THE ERDŐS-KO-RADO THEOREM FOR
INTERSECTING FAMILIES OF PERMUTATIONS

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Alison May Purdy

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SUPERVISORY AND EXAMINING COMMITTEE

Alison May Purdy, candidate for the degree of Master of Science in Mathematics, has presented a thesis titled, *The Erdős-Ko-Rado Theorem for Intersecting Families of Permutations*, in an oral examination held on July 22, 2010. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

External Examiner: Dr. Sandra Zilles, Department of Computer Science

Supervisor: Dr. Karen Meagher, Department of Mathematics and Statistics

Committee Member: Dr. Shaun Fallat, Department of Mathematics and Statistics

Committee Member: Dr. J. Chris Fisher, Adjunct Professor, Department of Mathematics and Statistics

Committee Member: Dr. Robert F. Bailey, Professional Associate, Department of Mathematics and Statistics

Chair of Defense: Dr. Laurie Sykes Tottenham, Department of Psychology
Abstract

The Erdős-Ko-Rado Theorem is a fundamental result in extremal set theory. It describes the size and structure of the largest collection of subsets of size $k$ from a set of size $n$ having the property that any two subsets have at least $t$ elements in common. Following the publication of the original theorem in 1961, many different proofs and extensions have appeared, culminating in the publication of the Complete Erdős-Ko-Rado Theorem by Ahlswede and Khachatrian in 1997. A number of similar results for families of permutations have appeared. These include proofs of the size and structure of the largest family of permutations having the property that any two permutations in the family agree on at least one element of the underlying set. In this thesis we apply techniques used in the proof of the Complete Erdős-Ko-Rado Theorem for set systems to prove a result for certain families of $t$-intersecting permutations. Specifically, we give the size and structure of a fixed $t$-intersecting family of permutations provided that $n \geq 2t + 1$ and show that this lower bound on $n$ is optimal.
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Dedication

This thesis is dedicated to the memory of my parents,

Herb and Eira Purdy,

whose love and encouragement supported me through many chapters of my life,

and to my husband, Steven Greve, for sharing this one with me.
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Chapter 1

Preliminaries

The Erdős-Ko-Rado Theorem [4] has been the focus of a great deal of interest since it was first published in 1961. This theorem describes the size and structure of the largest collection of subsets of size $k$ (or $k$-subsets) from a set of size $n$ having the property that any two subsets have at least $t$ elements in common. Such a collection is called $t$-intersecting. One statement of the theorem is as follows:

**Theorem 1.0.1.** Let $t \leq k \leq n$ be positive integers. Let $\mathcal{F}$ be a family of pairwise $t$-intersecting $k$-subsets of $\{1, \ldots, n\}$. There exists a function $n_0(k, t)$ such that for $n \geq n_0(k, t)$,

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Moreover, for $n > n_0(k, t)$, $\mathcal{F}$ meets this bound if and only if $\mathcal{F}$ is the collection of all $k$-subsets that contain a fixed $t$-subset.
In their original paper [4], Erdős, Ko and Rado showed that $n_0(k, 1) = 2k$ and gave $t + (k - t)\binom{k}{t}^3$ as a lower bound on $n$ for values of $t$ other than 1. In 1976, Frankl [5] proved that $n_0(k, t) = (t + 1)(k - t + 1)$ for $t \geq 15$ and in 1984, Wilson [18], showed that this bound was applicable for all values of $t$. In 1997, Ahlswede and Khachatrian published what is known as the Complete Erdős-Ko-Rado Theorem [1] which gives the maximum size and structure of $t$-intersecting set systems for all values of $n, k$ and $t$.

Another area of research generated by the Erdős-Ko-Rado Theorem concerns the extension of the theorem to mathematical objects other than sets. One such extension is to permutations. Two permutations from $S_n$, the symmetric group of permutations on a base set containing $n$ points, are intersecting if there is at least one element from the base set that has the same image under both permutations. Two permutations are $t$-intersecting if there are at least $t$ such elements. A family of permutations is intersecting if every pair of permutations is intersecting and the family is $t$-intersecting if every pair of permutations is $t$-intersecting. In 1977, Frankl and Deza [7] proved the following result:

Theorem 1.0.2. Let $\mathcal{A}$ be an intersecting family of permutations from $S_n$. Then $|\mathcal{A}| \leq (n - 1)!$.

An intersecting family that attains this upper bound can be constructed by including all the permutations that fix one common point.
Frankl and Deza conjectured that, for $n$ sufficiently large, the maximum possible size of a $t$-intersecting family of permutations is $(n-t)!$. Recently, Ellis, Friedgut and Pilpel [3] proved the following result:

**Theorem 1.0.3.** For any given $t \in \mathbb{N}$ and $n$ sufficiently large, if $\mathcal{A}$ is a $t$-intersecting family of permutations from $S_n$, then $|\mathcal{A}| \leq (n-t)!$, with equality if and only if there exist $t$ distinct elements of $\{1, \ldots, n\}$, $a_1, \ldots, a_t$ and $t$ distinct elements of $\{1, \ldots, n\}$, $b_1, \ldots, b_t$ such that $\sigma(a_i) = b_i$ for all $\sigma \in \mathcal{A}$ and $i \in \{1, \ldots, t\}$.

Their paper does not give any specific lower bound on $n$.

In this thesis we use the method of Ahlswede and Khachatrian [1] to prove this result and to provide an optimal lower bound on $n$ for certain families of permutations.

### 1.1 Outline of thesis

In the next chapter, we discuss several approaches that have been used to prove the Erdős-Ko-Rado Theorem for set systems and introduce the shifting technique used by Erdős, Ko and Rado and by Ahlswede and Khachatrian.

In Chapter 3 we present some known results concerning $t$-intersecting families of permutations.

In Chapter 4, two fixing operations on families of permutations are defined. These operations are designed to increase the number of points fixed by a given permutation.
We then attempt to modify one of these operations with the goal of developing an operation that can be used to transform any family of permutations into a family with the necessary properties for the proof of our main result.

In Chapter 5 we define a new operation on permutations that has the effect of left-shifting the set of fixed elements of the permutation and prove a number of results concerning this operation.

In Chapter 6, we introduce a number of concepts used by Ahlswede and Khachatrian in [1] and define analogous concepts for permutations. After establishing some properties of these concepts, we present the proof of our main result – an optimal lower bound for $n$ for $t$-intersecting families of permutations that are closed under the fixing operation defined in Chapter 4.

In Chapter 7, we will discuss the significance of our result and possible directions for future work.

1.2 Notation

The following definitions and conventions will be used throughout the thesis:

- The set of all positive integers from 1 to $n$ will be denoted by $[n]$. Similarly, the set of all positive integers from $x \neq 1$ to $y$ will be denoted by $[x,y]$. For example, the set $\{2, \ldots, n-1\}$ will be denoted by $[2,n-1]$. 
• The collection of all subsets of \([n]\) will be denoted by \(2^{[n]}\).

• The collection of all \(k\)-subsets of \([n]\) will be denoted by \(\binom{[n]}{k}\).

• A \(k\)-set system is a collection of \(k\)-subsets of \([n]\).

• A \(t\)-intersecting family of \(k\)-sets, \(\mathcal{F} \subseteq \binom{[n]}{k}\), is maximal if for any \(A \in \binom{[n]}{k} \setminus \mathcal{F}\), the \(k\)-set system \(\mathcal{F} \cup \{A\}\) is not \(t\)-intersecting. Similarly, a \(t\)-intersecting family of permutations, \(\mathcal{A} \subseteq S_n\), is maximal if for any \(\sigma \in S_n \setminus \mathcal{A}\), the family of permutations \(\mathcal{A} \cup \{\sigma\}\) is not \(t\)-intersecting.

• Two \(k\)-set systems are equivalent if one set system can be obtained from the other by reordering the underlying \(n\)-set. Similarly, two families of permutations are equivalent if one family can be obtained from the other by independent reordering of the domain and range.

• Permutations will generally be denoted as \(\sigma = \langle a_1 a_2 \ldots a_n \rangle\) where \(\sigma(i) = a_i\) for \(1 \leq i \leq n\). This is an abbreviation of the standard two-line notation. However, cycle notation will be used occasionally. Composition of permutations will be from right to left. For example,

\[
(12) \circ \langle 2 \ 3 \ 4 \ 1 \rangle = \langle 1 \ 3 \ 4 \ 2 \rangle.
\]

• Definitions of the graph terminology used in Sections 2.3 and 3.2 can be found in graph theory textbooks such as [10].
Chapter 2

Set Systems

In this chapter we give a brief overview of the results for intersecting set systems which began with the work of Erdős, Ko and Rado [4] and culminated in the publication of what is known as the Complete Erdős-Ko-Rado Theorem by Ahlswede and Khachatrian in 1997 [1]. We introduce a shifting technique for set systems and present a proof of the maximum size of an intersecting set system using this technique. We then define the Kneser graph and show how it can be used to prove the same result. Finally, we discuss improvements made to the lower bound on $n$ given in [4] and state the Complete Erdős-Ko-Rado Theorem.

A collection of sets is intersecting if every pair of sets in the collection is intersecting. It is $t$-intersecting if every pair of sets has at least $t$ points in common. We will call a $t$-intersecting $k$-set system trivially $t$-intersecting if it consists of all of the
\(k\)-subsets of \([n]\) that contain a given \(t\)-set. Such a collection has size

\[
\binom{n - t}{k - t}.
\]

When \(t = 1\), we will refer to such a set system as trivially intersecting.

### 2.1 The Erdős-Ko-Rado Theorem

The Erdős-Ko-Rado Theorem is a fundamental result in extremal set theory. Although it is usually expressed in terms of intersecting collections of \(k\)-subsets of \([n]\), the original paper is written in terms of intersecting collections of subsets with the conditions that the cardinality of the subsets is at most \(k\) and no subset in the collection is a subset of another subset in the collection. The paper contains two main theorems and a number of conjectures. The first theorem states that the maximum size of such a collection of subsets is \(\binom{n - 1}{k - 1}\), provided that \(n \geq 2k\), and this limit is only reached when all subsets are of size \(k\). The bound on \(n\) is necessary since if \(n < 2k\), all \(k\)-sets will be intersecting. The second theorem is equivalent to Theorem 1.0.1. As mentioned in Chapter 1, dramatic improvements have been made to the bound on \(n\) given in the original paper.

Many different proofs of the theorem have been published. For a review of some of these proofs, see [8].
2.2 The shifting operation for set systems

The initial proof of the Erdős-Ko-Rado theorem [4] and Ahlswede and Khachatrian’s proof of the complete theorem [1] make use of a technique called *shifting*. When applied to a $t$-intersecting collection of sets, this operation preserves the size of the sets, the size of the collection and the $t$-intersection (Frankl [6]).

For a collection $\mathcal{F}$ of subsets of $[n]$ and $i, j \in [n]$, the $(i, j)$-shift, $S_{i,j}$, is defined by

$$S_{i,j}(\mathcal{F}) = \{S_{i,j}(F) : F \in \mathcal{F}\},$$

where

$$S_{i,j}(F) = \begin{cases} (F\{j\}) \cup \{i\} & \text{if } i \notin F, j \in F, \ (F\{j\}) \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise}. \end{cases}$$

For example, if $\mathcal{F}$ consists of the sets

$$\{1,2,4\} \quad \{1,2,5\} \quad \{1,2,6\} \quad \{1,3,5\} \quad \{1,4,5\} \quad \{1,4,6\},$$

then $S_{3,4}(\mathcal{F})$ will consist of

$$\{1,2,3\} \quad \{1,2,5\} \quad \{1,2,6\} \quad \{1,3,5\} \quad \{1,4,5\} \quad \{1,3,6\}.$$

For any set $F \in \mathcal{F}$, if $i < j$ then either $S_{i,j}(F) = F$ or $S_{i,j}(F) \prec F$ where $\prec$ denotes the lexicographic order.

If a set system, $\mathcal{F}$, has the property that $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $i < j \leq n$, then $\mathcal{F}$ is called a *left-shifted*, *left compressed* or *stable* set system. It has been shown that
any set system can be transformed into a left-shifted set system by applying at most \( (\frac{n}{2}) \) successive shifting operations (Frankl [6]).

The following proof by Frankl and Graham [8] is based on the original proof by Erdős, Ko and Rado and illustrates the usefulness of the shifting operation.

**Theorem 2.2.1.** Let \( \mathcal{F} \subseteq \binom{[n]}{k} \) be an intersecting \( k \)-set system with \( n \geq 2k \). Then

\[
|\mathcal{F}| \leq \binom{n-1}{k-1}.
\]

**Proof.** We consider \( n = 2k \) and \( n > 2k \) as separate cases.

1. Case 1: \( n = 2k \).

Let \( A \) be any \( k \)-set in \( \mathcal{F} \). Its complement, \( \bar{A} \), will also be a \( k \)-set and since \( A \cap \bar{A} = \emptyset \), it follows that \( \bar{A} \notin \mathcal{F} \). Thus

\[
|\mathcal{F}| \leq \frac{1}{2} \binom{n}{k} = \binom{2k-1}{k-1}.
\]

2. Case 2: \( n > 2k \).

Since any intersecting set system can be transformed to a left-shifted intersecting set system of the same size, we assume that \( \mathcal{F} \) is left-shifted and then proceed by induction on \( n \).

Since \( n > 2k \), it follows that \( n \geq 3 \). If \( n = 3 \), then \( k = 1 \) and the theorem holds. Specifically,

\[
|\mathcal{F}| = 1 = \binom{n-1}{k-1}.
\]
Now assume the theorem holds for all \( n \in [3, m-1] \).

For \( n = m \), let \( \mathcal{G} = \{ F \in \mathcal{F} : m \notin F \} \) and let \( \mathcal{H} = \{ F \setminus \{m\} : m \in F \in \mathcal{F} \} \). Then \( |\mathcal{F}| = |\mathcal{G}| + |\mathcal{H}| \). Since \( \mathcal{G} \subseteq \mathcal{F} \) is an intersecting family of \( k \)-sets from the set \([m-1]\), by the induction hypothesis

\[
|\mathcal{G}| \leq \binom{m-2}{k-1}.
\]

We now use the fact that \( \mathcal{F} \) is left-shifted to prove that \( \mathcal{H} \) is an intersecting set system.

For any \( H_1, H_2 \in \mathcal{H} \),

\[
|H_1 \cup H_2| \leq 2k - 2 < m - 2.
\]

Hence, there is some \( x \in [m-1] \) such that \( x \notin H_1 \cup H_2 \). Clearly,

\[
x \notin (H_1 \cup \{m\}) \in \mathcal{F},
\]

and, since \( x < m \) and \( \mathcal{F} \) is left-shifted, it follows that \( (H_1 \cup \{x\}) \in \mathcal{G} \subseteq \mathcal{F} \). Also, \( H_2 \cup \{m\} \in \mathcal{F} \), so \((H_1 \cup \{x\}) \cap (H_2 \cup \{m\}) \neq \emptyset \). It then follows that \( H_1 \cap H_2 \neq \emptyset \) for all \( H_1, H_2 \in \mathcal{H} \).

Thus \( \mathcal{H} \) is an intersecting family of \((k-1)\)-sets from \([m-1]\) and applying the induction hypothesis to \( \mathcal{H} \) gives

\[
|\mathcal{H}| \leq \binom{m-2}{k-2}.
\]
Hence
\[ |\mathcal{F}| \leq \binom{m-2}{k-1} + \binom{m-2}{k-2} = \binom{m-1}{k-1} \]
as required. \qed

Another proof of this bound that uses the shifting technique can be found in [6].

2.3 The Kneser graph

For \( n \geq 2k \), the Kneser graph, \( K_{n,k} \), is the graph whose vertex set is the set of all \( k \)-subsets of \([n]\). Two vertices are adjacent if and only if the corresponding \( k \)-sets are disjoint. The Kneser graph is a vertex-transitive graph and an independent set of vertices forms an intersecting set system.

A proof of Theorem 2.2.1 using the Kneser graph appears in [13]. There Lovász shows that the eigenvalues of the adjacency matrix of the Kneser graph are given by
\[ \lambda_i = (-1)^i \binom{n-k-i}{k-i} \text{ for } i \in [0,k]. \]
The largest eigenvalue, which is equal to the degree of \( K_{n,k} \), is
\[ d = \binom{n-k}{k} \]
and the least eigenvalue is
\[ \tau = -\binom{n-k-1}{k-1}. \]
He also proves the following result which is often called Hoffman’s Ratio Bound.
**Theorem 2.3.1.** Let \( \alpha(G) \) be the maximum size of an independent set in a vertex-transitive graph \( G \) and let \( |V(G)| \) be the number of vertices in \( G \). Let \( d \) be the degree of \( G \) and let \( \tau \) be the least eigenvalue of the adjacency matrix of \( G \). Then

\[
\alpha(G) \leq \frac{|V(G)|}{1 - \frac{d}{\tau}}.
\]

Therefore, the maximum size of an independent set in the Kneser graph is

\[
\alpha(K_{n,k}) \leq \frac{\binom{n}{k}}{1 - \frac{\binom{n-k}{k-1}}{\binom{n}{k-1}}} = \binom{n-1}{k-1}.
\]

For a proof that uses the Kneser graph to show that only trivially intersecting \( k \)-set systems meet this bound, see Newman [14].

### 2.4 The lower bound on \( n \)

In their original paper, Erdős, Ko and Rado gave \( t + (k-t)\binom{k}{t}^3 \) as a lower bound for \( n \) for values of \( t > 1 \). They gave the following example to show that a lower bound is required.

Let \( n = 8 \), \( k = 4 \) and \( t = 2 \). Then

\[
\begin{align*}
\{1,2,3,5\} & \quad \{1,2,3,6\} & \quad \{1,2,3,7\} & \quad \{1,2,3,8\} & \quad \{1,2,4,5\} & \quad \{1,2,4,6\} \\
\{1,2,4,7\} & \quad \{1,2,4,8\} & \quad \{1,3,4,5\} & \quad \{1,3,4,6\} & \quad \{1,3,4,7\} \\
\{1,3,4,8\} & \quad \{2,3,4,5\} & \quad \{2,3,4,6\} & \quad \{2,3,4,7\} & \quad \{2,3,4,8\}
\end{align*}
\]
is a 2-intersecting 4-set system that is larger than the trivially 2-intersecting system which contains 15 sets. Each set in this set system contains exactly three elements from the set \{1, 2, 3, 4\}. (In fact, this set system is not maximal since \{1, 2, 3\} will 2-intersect with all of the sets in the system.)

In 1976, Frankl [5] proved that \(n_0(k, t) = (t + 1)(k - t + 1)\) for \(t \geq 15\) and in 1984, Wilson [18] showed that this bound was applicable for all values of \(t\) using linear algebraic methods.

The bound of Frankl and Wilson is a substantial improvement over that of Erdős, Ko and Rado. For example, if \(k = 4\) and \(t = 2\), then \(t + (k - t)\binom{k}{t}^3 = 434\) while \((t + 1)(k - t + 1) = 9\). In fact, \(n_0(k, t) = (t + 1)(k - t + 1)\) is the best possible lower bound for \(n\). This can be seen by comparing the size of a trivially \(t\)-intersecting family of \(k\)-sets, which we will call \(\mathcal{F}_0\), to the size of a family consisting of all \(k\)-sets which contain at least \(t + 1\) elements of a fixed subset of size \(t + 2\), which we will denote by \(\mathcal{F}_1\). A precise definition of \(\mathcal{F}_i\) will follow in Section 2.5.

The size of \(\mathcal{F}_1\) is given by

\[
|\mathcal{F}_1| = \binom{t + 2}{t + 1} \binom{n - (t + 2)}{k - (t + 1)} + \binom{n - (t + 2)}{k - (t + 2)}.
\]

When \(n = (t + 1)(k - t + 1)\),

\[
|\mathcal{F}_1| = \binom{n - t}{k - t} = |\mathcal{F}_0|,
\]
and when \( n < (t + 1)(k - t + 1) \),
\[
|\mathcal{F}_1| > |\mathcal{F}_0|.
\]

## 2.5 The Complete Erdős-Ko-Rado Theorem

The question of the maximum size and structure of a \( t \)-intersecting set system when \( n < (t + 1)(k - t + 1) \) was finally answered by Ahlswede and Khachatrian in 1997 [1]. Before stating the theorem, some further notation is needed.

