RHEOMETRY OF DENSE GRANULAR COLLAPSE ON INCLINED PLANES

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ABSTRACT

The continuum deformation of gravity-driven dense granular materials on steep inclined planes exhibit similar flow phenomenon to natural and industrial occurrences. This kind of granular flow dynamics corresponds to unsteady flows which currently lack any theoretical formulations for its kinematic properties.

The implementation of the $\mu(I)$ rheology model and the Moving Particle Semi-implicit (MPS) mesh-free method as a coupled set is used to predict quantitatively the flow properties and phenomena of the granular materials on inclined planes. During deformation of the granular materials, transition of flow regime shows that the velocity profile is maintained at the dense region but fluctuates when tending to the dilute region due to limited interaction between granular materials.

The flow pattern of the granular materials shows the development of multiple shear band behaviour resulting from the influence of gravity on the flowing granular materials depending on different Froude numbers. The granular materials shear stress is found to attain a yield point before reducing as the shear rate increases exhibiting a thixotropic behaviour.

We also compared the velocity profile, shear-band, surface profile and wave front of the flowing granular materials with physical studies on a two-dimensional configuration giving acceptable qualitative agreements.

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TABLE OF CONTENTS

Abstract	I
Acknowledgement	II
Table of contents	III
List of figures	VI
List of tables	IX
List of symbols	X
 CHAPTER 1: INTRODUCTION.....	 1
1.1 Background	1
1.2 Flow Regime	4
1.3 Developments	5
1.4 Scope of Thesis	9
1.5 Thesis Structure	10
 CHAPTER 2: EXPERIMENTAL SET UP AND THEORETICAL FORMULATION	 11
2.1 Experimental setup	11
2.2 Evolution of the flow body	12
2.3 Computational conditions	15

2.4 Theoretical formulations	16
2.5 The rheology $\mu(I)$	16
2.6 Regularisation technique	18
2.7 Governing equations	18
 CHAPTER 3: NUMERICAL METHOD	 20
3.1 MPS method	20
3.1.2 Requirement of the Kernel Function	22
3.2 Discretization technique	24
3.3 Boundary conditions	25
 CHAPTER 4: RESULT AND DISCUSSION	 27
4.1 Properties of the granular flow behaviour	27
4.2 Velocity profile	27
4.3 Velocity evolution	35
4.4 Velocity contour	38
4.5 Shear banding in unsteady states	41
4.6 Shear	42
4.7 Shear distribution	45
4.8 Surface profile	46

4.9 Wave front	50
4.10 General Discussion	51
 CHAPTER 5: CONCLUSION	 52
5.1 Summary	52
 References	 55

LIST OF FIGURES

Fig. 1.1: Examples of industrial and geophysical granular processes	3
Fig. 1.2: Typical granular materials	3
Fig. 2.1: Experimental setup	12
Fig. 2.2: The series of snapshots for an experiment with glass bead granules for 45° slope	13
Fig. 2.3: The series of snapshots for an experiment with glass bead granules for 30° slope	14
Fig. 3.1: Plotted kernel functions from table 3.1	22
Fig. 4.1 Horizontal velocity profile for 45° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations	29
Fig. 4.2 Vertical velocity profile for 45° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations	30
Fig. 4.3 Horizontal velocity profile for 30° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations	31
Fig. 4.4 Vertical velocity profile for 30° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations	32

Fig. 4.5 Velocity evolution during acceleration down a slope of 45° during initialisation when the granular materials start to deform at fixed sections	36
Fig. 4.6 Velocity evolution during acceleration down a slope of 30° during initialisation when the granular materials start to deform at fixed sections	37
Fig. 4.7 Velocity contour plot profile for 45° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s (e) 0.4s	39
Fig. 4.8 Velocity contour plot profile for 30° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s (e) 0.4s	40
Fig. 4.9 Multi-Shear band layers for 45° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.20s (e) 0.30s	43
Fig. 4.10 Multi-Shear band layers for 30.8° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.20s (e) 0.30s	44
Fig. 4.11 Rheological behaviour of the granular materials where (a) and (b) shows shear stress and shear rate as a function of time and (c) to (d) shows the friction coefficient and shear stress as a function of shear rate	45
Fig. 4.12 Shear stress and shear rate for 45° slope	47

Fig. 4.13: Surface profile for 45° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s	48
Fig. 4.14 Surface profile for 30° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s	49
Fig. 4.15 Wave front time evolution for 45° and 30° slope showing comparison between experimental and numerical results	51

LIST OF TABLES

Table 2.1 Geometry and simulation parameters	15
Table 3.1 List of other successful weighted functions	21

LIST OF SYMBOLS

I = inertia number

$\mu(I)$ = friction coefficient

P = pressure

d_g = grain size in diameter

ρ_s = density of granular materials

τ_i = local tangential shear stress

γ_{ij} = strain rate tensor

γ = second invariant

$\partial_j u_i$ = partial derivatives of velocity components in the i direction

$\partial_i u_j$ = partial derivatives of velocity components in the j direction

θ_s = static frictional angles

θ_2 = dynamic frictional angles

σ_{ij} = Cauchy stress tensor

ε = volume of fraction

δ_{ij} = unit tensor

τ_{ij} = deviatoric stress

I_0 = empirical coefficient constant

μ_s = static friction coefficient

μ_2 = dynamic friction coefficient

η_R = regularisation technique

α_r = regularisation convergence value

$\mathbf{u}$ = particle velocity vector

t = time

g = gravity

ρ = density for fluid

ρ_s = granular materials density

μ = dynamic viscosity

τ = stress tensor

$\omega(r)$ = the kernel function

r_e = radius of the interaction circle around each targeted particle

d_l = particle size

r = distance between a target particle i and one of its neighboring particle j

$\langle n \rangle_i$ = particle number density

$\langle \rho \rangle_i$ = real fluid density

m_i = is the mass of the fluid particle

ϕ = arbitrary scalar term

n^0 = initial particle density

n_i = under the incompressible flow

λ = coefficient used to scale the variance for consistency

β = threshold coefficient

ζ = coefficient used in the equation of state

c_0 = is the artificial sound speed in the reference medium

θ_r = angle of repose

(h) = depth of the granular flow

$\vec{u}$ = dimensionless velocity

$\vec{y}$ = dimensionless position of granular material

v_y = vertical velocity component

v_x = horizontal velocity component

$\langle \nabla \phi \rangle|_{ij}$ = gradient operator

$\langle \nabla^2 \phi \rangle|_{ij}$ = Laplacian operator

CHAPTER 1: INTRODUCTION

1.1 Background

The study of physical science shows to human existence how important it is to understand matter and energy. The complexities in the phase classification of matter which is unique in dense granular flows result in the need to further understand the rheological properties of the particulate system during geophysical transition and in formulating a general constitutive relationship ([Berzi et al. 2011](#); [Jop et al. 2005](#); [Jop, 2015](#); [Kumaran, 2006](#); [MiDi, 2004](#); [Pouliquen et al. 2009](#); [Shaninpoor and Siah, 1981](#); [Shahinpoor, 1983](#)). Natural hazardous events exhibit complex rheologies which are associated with geophysical granular flows like landslides, debris flows, avalanches and pyroclastic flows ([Forterre and Pouliquen 2008](#); [Pouliquen and Forterre 2002](#)). However, having a broad knowledge in this area would be significant for both scientific accomplishment and engineering application to mitigate risk from such hazards. Previous studies has been completed to gain a better understanding of the behaviour of dense granular flows on different flow configurations namely plane shear, coquette cell, vertical silo, inclined plane, heap flow and rotating drum ([Ancey, 2001](#); [Azanza et al. 1999](#); [Bagnold, 1954](#); [Cui and Gray, 2013](#); [Da Cruz et al. 2005](#); [Dunatunga et al. 2015](#); [Forterre and Pouliquen, 2008](#); [MiDi, 2004](#); [Pouliquen and Forterre 2009](#); [Shukla and Meheboob, 2013](#); [Xu and Jin, 2016](#)). Despite the study of these various configurations, the inclined plane represents a close relationship to real world situations as the collisions and frictional contacts exhibit complex properties during particulate flows. These kind of dry particulate flows phenomenon are centered on the effect of instantaneous application of gravitational force or other external forces and the internal friction force between granular materials.

