof. The first inequality is trivial. We only consider the second one. Let T be a tree with $c_1(T) = k$. Let S_1 be a 1-visibility cops' strategy to clear

T with k cops. We will describe a 0-visibility cops' strategy S_0 that clears T with $2k$ cops.

For each cop λ_i , $1 \leq i \leq k$, in the 1-visibility game, there are two corresponding cops λ_i^1 and λ_i^2 in the 0-visibility game. From Theorem 3.7 and Corollary 3.8, T must contain a hub, an avenue, or a street. Suppose that T contains a hub u and u is the root of T . Let v_i , $1 \leq i \leq d$, be the children of u . For $1 \leq i \leq d$, let w_{ij} be the children of v_i , $1 \leq j \leq m_i$. Let $X_{ij} = \{x \in V(T^{[u]}) \mid \text{dist}(x, v_i) = 3 \text{ and } \text{dist}(x, w_{ij}) = 2\}$, $1 \leq i \leq d$, $1 \leq j \leq m_i$. In S_1 , one cop, say λ_k , basically vibrates between v_i and w_{ij} while the other $k - 1$ cops clear each subtree $T^{[x]}$, $x \in X_{ij}$, individually. Then in the 0-visibility strategy S_0 , λ_k^1 vibrates between u and v_i and λ_k^2 vibrates between w_{ij} and its child between w_{ij} and $T^{[x]}$ (the case would be trivial if v_i or w_{ij} do not have children), while the other $2(k - 1)$ cops clear each subtree $T^{[x]}$, $x \in X_{ij}$, one by one. Thus, if T contains a hub, then S_0 clears T using $2k$ cops. Similarly, we can show that $2k$ cops can clear T if it contains an avenue or a street. Therefore $c_0(T) \leq 2k$. ■

A tree is called a *caterpillar* if removing all degree-one vertices produces a path or an empty graph. The next lemma shows that the lower bound in Theorem 5.1 is tight.

Lemma 5.2. *If T is a caterpillar, then $c_0(T) = c_1(T) = 1$.*

In Figure 2.1, we see that $c_1(T) = 3$ and $c_1(T') = 2$. This means that adding a pendant edge to a tree may cause the cop number to increase by

one. The next result gives a case where the cop number will not increase after adding a pendant edge.

Lemma 5.3. *Let T be a tree which has a path $v_1v_2v_3v_4u$ with $\deg_T(v_1) = 1$ and $\deg_T(v_i) = 2$, $2 \leq i \leq 4$. Let H be a tree obtained from T by adding a pendant edge v_0v_1 to the vertex v_1 . Then $c_1(H) = c_1(T)$.*

Proof. Pick u as the root in both T and H . From Algorithm 1, we know that v_4 has the same label in both T and H . Thus $c_1(H) = c_1(T)$. ■

From the above lemma, we can show that the lower bound in Theorem 5.1 is tight if for any two non-degree-2 vertices the distance between them is at least 8.

Lemma 5.4. *Let T be a tree such that the distance between any pair of non-degree-2 vertices is at least 8. Then $c_0(T) = c_1(T)$.*

Proof. Let T be a tree such that the distance between any two non-degree-2 vertices is at least 8. We first consider the case where $c_1(T) = 1$. If T contains a vertex u with $\deg_T(u) \geq 3$, then the forest $T - N_T^3[u]$ contains at least three components with cop number at least one. It follows from Theorem 3.5 that $c_1(T) \geq 2$. This is a contradiction. So T is a path, which implies that $c_0(T) = c_1(T) = 1$. Assume that if $c_1(T) \leq k$, then $c_0(T) = c_1(T)$.