For \( t, k, n, i \in \mathbb{N} \) and \( i \leq k - t/2 \), let
\[
\mathcal{F}_i = \left\{ A \in \binom{[n]}{k} : |A \cap [t + 2i]| \geq t + i \right\}.
\]

Note that \( \mathcal{F}_0 \) is a trivially \( t \)-intersecting set system. If a set system, \( \mathcal{F} \), can be obtained from \( \mathcal{F}_i \) by permutation of the base set, we say that \( \mathcal{F} \) is equivalent to \( \mathcal{F}_i \).

**Theorem 2.5.1.** Let \( t, k, n \) be positive integers with \( t \leq k \leq n \) and let \( r \) be a non-negative integer such that \( r \leq k - t \). If
\[
(k - t + 1) \left( 2 + \frac{t - 1}{r + 1} \right) < n < (k - t + 1) \left( 2 + \frac{t - 1}{r} \right),
\]
then \( \mathcal{F}_r \) is the unique (up to equivalence) \( t \)-intersecting \( k \)-set system with maximum size. (By convention, \( \frac{t - 1}{r} = \infty \) for \( r = 0 \).)

If \( n = (k - t + 1) \left( 2 + \frac{t - 1}{r+1} \right) \), then \( |\mathcal{F}_r| = |\mathcal{F}_{r+1}| \) and a system of maximum size will be equivalent to \( \mathcal{F}_r \) or \( \mathcal{F}_{r+1} \). \( \Box \)
Substituting $r = 0$ into the inequality, we see that the trivially $t$-intersecting set system, $\mathcal{F}_0$, is the unique largest system (up to equivalence) if $(k-t+1)(t+1) < n < \infty$. (All other trivially $t$-intersecting set systems can be obtained from $\mathcal{F}_0$ by reordering the underlying set $[n]$ and have the same cardinality as $\mathcal{F}_0$.) Thus the Complete Erdős-Ko-Rado Theorem proves that $n_0(k, t) = (t + 1)(k-t + 1)$ in addition to identifying the $t$-intersecting set systems of maximum size when $n < (t + 1)(k-t + 1)$.
Chapter 3

Permutations

In this chapter we consider intersecting permutations and present a number of different approaches for proving Erdős-Ko-Rado type results for intersecting families of permutations.

Two permutations, $\sigma, \pi \in S_n$, are intersecting if $\sigma(x) = \pi(x)$ for some $x \in [n]$. When $\sigma(x) = \pi(x)$ for some $x \in [n]$, we will say that $\sigma$ and $\pi$ agree on $x$. Let $t$ be a positive integer. Two permutations are $t$-intersecting if they agree on at least $t$ elements of $[n]$. For two permutations, $\sigma$ and $\pi$, we will use $|\sigma \cap \pi|$ to represent the number of elements on which they agree.

A family of permutations, $\mathcal{A} \subseteq S_n$, is intersecting if every pair of permutations in $\mathcal{A}$ is intersecting and it is $t$-intersecting if every pair of permutations in $\mathcal{A}$ is $t$-intersecting.
A family of permutations, $\mathcal{A} \subseteq S_n$, is *trivially intersecting* if it consists of all the permutations that agree on $x$ for some $x \in [n]$. This is equivalent to $\mathcal{A}$ being a coset of the stabilizer of $x$ in $S_n$. Such a family has size $(n - 1)!$. Similarly, a family of permutations, $\mathcal{A}$, is *trivially t-intersecting* if there exists a set $I \subseteq [n]$ with cardinality $t$ such that all permutations that agree on all elements of $I$ are in $\mathcal{A}$. A trivially $t$-intersecting family has size $(n - t)!$.

A point $x \in [n]$ is said to be *fixed* by $\sigma \in S_n$ if $\sigma(x) = x$. For any $\sigma \in S_n$, the *fixed point set* of $\sigma$ is defined as:

$$\text{fix}(\sigma) = \{x \in [n] : \sigma(x) = x\}.$$  

Note that if $n - 1$ of the $n$ elements are fixed, then the $n^{th}$ element is forced to be fixed. Therefore it is not possible to have $|\text{fix}(\sigma)| = n - 1$. If $S$ is a collection of permutations from $S_n$, then $\text{fix}(S) = \{\text{fix}(\sigma) : \sigma \in S\}$ is a set of subsets of $[n]$.

In 1977, Deza and Frankl [7] proved that the size of the largest possible intersecting family of permutations from $S_n$ is $(n - 1)!$. The fact that only trivially intersecting systems meet this bound was not proved until much later when Cameron and Ku [2] and Larose and Malvenuto [12] published independent proofs of this result. Since then, additional proofs by Godsil and Meagher [9] and Wang and Zhang [17] have been published. However, fewer significant results have appeared for $t$-intersecting families of permutations when $t \geq 2$. Deza and Frankl conjectured that for $n$ sufficiently large, the maximum size of a $t$-intersecting family of permutations is $(n - t)!$ and this
conjecture has recently been proved by Ellis, Friedgut and Pilpel [3].

3.1 Proofs of the Erdős-Ko-Rado Theorem for permutations

In this section, we present a proof of the bound on the size of an intersecting family of permutations due to Ellis, Friedgut and Pilpel [3] based on the original proof by Deza and Frankl [7] and a proof of the uniqueness of the sets that meet this bound due to Wang and Zhang [17]. Both proofs will use the following lemma. Cycle notation will be used for permutations throughout this section and \([i + j]_n\) will be used to represent \(a \in [n]\) such that \(a \equiv i + j \pmod{n}\).

Lemma 3.1.1. Let \(\rho \in S_n\) be an \(n\)-cycle and let \(H\) be the cyclic group of order \(n\) generated by \(\rho\). Any two distinct permutations in a left coset \(\sigma H\) of \(H\) will not be intersecting.

Proof. Let \(\rho = (x_1 x_2 \ldots x_n)\). Then for any \(i, j \in [n]\),

\[
\rho^i(x_j) = x_{[j+i]_n}.
\]

Suppose that there are two permutations \(\pi\) and \(\tau\) in \(\sigma H\) such that \(\pi(x_j) = \tau(x_j)\) for some \(j \in [n]\). Since \(\pi = \sigma \circ \rho^k\) and \(\tau = \sigma \circ \rho^\ell\) for some \(k, \ell \in [n]\), it follows that \(\rho^k(x_j) = \rho^\ell(x_j)\). This means that

\[
x_{[j+k]_n} = x_{[j+\ell]_n}
\]

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and that $k = \ell$. Therefore, if $\pi(x_j) = \tau(x_j)$ for some $j \in [n]$, then $\pi = \tau$. □

**Theorem 3.1.2.** Let $\mathcal{A}$ be an intersecting family of permutations from $S_n$. Then $|\mathcal{A}| \leq (n-1)!$.

*Proof.* Let $\mathcal{A}$ be an intersecting family of permutations from $S_n$. Let $\rho \in S_n$ be an $n$-cycle and let $H$ be the cyclic group of order $n$ generated by $\rho$. For any left coset $\sigma H$ of $H$, it follows from Lemma 3.1.1 that $\sigma H$ contains at most one permutation from $\mathcal{A}$. Since the left cosets of a subgroup partition the group and $|S_n| = n!$, it follows that $|\mathcal{A}| \leq (n-1)!$. □

**Theorem 3.1.3.** Let $\mathcal{A}$ be an intersecting family of permutations from $S_n$. If $|\mathcal{A}| = (n-1)!$, then $\mathcal{A}$ is a coset of a stabilizer of one point.

*Proof.* In this proof, we assume that $n \geq 6$. For $n \leq 5$, the theorem can be verified by computer.

Let $\mathcal{A}$ be an intersecting family of permutations from $S_n$ of size $(n-1)!$. We may assume that the identity permutation, $Id$, is in $\mathcal{A}$ since if it is not, taking a permutation $\pi \in \mathcal{A}$ and setting

$$\mathcal{A}' = \pi^{-1}\mathcal{A} = \{\pi^{-1}\sigma : \sigma \in \mathcal{A}\}$$

results in an intersecting set of permutations of the same size that contains the identity. Hence, assuming $Id \in \mathcal{A}$ and showing that $\mathcal{A}$ is the stabilizer of one point is sufficient to prove the theorem.

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Let $\rho = (x_1 x_2 \ldots x_n) \in S_n$ and let $H$ be the cyclic group of order $n$ generated by $\rho$. Then for any $i, j \in [n],$

$$\rho^i(x_j) = x_{[j+i]_n}.$$  

For a given $k \in [2, n - 1]$, let $\sigma_k = (x_1 x_2 \ldots x_k)$. We first show that exactly one of $\sigma_k$, $\sigma_k \circ \rho^{k-1}$ or $\sigma_k \circ \rho^{n-1}$ is in $A$.

Since $|A| = (n - 1)!$, it follows from Lemma 3.1.1 that each left coset of $H$ must contain exactly one permutation from $A$. Hence, for any given value of $k$, there is exactly one $i \in [n]$ such that $\sigma_k \circ \rho^i \in A$. Since $Id \in A$, it follows that $\sigma_k \circ \rho^i$ must have at least one fixed point.

Assume $\sigma_k \circ \rho^i(x_j) = x_j$ and let $\ell = [j + i]_n$. There are three cases to consider:

1. Case 1: If $\ell > k$, then $\sigma_k \circ \rho^i(x_j) = \sigma_k(x_\ell) = x_\ell$. Therefore $x_j = x_\ell$ and $i \equiv 0$ (mod $n$). It then follows that $\rho^i = Id$ and $\sigma_k \circ \rho^i = \sigma_k$.

2. Case 2: If $\ell = k$, then $\sigma_k \circ \rho^i(x_j) = \sigma_k(x_k) = x_1$. Thus $j = 1$ and $i = k - 1$.

3. Case 3: If $\ell < k$, then $\sigma_k \circ \rho^i(x_j) = x_{\ell+1}$. It then follows that $i \equiv -1$ (mod $n$).

Hence, for any given value of $k \in [2, n - 1]$, the permutation $\sigma_k \circ \rho^i$ will have fixed points only if $i \in \{k - 1, n - 1, n\}$. Since there is exactly one permutation from $\sigma_k H$ in $A$ and this permutation must intersect the identity permutation, it follows that exactly one of $\sigma_k$, $\sigma_k \circ \rho^{k-1} = (x_2 x_3 \ldots x_n)^{k-1}$ or $\sigma_k \circ \rho^{n-1} = (x_1 x_n x_{n-1} \ldots x_{k+1})$ will be in $A$. 

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Next, we prove by contradiction that not all 2-cycles are in $A$ and use this to show that there is at least one $(n-1)$-cycle in $A$.

Suppose that $A$ contains all 2-cycles in $S_n$. Any permutation that fixes only one or two points will not intersect with some 2-cycle therefore such a permutation will not be in $A$.

For $k = n-2$, exactly one of $(x_1 x_2 \ldots x_{n-2})$, $(x_2 x_3 \ldots x_n)^{n-3}$ or $(x_1 x_n x_{n-1})$ will be in $A$. Since $(x_1 x_2 \ldots x_{n-2})$ fixes only $x_{n-1}$ and $x_n$, and $(x_2 x_3 \ldots x_n)^{n-3}$ fixes only $x_1$, neither of these permutations will be in $A$. Therefore $(x_1 x_n x_{n-1})$ must be in $A$. Since $\rho$ is an arbitrary $n$-cycle, this implies that all 3-cycles will be in $A$.

If all 3-cycles are in $A$, then no permutation that fixes three or fewer points will be in $A$. Repeating the above argument for $k = n-3$ shows that all 4-cycles will be in $A$. Repeating this for successively lower values of $k$ shows that $A$ will contain all 5-cycles, all 6-cycles and so on. This process eventually leads to a contradiction.

If $n$ is even and $A$ contains all $\frac{n}{2}$-cycles, then $A$ will not contain any permutations that fix $\frac{n}{2}$ or fewer points which means that no $\frac{n}{2}$-cycles can be in $A$. If $n$ is odd and $A$ contains all $(\lfloor \frac{n}{2} \rfloor + 1)$-cycles, then $A$ will not contain any permutations that fix $(\lfloor \frac{n}{2} \rfloor + 1)$ or fewer points and so no $(\lfloor \frac{n}{2} \rfloor + 1)$-cycle will be in $A$. Therefore, we conclude that there is some 2-cycle in $S_n$ that is not in $A$.

Assume without loss of generality that the 2-cycle $(x_1 x_n) \notin A$. Since $\sigma_k \circ \rho^{n-1} = (x_1 x_n)$ for $k = n-1$, either $\sigma_{n-1} = (x_1 x_2 \ldots x_{n-1})$ or $\sigma_{n-1} \circ \rho^{n-2} = (x_n x_{n-1} \ldots x_2)$.
must be in \( A \). Both of these are \((n-1)\)-cycles so \( A \) will contain an \((n-1)\)-cycle.

Finally, we prove that \( A \) is the stabilizer of one point in \( S_n \).

Let \( \pi = (y_1 y_2 \ldots y_{n-1}) \) be an \((n-1)\)-cycle in \( A \). If \( n \) is even, set

\[
\rho = (y_n y_2 y_4 \ldots y_{n-2} y_1 y_3 y_5 \ldots y_{n-1})
\]

and if \( n \) is odd, set

\[
\rho = (y_n y_2 y_4 \ldots y_{n-1} y_3 y_1 y_5 y_7 \ldots y_{n-2}).
\]

Recall that \( \sigma_k \) is the \( k \)-cycle consisting of the first \( k \) elements of \( \rho \). In both cases, provided that \( n \geq 6 \), neither \( \sigma_k \) nor \( \sigma_k \rho^{n-1} \) will intersect with \( \pi \) for any \( k \in [2, n-1] \), so \( \sigma_k \rho^{k-1} \) must be in \( A \). Specifically, for \( n \) even,

\[
\sigma_k \rho^{k-1} = (y_2 y_4 \ldots y_{n-2} y_1 y_3 y_5 \ldots y_{n-1})^{k-1} \in A
\]

and for \( n \) odd,

\[
\sigma_k \rho^{k-1} = (y_2 y_4 \ldots y_{n-1} y_3 y_1 y_5 y_7 \ldots y_{n-2})^{k-1} \in A
\]

for all \( k \in [2, n-1] \). A permutation in \( S_n \) will intersect with the identity permutation and with \( \sigma_k \rho^{k-1} \) for all \( k \in [2, n-1] \) if and only if it fixes \( y_n \). Hence, all permutations in \( A \) must fix \( y_n \). Therefore, if \( A \) is an intersecting subset of maximum size containing the identity permutation, it is the stabilizer of one point. \( \square \)
3.2 The derangement graph

A derangement is a permutation that has no fixed points. That is, \( \sigma \in S_n \) is a derangement if \( \sigma(x) \neq x \) for all \( x \in [n] \).

For \( n \in \mathbb{N} \), the derangement graph, \( \Gamma_n \), is the graph whose vertex set is the set of all permutations in \( S_n \) and where two vertices are adjacent if and only if they are not intersecting. Thus, an independent set is an intersecting family of permutations. The derangement graph can also be defined as the normal Cayley graph whose vertices are the elements of \( S_n \) and whose connection set is the set of all derangements in \( S_n \).

(In a Cayley graph, the vertices are elements of a group and two vertices \( \sigma \) and \( \pi \) are adjacent if and only if \( \sigma^{-1} \circ \pi \) is in the connection set. A normal Cayley graph is a Cayley graph where the connection set is closed under conjugation.) The derangement graph was used by Godsil and Meagher [9] in their proof of Theorem 3.1.3 and its complement appears in the paper by Cameron and Ku [2]. In 2007, Renteln [16] proved that the least eigenvalue, \( \tau \), of the adjacency matrix of the derangement graph is given by

\[
\tau = \frac{-D_n}{n-1}
\]

where \( D_n \) is the number of derangements in \( S_n \). With this result, the ratio bound introduced in Chapter 2 can be used to calculate the size of the largest independent set in the graph and thus prove Theorem 3.1.2.
A graph is Hamilton-connected if every pair of distinct vertices is joined by a path that meets every vertex. We will use the following result from [15] in Chapter 5.

**Theorem 3.2.1.** The derangement graph is Hamilton-connected for $n \geq 4$. 

We now define a generalization of this graph in which an independent set is a $t$-intersecting family of permutations and show that, in some specific cases, this graph can be used to prove that the maximum size of a $t$-intersecting family is $(n - t)!$.

For $n, t \in \mathbb{N}$ with $t \leq n$, the generalized derangement graph, $\Gamma_{n,t}$, is the graph whose vertex set is the set of all permutations on $[n]$. Two vertices $\sigma$ and $\pi$ are adjacent if and only if $\sigma$ and $\pi$ agree on fewer than $t$ elements of $[n]$. Note that $\Gamma_{n,1} = \Gamma_n$.

**Proposition 3.2.2.** For $n, t \in \mathbb{N}$ with $t \leq n$, the generalized derangement graph, $\Gamma_{n,t}$, is a normal Cayley graph.

**Proof.** Let $X(S_n, C)$ be the Cayley graph whose vertices are the elements of $S_n$ with connection set given by

$$C = \{ \sigma \in S_n : |\text{fix}(\sigma)| < t \}.$$

Since $C$ is a union of conjugacy classes, it is closed under conjugation. Therefore, $X(S_n, C)$ is a normal Cayley graph.

By definition, $X(S_n, C)$ and $\Gamma_{n,t}$ have the same vertex set.
Suppose \( \sigma \) and \( \pi \) are adjacent vertices in \( \Gamma_{n,t} \). Then \( \sigma \) and \( \pi \) agree on fewer than \( t \) elements of \( [n] \). Since \( \sigma^{-1} \circ \pi(i) = i \) if and only if \( \sigma(i) = \pi(i) \), it follows that \( \sigma^{-1} \circ \pi(i) = i \) for fewer than \( t \) elements. Hence \( \sigma^{-1} \circ \pi \in C \) and so \( \sigma \) is adjacent to \( \pi \) in \( X(S_n, C) \).

Now suppose \( \sigma \) and \( \pi \) are adjacent vertices in \( X(S_n, C) \). Then \( \sigma^{-1} \pi \in C \) and thus \( |\text{fix}(\sigma^{-1} \pi)| < t \). Since \( \sigma^{-1} \circ \pi(i) = i \) whenever \( \sigma(i) = \pi(i) \),

\[
|\sigma \cap \pi| < t.
\]

Hence \( \sigma \) and \( \pi \) are adjacent in \( \Gamma_{n,t} \).

Therefore, two vertices \( \sigma \) and \( \pi \) are adjacent in \( X(S_n, C) \) if and only if they are adjacent in \( \Gamma_{n,t} \) and hence \( X(S_n, C) = \Gamma_{n,t} \). Since \( X(S_n, C) \) is a normal Cayley graph it follows that \( \Gamma_{n,t} \) is a normal Cayley graph. \( \square \)

**Corollary 3.2.3.** The derangement graph, \( \Gamma_n \), is a normal Cayley graph.

**Proof.** Corollary 3.2.3 follows immediately from Proposition 3.2.2 since \( \Gamma_n = \Gamma_{n,1} \) for all \( n \in \mathbb{N} \). \( \square \)

A subset \( H \subseteq S_n \) is transitive if for any \( x, y \in [n] \) there exists some \( \sigma \in H \) such that \( \sigma(x) = y \). The set \( H \) is called sharply transitive if there is only one such permutation in \( H \). The cyclic subgroup generated by an \( n \)-cycle that was introduced in Lemma 3.1.1 is an example of a sharply transitive subset.
Proposition 3.2.4. Let $H$ be a sharply transitive subset of $S_n$. The elements of $H$ form a clique of size $n$ in the derangement graph, $\Gamma_n$.

Proof. Let $\sigma$ and $\pi$ be distinct elements of $H$. Since $H$ is sharply transitive, $\sigma(i) \neq \pi(i)$ for all $i \in [n]$. Therefore $\sigma$ is adjacent to $\pi$ in $\Gamma_n$. This holds for any pair of elements of $H$, therefore the elements of $H$ form a clique in $\Gamma_n$. Since $H$ is sharply transitive, it must contain exactly $n$ permutations in order for an element of $[n]$ to be mapped exactly once to every element of $[n]$. \qed

The following result is proved for vertex-transitive graphs in [2].