The interaction between granular particles can be classified to exhibit plasticity behaviour during deformation. The kinetic theories which are dependent on a uniform arrest or infinite flow play a significant role in the description of the granular flow. Depending on the thickness of the flowing materials, the collisional flows are characterised with a certain local order depending on the density profile and distribution which are dense near the bed and very dilute at the free surface ([Azanza et al. 1999](#)). Several geophysical flows occur both in the industry and nature as shown in Fig. [1.1](#). In most cases the flows are destructive leading to loss of life and properties. Granular materials are found all around us in various forms, few examples of granular materials are shown in Fig. [1.2](#). The sensitivity of the granular flow varies for different configuration and parameters. Some of these parameters are the channel inclination, shape of the grains, granulometry, coupling with the interstitial fluid and to a lesser extent the flow rate and the kind of roughness ([Jop et al. 2006](#); [Azanza et al. 1999](#); [Ancey, 2002](#); [Pouliquen and Chevoir, 2002](#); [Goujon, 2003](#)). Depending on the configuration and sensitivity of parameters, deformation of the granular materials during shear may be classified as compact when the material becomes loose or dilatant when dense ([Pouliquen et al. 2002](#)) resulting into the development and transition of flow regimes.

Fig. 1.1: Examples of industrial and geophysical granular processes¹.

This is a public domain photo from <http://nsem.info/know-risks/avalanche>

Fig. 1.2: Typical granular materials¹.

- a. This is a public domain photo from <http://www.realfoodforlife.com/11-gluten-free-grains/>
- b. This is a public domain photo from <http://adnams.co.uk/about/news/hotel-news/nature-environment/southwold-snow-in-pictures/>

1.2 Flow Regime

The transfer of momentum and dissipation of energy from inter-particle collision leads to the formulation of a rheology from kinetic theories, where the resulting regime is termed as a collision or kinetic regime which is as a function of the inclination range and particle flux ([Azanza et al. 1999](#)). The inclined flow dynamics is related to two basic angles which are the angle of repose and maximum angle of stability, both known as the material properties ([Azanza et al. 1999](#); [Silbert et al. 2003](#)). The former is dependent on gravity and empirically observed from experimental or numerical studies which are the static state of the granular materials while the latter in most cases if greater than the angle of repose induces flow. This induced flow would stop only when force balance occur between gravity and the friction generated by energy transfer and dissipation between granular materials during multi-collision, otherwise the materials flow continuously until it approaches the end of the channel or when obstructed ([Ancey, 2001](#); [Pouliquen and Chevoir, 2002](#)). A flow to no flow or continuous flow is determined from the relativity of these angles which could result to a steady state flow, unsteady state flow or no flow ([Ancey, 2001](#); [Azanza et al. 1999](#); [Pouliquen and Chevoir, 2002](#)). A static or flow at rest when the inclination angle exceeds the angle of repose of the flowing granular materials termed as a steady uniform or continuous flow when the inclination angle exceeds the maximum angle of stability becoming rapidly unstable and producing a stationary wedge close to the boundary ([Ancey, 2001](#)). Since unstable flow is mostly determined by high flow rates or inclination, the free surface mostly experiences deformation in the transverse section and sometime may lead to longitudinal vortices as observed by [Forterre et al. \(2001\)](#) and [Shukla and Meheboob, \(2013\)](#).

The rate of transition between the solid and liquid phase is very short and done in substrate layers with uniform velocities until a minimum depth is reached. The dense flow regime is characterised as an inelastic collision where several interactions occurs between the particles which is mostly determined from the degree of slope and the angle of repose of the granular material. Inclined flows may behave as solid, liquid or gas ([Ancey, 2002; Pouliquen et al. 2002](#)). Under mild inclination or effect of microgravity during initialisation, the flow of granular material is termed a solid and a liquid for steeper inclinations. A solid flow regime is mostly referred to as the quasi-static as described from the plasticity characteristics where deformation of the material yields along slip plane as a result the ratio of tangential to normal stress to the internal friction of the granular material ([Pouliquen and Chevoir, 2002](#)).

1.3 Developments

Early mathematical formulations have been empirical and suggested by [Bagnold \(1954\)](#) since the mid 90's to be related to the mechanism found in the exchange of momentum between different particles collision occurring simultaneously for a specific time or infinitely depending on the flow type ([Bagnold, 1954](#)). Although this theory serves as the basis for several studies over the years, most recent studies show different flow properties while maintaining a similar configuration which might be linked to different experimental setup, sensitivity to the roughness conditions and flow instabilities in the form of vortices ([Forterre and Pouliquen, 2001; Lo et al. 2010](#)).

A complexity from the behaviour of the granular materials makes it easy to deal with arresting flows compared to those which flow continuously. The flow dynamics and characteristics at certain time for the continuous flow during

deformation are yet to be established. This brings about the friction between boundary layer and the use of two-dimensional geometry which would not capture the 3-d effect of the flow ([Azanza et al. 1999](#)). Rheology model sometimes don't capture dilute particles ([Xu and Jin, 2016](#)), and in some cases require fitting of parameters with the use of dimensionless variables ([Chauchat and Médale, 2014](#)). The formulation of varieties of model exists but with few contradictions, yet giving acceptable agreement with experimental studies for some flow phenomenon and profile characteristics ([Pouliquen and Chevoir, 2002](#)).

[Shaninpoor and Siah \(1981\)](#) and [Shaninpoor \(1983\)](#) further developed on Bagnold formulation for rapid flow using inter-granular collisions, friction and boundary conditions to describe the shear and normal stresses in the flow while comparing it with experimental observations. The given expression tends to classical results at a zero friction. It is expected that increase in inclination would present more complexities in the flow regime as instabilities generated leads to spontaneous pattern formulation resulting to longitudinal or spanwise vortices ([Börzsönyi, 2009](#); [Brodu et al. 2015](#); [Forterre and Pouliquen, 2001](#)). Since the instability of such flows makes it difficult to study, [Brodu et al. \(2015\)](#) introduces a frictional sidewall to initiate the addition of particle to wall frictional forces in order to balance the driving force using Discrete Element Method (DEM) method for simulating the flow. Coefficient of restitution and frictional coefficient was used to control the inelastic collision between particles and wall collision as the contact force using periodical boundary conditions. The investigation focused on flows on inclination greater than 30° generating a support flow regime with stabilised dense core within flustered regions close to the wall producing instabilities. This dense core exhibit constant properties over time and are located above highly agitated granular gaseous regime. At higher

inclination, the core transits to granular gas with an approximate constant depth. Overall a scaling law from the exploration of the angle and mass holdup was used to describe the velocity using a power law of $q \propto h^{1.25}$.