We now consider the case where $c_1(T) = k + 1$, $k \geq 1$. Since $c_1(T) \geq 2$, T must contain a vertex of degree at least 3. From Theorem 3.7 and the subsequent results in Section 3.2, T must contain a hub, an avenue, or a street. Suppose that T contains a hub, say v . If $\deg_T(v) = 2$, let u be a

non-degree-2 vertex that is nearest to v . It follows from Lemma 5.3 that u is also a hub. So we can suppose v is a hub of T such that $\deg_T(v) > 2$. In $T - N_T^3[v]$, the 1-visibility cop number of every component is at most k . Let H be a component in $T - N_T^3[v]$ and let $vv_1v_2 \cdots v_8$ be a path in T such that v_4 is a leaf in H and $\deg_T(v_i) = 2$, $1 \leq i \leq 7$. Let H' be a tree obtained from H by adding edges v_4v_3 , v_3v_2 and v_2v_1 . It follows from Lemma 5.3 that $c_1(H') = c_1(H) \leq k$. By the assumption, we have $c_0(H') = c_1(H') \leq k$. Thus, in $T - v$, the 0-visibility cop number of each component is at most k . So in the 0-visibility game, we can clear T by stationing one cop on v and using k cops to clear each component of $T - v$. Hence $c_0(T) \leq k + 1$. Since for any tree its 0-visibility cop number is always greater than or equal to its 1-visibility cop number, we have $c_0(T) = c_1(T) = k + 1$. If T contains an avenue or a street, since AVENUE-STRATEGY and STREET-STRATEGY are based on HUB-STRATEGY, it is not hard to see that $c_0(T) = c_1(T) = k + 1$. ■

Figure 5.1: A tree where each dashed line represents a path of length at least 8.

The distance condition in Lemma 5.4 can be tight. For example, consider the tree T in Figure 5.1, where every path between a pair of non-degree-

2 vertices is of length at least 8 except the path between u and v . Since $\text{dist}(u, v) = 7$, it is easy to see that $c_0(T) = 3$ and $c_1(T) = 2$.

The following lemma shows that the upper bound in Theorem 5.1 is also tight.

Lemma 5.5. *Let T be a caterpillar with at least one vertex of degree 3. Let H be a tree obtained from T by replacing every edge by a path of length 3. Then $c_0(H) = 2c_1(H) = 2$.*

5.2 Special Trees

This section focuses on some special trees. We show relations between zero-visibility cop number and one-visibility cop number of perfect binary trees and perfect k -ary trees.

In a rooted tree, the *height of a vertex* is the distance from the root to this vertex. The *height of a rooted tree* is the largest distance from the root to a leaf.

5.2.1 Perfect Binary Trees

A *perfect binary tree* T^2 is a rooted tree in which every internal vertex has exactly two children and all leaves have the same height. Here we calculate both one-visibility cop number and zero-visibility cop number for a perfect binary tree T as a function of the height of the tree. We use T_h^2 to denote a

perfect binary tree with height h .

Theorem 5.6. *Let T_h^2 be a perfect binary tree with $h \geq 0$. Then $c_1(T_h^2) = \lceil \frac{h+1}{5} \rceil$ and $c_0(T_h^2) = \lceil \frac{h+1}{3} \rceil$.*

Proof. We first show $c_1(T_h^2) = \lceil \frac{h+1}{5} \rceil$ using induction. If $0 \leq h \leq 4$, it is easy to see that one cop can clear the tree, and so the claim holds. Assume that the claim is true when $5m \leq h \leq 5m+4$, $m \geq 0$. We only need to show that the claim is also true when $5m+5 \leq h \leq 5m+9$.

Consider the case where $h = 5m+5$. Let v be a child of the root u in T_{5m+5}^2 . In the forest $T_{5m+5}^2 - N^3[v]$, there are at least two components which are isomorphic to T_{5m}^2 and there is at least one component which is isomorphic to T_{5m+2}^2 such that the path between any pair of the three components contains v (see figure 5.2). From the assumption and Theorem 3.5, we have $c_1(T_{5m+5}^2) \geq m+2$. Note that we can clear T_{5m+5}^2 with $m+2$ cops by utilizing HUB-STRATEGY where v is the hub. Thus $c_1(T_{5m+5}^2) = m+2$.

Figure 5.2: Tree T_{5m+5}^2 with root u .