Theorem 3.2.5. Let $C$ be a clique and $A$ an independent set in a vertex-transitive graph on $v$ vertices. Then $|C| \cdot |A| \leq v$. \qed

Since $S_n$ contains a sharply transitive subset for any value of $n$, the derangement graph has a clique of size $n$. Using Theorem 3.2.5, we obtain the following result for the size of an independent set, $A$,

$$|A| \leq (n - 1)!.$$ 

A subset $H \subseteq S_n$ is $t$-transitive if for any distinct $x_1, \ldots, x_t \in [n]$ and any distinct $y_1, \ldots, y_t \in [n]$ there exists some $\sigma \in S_n$ such that $\sigma(x_i) = y_i$ for all $i \in \{1, \ldots, t\}$. It is called sharply $t$-transitive if there is only one such element of $H$. 

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Proposition 3.2.6. Let $H$ be a sharply $t$-transitive subset of $S_n$. The elements of $H$ form a clique of size $n(n-1) \cdots (n-t+1)$ in the generalized derangement graph, $\Gamma_{n,t}$.

Proof. Let $H$ be a sharply $t$-transitive subset of $S_n$. Suppose $H$ is not a clique in $\Gamma_{n,t}$. Then there exist $\sigma, \pi \in H$, $\sigma \neq \pi$, such that $\sigma$ is not adjacent to $\pi$. Hence,

$$|\sigma \cap \pi| \geq t.$$ 

Choose $t$ distinct elements, $x_1, \ldots, x_t$, from $\{x \in [n] : \sigma(x) = \pi(x)\}$. For all $i \in \{1, \ldots, t\}$, we have that $\sigma(x_i) = \pi(x_i)$. Since $\sigma \neq \pi$, this means that $H$ is not sharply $t$-transitive. This is a contradiction, therefore $H$ must be a clique in $\Gamma_{n,t}$.

There are $n(n-1) \cdots (n-t+1)$ different sequences of elements of $[n]$ of length $t$. If $H$ is $t$-transitive, it must contain at least this number of permutations in order for $H$ to map a given sequence, $x_1, \ldots, x_t$, to every sequence of length $t$. If $|H| > n(n-1) \cdots (n-t+1)$, then (by the pigeonhole principle) there must be some $\sigma, \pi \in H$ such that $\sigma \neq \pi$ and $\sigma(x_i) = \pi(x_i)$ for all $i \in \{1, \ldots, t\}$. Hence, if $H$ is sharply $t$-transitive there can be no more than $n(n-1) \cdots (n-t+1)$ permutations in $H$.

Therefore, if $H$ is sharply $t$-transitive, the elements of $H$ form a clique of size $n(n-1) \cdots (n-t+1)$ in $\Gamma_{n,t}$.

If, for given values of $n$ and $t$, a sharply $t$-transitive subgroup exists, then Theorem 3.2.5 combined with Proposition 3.2.6 gives the result that the maximum size
of a $t$-intersecting family of permutations is $(n - t)!$. However, sharply $t$-transitive subgroups do not exist for all values of $n$ and $t$. Moreover, even when a sharply $t$-transitive subgroup of $S_n$ exists, there is no obvious way to apply the method of Wang and Zhang to show that only trivially $t$-intersecting families meet this bound. Specifically, the subgroup is unlikely to contain an element of order $n(n - 1) \cdots (n - t + 1)$ as is required for the proof.
Chapter 4

The Fixing Operation

In this chapter we introduce two different operations that increase the number of elements fixed by a given permutation. We discuss their properties with the goal of identifying an operation that, when applied to a family of permutations, will preserve the size and intersection properties of the family. We then further examine the operation used by Cameron and Ku [2], the fixing of the point \( x \) via \( \sigma \), and present several new results concerning this operation.

4.1 Fixing operations for permutations

For a point \( x \in [n] \) and a permutation \( \sigma \in S_n \), Cameron and Ku [2] define a permutation, \( \sigma_x \), called the fixing of the point \( x \) via \( \sigma \), as follows:

1. if \( \sigma(x) = x \), then \( \sigma_x = \sigma \);
2. if \( \sigma(x) \neq x \), then \( \sigma_x(y) = \begin{cases} x & \text{if } y = x, \\ \sigma(x) & \text{if } y = \sigma^{-1}(x), \\ \sigma(y) & \text{otherwise}. \end{cases} \)

Further, the permutation \( \sigma_{x_1,x_2,...,x_r} \) is defined recursively as the fixing of the point \( x_r \) via \( \sigma_{x_1,x_2,...,x_{r-1}} \). For example, if \( \sigma = \langle 45321 \rangle \), then \( \sigma_1 = \langle 15324 \rangle \) and \( \sigma_{1,4} = \langle 15342 \rangle \).

A set of permutations, \( S \), is said to be **closed under the fixing operation** if for each \( x \in [n] \) and each \( \sigma \in S \), it holds that \( \sigma_x \in S \). We will refer to such a family as a **fixed** family. It follows from this definition that every fixed family will contain the identity permutation. The next lemma extends a result of Cameron and Ku [2] from intersecting to \( t \)-intersecting families of permutations.

**Lemma 4.1.1.** Let \( \mathcal{A} \) be a fixed \( t \)-intersecting family of permutations. Then \( \text{fix}(\mathcal{A}) \) is a \( t \)-intersecting set system.

**Proof.** Assume that \( \text{fix}(\mathcal{A}) \) is not \( t \)-intersecting. Then there must be at least one pair of permutations, \( \sigma, \pi \in \mathcal{A} \) with \( \sigma \neq \pi \), such that

\[
| \text{fix}(\sigma) \cap \text{fix}(\pi) | < t.
\]

Choose such a pair so that the size of fix(\( \sigma \)) is as large as possible.

Since \( \sigma, \pi \in \mathcal{A} \), they must \( t \)-intersect. Therefore, there is at least one \( x \in [n] \) such that \( \sigma(x) = \pi(x) \neq x \). Since \( \mathcal{A} \) is closed under the fixing operation, \( \sigma_x \in \mathcal{A} \) and it...
follows that $\sigma_x$ will $t$-intersect with $\pi$. We now show that $|\text{fix}(\sigma_x)| > |\text{fix}(\sigma)|$ and that $\sigma_x$ intersects with $\pi$ at fewer than $t$ fixed points contradicting the maximality of $\text{fix}(\sigma)$.

From the definition of the fixing operation, it is clear that $\sigma_x$ will have more fixed points than $\sigma$. In fact, if $\sigma(x) = \sigma^{-1}(x)$, there will be two additional fixed points, $x$ and $\sigma^{-1}(x)$. If $\sigma(x) \neq \sigma^{-1}(x)$, the only additional fixed point will be $x$. But $\sigma_x(x) = x \neq \pi(x)$ and $\sigma_x(\sigma^{-1}(x)) = \sigma(x) = \pi(x) \neq \pi(\sigma^{-1}(x))$, so the number of fixed points at which $\sigma_x$ and $\pi$ intersect will not increase. Therefore,

$$|\text{fix}(\sigma_x) \cap \text{fix}(\pi)| < t \quad \text{and} \quad |\text{fix}(\sigma_x)| > |\text{fix}(\sigma)|$$

giving us the required contradiction. □

Cameron and Ku [2] showed that any intersecting family of permutations of size $(n-1)!$ is closed under the fixing operation. Thus they were able to use the properties of a fixed family without defining the operation for families of permutations.

For an arbitrary collection of permutations, $S$, we define the $x$-fixing, $F_x$, as follows:

$$F_x(S) = \{F_x(\sigma) : \sigma \in S\},$$

where

$$F_x(\sigma) = \begin{cases} 
\sigma_x & \text{if } \sigma_x \notin S, \\
\sigma & \text{otherwise}.
\end{cases}$$
Theorem 4.1.2. Let $\mathcal{A}$ be a $t$-intersecting family of permutations from $S_n$. Then $F_x(\mathcal{A})$ is a $t$-intersecting family of permutations for any $x \in [n]$.

Proof. Let $\sigma, \pi$ be any two distinct permutations from $\mathcal{A}$. For any given $x \in [n]$, let

$$X = [n] \setminus \{x, \sigma^{-1}(x), \pi^{-1}(x)\}.$$ 

Since $\sigma(y) = \sigma_x(y)$ for all $y \in [n] \setminus \{x, \sigma^{-1}(x)\}$ and $\pi(y) = \pi_x(y)$ for all $y \in \{x, \pi^{-1}(x)\}$, the number of elements from the set $X$ on which $\sigma$ and $\pi$ agree remains the same after the fixing operation. Application of the fixing operation to $\mathcal{A}$ can have three possible outcomes in terms of $\sigma$ and $\pi$: both $\sigma$ and $\pi$ are not changed by the fixing operation, both $\sigma$ and $\pi$ are changed by the fixing operation, or exactly one of these two permutations is changed. We will show that in all three cases $F_x(\sigma)$ and $F_x(\pi)$ will be $t$-intersecting.

1. Case 1: $F_x(\sigma) = \sigma$ and $F_x(\pi) = \pi$.

Since neither permutation is changed by the fixing operation, they will be $t$-intersecting after the fixing operation is applied to $\mathcal{A}$.

2. Case 2: $F_x(\sigma) \neq \sigma$ and $F_x(\pi) \neq \pi$.

This implies that $\sigma(x) \neq x$ and $\pi(x) \neq x$. If $\sigma$ and $\pi$ agree on $t$ elements from the set $X$, then $\sigma_x$ and $\pi_x$ will agree on the same $t$ elements and hence be $t$-intersecting, so assume that $\sigma$ and $\pi$ agree on fewer than $t$ elements of $X$.  

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Then they must agree on at least one of $x$, $\sigma^{-1}(x)$ or $\pi^{-1}(x)$. If they agree on either $\sigma^{-1}(x)$ or $\pi^{-1}(x)$, it follows that $\sigma^{-1}(x) = \pi^{-1}(x)$. Specifically, if $\sigma(\sigma^{-1}(x)) = \pi(\sigma^{-1}(x))$, then $x = \pi(\sigma^{-1}(x))$ and therefore $\pi^{-1}(x) = \sigma^{-1}(x)$.

Again there are three possibilities to consider:

(a) $\sigma(x) = \pi(x)$ and $\sigma^{-1}(x) \neq \pi^{-1}(x)$.

It follows from the definition of the fixing operation that $\sigma_x$ and $\pi_x$ agree on $x$. It also follows that $\sigma_x$ and $\pi_x$ do not agree on $\sigma^{-1}(x)$ or $\pi^{-1}(x)$ since

$$\sigma_x(\sigma^{-1}(x)) = \sigma(x) = \pi(x) = \pi_x(\pi^{-1}(x)) \neq \pi_x(\sigma^{-1}(x)).$$

Hence the size of the intersection between the two permutations will remain the same after the $x$-fixing of $\mathcal{A}$.

(b) $\sigma(x) \neq \pi(x)$ and $\sigma^{-1}(x) = \pi^{-1}(x)$.

In this case, the two permutations will agree on $x$ after the fixing operation and will not agree on $\sigma^{-1}(x) = \pi^{-1}(x)$ since

$$\sigma_x(\sigma^{-1}(x)) = \sigma(x) \neq \pi(x) = \pi_x(\pi^{-1}(x)) = \pi_x(\sigma^{-1}(x)).$$

Again, the size of the intersection between the two permutations will remain the same after the $x$-fixing of $\mathcal{A}$.

(c) $\sigma(x) = \pi(x)$ and $\sigma^{-1}(x) = \pi^{-1}(x)$.

In this case, using the definition of the fixing operation gives

$$\sigma_x(x) = \pi_x(x) \text{ and}$$

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\[ \sigma_x(\sigma^{-1}(x)) = \sigma(x) = \pi(x) = \pi_x(\pi^{-1}(x)) = \pi_x(\sigma^{-1}(x)). \]

Hence, the two permutations agree on the points \( \sigma^{-1}(x) = \pi^{-1}(x) \) and \( x \) both before and after the fixing operation and therefore the size of the intersection will remain the same.

3. Case 3: Either \( F_x(\sigma) \neq \sigma \) and \( F_x(\pi) = \pi \) or \( F_x(\sigma) = \sigma \) and \( F_x(\pi) \neq \pi \).

Assume without loss of generality that \( F_x(\sigma) \neq \sigma \) and \( F_x(\pi) = \pi \). Then \( \sigma(x) \neq x \) and either \( \pi(x) = x \) or \( \pi(x) \neq x \) and \( \pi_x \in A \).

(a) Assume that \( \pi(x) = x \). Then \( \sigma \) and \( \pi \) do not agree on \( x \) or on \( \sigma^{-1}(x) \).

Since \( \sigma \) and \( \pi \) are \( t \)-intersecting and \( x \) and \( \sigma^{-1}(x) \) are the only two points at which \( \sigma \) differs from \( \sigma_x \), it follows that \( \sigma_x \) and \( \pi = \pi_x \) must be \( t \)-intersecting.

(b) Assume that \( \pi(x) \neq x \) and \( \pi_x \in A \). In this case, \( \sigma \) and \( \pi_x \) are \( t \)-intersecting.

Since \( \sigma(x) \neq x \) and \( \pi_x(x) = x \), it follows that \( \sigma \) and \( \pi_x \) must agree on at least \( t \) points other than \( x \) and \( \sigma^{-1}(x) \). Furthermore, if \( \sigma \) and \( \pi_x \) agree on \( \pi^{-1}(x) \) and exactly \( t - 1 \) points from \( X \), then \( \sigma \) and \( \pi \) will not be \( t \)-intersecting. Therefore, \( \sigma \) and \( \pi_x \) must agree on at least \( t \) points from the set \( X \). Since \( \pi(y) = \pi_x(y) \) for all \( y \in X \), it follows that \( \sigma \) and \( \pi \) must also agree on at least \( t \) points from \( X \). Similarly, since \( \sigma(y) = \sigma_x(y) \) for
all \( y \in X \), the permutations \( \sigma_x \) and \( \pi \) must agree on at least \( t \) points in \( X \) and will therefore be \( t \)-intersecting.

In all cases, the permutations \( F_x(\sigma) \) and \( F_x(\pi) \) are \( t \)-intersecting. \( \square \)

Although \( F_x(\mathcal{A}) \) will be \( t \)-intersecting if \( \mathcal{A} \) is a \( t \)-intersecting family of permutations, the size of \( F_x(\mathcal{A}) \) may be less than that of \( \mathcal{A} \). An example of such a family is given in Table 4.1. In this example, \( \mathcal{A} \) is a maximal intersecting family of permutations from \( S_5 \) which includes the permutations \( \sigma = (31245) \) and \( \pi = (23145) \) but not the permutation \( (13245) \). Thus \( F_1(\sigma) = F_1(\pi) \) and \( |F_1(\mathcal{A})| < |\mathcal{A}| \). In fact, the size of the family decreases by two since there are two such pairs.

Table 4.1: Example of a maximal intersecting family of permutations where the \( x \)-fixing operation decreases the size of the family

<table>
<thead>
<tr>
<th>( \mathcal{A} )</th>
<th>( F_1(\mathcal{A}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(32145)</td>
<td>(12345)</td>
</tr>
<tr>
<td>(35142)</td>
<td>(15342)</td>
</tr>
<tr>
<td>(34125)</td>
<td>(14325)</td>
</tr>
<tr>
<td>(34152)</td>
<td>(14352)</td>
</tr>
<tr>
<td>(32154)</td>
<td>(12354)</td>
</tr>
<tr>
<td>(35124)</td>
<td>(15324)</td>
</tr>
<tr>
<td>(31245)</td>
<td>(13245) *</td>
</tr>
<tr>
<td>(31542)</td>
<td>(13542)</td>
</tr>
<tr>
<td>(32541)</td>
<td>(12543) *</td>
</tr>
<tr>
<td>(35241)</td>
<td>(15243)</td>
</tr>
<tr>
<td>(23145)</td>
<td>(13245) *</td>
</tr>
<tr>
<td>(53142)</td>
<td>(13542)</td>
</tr>
<tr>
<td>(24153)</td>
<td>(14253)</td>
</tr>
<tr>
<td>(52143)</td>
<td>(12543)</td>
</tr>
</tbody>
</table>

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An obvious way to avoid the decrease in the size of the family is to apply the fixing operation to each permutation in turn and to set $F_x(\sigma) = \sigma$ if $\sigma_x \in A'$ where $A'$ is the family consisting of all previously fixed permutations and all remaining unaltered permutations. However, this does not necessarily preserve the $t$-intersection. For instance, in the example in Table 4.1, neither $\sigma = \langle 3 1 2 4 5 \rangle$ nor $\pi = \langle 2 3 1 4 5 \rangle$ intersects with $\langle 1 4 3 5 2 \rangle \in F_1(A)$). Hence $F_1(A)$ will not be intersecting regardless of whether $\sigma$ or $\pi$ is fixed first.

A different fixing operation was introduced by Ku and Renshaw [11] for use with $t$-cycle-intersecting families of permutations. These are collections of permutations where any two elements written in their cycle decomposition form have at least $t$ cycles in common. For a permutation, $\sigma$, the $ij$-fixing, $[ij] \sigma$, is defined as follows:

1. if $\sigma(i) \neq j$, then $[ij] \sigma = \sigma$;

2. if $\sigma(i) = j$, then $[ij] \sigma(x) = \begin{cases} i & \text{if } x = i, \\ j & \text{if } x = \sigma^{-1}(i), \\ \sigma(x) & \text{otherwise}. \end{cases}$

Further, for a set of permutations, $A$, they define the $ij$-fixing of $A$, $\triangleleft_{ij}(A)$, as:

$$\triangleleft_{ij}(A) = \{ \triangleleft_{ij}(\sigma) : \sigma \in S \},$$

where

$$\triangleleft_{ij}(\sigma) = \begin{cases} [ij] \sigma & \text{if } [ij] \sigma \notin S, \\ \sigma & \text{otherwise}. \end{cases}$$
This operation does preserve the size of the family and has the property that \( \triangleleft_{ij}(\mathcal{A}) \) is \( t \)-cycle-intersecting if \( \mathcal{A} \) is \( t \)-cycle-intersecting. However, it does not follow that \( \triangleleft_{ij}(\mathcal{A}) \) is \( t \)-intersecting if \( \mathcal{A} \) is \( t \)-intersecting. For instance, performing the 1, 2-fixing on the family of permutations in Table 4.1 results in a family that is not intersecting.

### 4.2 More about the \( x \)-fixing operation

While the \( x \)-fixing operation of Cameron and Ku does preserve the \( t \)-intersection of a family of permutations, it does not necessarily preserve the size of the family.

We now examine in more detail when this problem occurs and how much smaller the family may become as a result of the fixing operation.

In order for a family of permutations, \( S \), to decrease in size after the \( x \)-fixing operation is applied for some \( x \in [n] \), the family must contain two distinct permutations, \( \sigma \neq \pi \), such that \( \sigma_x = \pi_x \notin S \). The example in Table 4.1 shows that it is possible to have such a pair in a maximal intersecting family of permutations. Similar examples can be constructed for larger values of \( t \) provided that \( n \geq t + 3 \).

The next proposition lists some properties of such a pair of permutations.

**Proposition 4.2.1.** If \( \sigma \neq \pi \in S \) and \( \sigma_x = \pi_x \notin S \), then:

1. \( \sigma(y) = \pi(y) \) for all \( y \in [n] \setminus \{x, \sigma^{-1}(x), \pi^{-1}(x)\} \),
2. \( \sigma(x) \neq x \) and \( \pi(x) \neq x \),

3. \( \sigma(x) \neq \pi(x) \) and \( \sigma^{-1}(x) \neq \pi^{-1}(x) \), and

4. \( \pi(\sigma^{-1}(x)) = \sigma(x) \) and \( \sigma(\pi^{-1}(x)) = \pi(x) \).