The higher the degree of inclination, the more difficulty it takes the flow in reaching a steady state. Since the frictional angle is material dependent, shallow slopes greater than the angle of repose are also classified as continuous flow which may attain a steady state. [Stanislav et al. \(2016\)](#) and [Jessop et al. \(2017\)](#) derived theoretical prediction for velocity, shear stress and shear rate for a constant thickness of the granular material on shallow inclined slope (17° to 25°). Although the frictional interaction is classified to be unsteady, energy from the collision of particles balances with the dissipation leading to a steady state and finally an arrest flow. The propositions was based on the gravitational and frictional forces acting as resistivity and driving force with rheological measurements sensitive to the flow characteristics while friction increases with shear rate under constant normal stress and linearly with the inertia number. The analytical model was validated using Discrete Element Method (DEM) numerical simulation method. [Jessop et al. \(2017\)](#) also proposed a mathematical formulation from a continuum description of two inter-penetrating phases interaction for unsteady developing flows on slopes for velocity, front wave and position while focusing on the transition from unsteady to uniform behaviour of fluidised current on shallow slopes.

Several studies related to dry granular flow on inclined planes were mostly completed by experimental and computer simulations. Experimental studies are limited in the literature due to difficulties in capturing rapid deformation of granular materials with constrained formulations. The behaviour of the granular materials during deformation have led to different propositions of its constitutive laws which

are then extended to simple rheology models used in predicting granular flows. [Lagrée et al \(2011\)](#) conducted comparative studies on some of the rheology models such as the $\mu(I)$ rheology, Newtonian, Bingham, Bagnold and the constant friction for granular column collapse and found that the $\mu(I)$ rheology model is capable of predicting most of its deformation dynamics. The incorporation of the $\mu(I)$ rheology model in an incompressible two or three dimensional fluid equation have inspired many numerical simulation methods for granular flows ([Barker et al. 2015 and 2017](#); [Ionescu, 2015](#); [Lagrée et al. 2011](#); [Tankeo et al. 2013](#)).

Most granular flow Simulations have mainly been carried out using a continuum or discrete modeling approach. One of the simulation method used is the particle method such as the Moving Particle Semi-implicit method (MPS) ([Koshizuka and Oka 1996](#)), Smoothed Particle Hydrodynamics (SPH) ([Lucy, 1977](#); [Gingold and Monaghan, 1977](#)) and the Material Point method (MPM) ([Sulsky et al. 1994](#)). The Smoothed Particle Hydrodynamics (SPH) method have been used to simulate granular collapse on an inclined bed ([Ikari and Gotoh, 2016](#)), free-surface flows ([Chambon et al. 2011](#)), and granular column collapse ([Szewc, 2017](#)). The Material Point Method has also been employed by [Dunatunga and Kamrin, \(2015\)](#) and [Fei et al. \(2012\)](#) to study granular flows on inclined plane. The MPS method has been used to study granular flows on limited configurations like the granular column collapse ([Xu et al. 2017](#)) and dry and submerged granular flows ([Jafari-Nodoushan et al. 2017](#)). To further explore more configurations using this method, the MPS method is employed with the $\mu(I)$ rheology as a coupled set to simulate the granular flow on steep inclined planes. These granular flows are either steady or unsteady depending on the flow regime. The coupled set have been used for granular column collapse on horizontal planes and now extended to inclined planes in this current study.

Previous studies were completed on steady granular flows leading to good understanding of its rheology, characteristics and proposition of empirical analytical methods compared to the unsteady state which is yet to be understood. Steady granular flows have been investigated using a range of inclination angles to show the flow kinematics (Fan et al. 2013; Ikari and Gotoh, 2016; Xu et al. 2017), run out distances (Ertas et al. 2001; Saingier et al. 2016), instabilities (Forterre and Pouliquen, 2003; Mollon, 2016; Shukla and Meheboob, 2013), deformation properties (Shojaee et al. 2012; Tordesillas et al. 2016), scaling (Pouliquen, 1999; Pouliquen and Forterre, 2002) and rheology (Berzi et al. 2011; Jop et al. 2006; MiDi, 2004; Pouliquen and Forterre, 2009; Pouliquen et al. 2006). Although sensitive parameters relative to the flow are experimentally not accessible making it difficult to predict the rheological behaviours without making reasonable assumptions, numerical methods give room for further studies and understanding of the granular matter.

1.4 Scope of Thesis

The MPS method is coupled with the $\mu(I)$ rheology to simulate the granular flow on steep inclined planes. These granular flows are either steady or unsteady depending on the flow regime. The coupled methods have been used for granular column collapse on horizontal planes. This study will extend the granular collapse using the coupled MPS and $\mu(I)$ rheology model to inclined planes. Since only few studies have focused on unsteady granular flow, a simple description of unsteady granular flow in two dimensional configurations to predict the flow characteristics using both physical and numerical methods by giving detailed microscopic and macroscopic observations will be provided. Detail observation includes the velocity (profiles, evolution, and

contours), surface profiles, wave front, shear, shear banding and distribution. The aim of this present study is to gain a greater understanding of the dynamics governing unsteady granular flows on inclined planes by incorporating the $\mu(I)$ inertial rheology model with the MPS particle method.

1.5 Thesis Structure

The thesis is organized according to the structure given below:

1. Chapter 2 gives a brief discussion on both experimental set up and results with the rheology model formulation for granular flows.
2. Chapter 3 describes the Moving particle semi-implicit method techniques and study of different available weighted function and trial on reviewing to state the most suitable in the MPS method for granular flows.
3. Chapter 4 presents both results and discussion for velocity profiles taking account of the horizontal and vertical components, velocity evolution, velocity contours, shear band, shear stress, and shear distribution, surface profile and wave fronts.
4. Chapter 5 gives a final remark on the conclusion of the work done.

CHAPTER 2: EXPERIMENTAL SET UP AND THEORETICAL FORMULATION

2.1 Experimental setup

Experiments of quasi-2D were performed in a narrow channel, where glass beads were confined between glasses sheets with a net width of 20 mm. Fig. 2.1 illustrates the experimental setup. The glass beads were initially held by a front gate, which is normal to the ground and top surface. The initial heap remains horizontal at the surface on 30.8° and 45.2° inclinations respectively. These angles are selected to achieve both stable and unstable flow while making it less difficult to extract physical measurements during experimental study. For all experiments, the total weight of the glass beads is maintained constant at 195 g, where the diameter of the used glass beads is 2 mm. For ease of PIV analysis, 20% of the glass beads are in black color while the remaining 80% are left plain. During the experimental runs, the front gate was lifted vertically, releasing the glass beads instantaneously (less than 0.04 s) to similarly represent natural behaviour down inclined plane. The whole process of the collapse was captured by a high-speed CMOS digital camera (IDT inc., X-Stream III) at a speed of 1000 frames per second (fps). Velocity fields in the collapsing masses are obtained by applying the particle image velocimetry (PIV) technique (e.g. [Lueptow et al. 2000](#)), where an open-source package, PIVlab (e.g. [Sarno et al. 2014](#); [Thielicke and Tamhuis, 2014](#)), is used.

The granular materials velocities are estimated using the particle image velocimetry (PIV) technique from snapshots taken by the CMOS digital camera. From these snapshots, surface profiles and wave front of the granular materials can also be determined.

Fig. 2.1: Experimental setup.