Let u be the root of T_{5m+9}^2 . In the forest $T_{5m+9}^2 - N^3[u]$, there are two components which are isomorphic to T_{5m+5}^2 such that the path between them contains u (see figure 5.3). So u is an avenue vertex and T_{5m+9}^2 can be cleared with $m + 2$ cops by utilizing AVENUE-STRATEGY.

Figure 5.3: Tree T_{5m+9}^2 with root u .

Therefore, $c_1(T_h^2) = \lceil \frac{h+1}{5} \rceil$ for any $h \geq 0$.

Similarly to the above, we can show $c_0(T_h^2) = \lceil \frac{h+1}{3} \rceil$ by Theorem 3.6 in [7].

■

5.2.2 Perfect k -ary Trees

A *perfect k -ary tree* is a rooted tree in which every internal vertex has k children and all leaves have the same height. We use T_h^k to denote a perfect k -ary tree with height h . Notice that the 1-visibility cop number of a perfect k -ary tree can be computed by Algorithm 1.

Theorem 5.7. *Let T_h^k be a perfect k -ary tree with $k \geq 3$ and $h \geq 0$. Then $c_1(T_h^k) = \lceil \frac{h+1}{4} \rceil$, and $c_0(T_h^k) = \lceil \frac{h+1}{2} \rceil$.*

Proof. We use induction to show $c_1(T_h^k) = \lceil \frac{h+1}{4} \rceil$. Observe that when $0 \leq h \leq 3$, one cop can clear the tree, and thus, the claim holds. Assume that the claim is true when $4m - 4 \leq h \leq 4m - 1$, $m \geq 1$. We will show that the claim is also true when $4m \leq h \leq 4m + 3$.

First consider the case where $h = 4m$. Let u be the root of T_{4m}^k . In the forest $T_{4m}^k - N^3[u]$, there are at least three components which are isomorphic to T_{4m-4}^k such that the path between any pair of the three components contains u (see Figure 5.4). From the assumption and Theorem 3.5, we have $c_1(T_{4m}^k) \geq m + 1$. Since we can clear T_{4m}^k with $m + 1$ cops by the HUB-STRATEGY where u is the hub. Thus $c_1(T_{4m}^k) = m + 1$. Similarly, we can show that $c_1(T_{4m+i}^k) = m + 1$ for $1 \leq i \leq 3$. Hence $c_1(T_h^k) = \lceil \frac{h+1}{4} \rceil$ for any T_h^k with $k \geq 3$ and $h \geq 0$.

Figure 5.4: Tree T_{4m}^3 with root u .

Similarly, from Theorem 3.6 in [7], we can show that $c_0(T_h^k) = \lceil \frac{h+1}{2} \rceil$. ■

Chapter 6

Conclusions and Future Work

6.1 Summary

In the original cops and robber game, the location of the robber is known throughout the game. In the zero-visibility cops and robber game, cops can only see the robber when both cops and robber occupy the same vertex. In this thesis, we explore another extreme, one-visibility cops and robber game where cops can see the robber when the distance between them is at most 1. Here we have considered one-visibility cops and robber game only on trees. We have described an essential theorem for the lower bound of cop number on trees. Here we give structure properties in terms of the hub, avenue and street and also provide strategies to clear trees according to their structures. Throughout the thesis, we have considered only one robber in trees. However if we extend the game with more than one robber the strategy provided here

can still clear trees. We have developed a linear algorithm for calculating the labels of a tree. Our focus was to compute the cop number of a tree and a road to clear the tree optimally. We have provided an optimal clearing guide to clear a tree. Moreover, we have established some relations between zero-visibility cop number and one-visibility cop number of a tree.

6.2 Future Work

There remains a considerable amount of future work concerning the one-visibility cops and robber game. We have considered only trees in this thesis, however, playing the one-visibility game on other graphs is yet to be explored. Finding efficient algorithms for computing the cop number of graphs like unicyclic graphs, CDGs, outerplanar graphs continue to be a challenge. Having optimal clearing strategies for 1-visibility cops and robber game on trees, its natural to consider other special classes of graphs such as planar graphs, cycle-disjoint graphs. Moreover, it would be very interesting to construct relationships between more general l -visibility games.

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