Proof. From the definition of the fixing operation, we have that \( \sigma_x(y) = \sigma(y) \) for all \( y \in [n] \) except \( x \) and \( \sigma^{-1}(x) \). Similarly, \( \pi_x(y) = \pi(y) \) for all \( y \in [n] \) except \( x \) and \( \pi^{-1}(x) \). Property 1 then follows from \( \sigma_x = \pi_x \). Property 2 follows easily from the requirement that \( \sigma_x, \pi_x \notin S \). Again by the definition, we have that \( \sigma(x) = \pi(x) \) if and only if \( \sigma_x(\sigma^{-1}(x)) = \pi_x(\pi^{-1}(x)) \). Since \( \sigma_x = \pi_x \), this will be true if and only if \( \sigma^{-1}(x) = \pi^{-1}(x) \). Combining this with Property 1 gives us that if \( \sigma(x) = \pi(x) \), then \( \sigma = \pi \). Hence if \( \sigma \neq \pi \), then \( \sigma(x) \neq \pi(x) \) and \( \sigma^{-1}(x) \neq \pi^{-1}(x) \). We now prove Property 4. From the second and third properties we have that \( \pi^{-1}(x) \neq x \) and \( \pi^{-1}(x) \neq \sigma^{-1}(x) \). The definition of the fixing operation then gives \( \sigma(\pi^{-1}(x)) = \sigma_x(\pi^{-1}(x)) \). Since \( \sigma_x = \pi_x \), it follows \( \sigma_x(\pi^{-1}(x)) = \pi_x(\pi^{-1}(x)) = \pi(x) \). A similar argument can be used to show that \( \pi_x(\sigma^{-1}(x)) = \sigma(x) \). \( \Box \)

Lemma 4.2.2. Let \( \mathcal{A} \) be a maximal \( t \)-intersecting system of permutations from \( S_n \). For any \( \sigma \in \mathcal{A} \) and \( x \in [n] \) such that \( \sigma_x \notin \mathcal{A} \), there can be at most one other permutation \( \pi \in \mathcal{A} \) such that \( \sigma_x = \pi_x \).

Proof. Suppose there exist three distinct permutations \( \sigma, \pi, \rho \) in \( \mathcal{A} \) such that \( \sigma_x \notin \mathcal{A} \) and \( \rho_x = \sigma_x = \pi_x \). By Proposition 4.2.1 we have the following:
\[
\sigma(x) \neq \pi(x), \quad \sigma^{-1}(x) \neq \pi^{-1}(x),
\]
\[
\sigma(x) \neq \rho(x), \quad \sigma^{-1}(x) \neq \rho^{-1}(x),
\]
\[
\pi(x) \neq \rho(x), \quad \pi^{-1}(x) \neq \rho^{-1}(x).
\]

Let \( \sigma(x) = y \), let \( \pi(x) = z \) and let \( \rho(x) = a \). The actions of the permutations \( \sigma, \pi, \rho \) and \( \sigma_x \) on the points \( x, \sigma^{-1}(x), \pi^{-1}(x) \) and \( \rho^{-1}(x) \) are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( \sigma^{-1}(x) )</th>
<th>( \pi^{-1}(x) )</th>
<th>( \rho^{-1}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>( y )</td>
<td>( x )</td>
<td>( z )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( z )</td>
<td>( y )</td>
<td>( x )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>( a )</td>
<td>( y )</td>
<td>( z )</td>
<td>( x )</td>
</tr>
<tr>
<td>( \sigma_x )</td>
<td>( x )</td>
<td>( y )</td>
<td>( z )</td>
<td>( a )</td>
</tr>
</tbody>
</table>

Let \( X = [n] \setminus \{x, \sigma^{-1}(x), \pi^{-1}(x), \rho^{-1}(x)\} \). Note that for all \( i \in X \),

\[
\sigma(i) = \pi(i) = \rho(i) = \sigma_x(i).
\]

If \( \sigma, \pi \) and \( \rho \) agree with all other permutations in \( \mathcal{A} \) on \( t \) elements of \( X \), then \( \sigma_x \) will also \( t \)-intersect with all permutations in \( \mathcal{A} \) and could be added to \( \mathcal{A} \) to give a larger \( t \)-intersecting family containing \( \mathcal{A} \). This contradicts the maximality of \( \mathcal{A} \). Therefore, there must be some permutation, \( \psi \), in \( \mathcal{A} \) that agrees with \( \sigma, \pi \) and \( \rho \) on fewer than \( t \) elements of \( X \).

Any permutation in \( \mathcal{A} \) must agree with \( \sigma, \pi \) and \( \rho \) on \( t-2 \) or more elements of \( X \) since if a permutation agrees with \( \sigma, \pi \) and \( \rho \) on fewer than \( t-2 \) elements of \( X \),
it will not $t$-intersect with all three of $\sigma$, $\pi$ and $\rho$. Hence, there must be some $\psi \in A$ that agrees with $\sigma$, $\pi$ and $\rho$ either on $t-2$ elements of $X$ and two of the points $x$, $\sigma^{-1}(x)$, $\pi^{-1}(x)$ or $\rho^{-1}(x)$ or on $t-1$ elements of $X$ and at least one of the points $x$, $\sigma^{-1}(x)$, $\pi^{-1}(x)$ or $\rho^{-1}(x)$.

There are four possible ways of constructing $\psi$ to meet these conditions:

1. $\psi(\sigma^{-1}(x)) = y$ and $\psi(\rho^{-1}(x)) = a$ (with $t-1$ intersections in $X$);

2. $\psi(\pi^{-1}(x)) = z$ and $\psi(\rho^{-1}(x)) = a$ (with $t-1$ intersections in $X$);

3. $\psi(\sigma^{-1}(x)) = y$, $\psi(\pi^{-1}(x)) = z$ and $\psi(\rho^{-1}(x)) \neq a$ (with $t-1$ intersections in $X$);

4. $\psi(\sigma^{-1}(x)) = y$, $\psi(\pi^{-1}(x)) = z$ and $\psi(\rho^{-1}(x)) = a$ (with $t-1$ or $t-2$ intersections in $X$).

The actions of $\psi$ on $x$, $\sigma^{-1}(x)$, $\pi^{-1}(x)$ and $\rho^{-1}(x)$ in each of these cases are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>$\psi(x)$</th>
<th>$\psi(\sigma^{-1}(x))$</th>
<th>$\psi(\pi^{-1}(x))$</th>
<th>$\psi(\rho^{-1}(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1.</td>
<td>-</td>
<td>$y$</td>
<td>-</td>
<td>$a$</td>
</tr>
<tr>
<td>Case 2.</td>
<td>-</td>
<td>-</td>
<td>$z$</td>
<td>$a$</td>
</tr>
<tr>
<td>Case 3.</td>
<td>-</td>
<td>$y$</td>
<td>$z$</td>
<td>-</td>
</tr>
<tr>
<td>Case 4.</td>
<td>-</td>
<td>$y$</td>
<td>$z$</td>
<td>$a$</td>
</tr>
</tbody>
</table>
Comparing the actions of $\psi$ to those of $\sigma_x$ on $x$, $\sigma^{-1}(x)$, $\pi^{-1}(x)$ and $\rho^{-1}(x)$, it is clear that in all four cases, $\psi$ will $t$-intersect with $\sigma_x$, again contradicting the maximality of $\mathcal{A}$. Therefore it is not possible to have three distinct permutations, $\sigma$, $\pi$ and $\rho$, in $S$ such that $\sigma_x = \pi_x = \rho_x$. □

Although this does place some limits on the decrease in the size that may occur as a result of the application of the fixing operation, the problem is not limited to one such pair as was seen in the example in Table 4.1.

4.3 Modifying the $x$-fixing operation

In the remainder of this chapter, we prove several propositions concerning the family of permutations resulting from the application of the $x$-fixing operation to a maximal $t$-intersecting family. In particular, we consider the permutations $(yz) \circ \sigma$ and $(yz) \circ \pi$ where $\sigma$ and $\pi$ are defined as in Proposition 4.2.1, and $y = \sigma(x)$ and $z = \pi(x)$ as in Lemma 4.2.2. Here $(yz) \circ \sigma$ and $(yz) \circ \pi$ represent the permutations formed by transposing $y$ and $z$ in $\sigma$ and $\pi$. The goal is to modify the $x$-fixing operation to enable the transformation of any maximal $t$-intersecting family of permutations into a fixed family while maintaining the size of the family and the $t$-intersection.

Proposition 4.3.1. Let $\mathcal{A}$ be a maximal $t$-intersecting family of permutations. Let $\sigma$ and $\pi$ be two distinct permutations in $\mathcal{A}$ such that $\sigma_x = \pi_x$ for some $x \in [n]$ and
\(\sigma_x \notin A\). Let \(\sigma(x) = y\) and \(\pi(x) = z\). Then \((yz) \circ \sigma\) and \((yz) \circ \pi\) cannot both be in \(A\).

**Proof.** From Proposition 4.2.1 we have that \(y \neq x\), \(z \neq x\) and \(y \neq z\).

Since \(A\) is maximal, if \(\sigma_x \notin A\) there must exist some permutation, \(\psi \in A\) such that \(|\sigma_x \cap \psi| < t\). This permutation must agree with \(\sigma\) and \(\pi\) on exactly \(t - 1\) elements of \(X = [n] \{ x, \sigma^{-1}(x), \pi^{-1}(x) \}\) and at least one of \(x, \sigma^{-1}(x)\) or \(\pi^{-1}(x)\). There are three possible ways this could happen:

1. \(\psi(x) = y\) and \(\psi(\pi^{-1}(x)) = x\),

2. \(\psi(x) = z\) and \(\psi(\sigma^{-1}(x)) = x\),

3. \(\psi(\sigma^{-1}(x)) = y\) and \(\psi(\pi^{-1}(x)) = z\).

Now consider the permutations \((yz) \circ \sigma\) and \((yz) \circ \pi\). The actions of the permutations \(\sigma, \pi, (yz) \circ \sigma\) and \((yz) \circ \pi\) on the elements \(x, \sigma^{-1}(x)\) and \(\pi^{-1}(x)\) are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(\sigma^{-1}(x))</th>
<th>(\pi^{-1}(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma)</td>
<td>(y)</td>
<td>(x)</td>
<td>(z)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>(z)</td>
<td>(y)</td>
<td>(x)</td>
</tr>
<tr>
<td>((yz) \circ \sigma)</td>
<td>(z)</td>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>((yz) \circ \pi)</td>
<td>(y)</td>
<td>(z)</td>
<td>(x)</td>
</tr>
</tbody>
</table>
Comparing these mappings with the three possibilities for $\psi$, it is clear that at most one of $(yz) \circ \sigma$ and $(yz) \circ \pi$ will $t$-intersect with $\psi$ and thus both cannot be in $A$. □

**Proposition 4.3.2.** Let $A$ be a maximal $t$-intersecting family of permutations. Let $\sigma$ and $\pi$ be two distinct permutations in $A$ such that $\sigma_x = \pi_x$ for some $x \in [n]$ and $\sigma_x \notin A$. Let $\sigma(x) = y$ and $\pi(x) = z$. If $A$ contains $(yz) \circ \sigma$, then $(yz) \circ \pi \notin F_x(A)$ and $F_x(A) \cup \{(yz) \circ \pi\}$ is a $t$-intersecting family of permutations. Alternatively, if $A$ contains $(yz) \circ \pi$, then $(yz) \circ \sigma \notin F_x(A)$ and $F_x(A) \cup \{(yz) \circ \sigma\}$ is a $t$-intersecting family of permutations.

**Proof.** We prove the case when $(yz) \circ \sigma \in A$ only, since the proof of the second statement proceeds analogously.

Assume that $(yz) \circ \sigma \in A$. By Proposition 4.2.1, we have that $((yz) \circ \pi)(x) \neq x$. It then follows that $(yz) \circ \pi \in F_x(A)$ only if $(yz) \circ \pi \in A$. Since $(yz) \circ \sigma \in A$, by Proposition 4.3.1 $(yz) \circ \pi \notin A$. Therefore $(yz) \circ \pi \notin F_x(A)$.

We now prove by contradiction that $F_x(A) \cup \{(yz) \circ \pi\}$ is a $t$-intersecting family. Assume that $F_x(A) \cup \{(yz) \circ \pi\}$ is not $t$-intersecting. Since $F_x(A)$ is $t$-intersecting, there must exist some permutation $\psi \in F_x(A)$ such that

$$|\psi \cap (yz) \circ \pi| < t.$$

Also, since $\sigma_x$ and $((yz) \circ \sigma)_x$ are in $F_x(A)$, both must $t$-intersect with $\psi$. 44
Let $X = [n] \setminus \{x, \sigma^{-1}(x), \pi^{-1}(x)\}$. Note that $\sigma$, $\pi$, $(yz) \circ \sigma$, $(yz) \circ \pi$, $\sigma_x$ and $((yz) \circ \sigma)_x$ agree on all elements of $X$. If $\psi$ agrees with these permutations on $t$ or more elements of $X$, then $|\psi \cap ((yz) \circ \pi)| \geq t$. Thus $\psi$ must agree with $\sigma_x$ and $((yz) \circ \sigma)_x$ on at least one of $x$, $\sigma^{-1}(x)$ or $\pi^{-1}(x)$. This can only happen if $\psi(x) = x$. It then follows that $\psi$ cannot $t$-intersect with both $\sigma$ and $(yz) \circ \sigma$ and so $\psi \notin A$. Therefore there must exist some permutation $\tau \in A$ such that $F_x(\tau) = \psi$. If $\tau$ agrees with $\sigma$ on $t$ or more elements of $X$, then $\psi$ will $t$-intersect with $(yz) \circ \pi$. Hence $\tau$ must agree with each of $\sigma$, $\pi$ and $(yz) \circ \sigma$ on at least one of $x$, $\sigma^{-1}(x)$ and $\pi^{-1}(x)$ and on fewer than $t$ elements of $X$. This requires that $\tau(x) = z$ and $\tau(\sigma^{-1}(x)) = x$ and $\tau$ must agree with $\sigma$ on exactly $t - 1$ elements of $X$. Then $\psi$ will agree with $(yz) \circ \pi$ on $t - 1$ elements of $X$ and $\psi(\sigma^{-1}(x)) = z = ((yz) \circ \pi)(\sigma^{-1}(x))$. Hence $\psi$ and $(yz) \circ \pi$ will be $t$-intersecting which contradicts the assumption that $F_x(A) \cup \{(yz) \circ \pi\}$ is not $t$-intersecting.

This proves that if $\sigma$ and $\pi$ are the only two permutations in $A$ that are changed to the same permutation by the application of the $x$-fixing operation, then the addition of $(yz) \circ \sigma$ or $(yz) \circ \pi$ to $F_x(A)$ will give a $t$-intersecting family of the same size as $A$. However, if $A$ contains more than one such pair, it remains to be shown that the permutations added to $F_x(A)$ will $t$-intersect with each other.

**Proposition 4.3.3.** Suppose that $A$ is a maximal $t$-intersecting family of permutations. Let $\sigma$, $\pi$, $\tau$ and $\rho$ be distinct permutations in $A$ such that $\sigma_x = \pi_x$ and $\tau_x = \rho_x$ for some $x \in [n]$ and $\sigma_x, \tau_x \notin A$. Let $\sigma(x) = y$, $\pi(x) = z$, $\tau(x) = a$ and $\rho(x) = b$. 


Now suppose that \((yz) \circ \sigma\) and \((ab) \circ \tau\) are in \(A\). Then \((yz) \circ \pi\) and \((ab) \circ \rho\) are \(t\)-intersecting.

**Proof.** The actions of the permutations \(\sigma, \pi, (yz) \circ \sigma, \tau, \rho\) and \((ab) \circ \tau\) on \(x, \sigma^{-1}(x), \pi^{-1}(x), \tau^{-1}(x)\) and \(\rho^{-1}(x)\) are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(\sigma^{-1}(x))</th>
<th>(\pi^{-1}(x))</th>
<th>(\tau^{-1}(x))</th>
<th>(\rho^{-1}(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma)</td>
<td>(y)</td>
<td>(x)</td>
<td>(z)</td>
<td>(\alpha)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>(z)</td>
<td>(y)</td>
<td>(x)</td>
<td>(\alpha)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>((yz) \circ \sigma)</td>
<td>(z)</td>
<td>(x)</td>
<td>(y)</td>
<td>(\alpha)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>(\tau)</td>
<td>(a)</td>
<td>(\delta)</td>
<td>(\gamma)</td>
<td>(\tau^{-1}(x))</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>(\rho)</td>
<td>(b)</td>
<td>(\delta)</td>
<td>(\gamma)</td>
<td>(\alpha)</td>
<td>(x)</td>
</tr>
<tr>
<td>((ab) \circ \tau)</td>
<td>(b)</td>
<td>(\delta)</td>
<td>(\gamma)</td>
<td>(\alpha)</td>
<td>(x)</td>
</tr>
</tbody>
</table>

The permutations \(\pi\) and \(\rho\) are in \(A\), so they are \(t\)-intersecting. The permutations \(\pi\) and \((yz) \circ \pi\) agree on all elements of \([n]\) except \(x\) and \(\sigma^{-1}(x)\). Similarly, the permutations \(\rho\) and \((ab) \circ \rho\) agree on all elements of \([n]\) except \(x\) and \(\tau^{-1}(x)\). Hence, if \(\pi\) and \(\rho\) agree on \(t\) or more elements of \([n]\) \(\setminus\{x, \sigma^{-1}(x), \tau^{-1}(x)\}\), then \((yz) \circ \pi\) and \((ab) \circ \rho\) will be \(t\)-intersecting and we are done. Therefore, we assume that \(\pi\) and \(\rho\) agree on fewer than \(t\) elements of \([n]\) \(\setminus\{x, \sigma^{-1}(x), \tau^{-1}(x)\}\). Then \(\pi\) and \(\rho\) must agree on at least one of \(x, \sigma^{-1}(x)\) or \(\tau^{-1}(x)\).

Let \(Y = [n] \setminus \{x, \sigma^{-1}(x), \pi^{-1}(x), \tau^{-1}(x), \rho^{-1}(x)\}\). Suppose that \(\pi^{-1}(x) = \rho^{-1}(x)\). Then \(\pi\) and \(\rho\) agree on fewer than \(t - 1\) elements of \(Y\). Suppose they agree on exactly
$t-2$ elements of $Y$. Then $\sigma$ and $\rho$ agree on $t-2$ elements of $Y$ also. Then $\sigma$ and $\rho$ must agree on two elements in $\{x, \sigma^{-1}(x), \pi^{-1}(x), \tau^{-1}(x), \rho^{-1}(x)\}$. Since $\sigma^{-1}(x) \neq \pi^{-1}(x)$ by Proposition 4.2.1, this implies that $y = b$ and $\alpha = a$. Similarly, $(yz) \circ \sigma$ and $\rho$ must 2-intersect in $\{x, \sigma^{-1}(x), \pi^{-1}(x), \tau^{-1}(x), \rho^{-1}(x)\}$ which implies that $z = b$ and $\alpha = a$. This is a contradiction since $z \neq y$, therefore $\pi^{-1}(x) \neq \rho^{-1}(x)$.

With $\pi^{-1}(x) \neq \rho^{-1}(x)$ and $\pi$ and $\rho$ agreeing on fewer than $t$ elements of $[n] \setminus \{x, \sigma^{-1}(x), \tau^{-1}(x)\}$, it follows that $\pi$ and $\rho$ will agree on $t-1$ or fewer elements of $Y$. Assume they agree on exactly $t-1$ elements of $Y$ and at least one of $x, \sigma^{-1}(x),$ and $\tau^{-1}(x)$. We consider these three cases separately.

1. Case 1: $\pi(x) = \rho(x)$.

If $\pi$ and $\rho$ agree on $t-1$ elements of $Y$, then $\sigma$ will agree with $\rho$, $\tau$ and $(ab) \circ \tau$ on the same $t-1$ elements of $Y$ and must agree with each of these three permutations on at least one element in $\{x, \sigma^{-1}(x), \pi^{-1}(x), \tau^{-1}(x), \rho^{-1}(x)\}$. Since $y \neq z$, it follows that $\sigma(x) \neq \rho(x)$ and $\sigma(x) \neq ((ab) \circ \tau)(x)$. Also, since $\pi^{-1}(x) \neq x$, it follows that $\gamma \neq z$. Thus $\sigma$ and $\rho$ must agree on $\sigma^{-1}(x)$ or $\tau^{-1}(x)$; that is, either $\sigma^{-1}(x) = \rho^{-1}(x)$ or $\alpha = a$. In either case, $\sigma$ cannot $t$-intersect with $(ab) \circ \tau$.

This contradicts the $t$-intersection of $\mathcal{A}$.