2.2 Evolution of the flow body

The deformed profile of glass bead granule from the experiment is shown in Figs. 2.2 and 2.3 above for both 45.2° and 30.8° slope. This snapshot has been rotated to the horizontal plane after the processing of the images captured by the high speed camera. The avalanche starts to flow from rest due to the low basal friction and flows infinitely as a result of the flow regime. Due to experimental limitations from the capturing of the rapid flow, only an approximate length is considered in this study. For simplicity, the two angles mentioned above would be referred to as 45° and 30° in other sections of this thesis.

Fig. 2.2: The series of snapshots for an experiment with glass bead granules for 45° slope.

(The single red dash-line indicates the initial flow front at 0s while others are selected positions for the velocity profile)

Fig. 2.3: The series of snapshots for an experiment with glass bead granules for 30° slope.

It is observed that the initial profile of the granular materials on the inclined plane differs but transits to a similar profile at a later time step. Although the rate of deformation is much slower for the 30° inclination, it follows similar flow dynamics with the 45°. At the early stages of the granular materials flow ($t = 0-0.10s$), the rate of deformation for both angles are observed to be nearly constant before the rate of deformation for the 45° increased at around 0.2s. The velocity profiles are averaged at transverse section to properly interpolate missing points for the accuracy of the measured properties. This is shown by the thick red lines while the centers are the measured profile used in this study.

2.3 Computational conditions

In this current study, MPS coupled with the local rheology model $\mu(I)$ is employed to validate experimental studies for unsteady granular materials flow. Inclinations greater than the maximum angle of stability were used as shown in Table 2.1 (Silbert et al. 2003) to simulate the deformation of the granular materials on inclined planes.

Table 2.1 Geometry and simulation parameters

Case study parameters					
Case	θ	La (cm)	Lb (cm)	Lc (cm)	No's of Particles
1	45.2°	11.30	11.28	16.09	1711
2	30.8°	14.44	08.51	16.90	1548
Simulation parameters					
θ_s	θ_2	ρ_s (kg/m ³)	ε	I_0	
19.6°	28.5°	2500	0.6	0.279	

The mass of the granular materials was held by a moving gate, and the simulation section considered was similar to the experimental with a span of 0.4m.

2.4 Theoretical formulations

The introduction of several rheology models have led to questions on determining a more efficient model in explaining the behaviour and characteristics of granular flow. Models have been based on theoretical formulations and generated empirical theories from experimental studies. [Lagrée et al. \(2011\)](#) conducted a review on different rheological formulations such as the Bagnold, Bingham, constant friction models. The results showed the effectiveness in the capturing of the dynamics of high transient granular flows using the $\mu(I)$ -rheology. In this thesis the $\mu(I)$ -rheology model is adopted to study unsteady granular flow on inclined plane.

2.5 The rheology $\mu(I)$

The dynamical behaviour of dense granular flow have been described using constitutive law based on the dimensional analysis relating the shear stress to the normal stress with coefficient of friction as the proportionality, a term controlled by the inertia number ([Da Cruz et al. 2005](#); [MiDi, 2004](#); [Jop et al. 2006](#)).

$$I = \frac{\gamma d_g}{\sqrt{\frac{P}{\rho_s}}} \quad \text{with} \quad \mu(I) = \frac{\tau_i}{P} \quad (2.1a)$$

where I , the inertia number is defined as the square root of the Salvage number or Coulomb number ([Ancey et al. 1999](#); [Forterre and Pouliquen, 2008](#); [Savage, 1984](#)),

$\mu(I)$ is the friction coefficient, P is the pressure, d_g is the grain size in diameter, ρ_s is the density of granular materials, τ_i is the local tangential shear stress, γ_{ij} is the strain rate tensor and its second invariant γ with $\partial_j u_i$ and $\partial_i u_j$ taken to be the partial derivatives of velocity components in the i and j directions as defined below:

$$\gamma_{ij} \equiv (\partial_j u_i + \partial_i u_j) \quad \gamma = \sqrt{0.5 \gamma_{ij} \gamma_{ij}} \quad (2.1b)$$

The inertia number is very small for slow flows still in the quasi-static phase and large for rapid flows where the confinement force becomes diluted due to the high shear rates. This dimensional analysis demonstrates the collisional and frictional flow regime. The frictional rheology expressed with the internal stress tensor and phenomenological friction coefficient as expressed by [Jop et al. \(2006\)](#) and [Schaeffer, \(1987\)](#) is given below:

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{\frac{I_0}{I} + 1} \quad \text{and} \quad \varepsilon = \varepsilon_{max} + (\varepsilon_{min} - \varepsilon_{max})I \quad (2.2)$$

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}, \quad \text{with} \quad \tau_{ij} = \frac{\mu(I)P}{\gamma} \gamma_{ij} \quad (2.3)$$

The granular materials only flow if the following condition is satisfied:

$$|\tau| > \mu_s P \quad \text{where} \quad |\tau| = \sqrt{0.5 \tau_{ij} \tau_{ij}} \quad (2.4)$$

where the static friction coefficient $\mu_s = \tan \theta_s$, and the dynamic friction coefficient $\mu_2 = \tan \theta_2$ are material dependent coefficients (note that θ_s and θ_2 are static and dynamic

frictional angles), I_0 is the empirical coefficient constant, τ_{ij} is the deviatoric stress, σ_{ij} is the Cauchy stress tensor, δ_{ij} a unit tensor, ε the volume of fraction. The variation in the volume of fraction of granular particles when shear is applied for large inertia number is also given above. Typical values used are $\varepsilon_{min} = 0.4$ and $\varepsilon_{max} = 0.6$ (Forterre and Pouliquen, 2008; MiDi, 2004) indicating a weak dependence on the inertia number.

2.6 Regularisation technique

To minimize the non differential errors from the dimensional analysis relationship, a simple regularisation technique η_R is used to smoothing the results. Depending on high or low shear approximations, an appropriate small value α_r is introduced to yield an order of convergence (Chauchat and Médale, 2014; Frigaard and Nouar, 2005). In this study α_r is taken to be 0.000001 as used by Xu and Jin (2016).

$$\eta_R = \left(\mu_s + \frac{(\mu_2 - \mu_s)|\dot{\gamma}|}{I_0\sqrt{\varepsilon P} + |\dot{\gamma}|} \right) \frac{P}{(|\dot{\gamma}|^2 + \alpha_r^2)^{\frac{1}{2}}} \quad (2.5)$$

The granular flow characteristic is relatively similar to the visco-plastic (Chauchat and Médale, 2014).

2.7 Governing equations

The governing equations used for the particle type method are the continuity equation and the Navier-Stokes momentum equation describing the incompressible fluid flow for mass and momentum conservation. This equation is mostly represented in Lagrangian form in the MPS method:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} = 0 \quad (2.6)$$

$$\frac{\rho D\mathbf{u}}{Dt} = -\nabla P + \nabla \cdot \boldsymbol{\tau} + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} \quad (2.7)$$

Where $\mathbf{u}$ is the particle velocity vector, t is time, g denotes the body force which is usually gravity, and ρ represents density for fluid which is converted to the granular materials density ρ_s by multiplying the fluid density with the volume fraction, P is pressure and $\boldsymbol{\tau}$ stress tensor.