The other two cases can be proved in a similar manner. □

Thus in the case where either $(yz) \circ \sigma$ or $(yz) \circ \pi$ is in $\mathcal{A}$, we can modify the
$x$-fixing operation so that the size and the $t$-intersection of the family is maintained. However, the case when neither $(yz) \circ \sigma$ nor $(yz) \circ \pi$ is in $\mathcal{A}$ remains to be dealt with. It seems unlikely that such a family would have the maximum possible size for any given values of $n$ and $t$, but this has yet to be proved.
Chapter 5

The Compression Operation

We now define a new operation on families of permutations that is designed to make the elements fixed by a permutation as small as possible while retaining the size and structure of the family. We will use the term compression to describe this operation. Although the definition of this operation and the proofs of some of its properties apply to any family of permutations, others are restricted to fixed families. When applied to a permutation, $\sigma$, this operation has the effect of left shifting the set $\text{fix}(\sigma)$.

5.1 Definition of the compression operation for permutations

For a permutation, $\sigma \in S_n$, and $i, j \in [n]$ with $i < j$, the $(i,j)$-compression of $\sigma$, denoted by $\sigma_{i,j}$, is the permutation defined as:
1. if \( \sigma(i) = i \) or \( \sigma(j) \neq j \), then \( \sigma_{i,j} = \sigma \);

2. if \( \sigma(i) \neq i \) and \( \sigma(j) = j \), then \( \sigma_{i,j}(y) = \begin{cases} 
i & \text{if } y = i; \\
\sigma(i) & \text{if } y = j; \\
j & \text{if } y = \sigma^{-1}(i); \\
\sigma(y) & \text{otherwise}. 
\end{cases} \)

For example, if \( \sigma = (4\ 5\ 3\ 2\ 1) \), then \( \sigma_{1,3} = (1\ 5\ 4\ 3\ 2) \).

The effect of the compression operation on the set \( \text{fix}(\sigma) \) when \( j \) is fixed under \( \sigma \) and \( i \) is not, is to remove \( j \) from \( \text{fix}(\sigma) \) and replace it with \( i \). Note that in this case \( \sigma^{-1}(i) \) is not fixed by either \( \sigma \) or \( \sigma_{i,j} \).

Let \( \mathcal{S} \) be a collection of permutations. We define the \((i, j)\)-compression, \( \mathcal{C}_{i,j} \), as follows:

\[
\mathcal{C}_{i,j}(\mathcal{S}) = \{ \mathcal{C}_{i,j}(\sigma) : \sigma \in \mathcal{S} \}, \\
where \\
\mathcal{C}_{i,j}(\sigma) = \begin{cases} 
\sigma_{i,j} & \text{if } \sigma_{i,j} \notin \mathcal{S}; \\
\sigma & \text{otherwise}. 
\end{cases}
\]

If a family of permutations, \( \mathcal{S} \), has the property that \( \mathcal{C}_{i,j}(\mathcal{S}) = \mathcal{S} \) for all \( 1 \leq i < j \leq n \), then \( \mathcal{S} \) is called a \textit{compressed} family.

The condition that \( i \neq j \) could be used in place of \( i < j \) in the definition of the compression operation and the terms \textit{left-compression} and \textit{left-compressed} used when the condition \( i < j \) is added. Although the results presented in the next section hold for all \( i \neq j \), we will use the operation as defined with \( i < j \).
5.2 Properties of the compression operation

Theorem 5.2.1. Let $S$ be any collection of permutations from $S_n$. Then for any $i,j \in [n]$,

$$|C_{i,j}(S)| = |S|.$$

Proof. It is clear from the definition of the compression operation that $C_{i,j}(S)$ will not be larger than $S$ and that the size will decrease only if there are two distinct permutations, $\sigma$ and $\pi$, in $S$ such that $\sigma_{i,j} = \pi_{i,j}$ and $\sigma_{i,j} \notin S$. Therefore, to prove the lemma it is sufficient to show that $\sigma_{i,j} = \pi_{i,j}$ only if $\sigma = \pi$.

Suppose that there are two permutations, $\sigma, \pi \in S$, such that $\sigma_{i,j} = \pi_{i,j}$ and $\sigma_{i,j} \notin S$. Then $\sigma \neq \sigma_{i,j}$ and thus $\sigma(i) \neq i$ and $\sigma(j) = j$. Similarly, $\pi(i) \neq i$ and $\pi(j) = j$. Also,

$$\sigma_{i,j}(\sigma^{-1}(i)) = j \quad \text{and} \quad \pi_{i,j}(\pi^{-1}(i)) = j$$

and thus it follows from $\sigma_{i,j} = \pi_{i,j}$ that $\sigma^{-1}(i) = \pi^{-1}(i)$. Using this and the definition of the compression operation we have that

$$\sigma(x) = \sigma_{i,j}(x) = \pi_{i,j}(x) = \pi(x)$$

for all $x \in [n] \setminus \{i,j, \sigma^{-1}(i) = \pi^{-1}(i)\}$. Therefore, it only remains to show that $\sigma$ and $\pi$ agree on $i$, $j$ and $\sigma^{-1}(i)$.

We have already seen that $\sigma(j) = j = \pi(j)$ and that $\sigma^{-1}(i) = \pi^{-1}(i)$. It then follows
that $\sigma(\sigma^{-1}(i)) = \pi(\sigma^{-1}(i))$. Since $\sigma$ and $\pi$ have been shown to agree on all elements of $[n]$ except $i$, they must agree on $i$ also. Hence, if $\sigma_{i,j} = \pi_{i,j}$, then $\sigma = \pi$.  

\textbf{Theorem 5.2.2.} Let $S$ be any collection of permutations from $S_n$. Then $S$ can be transformed into a compressed family by the application of a finite number of compression operations.

\textit{Proof.} First we consider the effect of the compression operation on a single permutation. Let $\sigma$ be a permutation in $S_n$ such that $\sigma(j) = j$ and $\sigma(i) \neq i$ for some $i < j \in [n]$. Then the $(i,j)$-compression operation takes the element, $j$, that is fixed by $\sigma$ and the element $i$ that is not fixed by $\sigma$ and changes the permutation so it fixes $i$ and not $j$. The compression operation also changes the image of $\sigma^{-1}(i)$, but $\sigma^{-1}(i)$ will not be fixed either before or after the compression. The effect of the $(i,j)$-compression on the set $\text{fix}(\sigma)$ is the same as that of performing the $(i,j)$-shifting operation on $\text{fix}(\sigma)$. Even after multiple compression operations for different values of $i$ and $j$, the size of $\text{fix}(\sigma)$ will remain the same. After each compression operation, either $\text{fix}(\sigma_{i,j}) = \text{fix}(\sigma)$ or $\text{fix}(\sigma_{i,j}) \prec \text{fix}(\sigma)$. Thus it is clear that a finite number of compression operations can reduce elements in the fixed point set so that no further changes are possible.

Now consider a family of permutations, $S$. Since different permutations may have the same set of fixed points, looking at the fixed point sets of the permutations in $S$
is not sufficient to identify which permutations will be changed by a particular \((i,j)\)-compression. However, if any permutation is changed by the compression operation, its fixed point set will change. Suppose that we apply in succession all possible \((i,j)\)-compressions to \(S\). If \(C_{i,j}(S) = S\) for all \(i < j \in \lbrack n \rbrack\), then \(S\) is compressed. Otherwise, some permutations will have been changed and as a result their fixed point sets will have been left-shifted; that is, for each \(\sigma \in S\) and \(i < j \in \lbrack n \rbrack\) such that \(C_{i,j}(\sigma) \neq \sigma\), we have \(\text{fix}(C_{i,j}(\sigma)) \prec \text{fix}(\sigma)\). If we then repeat all possible \((i,j)\)-compression operations on our new family of permutations, \(S'\), we get the same result; either \(S'\) is compressed or the fixed point sets of some permutations change. Since the fixed point sets are always changed to sets that appear earlier in the lexicographic order, this process is necessarily finite.

Unlike the shifting operation for set systems, this operation does not necessarily preserve the \(t\)-intersection for an arbitrary \(t\)-intersecting family of permutations. For example, let \(\mathcal{A}\) be a 2-intersecting family of permutations from \(S_5\) containing the permutations \(\sigma = \langle 3 4 5 1 2 \rangle\) and \(\pi = \langle 3 2 4 1 5 \rangle\). Then \(C_{1,2}(\mathcal{A})\) will contain the permutations \(\sigma = \langle 3 4 5 1 2 \rangle\) and \(\pi_{1,2} = \langle 1 3 4 2 5 \rangle\) which are not 2-intersecting. However, if the \(t\)-intersecting family of permutations is a fixed family, then the compression operation will preserve the \(t\)-intersection. Since the application of the compression operation to a maximal, fixed \(t\)-intersecting family of permutations will result in a
fixed family of permutations (Theorem 5.2.5), this result holds for repeated applications of the compression operation.

**Theorem 5.2.3.** Let $A$ be a fixed $t$-intersecting family of permutations from $S_n$. Then $C_{i,j}(A)$ is a $t$-intersecting family of permutations for any $i,j \in [n]$.

*Proof.* Let $\sigma$ and $\pi$ be any two distinct permutations in $A$. Let $i,j$ be any two elements of $[n]$ such that $i \neq j$. Let $X = \{i, j, \sigma^{-1}(i), \pi^{-1}(i)\}$ and let $Y = [n] \setminus X$. If $\sigma$ and $\pi$ agree on $t$ or more elements of $Y$, then $C_{i,j}(\sigma)$ and $C_{i,j}(\pi)$ will also agree on the same elements of $Y$.

Applying the $(i,j)$-compression operation to $A$ can have three possible outcomes for $\sigma$ and $\pi$: both $\sigma$ and $\pi$ remain the same after the compression operation, both $\sigma$ and $\pi$ are changed by the compression operation, and exactly one of the two permutations is changed. We show that in all three cases $C_{i,j}(\sigma)$ and $C_{i,j}(\pi)$ are $t$-intersecting.

1. Case 1: $C_{i,j}(\sigma) = \sigma$ and $C_{i,j}(\pi) = \pi$.

   Since neither permutation is changed, they will be $t$-intersecting after the compression operation is applied to $A$.

2. Case 2: $C_{i,j}(\sigma) \neq \sigma$ and $C_{i,j}(\pi) \neq \pi$.

   In this case $\sigma$ and $\pi$ agree on $j$ while $C_{i,j}(\sigma)$ and $C_{i,j}(\pi)$ agree on $i$. Furthermore, if $\sigma(i) = \pi(i)$ then $(C_{i,j}(\sigma))(j) = (C_{i,j}(\pi))(j)$ and if $\sigma^{-1}(i) = \pi^{-1}(i)$ then
\((C_{i,j}(\sigma))(\sigma^{-1}(i)) = j = (C_{i,j}(\pi))(\pi^{-1}(i))\). Hence the size of the intersection remains unchanged after the compression operation is applied to \(\sigma\) and \(\pi\).

3. Case 3: Exactly one of \(\sigma\) or \(\pi\) changes.

Assume without loss of generality that \(\sigma \neq C_{i,j}(\sigma)\) and that \(\pi = C_{i,j}(\pi)\). This implies that \(\sigma(j) = j\) and \(\sigma(i) \neq i\). Since \(\pi\) is unchanged by the compression operation, we have one of the following three cases:

(a) \(\pi(i) = i\).

In this case, \(\sigma\) and \(\pi\) do not agree on \(i = \pi^{-1}(i)\) or \(\sigma^{-1}(i)\). Hence \(\sigma\) and \(\pi\) agree on at most one element of \(X\). Since \((C_{i,j}(\sigma))(i) = i\), the two permutations agree on at least one element of \(X\) after the compression operation.

(b) \(\pi(j) \neq j\).

If \(\sigma\) and \(\pi\) intersect at fewer than \(t\) points in \(Y\), then either \(\sigma(i) = \pi(i)\) or \(\sigma^{-1}(i) = \pi^{-1}(i)\). Since \(\mathcal{A}\) is fixed and \(\sigma(i) \neq i\), the permutation \(\sigma_i\) must be in \(\mathcal{A}\). Recall that \(\sigma_i(x) = \sigma(x)\) for all \(x \in [n]\) except for \(i\) and \(\sigma^{-1}(i)\) and that \(\sigma(i) = \sigma_i(\sigma^{-1}(i))\). Thus, if \(\sigma\) and \(\pi\) agree on one or both of \(i\) and \(\sigma^{-1}(i)\), then \(\sigma_i\) and \(\pi\) do not agree on either. Therefore \(\sigma\) and \(\pi\) must agree on at least \(t\) elements of \(Y\).

(c) \(\pi(i) \neq i, \pi(j) = j\) and \(\pi_{i,j} \in \mathcal{A}\).

In this case \(\sigma\) and \(\pi_{i,j}\) do not agree on \(i, j, \sigma^{-1}(i)\) or \(\pi^{-1}(i)\) and so they
must agree on at least $t$ elements of $Y$. Since $\pi_{i,j}(x) = \pi(x)$ for all $x \in Y$, it follows that $\sigma$ and $\pi$ agree on at least $t$ elements of $Y$. $\square$

**Lemma 5.2.4.** Let $i, j \in [n]$ with $i \neq j$. For any permutation $\sigma \in S_n$ such that $\sigma(j) = j$,

$$(\sigma_{i,j})_x = (\sigma_x)_{i,j} \text{ for all } x \in [n] \setminus \{i, j\}.$$

**Proof.** If $\sigma(i) = i$ then $\sigma_{i,j} = \sigma$ and $\sigma_x(i) = i$. It then follows that $(\sigma_x)_{i,j} = \sigma_x = (\sigma_{i,j})_x$.

Also, since $x \neq j$, if $\sigma(x) = x$ then $\sigma_x = \sigma$ and $\sigma_{i,j}(x) = x$ so $(\sigma_{i,j})_x = \sigma_{i,j} = (\sigma_x)_{i,j}$.

Hence we assume that $\sigma(i) \neq i$ and $\sigma(x) \neq x$.

From the definitions of the $(i, j)$-compression operation and the $x$-fixing operation, we have that

$$\sigma_{i,j}(y) = \sigma(y) \text{ for all } y \in [n] \setminus \{i, j, \sigma^{-1}(i)\} \quad \text{and}$$

$$\sigma_x(y) = \sigma(y) \text{ for all } y \in [n] \setminus \{x, \sigma^{-1}(x)\}.$$

If $\{x, \sigma^{-1}(x)\} \cap \{i, j, \sigma^{-1}(i)\} = \emptyset$, the $(i, j)$-compression and the $x$-fixing will act on different elements of $[n]$. If this is the case, the order in which the operations are applied will not affect the final result and so $(\sigma_{i,j})_x = (\sigma_x)_{i,j}$. Since $x \in [n] \setminus \{i, j\}$ and $\sigma(j) = j$, it follows that $x \neq i$ or $j$, and $\sigma^{-1}(x) \neq j$. Hence $\{x, \sigma^{-1}(x)\} \cap \{i, j, \sigma^{-1}(i)\} = \emptyset$ unless $\sigma^{-1}(x) = i$ or $\sigma^{-1}(i) = x$.

It can be shown that $(\sigma_{i,j})_x = (\sigma_x)_{i,j}$ when we have one or both of $\sigma^{-1}(x) = i$ and $\sigma^{-1}(i) = x$ by directly comparing $(\sigma_{i,j})_x(z)$ to $(\sigma_x)_{i,j}(z)$ for all $z \in \{i, j, \sigma^{-1}(i), x, \sigma^{-1}(x)\}$. 56
For example, suppose $\sigma^{-1}(x) = i$ and $\sigma^{-1}(i) = x$. The actions of the permutations $\sigma$, $\sigma_x$, $\sigma_{i,j}$, $(\sigma_x)_{i,j}$ and $(\sigma_{i,j})_x$ on $x = \sigma^{-1}(i)$, $i = \sigma^{-1}(x)$ and $j$ are summarized below.

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<tr>
<td>$\sigma$</td>
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<td>$x$</td>
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<tr>
<td>$\sigma_x$</td>
<td>$x$</td>
<td>$i$</td>
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</tr>
<tr>
<td>$(\sigma_x)_{i,j}$</td>
<td>$x$</td>
<td>$i$</td>
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<td>$\sigma_{i,j}$</td>
<td>$j$</td>
<td>$i$</td>
<td>$x$</td>
</tr>
<tr>
<td>$(\sigma_{i,j})_x$</td>
<td>$x$</td>
<td>$i$</td>
<td>$j$</td>
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The remaining two cases can be proved in a similar manner. □

**Theorem 5.2.5.** Let $\mathcal{A}$ be a maximal fixed $t$-intersecting family of permutations from $S_n$. Then for any $i \neq j \in [n]$, the family $\mathcal{C}_{i,j}(\mathcal{A})$ is a fixed family of permutations.

**Proof.** Let $\sigma$ be a permutation in $\mathcal{C}_{i,j}(\mathcal{A})$ for some $i \neq j \in [n]$. In order to show that $\mathcal{C}_{i,j}(\mathcal{A})$ is fixed, it is sufficient to show that $\sigma_x \in \mathcal{C}_{i,j}(\mathcal{A})$ for all $x \in [n]$. We consider the cases when $\sigma \notin \mathcal{A}$ and when $\sigma \in \mathcal{A}$ separately.

1. Case 1: $\sigma \in \mathcal{C}_{i,j}(\mathcal{A})$ and $\sigma \notin \mathcal{A}$.

   Since $\sigma \notin \mathcal{A}$, there is a permutation $\pi \in \mathcal{A}$ such that $\pi_{i,j} = \sigma$ and $\pi \neq \pi_{i,j}$.

   Again we consider three cases: $x \in [n]\{i,j\}$, $x = i$ and $x = j$.

   (a) $x \in [n]\{i,j\}$.

   Since $\pi \neq \pi_{i,j}$, we have that $\pi(j) = j$, and so $(\pi_x)_{i,j} = (\pi_{i,j})_x$ for all
\( x \in [n] \setminus \{i, j\} \) by Lemma 5.2.4. Since \( A \) is fixed, it follows that \( \pi_x \in A \) and hence that \( (\pi_x)_{i,j} \in C_{i,j}(A) \) for all \( x \in [n] \). It then follows that for all \( x \in [n] \setminus \{i, j\} \),

\[
\sigma_x = (\pi_{i,j})_x = (\pi_x)_{i,j} \in C_{i,j}(A).
\]

(b) \( x = i \).

Since \( \sigma \) is not an element of \( A \) but is in \( C_{i,j}(A) \), it follows that \( \sigma(i) = i \).

Therefore, \( \sigma_i = \sigma \in C_{i,j}(A) \) as required.

(c) \( x = j \).

Since \( A \) is fixed, \( \pi_i \in A \) and since \( \pi_i(i) = i \), it follows that \( (\pi_i)_{i,j} = \pi_i \). Therefore \( \pi_i \in C_{i,j}(A) \). We now show that \( \pi_i = \sigma_j \) and thus that \( \sigma_j \in C_{i,j}(A) \). Since \( \sigma = \pi_{i,j} \neq \pi \), we have that \( \sigma^{-1}(j) = \pi^{-1}(i) \) and that \( \sigma(y) = \pi(y) \) for all \( y \in [n] \setminus \{i, j, \pi^{-1}(i)\} \). Also, \( \sigma_j(y) = \sigma(y) \) for all \( y \in [n] \setminus \{i, \pi^{-1}(i)\} \).

Therefore,

\[
\sigma_j(y) = \pi_i(y) \quad \text{for all} \quad y \in [n] \setminus \{i, j, \pi^{-1}(i)\}.
\]

From the definitions of the \((i, j)\)-compression and the \(x\)-fixing operations,

\[
\sigma_j(i) = i = \pi_i(i),
\]

\[
\sigma_j(j) = j = \pi_i(j), \quad \text{and}
\]

\[
\sigma_j(\pi^{-1}(i)) = \sigma(j) = \pi(i) = \pi_i(\pi^{-1}(i)).
\]

Hence \( \pi_i = \sigma_j \in C_{i,j}(A) \).
2. Case 2: \( \sigma \in \mathcal{A} \).

In this case, either \( \sigma \neq \sigma_{i,j} \) and \( \sigma_{i,j} \in \mathcal{A} \), or \( \sigma = \sigma_{i,j} \). We consider these two possibilities separately.

(a) \( \sigma \neq \sigma_{i,j} \) and \( \sigma_{i,j} \in \mathcal{A} \).