CHAPTER 3: NUMERICAL METHOD

3.1 MPS method

Moving Particle Semi-implicit (MPS) method is a gridless approach used to describe the motion of fluid in continuum mechanics based on fully Lagrangian description, where fluid characterization occurs on discrete particle interaction without numerical diffusion. Since its inception in 1996 by Koshizuka and Oka, the method has been devoted to solve several simulation problems across several fields and popularly known for its advancements in computational fluid dynamics. The meshless method for solving partial differential equations involves discrete particle interaction that employs the use of kernels for approximation and interpolation numerically. Particles are scaled to appropriate mapping features with each carrying a physical property. As motion takes place interaction of particles occur and is smoothened with scaled kernel functions whilst updating the properties for surrounding particles. The particle equation is generated based on kernel functions estimated from derivatives of kernel density satisfying certain requirements which are mostly positive definiteness meaning symmetrically positive over the domain which generates metric distances, optimum searching radius, compaction over the domain and unified relative to zero interaction. They are based on summation of physical properties relative to the entire particles in a given domain. Several kernel functions have been used in several studies as shown in Table 3.1. The kernel function ω_{r_0} adopted in this study is stated below (Shakibaeinia and Jin 2010).

$$\omega(r) = \begin{cases} \left(1 - \frac{r}{r_e}\right)^3, & 0 \leq \frac{r}{r_e} < 1 \\ 0, & \frac{r}{r_e} \geq 1 \end{cases} \quad (3.1)$$

Table 3.1 List of other successful weighted functions

S/N	Functions	Conditions
W1	$\omega(r) = \begin{cases} \frac{r_e}{r} - 1 \\ 0 \end{cases}$	$0 \leq r < r_e$ $r_e \leq r$ Koshizuka et al. (1998)
W2	$\omega(r) = \begin{cases} \frac{40}{7\pi r_e^2} \left(1 - 6 \left(\frac{r}{r_e} \right)^2 + 6 \left(\frac{r}{r_e} \right)^3 \right) \\ \frac{10}{7\pi r_e^2} \left(2 - 2 \left(\frac{r}{r_e} \right) \right)^3 \\ 0 \end{cases}$	$0 \leq r < \frac{r_e}{2}$ $\frac{r_e}{2} < r < r_e$ $r > r_e$ Shao and Lo (2003)
W3	$\omega(r) = \begin{cases} \frac{480\sqrt{2} - 705}{512\sqrt{2} - 745} \frac{1}{r_e} + \frac{-960\sqrt{2} - 1515}{512\sqrt{2} - 745} \frac{1}{r_e} \left(\frac{r}{r_e} \right)^2 + \\ \frac{-210}{512\sqrt{2} - 745} \frac{1}{r_e} \left(\frac{r}{r_e} \right)^2 + \frac{480\sqrt{2} - 600}{512\sqrt{2} - 745} \frac{1}{r_e} \left(\frac{r}{r_e} \right)^4 \\ 0 \end{cases}$	$0 \leq r \leq r_e$ $r_e < r$ Sheu et al. 2011
W4	$\omega(r) = \begin{cases} \left(1 - \frac{r}{r_e} \right)^3 & \left(1 + \frac{r}{r_e} \right)^3 \\ 0 \end{cases}$	$0 \leq r < r_e$ $r_e < r$ Kyung et al. 2014
W5	$\omega(r) = \begin{cases} \frac{r_e}{0.85r + 0.15r_e} - 1 \\ 0 \end{cases}$	$0 \leq r < r_e$ $r_e \leq r$ Zhang and Wan 2012

3.1.2 Requirement of the Kernel Function

The two major requirements for smoothing functions used in the mps method are stated below based on literature reviews ([Tony et al. 2011](#) and [Souto-Iglesias et al. 2013](#)) and are used to determine the appropriate kernel function to be used in this present study.

Property 1

The smoothing function should satisfy the smoothed Dirac delta function also sometimes called nascent delta functions. This function can also be viewed as a limit Gaussian or a Lorentzian.

$$\lim_{r_e \rightarrow 0} \omega(r, r_e) dr = 1 \quad \dots \quad \int_{-\infty}^{\infty} \omega(r, r_e) dr = 1$$

Property 2

The smoothing function should satisfy the Mercer's condition stated as positive-definite symmetric.

The ratio between the relative distance between particles and the searching radius is to scale the physical parameters of the kernel functions respective to the particle interaction range. Fig. 3.1 below shows the relationship between the kernels and the scaled ratio of the kernels listed on Table 3.1.

In this current study the weighted function employed is a third-order polynomial spiky kernel function developed by [Shakibaeinia and Jin \(2010\)](#), which is defined as taken as ωr_0 from Equation 3.1. Although ωr_0 and W4 meet the stated conditions for the smoothing function, ωr_0 is said to prevent clustering of particles and avoids an infinity value when neighboring particles are relatively close to each other ([Shakibaeinia and Jin, 2010](#)).

Figure 3.1: Plotted kernel functions from table 3.1

where $\omega(r)$ the kernel function, r_e is the radius of the interaction circle around each targeted particle, taken as $4.0d_l$ (d_l is the particle size) and r is the distance between a target particle i and one of its neighboring particles j as shown below:

$$r = \sqrt{|x_j - x_i|^2 + |y_j - y_i|^2} \quad (3.1)$$

The particle number density at a position i is defined as follows

$$\langle n \rangle_i = \sum_{j \neq i} \omega(|r_j - r_i|, r_e) \quad (3.2)$$

Assuming a constant mass for all particles, the relationship between the particle number density and real fluid density is stated below:

$$\langle \rho \rangle_i = \frac{\sum_{j \neq i} m_i \omega(|r_j - r_i|, r_e)}{\int_v \omega(|r_j - r_i|, r_e) dv} \quad (3.3)$$

where m_i is the mass of the fluid particle, hence the continuity equation is satisfied when the particle number density is kept constant.

3.2 Discretization technique

The interaction of particles with the help of the kernel function is used to define the gradient and Laplacian operator with the use of arbitrary scalar term ϕ ([Koshizuka and Oka, 1996](#)) as expressed below.

$$\langle \nabla \phi \rangle_{ij} = \frac{d}{n^0} \sum_{j \neq i} \frac{(\phi_j - \phi_i)}{|\vec{r}_j - \vec{r}_i|^2} (\vec{r}_j - \vec{r}_i) \omega(\vec{r}_j - \vec{r}_i) \quad (3.4)$$

$$\langle \nabla^2 \phi \rangle_{ij} = \frac{2d}{\lambda n^0} \sum_{j \neq i} (\phi_j - \phi_i) \omega(\vec{r}_j - \vec{r}_i) \quad (3.5)$$

where n^0 can also be taken as n_i under the incompressible flow condition and λ is a coefficient used to scale the variance for consistency with the analytical given below (Koshizuka and Oka, 1996):

$$\lambda = \frac{\int_v \omega(|\vec{r}_j - \vec{r}_i|) |\vec{r}_j - \vec{r}_i|^2 dv}{\int_v \omega(|\vec{r}_j - \vec{r}_i|) dv} \cong \frac{\sum_{j \neq i} \omega(|r_j - r_i|) |\vec{r}_j - \vec{r}_i|^2}{\sum_{j \neq i} \omega(|r_j - r_i|)} \quad (3.6)$$

To calculate the pressure field in the MPS model, the equation of state which was first introduced by Monaghan (1994) to reduce the computing time from computing the pressure term by replacing the fluid density with particle number density as expressed below.

$$p_i = \frac{\rho c_0^2}{\zeta} \left(\left(\frac{\langle n \rangle_i^*}{n^0} \right)^\zeta - 1.0 \right) \quad (3.7)$$

where ζ is a coefficient taken as 7 (Monaghan 1994, Shakibaeinia and Jin 2010), c_0 is the artificial sound speed in the reference medium. The limitation of the compressibility of the granular materials to less than 1% is based on the artificial sound of speed at 10 times the maximum flow velocity.