Because \( \sigma(j) = j \), by Lemma 5.2.4 we have that \( (\sigma_x)_{i,j} = (\sigma_{i,j})_x \) for all \( x \in [n]\setminus\{i,j\} \). Since \( \mathcal{A} \) is fixed and \( \sigma, \sigma_{i,j} \in \mathcal{A} \), it follows that \( \sigma_x \in \mathcal{A} \) and \( (\sigma_{i,j})_x \in \mathcal{A} \) for all \( x \in [n] \). Therefore, for all \( x \in [n]\setminus\{i,j\} \),

\[
(\sigma_x)_{i,j} \in \mathcal{A} \quad \text{and} \quad \sigma_x \in \mathcal{C}_{i,j}(\mathcal{A}).
\]

Suppose \( x = i \). The permutation \( \sigma_i \) is in \( \mathcal{A} \). Since \( (\sigma_i)_{i,j} = \sigma_i \), it follows that \( \sigma_i \in \mathcal{C}_{i,j}(\mathcal{A}) \).

If \( x = j \), it follows from \( \sigma(j) = j \) that \( \sigma_j = \sigma \in \mathcal{C}_{i,j}(\mathcal{A}) \).

(b) \( \sigma = \sigma_{i,j} \).

Since \( \mathcal{A} \) is fixed, the permutation \( \sigma_x \) is in \( \mathcal{A} \) for all \( x \in [n] \). If \( \mathcal{C}_{i,j}(\sigma_x) = \sigma_x \) (as is the case if \( x = i \) or if \( \sigma(x) = x \)), we are done, so assume that \( \mathcal{C}_{i,j}(\sigma_x) \neq \sigma_x \), i.e. that \( (\sigma_x)_{i,j} \neq \sigma_x \) and \( (\sigma_x)_{i,j} \notin \mathcal{A} \). From this it follows that \( \sigma_x(j) = j \) and \( \sigma_x(i) \neq i \) which in turn implies that \( \sigma(i) \neq i \), since if \( \sigma(i) = i \), then \( \sigma_x(i) = i \). If \( \sigma(j) = j \) with \( \sigma(i) \neq i \), then \( \sigma_{i,j} \neq \sigma \). Therefore \( \sigma(j) \neq j \).

There are two situations in which it is possible to have both \( \sigma_x(j) = j \) and
σ( j) ≠ j. The first is when σ( j) = x and σ( x) = j with j ≠ x and the second is when x = j. In both cases we show that (σ x)i, j ∉ A contradicts the maximality of A.

If σ( j) = x and σ( x) = j with x ≠ j, the permutations σ x, σ, σ i, (σ x)i ∈ A will act on the points x, i, j, σ x⁻¹(i) ∈ [n] as follows:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
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<th>σ⁻¹(i)</th>
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<tr>
<td>σ x</td>
<td>x</td>
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<tr>
<td>σ</td>
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<td>σ i</td>
<td>j</td>
<td>i</td>
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<td>a</td>
</tr>
<tr>
<td>(σ x)i</td>
<td>x</td>
<td>i</td>
<td>j</td>
<td>a</td>
</tr>
</tbody>
</table>

Note that σ x(y) = σ(y) = σ i(y) = (σ x)i(y) for all y ∈ [n] \ {x, i, j, σ x⁻¹(i)}.

Since it is not possible to intersect all four permutations at points in {x, i, j, σ x⁻¹(i)}, all other permutations in A must agree with σ x on at least t elements of [n] \ {x, i, j, σ x⁻¹(i)}. However, (σ x)i, j agrees with σ x on all elements of [n] \ {x, i, j, σ x⁻¹(i)}. Therefore, (σ x)i, j will t-intersect with all permutations in A contradicting the maximality of A.

If x = j and either σ( j) = i or σ⁻¹(j) = i, then (σ j)i, j = σ i ∈ A follows from the definitions of the x-fixing and the (i, j)-compression. Hence we assume that σ( j) ≠ i and σ⁻¹(j) ≠ i. Then σ j, σ, σ i, (σ j)i ∈ A will act on i, j, σ j⁻¹(i), σ⁻¹(j) ∈ [n] as follows:
Using an argument analogous to that used for the case when $\sigma(j) = x$ and $\sigma(x) = j$, it can be shown that $(\sigma_x)_{i,j}$ will $t$-intersect with all permutations in $A$ thus contradicting the maximality of $A$. □

**Theorem 5.2.6.** Let $A$ be a fixed $t$-intersecting family of permutations. If $C_{i,j}(A)$ is a fixed trivially $t$-intersecting family of permutations, then $A$ is trivially $t$-intersecting.

*Proof.* The identity permutation is an element of $C_{i,j}(A)$ since it is a fixed family of permutations. Therefore there is a set $I \subseteq [n]$ such that $|I| = t$ and $I \subseteq \text{fix}(\sigma)$ for all $\sigma \in C_{i,j}(A)$.

If $j \in I$, then $\sigma(j) = j$ for all $\sigma \in C_{i,j}(A)$. Then $A = C_{i,j}(A)$ and the theorem follows immediately so we assume that $j \notin I$. We consider the cases when $i \notin I$ and when $i \in I$ separately.

1. Case 1: $j \notin I$ and $i \notin I$.

For all $\pi \in A$, either $\pi = C_{i,j}(\pi)$ and $\text{fix}(\pi) = \text{fix}(C_{i,j}(\pi))$, or $\pi \neq C_{i,j}(\pi)$ and
\( \text{fix}(\pi) = (\text{fix}(C_{i,j}(\pi)) \setminus \{i\}) \cup \{j\} \). Since \( i \notin I \), it follows that \( I \subseteq \text{fix}(\pi) \) for all \( \pi \in \mathcal{A} \). Since \( |\mathcal{A}| = |C_{i,j}(\mathcal{A})| \) by Theorem 5.2.1, it follows that \( \mathcal{A} \) is a trivially \( t \)-intersecting family.

2. Case 2: \( j \notin I \) and \( i \in I \).

We first consider the case when \( n - t \geq 5 \) in order to make use of Theorem 3.2.1.

Partition \( C_{i,j}(\mathcal{A}) \) according to the image of \( j \) under each permutation. There will be one partition for each of the \( n - t \) elements in \([n] \setminus I\) and each partition will contain \( (n - t - 1)! \) permutations. Let \([n] \setminus I = \{j, x_1, x_2, \ldots, x_m\}\) where \( m = n - t - 1 \) and denote the partition containing all the permutations that map \( j \) to \( x_\ell \) as \( X_{x_\ell} \) for \( \ell \in [m] \). Let \( X_j \) be the partition containing all permutations that fix \( j \). From the definition of the compression operation, it is clear that \( \sigma \in \mathcal{A} \) for all \( \sigma \in X_j \).

Next, we show that any two permutations within a partition that agree on exactly \( t + 1 \) elements of \([n]\) either were both changed by the \((i,j)\)-compression operation or were both unchanged in which case they are both in \( \mathcal{A} \). Without loss of generality, we consider two permutations, \( \sigma \) and \( \pi \), from \( X_{x_1} \). Suppose that they agree on exactly \( t + 1 \) points. Then \( \sigma(y) = \pi(y) \) for all \( y \in I \cup \{j\} \).

Let \( \sigma' \neq \sigma \) and \( \pi' \neq \pi \) be permutations such that \( \sigma'_{i,j} = \sigma \) and \( \pi'_{i,j} = \pi \). The actions of the permutations \( \sigma, \pi, \sigma' \) and \( \pi' \) on the points \( i, j, \sigma^{-1}(j) \) and \( \pi^{-1}(j) \) are shown below.
We can see from this that $\sigma$ and $\pi'$ will agree on only $t - 1$ elements of $[n]$. The same is true for $\pi$ and $\sigma'$. Since $A$ is $t$-intersecting, either $\sigma, \pi \in A$ or $\sigma', \pi', \in A$.

We now consider two permutations from two different partitions (excluding $X_j$) and show that if they agree on exactly $t$ elements of $[n]$ either both were changed or both are in $A$. We exclude $X_j$ since every permutation in $X_j$ has both $i$ and $j$ fixed and thus will $t$-intersect with any permutation in $A$ regardless of whether or not it is changed by the compression operation. Assume without loss of generality that $\rho \in X_{x_1}$ and $\tau \in X_{x_2}$ and suppose that $\rho$ and $\tau$ agree on exactly $t$ elements, specifically, on the $t$ elements in $I$. Note that for all $n - t \geq 2$, such a pair will exist for any two partitions. Let $\rho' \neq \rho$ and $\tau' \neq \tau$ be permutations such that $\rho'_{i,j} = \rho$ and $\tau'_{i,j} = \tau$. The actions of the permutations $\rho, \tau, \rho'$ and $\tau'$ on the points $i, j, \rho^{-1}(j)$ and $\tau^{-1}(j)$ are shown below.
Again, this illustrates that either $\rho, \tau \in \mathcal{A}$ or $\rho', \tau' \in \mathcal{A}$.

We now use these two results to show that either all the permutations in $C_{i,j}(\mathcal{A}) \setminus X_j$ are in $\mathcal{A}$ or all were changed by the $(i,j)$-compression operation. For each partition $X_x$, construct a graph, $G_x$, whose vertices are the permutations in the partition and where two vertices are adjacent if and only if the permutations agree on exactly $t+1$ points. These graphs are isomorphic to the derangement graph on $n-t-1$ points, $\Gamma_{(n-t-1)}$, and so by Theorem 3.2.1 they will be Hamilton-connected provided that $n-t-1 \geq 4$. This means that any two vertices in a given graph, $G_x$, can be connected by a path that visits each vertex in the graph exactly once. Since adjacent vertices agree on exactly $t+1$ points, either all the permutations in the partition are in $\mathcal{A}$ or all were changed by the compression operation.

To extend this result to all the permutations in $C_{i,j}(\mathcal{A}) \setminus X_j$, note that for any given a permutation in $X_x$, there will be a permutation in $X_{x_{t+1}}$ such that the two agree on exactly $t$ points. Using the fact that any two such permutations
must either be in \( A \) or must have been changed, we can easily show that the result applies to all permutations in \( C_{i,j}(A) \setminus X_j \).

It now remains to be shown that \( A \) is trivially \( t \)-intersecting. This is obvious if all of the permutations in \( C_{i,j}(A) \setminus X_j \) are in \( A \) since then \( A = C_{i,j}(A) \). If all of the permutations in \( C_{i,j}(A) \setminus X_j \) were changed by the compression operation, it follows that \( j \) is fixed for all permutations in \( A \). Then \( (I \setminus \{i\}) \cup \{j\} \subset \text{fix}(\sigma) \) for all \( \sigma \in A \). Since \( |A| = |C_{i,j}(A)| \) by Theorem 5.2.1, it follows that \( A \) is a trivially \( t \)-intersecting family.

Finally, we consider the case when \( n - t < 5 \). For \( n - t = 1 \) and \( n - t = 2 \), the result is trivial. For \( n - t = 3 \), the theorem can be verified by direct examination of the six permutations in the family. For \( n - t = 4 \), it is possible to list all the permutations that do not fix \( j \) in such a manner that each permutation exactly \( t \)-intersects with the permutations immediately preceding and following it in the list. An example of such an ordering is given in Table 5.1. The reasoning used for \( n - t \geq 5 \) can then be applied to show that \( A \) is trivially \( t \)-intersecting. \( \square \)
Table 5.1: List of permutations from a trivially $t$-intersecting family of size $n-t=4$
ordered so that consecutive permutations exactly $t$-intersect

<table>
<thead>
<tr>
<th>$j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
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<td>$x_3$</td>
<td>$x_2$</td>
<td>$x_1$</td>
<td>$j$</td>
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</table>

(Note that the permutations which fix $j$ are not included in the list and the elements whose images are the same under all permutations (i.e. the elements of $I$) are not shown.)
Chapter 6

An Optimal Bound for Fixed Families

In this chapter we show that if \( n \geq 2t + 1 \), a fixed \( t \)-intersecting family of permutations will have size at most \((n - t)!\) and that only trivially \( t \)-intersecting families achieve this bound. We also show that \( n \geq 2t + 1 \) is the optimal lower bound on \( n \).

Our proof is modelled on the method of Ahlswede and Khachatrian [1]. We introduce the concept of generating sets which is used in the proof of the Complete Erdős-Ko-Rado theorem for set systems and then define a similar concept for permutations. After proving some lemmas concerning this and related concepts, we proceed to the proof of our main result.
6.1 Generating sets

The concept of generating sets for set systems was introduced by Ahlswede and Khachatrian [1] and is an integral component of their proof of the complete Erdős-Ko-Rado theorem. This concept allows many $k$-set systems to be defined by a smaller number of sets using the following operation.

Let $B$ be a subset of $[n]$. The upset of $B$, denoted $\mathcal{U}(B)$, is defined as follows:

$$\mathcal{U}(B) = \{B' \in 2^{[n]} : B \subseteq B'\}.$$  

For a set system, $\mathcal{B} \subseteq 2^{[n]}$, the upset is defined as

$$\mathcal{U}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{U}(B).$$

Let $\mathcal{B} \subseteq \binom{[n]}{k}$ be a $k$-set system from $[n]$. A collection of sets, $g(\mathcal{B})$, is called a generating set of $\mathcal{B}$ if $|B| \leq k$ for all $B \in g(\mathcal{B})$ and

$$\mathcal{U}(g(\mathcal{B})) \cap \binom{[n]}{k} = \mathcal{B}.$$  

The set of all generating sets of $\mathcal{B}$ is denoted by $G(\mathcal{B})$. Note that $G(\mathcal{B}) \neq \emptyset$ since $\mathcal{B} \in G(\mathcal{B})$. The trivially intersecting $k$-set system, $\mathcal{F}_0$, is an example of a set system that can be defined by a smaller number of sets. All the sets in $\mathcal{F}_0$ will contain the element 1, and $\{\{1\}\}$ will be a generating set for $\mathcal{F}_0$.

Some additional definitions are needed in order to describe the specific type of generating set required for the proof of the main theorem.
For \( B = \{b_1, b_2, \ldots, b_k\} \in \binom{[n]}{k} \), where \( b_1 < b_2 < \cdots < b_k \), the collection of all sets that can be obtained from \( B \) by left-shifting is denoted by \( \mathcal{L}(B) \) and is formally defined as follows:

\[
\mathcal{L}(B) = \{ A = \{a_1, a_2, \ldots, a_k\} \in \binom{[n]}{k} : a_i \leq b_i \text{ for all } i \in [k] \}.
\]

For a set system, \( \mathcal{B} \subseteq 2^{[n]} \),

\[
\mathcal{L}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{L}(B).
\]

If \( \mathcal{L}(\mathcal{B}) = \mathcal{B} \), then \( \mathcal{B} \) is said to be left-compressed. Unlike the left-shifting operation for collections of subsets described in Chapter 2, the number of sets in \( \mathcal{L}(\mathcal{B}) \) may be greater than the number in \( \mathcal{B} \). However, a collection of sets will satisfy \( \mathcal{L}(\mathcal{B}) = \mathcal{B} \) if and only if \( S_{i,j}(\mathcal{B}) = \mathcal{B} \) for all \( i < j \leq n \), that is, when \( \mathcal{B} \) is a stable (or left-shifted) set system as defined for the left-shifting operation.

For a generating set \( g(\mathcal{B}) \in G(\mathcal{B}) \), consider \( \mathcal{L}(g(\mathcal{B})) \). Now define \( \mathcal{L}_*(g(\mathcal{B})) \) as the set of minimal (in the sense of set inclusion) elements of \( \mathcal{L}(g(\mathcal{B})) \). Let \( G_*(\mathcal{B}) \) be the set of all generating sets of \( \mathcal{B} \) such that \( \mathcal{L}_*(g(\mathcal{B})) = g(\mathcal{B}) \). Note that if \( \mathcal{B} \) is left-compressed, then \( G_*(\mathcal{B}) \neq \emptyset \) since \( \mathcal{B} \in G_*(\mathcal{B}) \).

An obvious adaptation of the generating set concept for permutations would be to define a generating set as a collection of sets of mappings which define a bijection from a subset of \([n]\) to another subset of \([n]\). The analogue of the upset for such a set of mappings would be the set of all permutations on \([n]\) that include all of the
mappings in the set. With this definition, any family of permutations would have a generating set since it would be a generating set for itself. However, other concepts such as left-shifting that are required for the proof of our main result become difficult to define.

Instead, we will begin with subsets of \([n]\) and define the up-permutation of a set, \(B \subseteq [n]\), as

\[
\mathcal{U}_p(B) = \{ \sigma \in S_n : B \subseteq \text{fix}(\sigma) \}.
\]

For example, if \(B = \{1, 2\}\) and \(n = 4\), then \(\mathcal{U}_p(B) = \{\langle 1\ 2\ 3\ 4 \rangle, \langle 1\ 2\ 4\ 3 \rangle\}\).

For a collection of sets, \(\mathcal{B}\), we define the up-permutation to be

\[
\mathcal{U}_p(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{U}_p(B).
\]

Finally, a collection of sets, \(g(\mathcal{S})\), is a generating set for \(\mathcal{S} \subseteq S_n\), if

\[
\mathcal{U}_p(g(\mathcal{S})) = \mathcal{S}.
\]

Since these generating sets are subsets of \([n]\), the definitions of \(\mathcal{L}(g(\mathcal{B}))\), \(\mathcal{L}_*(g(\mathcal{B}))\) and \(G_*(\mathcal{B})\) as written for generating sets of \(k\)-set system, \(\mathcal{B}\), will apply directly to generating sets for families of permutations.

One drawback of this definition of generating sets is that not all \(t\)-intersecting families of permutations will have a generating set. However, the following lemma will show that any maximal fixed \(t\)-intersecting family will have a generating set.
Lemma 6.1.1. Let $\mathcal{A}$ be a maximal fixed $t$-intersecting family of permutations. Then $\text{fix}(\mathcal{A})$ is a generating set for $\mathcal{A}$.

Proof. Since $\mathcal{A}$ is $t$-intersecting and fixed, by Lemma 4.1.1 the set system $\text{fix}(\mathcal{A})$ will be $t$-intersecting. Then $\mathcal{U}_p(\text{fix}(\mathcal{A}))$ will be a $t$-intersecting family of permutations since every pair of permutations will agree on at least $t$ fixed points.

For all $\sigma \in \mathcal{A}$, by definition $\text{fix}(\sigma) \in \text{fix}(\mathcal{A})$ and $\sigma \in \mathcal{U}_p(\text{fix}(\mathcal{A}))$. Therefore,

$$\mathcal{A} \subseteq \mathcal{U}_p(\text{fix}(\mathcal{A})).$$

Since $\mathcal{U}_p(\text{fix}(\mathcal{A}))$ is $t$-intersecting and $\mathcal{A}$ is maximal, it follows that $\mathcal{A} = \mathcal{U}_p(\text{fix}(\mathcal{A}))$ and therefore that $\text{fix}(\mathcal{A})$ is a generating set for $\mathcal{A}$. $\square$

We will now show that any generating set of a $t$-intersecting family of permutations from $S_n$ will be a $t$-intersecting set system provided that $n > t + 1$. This restriction on $n$ is necessary since if $n = t$ or $n = t + 1$, the identity permutation will be the only element in a maximal fixed $t$-intersecting family. Since the up-permutation of any set of size $n - 1$ will be the identity permutation, it is possible to have a generating set containing two sets, $A \neq B$, such that $|A| = |B| = n - 1$ and $|A \cap B| = n - 2$.

Lemma 6.1.2. Let $\mathcal{A}$ be a $t$-intersecting family of permutations from $S_n$ with $n > t + 1$. Then any $g(\mathcal{A}) \in G(\mathcal{A})$ will be a $t$-intersecting set system.

Proof. Assume that there exists some $g(\mathcal{A}) \in G(\mathcal{A})$ such that $g(\mathcal{A})$ is not a $t$-intersecting set system. Then there will be at least two sets, $A, B \in g(\mathcal{A})$ such
that $|A \cap B| < t$.

First, we claim that either $|A| < n - 1$, or $|B| < n - 1$. If this is not the case, then $|A \cap B| \geq n - 2$. Since $A$ and $B$ are not $t$-intersecting, this implies that $t \geq n - 1$ which contradicts the requirement that $n > t + 1$. Thus at least one of $A$ or $B$ will have cardinality less than $n - 1$.