3.3 Boundary conditions

At the free surface no particle exist outside the region, the particle number density decreases for particles on the free surface. To identify the free surface particles, the condition stated below must be satisfied.

$$\langle n \rangle_i^* < \beta n^0 \quad (3.8)$$

where β is the threshold coefficient, it value ranges between 0.8 and 0.99 (Shakibaeinia and Jin 2010). In this study β is taken as 0.99. A similar approach is

used for the solid boundary where several evenly distributed ghost particles are added beyond every solid particle to avoid density deficiency close to the boundary. The radius of interaction of particles with a minimum of four layers of ghost particles are used beyond all solid boundary sections in this study to enable more interaction between particles within the searching radius a study also conducted by (Ataie-Ashtiani, B., and Farhadi, 2006). Throughout the simulation, the fluid, solid and ghost particles are the three particle types used while maintaining a constant particle searching radius.

Also in calculating the motion of particles and velocity, a predictor and corrector approach is employed to solve the governing equation. A prediction is made regarding particle positions and velocity u^* and r^* resulting to n^* which is a particle number density using the new location from the temporal velocity and position of particles as shown below:

$$u^* = u^n + \Delta t(\nabla\tau + f) \quad (3.9)$$

$$r^* = r^n + u^* \Delta t \quad (3.10)$$

$$\langle n \rangle_i^* = \sum_{j \neq i} \omega(|r_j - r_i|^*, r_e) \quad (3.11)$$

The velocity and particle correction is then stated below for new particles

$$u^{n+1} = u^* - \frac{\Delta t}{\rho} \nabla p^{n+1} \quad (3.12)$$

$$r^{n+1} = r^n + u^{n+1} \Delta t. \quad (3.13)$$

where Δt is taken as the time step, depending on the numbers of particles the above is calculated for all particles at every time steps.

CHAPTER 4: RESULT AND DISCUSSION

4.1 Properties of the granular flow behaviour

Experimental studies from the instantaneous collapse of the glass beads on inclined planes were used to validate results from numerical model, while extensive microscopic observations beyond the scope of physical measurements were captured and further discussed using the numerical model.

4.2 Velocity profile

The rheology model is applied to simulate unsteady granular flows on inclined planes. Inclinations greater than the angle of repose θ_r are used to quantitatively predict the velocity profile by simulation and physical observations. Although both experimental setups and numerical configuration are made identical, the effect of the sliding gate leading to instantaneous effect of gravity differ as granular particles close to the gate deflects upward with larger velocities in the simulation compared to the experimental observation. This led to fluctuations at the wave front locations at time 0.05s at both inclination angles for the numerical results due to the treatment of the interface between the sliding gate and the granular material. Figs. 4.1 – 4.4 shows the horizontal and vertical velocity profile for 45° and 30° slope where the flow condition is $\theta_s < \theta$, $\theta > \theta_2$. The dense nature of the granular flow is maintained for a very short time before transiting into a dilute state with the front region becoming agitated. The dense to dilute state can be distinguished based on the vector interaction between neighbouring particles in the MPS method. Dimensionless parameters $\vec{u} = u/\sqrt{gd_g}$, $\vec{v} = v/\sqrt{gd_g}$ and $\vec{y} = y/d_g$ are used to denote both the horizontal and vertical components

of the velocity and the position of the granular materials during deformation, where g is acceleration due to gravity and d_g is the grain size. To describe the granular flow dynamics and profile, the positions where the depth of the granular flow (h) is observed to be at its peak are selected from the rear, front, and the intermediate section of the shearing granular materials. Granular particles located before 0.108m remained in a quasi-static state with little variation in the surface profile caused by the differentials in the rate of deformation of the granular materials at 0.05s as shown in Fig. 4.1. At 0.108m the increase in the plasticity effect leads to a convex shape profile similar to observations from (Ancy, 2001; Jop et al. 2006; MiDi, 2004; Schaefer, 1987; Schaefer et al. 2010). Also there exists a slight increase in the free surface depth at 0.05s at location 0.148m, this is predicted to have occurred from the different boundary conditions found in the experiment and the assumed for the simulation as also stated in (Pouliquen, 1999; Silbert et al. 2003). The velocity profile transition is observed from the quasi-static state into a developed granular flow with trends of a power law and gradually into a linear relationship between $\dot{u}$ and $\dot{y}$. The temporal and deformation of the vertical velocity profile at 45° inclination plane is illustrated in Fig. 4.2. At $t = 0.05$ s, a quasi static state can be observed in Fig. 4.2a, with gradual deviation of the granular materials from location $x = 0.108$ m approximately midway of the depth profile.

Fig. 4.1 Horizontal velocity profile for 45° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations

Fig. 4.2 Vertical velocity profile for 45° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations

Fig. 4.3 Horizontal velocity profile for 30° slope showing comparison between experimental and numerical results at a time $t(s)$ from specific locations

Fig. 4.4 Vertical velocity profile for 30° slope showing comparison between experimental and numerical results at a time $t(\text{s})$ from specific locations

The velocities of the physical study remained positive maintaining only the directional change close to the boundary with that of the MPS remaining in the negative path, meaning no change of direction besides the one which occurred close to the boundary. This change in direction is caused by the effect of gravitational force and the v_y being zero at the surface profile and at region where the granular materials is highly dilated. The granular flow is observed to be continuous with the minimum flow depth h_{min} after full transition to the maximum velocity region becomes uniform until the end of the channel is reached. Particles at location 0.048m are maintained in the quasi-static phase. At 0.2s at location 0.168m, the velocity profiles take on a concave shape close to the boundary and after the inflection point to a convex-like shape for the physical observation and a linear trend for the MPS. At location 0.218m, MPS and the physical observation both display a concave shape with varying velocity distribution. At 0.4s, the velocity distribution reduces uniformly close to zero before other neighbouring materials collision effects replaces the granular particles at the frontal region of the granular flow. Fig. 4.3 shows that after the opening of the sliding gate at 0.05s particles located after 0.095m started accelerating slowly while the one before it remained in quasi-static state for 30 ° slope. Granular particles located above the boundary region close to the surface tend to attain more magnitude throughout the granular flow. At location 0.15m the MPS direction changed around a depth close to the surface profile, while the physical observation maintained the same direction from the boundary with little variation in the uniform velocity distribution. The depth of the granular materials for both physical and MPS at 0.05s are well predicted unlike that of 0.2s when the flow is developed with the magnitude of the velocity v_x increases along the channel at different fixed positions. Decrease in the inclination angle results into reduction in

the shear rate of the granular materials as observed by comparing 30° to 45° . This decrease results into longer time for the quasi-static region to transit to the kinetic phase. At 0.05s, only the wave front region shows visible deformation while other region before 0.115m remained in a static. After 0.2s for positions greater than 0.279m the MPS transits to an approximate uniform velocity distribution with a linear relationship while the physical study maintained a power law trend resulting from discrepancies in the velocity distribution close to the boundary. After the flow is developed at 0.2s, the physical observation at the rear of the flow have a convex shape and after the inflection point shows a linear relationship while the numerical show a convex shape at locations 0.179m and 0.229m before showing a linear trend at the wave front region at location 0.279m. At time 0.4s, the velocity distribution behaves linearly and also tends to a power law. The physical study shows a concave shape after the inflection point close to the surface profile. This is predicted to have occurred from the instabilities found in the flow. Fig. 4.4 shows good prediction during quasi-static state aside at the peak region of the flow at 0.05s from 0.15m where the v_y component for MPS changed direction towards the boundary and finally returned positive at the surface profile resulting to a concave shape from a depth of 0.025m during inflection with a linear trend from the boundary. The depth of the granular materials at this state is approximately equal until time 0.2s when the collision interaction between particles differs in resultant direction resulting from the difference in the densely packed neighbouring particles interaction. At 4.0s, the v_y for both cases shows good prediction with similar projection. The 45° and 30° slopes are observed to share similar properties with very little variations giving good prediction with physical observation. In general it can be observed that the horizontal velocity gave better prediction when compared to the vertical velocity. This may result from

the influence of gravity from the vertical direction and the assumption of the z component of the coordinates to be zero.