From this it follows that there will be permutations in $\mathcal{U}_p(A)$ and $\mathcal{U}_p(B)$ that agree only on the elements in $A \cap B$ and hence are not $t$-intersecting. This is a contradiction since all permutations in $\mathcal{U}_p(A)$ and $\mathcal{U}_p(B)$ are in $A$ and $A$ is $t$-intersecting. \qed

For a set $S \subseteq [n]$, let $s^+(S)$ denote the largest element in $S$. For a collection of sets, $S \subseteq 2^{[n]}$, let $s^+(S) = \max\{s^+(S) : S \in S\}$.

Finally, for a maximal fixed $t$-intersecting family of permutations, $\mathcal{A}$, let

$$s_{\min}(G(\mathcal{A})) = \min\{s^+(g(\mathcal{A})) : g(\mathcal{A}) \in G(\mathcal{A})\}.$$ 

**Lemma 6.1.3.** Let $\mathcal{A}$ be a maximal fixed $t$-intersecting family of permutations from $S_n$ with $n \geq t + 1$. Then

$$s_{\min}(G(\mathcal{A})) \geq t.$$ 

Furthermore, if $s_{\min}(G(\mathcal{A})) = t$, then $\mathcal{A}$ is a trivially $t$-intersecting family of permutations.

**Proof.** Suppose that $s_{\min}(G(\mathcal{A})) < t$. Then there is some $g(\mathcal{A}) \in G(\mathcal{A})$ such that $s^+(g(\mathcal{A})) < t$. Let $A \in g(\mathcal{A})$. Then $|A| < t$ and $\mathcal{U}_p(A)$ is not $t$-intersecting when
n ≥ t + 1. Since \( \mathcal{U}_p(A) \subseteq \mathcal{U}_p(g(A)) = A \), this contradicts the \( t \)-intersection of \( A \).

Now suppose that \( s_{\min}(G(A)) = t \). Then there exists some \( g(A) \in G(A) \) such that \( s^+(g(A)) = t \). For all \( A \in g(A) \), it follows that \( |A| = t \) and hence \( g(A) = \{[t]\} \). Therefore \( \mathcal{U}_p(g(A)) = A \) is trivially \( t \)-intersecting. \( \square \)

Before proceeding to the main theorem we require some additional lemmas concerning generating sets. These lemmas are modelled on lemmas in [1].

**Lemma 6.1.4.** Let \( A \) be a compressed family of permutations from \( S_n \) and let \( g(A) \in G(A) \) be any generating set of \( A \). Then:

1. \( \mathcal{L}_s(g(A)) \in G(A) \);

2. \( s^+(\mathcal{L}_s(g(A))) \leq s^+(g(A)) \).

**Proof.** The second statement follows easily from the definition of \( \mathcal{L}_s(g(A)) \).

In order to prove the first statement, we need to show that \( \mathcal{U}_p(\mathcal{L}_s(g(A))) = A \) for any \( g(A) \in G(A) \).

Let \( g(A) \in G(A) \). Since \( g(A) \subseteq \mathcal{L}(g(A)) \), it follows that \( \mathcal{U}_p(g(A)) = A \subseteq \mathcal{U}_p(\mathcal{L}(g(A))) \). That \( \mathcal{U}_p(\mathcal{L}(g(A))) = \mathcal{U}_p(\mathcal{L}_s(g(A))) \) follows from the definitions of \( \mathcal{L}_s(g(A)) \) and the up-permutation operation. Therefore,

\[
A \subseteq \mathcal{U}_p(\mathcal{L}_s(g(A))) .
\]
To complete the proof, we need to show that $\mathcal{U}_p(L_*(g(A))) \subseteq A$. We will do this by showing that any permutation in $\mathcal{U}_p(L_*(g(A)))$ is an element of $A$.

Let $\sigma \in \mathcal{U}_p(L_*(g(A)))$. Then $\sigma \in \mathcal{U}_p(L(g(A)))$ and there will be some set $B \in L(g(A))$ such that $B \subseteq \text{fix}(\sigma)$. If $B \in g(A)$, then $\sigma \in A$ and we are done, so assume $B \notin g(A)$. Then there will be some $B' \in g(A)$ such that $B \in L(B')$. Note that $|B| = |B'|$.

Let $B = \{x_1, x_2, \ldots, x_\ell\}$ and let $B' = \{y_1, y_2, \ldots, y_\ell\}$. Then $x_i \leq y_i$ for all $i \in \ell$ and, since $B \notin g(A)$, there must be at least one $i \in \ell$ such that $x_i \neq y_i$.

Let $\text{fix}(\sigma) = \{x_1, \ldots, x_\ell, z_1, \ldots, z_m\}$ and choose a permutation, $\pi$, such that $\text{fix}(\pi) = B' \cup Z$, where $Z = \{z_1, \ldots, z_m\}$. Since $B' \in g(A)$, it follows that $\pi \in A$. Note that $B'$ and $Z$ are not necessarily disjoint sets. If $x_1 = y_1$, then $\text{fix}(\pi) = \{x_1, y_2, \ldots, y_\ell\} \cup Z$. If $x_1 \neq y_1$, then $\pi_{x_1, y_1} \in A$ since $A$ is left-compressed. Then $\text{fix}(\pi_{x_1, y_1}) = \{x_1, y_2, \ldots, y_\ell\} \cup \{z \in Z : z \neq y_1\}$. Repeating this in order of increasing $i$ for all pairs $(x_i, y_i)$ gives a permutation $\pi' \in A$ such that $\text{fix}(\pi') = \{x_1, x_2, \ldots, x_\ell\} \cup \{z \in Z : z \neq y_i \text{ for any } i \in \ell\}$. It follows that $\text{fix}(\pi') \subseteq \text{fix}(\sigma)$. Since $\pi' \in A$, there exists some set $B'' \in g(A)$ such that $\pi' \in \mathcal{U}_p(B'')$. It then follows that $B'' \subseteq \text{fix}(\pi') \subseteq \text{fix}(\sigma)$ and that $\sigma \in \mathcal{U}_p(B'')$. Hence $\sigma \in A$. □
Lemma 6.1.5. Let $\mathcal{A}$ be a family of permutations from $S_n$ and let $g(\mathcal{A})$ be a generating set in $G_*(\mathcal{A})$. For a set $E \in g(\mathcal{A})$, define

$$\mathcal{D}(E) = \{ \sigma \in S_n : \text{fix}(\sigma) \cap [s^+(E)] = E \}.$$ 

Then $\mathcal{A}$ is a disjoint union

$$\mathcal{A} = \bigcup_{E \in g(\mathcal{A})} \mathcal{D}(E).$$

Proof. In order to prove that $\mathcal{A}$ is a disjoint union of $\mathcal{D}(E)$'s, we will begin by showing that every permutation in $\mathcal{A}$ is contained in $\mathcal{D}(E)$ for at least one $E \in g(\mathcal{A})$ and then show that $\mathcal{D}(E_1) \cap \mathcal{D}(E_2) = \emptyset$ for any $E_1 \neq E_2 \in g(\mathcal{A})$.

Let $\sigma$ be any permutation in $\mathcal{A}$. Then there will be some $E \in g(\mathcal{A})$ such that $\sigma \in \mathcal{U}_p(E)$. If $\text{fix}(\sigma) \cap [s^+(E)] = E$, then $\sigma \in \mathcal{D}(E)$ and we are done, so assume that $\text{fix}(\sigma) \cap [s^+(E)] \neq E$. Let $\text{fix}(\sigma) \cap [s^+(E)] = \{b_1, b_2, \ldots, b_\ell\}$ where $b_1 < b_2 < \cdots < b_\ell$, and let $|E| = k$. Since $g(\mathcal{A}) \in G_*(\mathcal{A})$, there will be some $E' \in g(\mathcal{A})$ such that $E' = \{b_1, b_2, \ldots, b_m\}$ for some $m \leq k < \ell$. Then $\sigma \in \mathcal{D}(E')$.

Suppose that $\mathcal{D}(E_1) \cap \mathcal{D}(E_2) \neq \emptyset$ for some $E_1 \neq E_2 \in g(\mathcal{A})$. Then there will be some $\sigma \in \mathcal{A}$ such that $\sigma \in \mathcal{D}(E_1) \cap \mathcal{D}(E_2)$. Thus

$$\text{fix}(\sigma) \cap [s^+(E_1)] = E_1 \quad \text{and} \quad \text{fix}(\sigma) \cap [s^+(E_2)] = E_2.$$ 

If $s^+(E_1) = s^+(E_2)$, it follows that $E_1 = E_2$ so we will assume without loss of generality that $s^+(E_1) > s^+(E_2)$. Then $E_2 \subsetneq E_1$ and hence $g(\mathcal{A})$ is not minimal with respect to set inclusion which is a contradiction of $g(\mathcal{A}) \in G_*(\mathcal{A})$. \qed
Lemma 6.1.6. Let $\mathcal{A}$ be a family of permutations from $S_n$ and let $g(\mathcal{A})$ be a generating set in $G_*(\mathcal{A})$. Choose a set $\hat{E} \in g(\mathcal{A})$ such that $s^+(\hat{E}) = s^+(g(\mathcal{A}))$. Then $\mathcal{D}(\hat{E})$, as defined in Lemma 6.1.5, is the set of all permutations in $\mathcal{A}$ which are generated by $\hat{E}$ alone. That is,

$$\mathcal{D}(\hat{E}) = \mathcal{U}_p(\hat{E}) \setminus \mathcal{U}_p \left( g(\mathcal{A})\{\hat{E}\} \right).$$

Further,

$$|\mathcal{D}(\hat{E})| = \sum_{j=0}^{s^+(\hat{E}) - |\hat{E}|} (-1)^j \binom{s^+(\hat{E}) - |\hat{E}|}{j} (n - |\hat{E}| - j)!.$$  

Proof. By definition $\mathcal{D}(E) \subseteq \mathcal{U}_p(E)$ for any $E \in g(\mathcal{A})$. Thus since $\mathcal{A}$ is a disjoint union of the $\mathcal{D}(E)$'s (Lemma 6.1.5), it follows that any permutation in $\mathcal{A}$ generated only by $\hat{E}$ will be in $\mathcal{D}(\hat{E})$. Since $g(\mathcal{A})$ is minimal with respect to set inclusion, any $E \in g(\mathcal{A})\{\hat{E}\}$ will contain some $x \in [n]$ such that $x \notin \hat{E}$ and $x < s^+(\hat{E})$. Hence $\mathcal{D}(\hat{E})$ will not contain any permutations generated by $g(\mathcal{A})\{\hat{E}\}$.

The formula for the cardinality of $\mathcal{D}(\hat{E})$ is derived by using the principle of inclusion-exclusion to count the number of permutations in $S_n$ that have all the elements of $\hat{E}$ fixed and none of the elements of $[s^+(\hat{E})]\setminus \hat{E}$ fixed. \hfill $\square$

Lemma 6.1.7. Let $\mathcal{A}$ be a family of permutations from $S_n$ and let $g(\mathcal{A})$ be a generating set in $G_*(\mathcal{A})$. Let $\hat{E}$ be a set in $g(\mathcal{A})$ such that $s^+(\hat{E}) = s^+(g(\mathcal{A}))$ and let $\hat{E}' = \hat{E}\{s^+(\hat{E})\}$. Define

$$\mathcal{D}'(\hat{E}) = \left\{ \sigma \in S_n : \text{fix}(\sigma) \cap [s^+(\hat{E})-1] = \hat{E}' \right\}.$$
Then $\mathcal{D}'(\hat{E})$ will be the set of all permutations which are generated by $\hat{E}'$ and not by $g(A) \setminus \{\hat{E}\}$. Furthermore,

$$|\mathcal{D}'(\hat{E})| \geq (n - |\hat{E}| + 1) |\mathcal{D}(\hat{E})|.$$  

Proof. From the definitions of $\mathcal{D}(\hat{E})$ and $\mathcal{D}'(\hat{E})$, it is easy to see that

$$\mathcal{D}'(\hat{E}) = \mathcal{D}(\hat{E}) \cup \left\{ \sigma \in S_n : \text{fix}(\sigma) \cap [s^+(\hat{E})] = \hat{E}' \right\}.$$ 

In other words, we can partition $\mathcal{D}'(\hat{E})$ based on whether or not $s^+(\hat{E})$ is fixed.

We know from Lemma 6.1.6 that $\mathcal{D}(\hat{E})$ does not contain any permutations generated by $g(A) \setminus \{\hat{E}\}$. To show that $\{\sigma \in S_n : \text{fix}(\sigma) \cap [s^+(\hat{E})] = \hat{E}'\}$ does not contain any permutations generated by $g(A) \setminus \{\hat{E}\}$, recall from the proof of Lemma 6.1.6 that any $E \in g(A) \setminus \{\hat{E}\}$ will contain some $x \in [n]$ such that $x \notin \hat{E}$ and $x < s^+(\hat{E})$. Clearly $x \notin \hat{E}$ implies that $x \notin \hat{E}'$ and thus $\{\sigma \in S_n : \text{fix}(\sigma) \cap [s^+(\hat{E})] = \hat{E}'\}$ does not contain any permutations generated by $g(A) \setminus \{\hat{E}\}$.

It remains to be shown that all of the permutations generated by $\hat{E}'$ and not by $g(A) \setminus \{\hat{E}\}$ are in $\mathcal{D}'(\hat{E})$. Suppose there is a permutation, $\sigma$, such that $\sigma \notin \mathcal{D}'(\hat{E})$ and $\sigma \in \mathcal{U}_p(\hat{E}') \setminus \mathcal{U}_p(g(A) \setminus \{\hat{E}\})$. Then $(\text{fix}(\sigma) \cap [s^+(\hat{E}) - 1]) \neq \hat{E}'$ and $\hat{E}' \subseteq \text{fix}(\sigma)$. Let $\text{fix}(\sigma) \cap [s^+(\hat{E}) - 1] = \{b_1, b_2, \ldots, b_\ell\}$ where $b_1 < b_2 < \cdots < b_\ell$ and let $|\hat{E}| = k$. Then $k \leq \ell$ since $\text{fix}(\sigma)$ must contain some $x \in [s^+(\hat{E}) - 1]$ such that $x \notin \hat{E}'$. Since $g(A) \in G_s(A)$, there is some $A \subseteq \{b_1, b_2, \ldots, b_\ell\} \subseteq \text{fix}(\sigma)$ such that $A \in g(A)$. This implies that $\sigma \in \mathcal{U}_p(A)$ which contradicts the assumption that $\sigma \in \mathcal{U}_p(\hat{E}') \setminus \mathcal{U}_p(g(A) \setminus \{\hat{E}\})$.
Let $\sigma$ be any permutation in $\mathcal{D}(\hat{E})$. Then $\sigma \in \mathcal{D}'(\hat{E})$. For any $x \in [n] \setminus \hat{E}$, the permutation formed by transposing $\sigma(x)$ and $s^+(\hat{E})$ will be an element of $\mathcal{D}'(\hat{E}) \setminus \mathcal{D}(\hat{E})$. Therefore,$$
mid \mathcal{D}'(\hat{E}) \nmid \geq \nmid \mathcal{D}(\hat{E}) \nmid + (n - \nmid \hat{E} \nmid) \nmid \mathcal{D}(\hat{E}) \nmid = \left( n - \nmid \hat{E} \nmid + 1 \right) \nmid \mathcal{D}(\hat{E}) \nmid. \quad \Box$$

**Corollary 6.1.8.** Let $\mathcal{A}$ be a family of permutations from $S_n$ and let $g(\mathcal{A})$ be a generating set in $G_\ast(\mathcal{A})$. Let $\hat{E}$ be a set in $g(\mathcal{A})$ such that $s^+(\hat{E}) = s^+(g(\mathcal{A}))$ and let $\hat{E}' = \hat{E} \setminus \{s^+(\hat{E})\}$ and define $\mathcal{D}'(\hat{E})$ as in Lemma 6.1.7. If $s^+(\hat{E}) - \nmid \hat{E} \nmid \geq 1$, then$$
mid \mathcal{D}'(\hat{E}) \nmid > \left( n - \nmid \hat{E} \nmid + 1 \right) \nmid \mathcal{D}(\hat{E}) \nmid. \tag{6.1.1}$$

**Proof.** We will show that the inequality is strict when $s^+(\hat{E}) - \nmid \hat{E} \nmid \geq 1$ by identifying a permutation in $\mathcal{D}'(\hat{E})$ that is not formed by transposing two elements of a permutation from $\mathcal{D}(\hat{E})$ and so is not counted in Lemma 6.1.7.

If $s^+(\hat{E}) - \nmid \hat{E} \nmid \geq 1$, there will be some $y \in [s^+(\hat{E})]$ such that $y \notin \hat{E}$. Let $\pi$ be a permutation from $S_n$ such that $\text{fix}(\pi) = \hat{E} \cup \{y\}$. Then $\pi \notin \mathcal{D}(\hat{E})$. Now consider the permutation $\tilde{\pi} = (y, s^+(\hat{E})) \circ \pi$. Then $\tilde{\pi}(y) = s^+(\hat{E})$ and $\tilde{\pi}(s^+(\hat{E})) = y$. Thus fix($\tilde{\pi}$) = $\hat{E}'$ and so $\tilde{\pi} \in \mathcal{D}'(\hat{E})$ but $\tilde{\pi} \notin \mathcal{D}(\hat{E})$. Equation 6.1.1 then follows. \quad \Box

**Lemma 6.1.9.** Let $\mathcal{A}$ be a compressed fixed $t$-intersecting family of permutations from $S_n$ with $n > t + 1$. Let $g(\mathcal{A})$ be a generating set in $G_\ast(\mathcal{A})$ and let $E_1$ and $E_2$ be sets in $g(\mathcal{A})$. If there exist some $i < j \in [n]$ such that $i \notin E_1 \cup E_2$ and $j \in E_1 \cap E_2$,
then
\[ |E_1 \cap E_2| \geq t + 1. \]

**Proof.** If there exist \( i, j \in [n] \) as described above, then either the set \( S_{i,j}(E_1) = (E_1 \setminus \{j\}) \cup \{i\} \) or a subset of \( S_{i,j}(E_1) \) will be in \( g(A) \in G_*(A) \). Call this set \( A \). If \( i \notin A \), then \( A \) will be a subset of \( E_1 \) which is a contradiction since \( g(A) \) is minimal by inclusion. Therefore, it follows that \( i \in A \). By Lemma 6.1.2 we have that \( g(A) \) is a \( t \)-intersecting set system and so \( A \) and \( E_2 \) must \( t \)-intersect. Since \( j \in E_1 \cap E_2 \) but \( j \notin A \) and \( i \notin E_2 \),
\[ |E_1 \cap E_2| \geq t + 1. \]
\[ \square \]

### 6.2 Proof of main result

**Lemma 6.2.1.** Let \( \mathcal{A} \) be a compressed fixed \( t \)-intersecting family of permutations from \( S_n \) of maximum possible size. If \( n \geq 2t + 1 \), then
\[ s_{\min}(G(\mathcal{A})) \leq t. \]

**Proof.** If \( n \geq 2t + 1 \), then \( n > t + 1 \) for all \( t \in \mathbb{N} \).

By Lemma 6.1.1, we have that \( G(\mathcal{A}) \neq \emptyset \) and since \( \mathcal{A} \) is compressed, it follows from Lemma 6.1.4 that \( s_{\min}(G(\mathcal{A})) = s^+(g(\mathcal{A})) \) for some \( g(\mathcal{A}) \in G_*(\mathcal{A}) \). We will use
this $g(A)$ to show that if $s^+(g(A)) > t$, then $A$ does not have the maximum possible size.

Assume that $s^+(g(A)) = t + \delta$ for some positive integer $\delta$. Obviously it is necessary that $t + \delta \leq n$. Now partition $g(A)$ into two disjoint collections of sets,

$$g_0(A) = \{ B \in g(A) : s^+(B) = t + \delta \},$$

and

$$g_1(A) = g(A) \setminus g_0(A).$$

In other words, all sets that contain $t + \delta$ will be in $g_0(A)$ and all sets that do not contain $t + \delta$ will be in $g_1(A)$. For every $B_0 \in g_0(A)$ and $B_1 \in g_1(A)$, it is clear that $|(B_0 \setminus \{t + \delta\}) \cap B_1| \geq t$.