4.3 Velocity evolution

The study of the flow phenomena at transverse fixed position along the channel as the velocity distribution changes with time is illustrated in Figs. 4.5 and 4.6 for the experimental and numerical studies. The selected locations are fixed positions along the channel at the rear, front and intermediate regions of the flowing granular materials. The time selected shows the flow phenomena during the time of the initial development and transition to a fully developed flow phase with limited instabilities. Fig. 4.5a shows that at $t = 0.05\text{s}$ - 0.15s , the velocity increased linearly from the boundary to about 70% reduction in the flow depth. At $t = 0.15\text{s}$, after the flow is initiated and where slippage of the granular materials is partially developed, the velocity magnitude increases at every corresponding locations. Increase in the velocity magnitude at the peak depth of the granular materials flow is predicted to continue until the flow depth or thickness reduces to minimum h_{min} at fixed location. At $x = 0.148\text{m}$ close to the flow front of the flowing granular materials, the magnitude of the velocity reduced indicating that the driving force of the flow is minimal at the location close to the end of the quasi-static phase. For the two locations ($x = 0.048\text{m}$, $x = 0.148\text{m}$) from $t = 0.05\text{s}$ to $t = 0.15\text{s}$ result shows that the magnitude of the velocity increases at the rear at about $\frac{1}{3}b$ (b is the bottom span of the flowing granular materials) and reduces gradually to the flow front area as shown in Figs. 4.5 (b, c). The velocity profiles show a linear trend with little variations in magnitude at $x = 0.148\text{m}$. Fig. 4.6 illustrates the velocity evolution at 30° inclination angle. The results indicate a similar trend as in the 45° inclination slope to half of the

flow depth with a sudden drop and later a linear increase to the free surface at $x = 0.15\text{m}$. The velocity profiles may differ close to the free surface due to particles interaction in the dilute phase region. Both inclinations show similar phenomena for both physical and numerical studies at fixed position for different time showing the extent at which the flow is time dependent.

Fig. 4.5 Velocity evolution during acceleration down a slope of 45° during initialisation when the granular materials start to deform at fixed sections.

Fig. 4.6 Velocity evolution during acceleration down a slope of 30° during initialisation when the granular materials start to deform at fixed sections.

4.4 Velocity contour

The velocity contour shows the size and magnitude of the velocity distribution along the sheared granular materials from time $t = 0.05\text{s}$ to $t = 0.4\text{s}$. Fig. 4.7 illustrates the simulated and experimental velocity contour profile at 45° inclination plane. The gradual decline in the size of the static region occurs until it attains minimal velocity at $t = 0.4\text{s}$. The maximum velocity is found at the frontal region with the least thickness while the minimum velocity is found at the rear portion close to the boundary. The rate of transformation from the distribution of velocity occurs very fast for steep inclinations. Although the granular materials continue to flow if not interrupted, the velocity distribution in the frontal top region is maximum while reducing gradually to the rear throughout the deformation of the granular materials. Fig. 4.8 shows that with an inclination angle of 30° which has a larger static region as compared to 45° but with the same similar velocity distribution phenomena. Although the thickness of the quasi-static phase reduces gradually during deformation, the bottom particles at this state maintain a minimal depth before dissipation due to increase in its velocity. The velocity distribution range is found to increase with time until its minimal depth is attained forming a uniform strip of velocity distribution. The numerical simulation showed good agreement to the experimental findings. Slight discrepancy can be observed at $t = 0.10\text{s}$ and $t = 0.2\text{s}$ where the particles dissipate from the static region in the simulated results. This observation is caused by the boundary and interface control between the sliding gate and the deforming granular material in the numerical studies. These deflected particles can be seen at the top free surface of the flowing granular materials which would later disappear at a later time step in the numerical study. These deflected particles were not used when estimating the results for the flow properties.

Fig. 4.7 Velocity contour plot profile for 45° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s (e) 0.4s

Fig. 4.8 Velocity contour plot profile for 30° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s (e) 0.4s

4.5 Shear banding in unsteady states

Granular deformation results from the localisation of strain formed during imbalance of forces of particles from initialisation. The frictional resistance from the interaction between particles leads to vortex formation instantaneously from the effect of gravitational force which exists from the quasi-static phase as shown in Figs. 4.9 and 4.10. The shear band is estimated by using the Froude number ($u/\sqrt{gd_g}$). The connection between vortices and shear band was stated to be from the linear combination of pure shear and particle rotation around the core regions (Crosta et al. 2015). The characterisation of these vortex cores determines the failure phenomena of the granular materials. The areas of failure form isolines at all substrate layers at an approximate angle relative to the channel boundary. These isolines correspond to layers with different Froude distribution influenced by gravitational force giving anisotropic behaviour, which results into attractive interaction as suggested by Charles Campbell (1989). The thickness of the granular materials between the isolines decreases with time and in height developing into multiple shears banding as the material deform. This nature of localisation is dependent on the transformation from the quasi-static state to the kinetic state. The behaviour of deformation of several materials differ, the rheology model produces the characteristics of the macroscopic constitutive description of the granular materials but failed to show instability phenomena during deformation. While studying granular flows on erodible and non-erodible inclines, Crosta et al. (2015) results showed a similar phenomenon on a 22° slope. Before the arrest of the flow a stepped profile arises from localized shear bands which increase in number over time. The decrease in the static region transits into multiple shear bands during the collapse of the granular materials down the slope.

4.6 Shear

The rheology from microscopic observation shows the shear stress, shear rate and coefficient of friction in the granular flow during slippage as shown in Figs. 4.11 (a-d) for 45° slope. It should be noted that shear is not measured but calculated based on the velocity gradient from the model results. The evolution of the shear stress shows a rapid increase from initialisation to a yielding point when the stress began to decrease over time after attaining its peak stress Fig. 4.11(a). The shear stress in the material had a sudden increase from initialisation point and subsequently reduced as shear rate increases. The increase in the shear rate as time increases led to reduction in the height of the flowing granular materials which was approximately steady at height 0.018m. At time 0.1s the shear stress in the material gradually reduces as time increases giving the granular materials more resistivity to limit further segregation, hence the maintenance of the uniform minimum height of flow as the material propagates infinitely. The relative displacement of these particles at respective shear zone increases and causes dilation in the granular flow. This dilatancy leads to dispersion and transfer of shear zones as a result of resistance from neighbouring particles during collision. Several fluctuations as a result of dynamic equilibrium between the local contraction of particles and dilation during granular flow show negligible shear stress at the surface profiles. The properties of the granular materials exhibit frictional angles ranging from θ_s to θ_2 which is relatively uniform as time and shear rate increases corresponding to a very loose material shown in Fig. 4.11(b). A solid to liquid behaviour is finally observed after 0.3s with an approximate uniform friction coefficient, a process perfectly reversible as the rate of shear in the material is increased as shown in Fig. 4.11(c). The shear stress increased linearly at initialisation as the shear rate increases and later decreases while remaining constant as shown in

Fig. 4.11(d). In general the granular flow exhibits a thixotropic behaviour observed from Figs. 4.11(a- d).

Fig. 4.9 Multi-Shear band layers for 45° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.20s (e) 0.30s

Fig. 4.10 Multi-Shear band layers for 30° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.20s (e) 0.30s

Fig. 4.11 Rheological behaviour of the granular materials where (a) and (b) shows shear stress and shear rate as a function of time and (c) to (d) shows the friction coefficient and shear stress as a function of shear rate

4.7 Shear distribution

The distribution of shear stress shows the yielding stress behaviour of the granular materials. During the quasi-static regime, the granular materials deforms slowly with the high magnitude of the shear stress found at half left and the least at the free surface of the granular materials as shown in Fig. 4.12(a). This shows that the particles close to the free surface are in the kinetic mode where individual particles

possess the momentum of their motion while in the frictional and collisional mode, momentum is transferred from particle to particle interactions [Campbell, 1989](#). This region is mostly the dense region of the bulk granular materials. The shear stress in the granular materials at the front wave region is minimal with slippage from the front bottom region having maximum shear rate as shown in Fig. [4.12\(b\)](#). As the effect of the shear stress reduces, the shear rate increases at the boundary of the granular materials.

4.8 Surface profile

The free surface dynamics is dependent on the rate of flow and the thickness $h(t)$ of the profile related to the time evolution of internal stresses of the granular materials. Having found good agreement between physical and the numerical results, the increase in the inclination from 30° to 45° led to higher acceleration but yet the result showed that from 0s to 0.10s, the rate of deformation of the material with respect to the material thickness $h(t)$ for 45° was slower compared to the 30° as shown in Figs. [4.13](#) and [4.14](#). It was observed that the rate of reduction at the peak of the granular materials increases after the deformed material approaches the angle of repose on the inclined plane for 30° as opposed to the expectation of assuming that granular materials on the 45° would flow faster from initialisation. This might be due to the differences in the setup and initial granular material profile when at rest.

The rear free surface of the flowing mass at 0.5s to 0.2s maintained acceptable positions besides the top of the free surface and the frontal region of the flowing mass as shown in Fig. [4.13](#). The depth of the flow for the simulation in 45° is slightly underestimated unlike that of the 30° as shown in Fig. [4.14](#).

Fig. 4.12 Shear stress and shear rate for 45° slope.

Fig. 4.13: Surface profile for 45° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s

Fig. 4.14 Surface profile for 30° slope showing comparison between experimental and numerical results for time (a) 0.05s (b) 0.10s (c) 0.15s (d) 0.2s

At 0.2s a sudden increase in the rate of shear at the wave front was observed for the physical study. The frontal positions of granular particles increased with an average of 3% from 0s to 0.2s in both studies.

4.9 Wave front

The wave front from the propagation of the material occurs at the frontal region of the material when the depth of the granular materials is approximately zero ($h \approx 0$). Particle dispersion are forced towards areas of least shear which also is a limitation in the use of the rheology model, especially having that the flow front is unsteady with segregation of particles with higher velocities (Xu and Jin 2016). Particles at the wave front have maximum velocities. The length of the inclined channel limits the experimental results to four time steps with 0.05 increments (0.05s-0.20s) while the numerical shows from (0.05s to 0.4s) as shown in Fig. 4.15. The results are plotted and compared for both 45° and 30° giving acceptable prediction. Unlike arresting granular flows on slopes where existing mathematical formulations can be used to predict the flow front, run-out and arrest time, continuous flowing granular materials on inclined plane have no existing formulations to predict the flow front as the run-out and arrest time does not exist for such flows. It can be observed from Fig. 4.15 that the difference in slope between 45° and 30° show similar flow front at the early stages of the granular flow similar to observations observed by Lee and Yang 2016 for arresting flows at the earliest stages for different slopes.

Fig. 4.15 Wave front time evolution for 45° and 30° slope showing comparison between experimental and numerical results

4.10 General Discussion

The approach used in this study is able to show acceptable results with physical observations for unsteady granular flow for both microscopic and macroscopic properties. Although the model did not capture the granular behaviour at the frictional phase, measured properties were similar to observations from the physical study. This limitation from the frictional rheological model may be from the friction and dilatancy law where the sheared granular flow is controlled by the inertia number which is relatively small when the flow is initiated.

CHAPTER 5: CONCLUSION

5.1 Summary

In this thesis, we have presented experimental and numerical studies on the rheological properties of granular collapse on inclined plane. These rheological properties deal with complexities in the granular fluids, constituting the need to properly understand granular matter. The introduction of the empirical $\mu(I)$ rheology model into MPS as a coupled set is used to describe the properties of the granular behavior for steady and unsteady flows. This granular behavior is governed by the contact dynamics between particles at the dense state where momentum transport is developed from inter-particle collision and the effect of gravity. A major part of this thesis is concerned with the velocity distribution and effect of gravity considered in its interrelation with the formation of shear bands. The two inclination angles discussed in this thesis show no major differences in the deformation of the granular flow properties. Although the initial geometry of the static granular material slightly differs, it converges after a period of time to a similar flow dynamics for different inclination angles. This indicates that granular collapses under the same configuration are governed by similar constitutive law.

The rate of deformation and transition of the dense granular materials affects the velocity distribution at the inflection points between the quasi-static and kinetic phase resulting into instabilities. Despite these instabilities a linear relationship exists for the velocity at several transverse sections. At fixed transverse locations, the velocity profile is observed to maintain a linear relationship between dimensionless velocity $\tilde{u} = u/\sqrt{gd_g}$ and position $\tilde{y} = y/d_g$ at different time. The selections of different fixed positions limit the effect of instability in the granular flow properties when

considering the changes in time during the deformation of the granular materials. Constant changes in the depth of unsteady granular flows over time makes it difficult to give similar flow pattern observed in steady flow, where there exist relationship between the height and velocity of the granular flow.

The functional rheological parameters used in the simulation of the granular collapse captured some of the complex behavior at larger values of the inertia number I , after the shear stress yields as the shear rate increases at the early stages of the granular collapse. Although the high value of I contributes to particle dispersion at the flow front, the effect when compared to the experimental results is observed to be negligible. The deformation of the granular materials was observed to develop multiple shear bands linked by isolines at substrate layers giving anisotropic behaviour. The instantaneous effect of gravity initiates the frictional resistance from the cohesive interaction between particles resulting into the shear band formation. The localisation of strain is subject to the lack of force balance in the granular materials during deformation. Although there seems to be a relationship between gravity and viscous stresses, it is uncertain if this behavior is associated with the development of shear bands. The thickness of the shear bands increases and later diminishes as the depth of the granular materials tends to its minimum. The evolution of the shear stress increases linearly and reduces after attaining its peak stress. The distribution of the shear stress is found at the dense state of the granular materials. The free surface dynamics depends on the thickness of the granular profile and inclination angle related to the time evolution of internal stresses in the granular materials. Early stages of the granular flow are found to exhibit similar flow front for different slopes which might slightly differ at a later state.

The coupled model used in this study is capable of capturing some complex behaviour and flow dynamics of the granular materials, suggesting that further empirical formulations can be used to enhance the friction laws of other configurations. Although the unsteady state of the granular flow still lacks theoretical description, the results presented in this thesis would aid in its formulation. The results from the coupled model show good agreement with experimental investigations recommending the method as a powerful simulation technique for granular avalanche.

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