Next, we partition the sets in $g_0(A)$ according to their cardinality. Let

$$\mathcal{R}_i = \{ B \in g_0(A) : |B| = i \}.$$

Note that $\mathcal{R}_i = \emptyset$ when $i \leq t$. If $i < t$, then $\mathcal{U}_p(\mathcal{R}_i)$ will not be $t$-intersecting and so the family generated by $g(A)$ will not be $t$-intersecting. If $i = t$, then $[t] \in \mathcal{L}(B)$ for any $B \in \mathcal{R}_t$. Then $\mathcal{U}_p(g(A))$ will not be $t$-intersecting since $B$ and $[t]$ are not $t$-intersecting. Also, $\mathcal{R}_i = \emptyset$ when $i \geq t + \delta$. If $i = t + \delta$, then $g(A) = \{[t + \delta]\}$ and $A$ is clearly not a $t$-intersecting family of maximum size, and $i > t + \delta$ is not possible since $t + \delta$ is the largest element in the set. Hence,

$$g_0(A) = \bigcup_{t < i < t + \delta} \mathcal{R}_i.$$
If $R_n \neq \emptyset$ or $R_{n-1} \neq \emptyset$, the only permutation in $\mathcal{A}$ will be the identity permutation since $g(\mathcal{A})$ is minimal by inclusion. Thus $\mathcal{A}$ would not be maximal for $n > t + 1$. Therefore, if $R_i \neq \emptyset$, then $i \leq n - 2$.

We now consider the set system formed by removing $t + \delta$ from the sets in $g_0(\mathcal{A})$. Let

$$R'_i = \{E \setminus \{t + \delta\} : E \in R_i\}.$$ 

Clearly $|R_i| = |R'_i|$ and $|E'| = i - 1$ for $E' = E \setminus \{t + \delta\} \in R'_i$.

For any $E_1, E_2 \in g_0(\mathcal{A})$ with $|E_1 \cap E_2| = t$, it follows from Lemma 6.1.9 that $i \in E_1 \cup E_2$ for all $i < t + \delta$. A simple counting argument then gives us

$$|E_1| + |E_2| = |E_1 \cup E_2| + |E_1 \cap E_2| = 2t + \delta.$$ 

Therefore, if $|E_1| + |E_2| \neq 2t + \delta$, then $|E_1 \cap E_2| \neq t$. Since $E_1$ and $E_2$ are $t$-intersecting, this means that $|E_1 \cap E_2| \geq t + 1$. It then follows that for any $E'_1 \in R'_i$ and $E'_2 \in R'_j$ with $i + j \neq 2t + \delta$,

$$|E'_1 \cap E'_2| \geq t.$$ 

We claim that for $i, j \in \mathbb{N}$ such that $i + j = 2t + \delta$, if $R_i \neq \emptyset$, then $R_j \neq \emptyset$.

To prove this claim, suppose that there is a set $E_1$ in $R_i$ such that $|E_1 \cap E| \geq t + 1$ for all $E \in g_0(\mathcal{A})$. Then $(E_1 \setminus \{t + \delta\}) \cap E \geq t$ for all $E \in g_0(\mathcal{A})$. Since $E_1 \setminus \{t + \delta\}$ also $t$-intersects with the sets in $g_1(\mathcal{A})$ and $\mathcal{A}$ is maximal, we must have $E_1 \setminus \{t + \delta\} \in g(\mathcal{A})$. 

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But this implies that $E_1 \notin g(A)$ since $g(A)$ is minimal by inclusion, a contradiction.

Therefore, for any $E_1 \in R_i$, there must exist at least one other set, $E_2 \in g_0(A)$, such that $|E_1 \cap E_2| = t$. It then follows that $|E_1| + |E_2| = 2t + \delta$. Thus, if $R_i \neq \emptyset$, then $R_j \neq \emptyset$ where $j = 2t + \delta - i$.

We have already seen that $R_i = \emptyset$ when $i \leq t$ or $i \geq t + \delta$. We now show that if $A$ is as large as possible and $n \geq 2t + 1$, then $R_i = \emptyset$ for all $i$ and hence $s^+(g(A)) = t$.

We will do this by assuming that $R_i \neq \emptyset$ for some $i \in \{t + 1, \ldots, t + \delta - 1\}$ and then constructing a family of permutations that is larger than $A$. We consider the cases where $i \neq 2t + \delta - i$ and where $i = 2t + \delta - i$ separately.

Assume $R_i \neq \emptyset$ for some $i \in \{t + 1, \ldots, t + \delta - 1\}$. Then $R_{2t+\delta-i} \neq \emptyset$.

1. Case 1: $i \neq 2t + \delta - i$.

Consider the sets

$$f_1 = (g(A) \setminus (R_i \cup R_{2t+\delta-i})) \cup R'_i$$

and

$$f_2 = (g(A) \setminus (R_i \cup R_{2t+\delta-i})) \cup R'_{2t+\delta-i}.$$

We first show that $f_1$ is $t$-intersecting. Clearly, $g(A) \setminus (R_i \cup R_{2t+\delta-i})$ is a $t$-intersecting set system since by Lemma 6.1.2, $g(A)$ is $t$-intersecting. Let $E'_i$ be a set in $R'_i$. Then $E_1 = E'_i \cup \{t + \delta\}$ is a set in $R_i$. As shown previously, $|E_1 \cap E_j| \geq t + 1$ for all $E_j \in R_j$ where $j \neq 2t + \delta - i$. Hence, $E'_i$ will $t$-intersect
with all sets in \( g_1(A), g_0(A) \setminus \mathcal{R}_{2t+\delta-i} \) and \( \mathcal{R}'_i \). Thus \( f_1 \) is a \( t \)-intersecting set system.

A similar argument can be used to show that \( f_2 \) is \( t \)-intersecting.

Let \( \mathcal{B}_1 = \mathcal{U}_p(f_1) \) and let \( \mathcal{B}_2 = \mathcal{U}_p(f_2) \). Then \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are \( t \)-intersecting families of permutations. We claim that

\[
\max_{i=1,2} |\mathcal{B}_i| > |A|.
\]

For any set \( E \in \mathcal{R}_i \subseteq g_0(A) \), consider \( \mathcal{D}(E) \) as defined in Lemma 6.1.5. Recall that \( \mathcal{D}(E_1) \cap \mathcal{D}(E_2) = \emptyset \) for all \( E_1, E_2 \in g(A) \). By Lemma 6.1.6 we have that \( \mathcal{D}(E) \) is the set of all permutations generated only by \( E \). Therefore, the permutations generated by \( \mathcal{R}_i \) and not by \( g(A) \setminus \mathcal{R}_i \) will be given by

\[
\mathcal{D}(\mathcal{R}_i) = \bigcup_{E \in \mathcal{R}_i} \mathcal{D}(E).
\]

Again by Lemma 6.1.6,

\[
|\mathcal{D}(E)| = \sum_{j=0}^{s^+(E) - |E|} (-1)^j \binom{s^+(E) - |E|}{j} (n - |E| - j)!.
\]

Since \( |E| = i \) and \( s^+(E) = t + \delta \) for all \( E \in \mathcal{R}_i \),

\[
|\mathcal{D}(\mathcal{R}_i)| = |\mathcal{R}_i| \cdot |\mathcal{D}(E)| = |\mathcal{R}_i| \sum_{j=0}^{t+\delta-i} (-1)^j \binom{t+\delta-i}{j} (n - i - j)!.
\]

Similarly,

\[
\mathcal{D}(\mathcal{R}_{t+\delta-i}) = \bigcup_{E \in \mathcal{R}_{t+\delta-i}} \mathcal{D}(E).
\]
is the set of permutations generated by \( D(R_{t+\delta-i}) \) and not by \( g(A) \setminus D(R_{t+\delta-i}) \).

It follows from Lemma 6.1.7 that
\[
D'(R_i) = \bigcup_{E \in \mathcal{R}_i} D'(E)
\]
is the set of all permutations generated by \( R'_i \) and not by \( g(A) \setminus \mathcal{R}_i \). Since 
\( i < t+\delta \), we can use Equation 6.1.1 from Corollary 6.1.8 to show that
\[
|B_1| = |A| - (|D(R_i)| + |D(R_{2t+\delta-i})|) + |D'(R_i)|
\]
\[
> |A| - |D(R_{2t+\delta-i})| + (n-i) \cdot |D(R_i)|
\]
and
\[
|B_2| = |A| - (|D(R_i)| + |D(R_{2t+\delta-i})|) + |D'(R_{2t+\delta-i})|
\]
\[
> |A| - |D(R_i)| + (n-2t-\delta + i) \cdot |D(R_{2t+\delta-i})|.
\]

We now prove by contradiction that either \( B_1 \) or \( B_2 \) is larger than \( A \).

If \( |A| \geq |B_1| \), then
\[
|D(R_{2t+\delta-i})| > (n-i)|D(R_i)|
\]
and if \( |A| \geq |B_2| \), then
\[
|D(R_i)| > (n-2t-\delta + i)|D(R_{2t+\delta-i})|.
\]

Combining these equations we have
\[
\frac{|D(R_{2t+\delta-i})|}{n-i} > |D(R_i)| > (n-2t-\delta + i) \cdot |D(R_{2t+\delta-i})|.
\]
Recall that \( i \leq n - 2 \) if \( R_i \neq \emptyset \), so \( n - i \neq 0 \).

Thus

\[
\left| \mathcal{D}(R_{2t+\delta-i}) \right| > (n-2t-\delta+i)(n-i) \cdot \left| \mathcal{D}(R_{2t+\delta-i}) \right|
\]

and

\[
1 > (n-2t-\delta+i)(n-i).
\]

But \( n - i \geq 2 \) and \( n - (2t + \delta - i) \geq 2 \) if \( R_i \) and \( R_{2t+\delta-i} \) are not empty sets. Therefore at least one of \( B_1 \) or \( B_2 \) will be larger than \( A \).

2. Case 2: \( i = 2t + \delta - i \).

In this case, \( i = t + \frac{\delta}{2} \) and hence \( \delta \) must be divisible by 2. Consider \( R'_{t+\delta/2} \). For any \( B' \in R'_{t+\delta/2} \), note that \( |B'| = t + \frac{\delta}{2} - 1 \) and \( B' \subset [t+\delta-1] \).

First we claim that there exists some \( a \in [t+\delta-1] \) and some collection of subsets, \( \mathcal{T}' \subseteq R'_{t+\delta/2} \), such that for all \( B' \in \mathcal{T}' \), we have that \( a \notin B' \) and

\[
|\mathcal{T}'| \geq \frac{|R'_{t+\delta/2}| \cdot \frac{\delta/2}{t+\delta-1}}{2(t+\delta-1)} = \frac{|R_{t+\delta/2}| \cdot \frac{\delta}{2(t+\delta-1)}}{2(t+\delta-1)}.
\] (6.2.1)

To prove this claim we will consider the complements of the sets in \( \mathcal{T}' \) in \([t+\delta-1]\). For all \( B'_j \in R'_{t+\delta/2} \), let \( C'_j = [t+\delta-1] \setminus B'_j \). Then \( |C'_j| = \frac{\delta}{2} \) and there will be \( |R'_{t+\delta/2}| \) distinct sets of size \( \frac{\delta}{2} \) with entries from \([t+\delta-1]\). By the pigeonhole principle, there exists some \( a \in [t+\delta-1] \) such that \( a \) is in at least \( \frac{|R'_{t+\delta/2}|}{t+\delta-1} \cdot \frac{\delta}{2} \) of the \( C'_j \)'s. The claim then follows easily.
For any \( B_1', B_2' \in \mathcal{T}' \), Lemma 6.1.9 gives

\[
|B_1' \cap B_2'| \geq t.
\]

From Case 1 we have that \( \mathcal{R}_i = \emptyset \) for \( i \neq t + \frac{\delta}{2} \). Hence

\[
f' = (g(A) \setminus \mathcal{R}_{t+\delta/2}) \cup \mathcal{T}'
\]
is a \( t \)-intersecting set system and therefore \( \mathcal{U}_p(f') \) will be a \( t \)-intersecting family of permutations.

We now show that \( |\mathcal{U}_p(f')| > |A| \) when \( n \geq 2t + 1 \).

Let

\[
\mathcal{D}_1 = \mathcal{U}_p(g(A) \setminus \mathcal{R}_{t+\delta/2}),
\]

\[
\mathcal{D}_2 = \mathcal{U}_p(\mathcal{R}_{t+\delta/2}) \setminus \mathcal{D}_1,
\]

\[
\mathcal{D}_3 = \mathcal{U}_p(\mathcal{T}') \setminus \mathcal{D}_1.
\]

Then \( A = \mathcal{D}_1 \cup \mathcal{D}_2 \) and \( \mathcal{U}_p(f') = \mathcal{D}_1 \cup \mathcal{D}_3 \). Hence, to show that \( |\mathcal{U}_p(f')| > |A| \),

it is sufficient to show that \( |\mathcal{D}_3| > |\mathcal{D}_2| \).

From Lemma 6.1.6 we have

\[
|\mathcal{D}_2| = |\mathcal{R}_{t+\delta/2}| \sum_{j=0}^{\delta/2} (-1)^{j} \binom{\delta/2}{j} (n-(t+\delta/2)-j)! (6.2.2)
\]

Now we estimate the size of \( \mathcal{D}_3 \). Let \( B \in \mathcal{R}_{t+\delta/2} \). Recall that \( \mathcal{D}(B) \) is the set of all permutations generated by \( B \in \mathcal{R}_{t+\delta/2} \) and not by \( g(A) \setminus B \). As
shown in Lemma 6.1.7, for each permutation in $\mathcal{D}(B)$ there will be at least $n - i + 1 = n - (t + \delta/2) + 1$ permutations generated by $B' = B \setminus \{t + \delta\}$ and not by $g(A) \setminus B$. From Lemma 6.1.6 we have

$$\mathcal{D}(B) = \sum_{j=0}^{(t+\delta)-(t+\delta/2)} (-1)^j \binom{(t+\delta)-(t+\delta/2)}{j} (n-(t+\delta/2)-j)!$$

As in Case 1, we use Equation 6.1.1 from Corollary 6.1.8, to show that

$$|\mathcal{D}_3| > |\mathcal{F}'| (n-(t+\delta/2) + 1) \sum_{j=0}^{\delta/2} (-1)^j \binom{\delta/2}{j} (n-(t+\delta/2)-j)!.$$ 

Using Equation 6.2.1 gives

$$|\mathcal{D}_3| > |\mathcal{R}_{t+\delta}| \cdot \frac{\delta}{2(t+\delta-1)} (n-(t+\delta/2) + 1) \sum_{j=0}^{\delta/2} (-1)^j \binom{\delta/2}{j} (n-(t+\delta/2)-j)!.$$ 

Combining Equations 6.2.3 and 6.2.2, we have

$$|\mathcal{D}_3| > \frac{\delta}{2(t+\delta-1)} (n-t-\delta/2 + 1) \cdot |\mathcal{D}_2|.$$ 

Hence it is sufficient to show that

$$\frac{\delta}{2(t+\delta-1)} (n-t-\delta/2 + 1) \geq 1$$

or equivalently that

$$n \geq \frac{\delta^2 + 2\delta + 2t\delta + 4t - 4}{2\delta}.$$ 

Since $n \geq 2t + 1$, Equation 6.2.4 will hold if

$$2t + 1 \geq \frac{\delta^2 + 2\delta + 2t\delta + 4t - 4}{2\delta}.$$
This is true if $t > \delta/2$.

Now suppose $t \leq \delta/2$. Since $n \geq t + \delta$, we have

$$2t \leq \delta \leq n - t.$$  

This also implies that $n \geq 3t$.

Equation 6.2.4 can be rewritten as

$$0 \geq \delta^2 + \delta(2 + 2t - 2n) + 4t - 4.$$  

Since $\delta^2 + \delta(2 + 2t - 2n) + 4t - 4$ is a quadratic in $\delta$, if the inequality holds for $\delta = 2t$ and $\delta = n - t$, it will hold for all possible values of $\delta$.

Let $\delta = 2t$. Then Equation 6.2.4 becomes

$$0 \geq 8t^2 + 8t - 4tn - 4$$

or equivalently

$$tn \geq 2t^2 + 2t - 1.$$  

Since $n \geq 3t$, it is sufficient to show that

$$t^2 \geq 2t - 1$$

which is clearly true for all $t \in \mathbb{N}$.

Now let $\delta = n - t$. Then Equation 6.2.4 becomes

$$0 \geq n(-n + 2t + 2) + (2t - t^2 - 4).$$  

(6.2.5)
We consider the cases when \( t \geq 2 \) and when \( t = 1 \) separately.

If \( t \geq 2 \), then \( n(-n + 2t + 2) \leq 0 \) provided that \( 2t + 2 \leq n \). Since \( n \geq 3t \), this holds for all \( t \geq 2 \). Thus Equation 6.2.5 holds for all \( t \geq 2 \) since \( (2t - t^2 - 4) \) is clearly negative for all \( t \in \mathbb{N} \).

Now we consider the case when \( t = 1 \). Then Equation 6.2.5 becomes

\[
0 \geq -(n^2 - 4n + 3) = -(n - 3)(n - 1).
\]

This inequality holds for all possible values of \( n \) since \( n \geq 3t \) implies that \( n \geq 3 \).

Thus Equation 6.2.4 holds for all \( t \in \mathbb{N} \). Hence \( |\mathcal{D}_3| > |\mathcal{D}_2| \) and therefore \( |\mathcal{U}_p(f')| > |\mathcal{A}| \) when \( n \geq 2t + 1 \).

In both cases we have shown that if \( s_{\text{min}}(G(\mathcal{A})) > t \), then \( \mathcal{A} \) does not have the maximum possible size. \( \square \)

We now proceed to our main theorem.

**Theorem 6.2.2.** Let \( \mathcal{A} \) be a fixed \( t \)-intersecting family of permutations from \( S_n \). If \( n \geq 2t + 1 \), then

\[
|\mathcal{A}| \leq (n - t)!.
\]

Further, if \( |\mathcal{A}| = (n - t)! \), then \( \mathcal{A} \) is trivially \( t \)-intersecting.

**Proof.** Combining Lemmas 6.2.1 and 6.1.3 we have that if \( n \geq 2t + 1 \) and \( \mathcal{A} \) is a compressed fixed \( t \)-intersecting family of permutations from \( S_n \) of maximum size, then
\( s_{\min}(G(A)) = t \) which implies that \( A \) is trivially \( t \)-intersecting. Thus, for compressed fixed \( t \)-intersecting families of permutations, the maximum size is \((n - t)!\) and this size is only attained by trivially \( t \)-intersecting families.

Now suppose that \( A \) is a fixed \( t \)-intersecting family of permutations from \( S_n \). By Theorems 5.2.2, 5.2.5, 5.2.3 and 5.2.1, repeated applications of the compression operation will give a compressed fixed \( t \)-intersecting family of permutations of the same size as \( A \). If the resulting compressed fixed \( t \)-intersecting family is trivially \( t \)-intersecting, then \( A \) is trivially \( t \)-intersecting by Lemma 5.2.6. Hence the above result for compressed fixed \( t \)-intersecting families will apply to all fixed \( t \)-intersecting families. \( \Box \)

### 6.3 Significance of Theorem 6.2.2

Unlike the result of Ellis, Friedgut and Pilpel (Theorem 1.0.3) which applies to all \( t \)-intersecting families of permutations from \( S_n \) provided that \( n \) is sufficiently large relative to \( t \), our proof of the same result is restricted to fixed \( t \)-intersecting families. However, we have proved a specific lower bound on \( n \) and claim that it is the best possible. In order to prove this claim, we will look at families of permutations that are analogous to the set systems, \( F_i \) as defined in Section 2.5.
Recall that for set systems,
\[ F_i = \left\{ A \in \binom{[n]}{k} : |A \cap [t+2i]| \geq t+i \right\} \]
for \( t, k, n, i \in \mathbb{N} \) and \( i \leq k-t/2 \). These set systems consist of all of the \( k \)-sets that contain at least \( t+i \) elements from the set \([t+2i]\). We will now define similar families of permutations where each family consists of all the permutations that fix at least \( t+i \) of the elements in \([t+2i]\). For \( n, t, i \in \mathbb{N} \) and \( i \leq n-t/2 \), let
\[ A_i = \{ \sigma \in S_n : |\text{fix}(\sigma) \cap [t+2i]| \geq t+i \} . \]
As defined, these families are fixed and thus are covered by Theorem 6.2.2.

For \( A_0 \) and \( A_1 \), straightforward counting arguments give
\[ |A_0| = (n-t)! , \]
and
\[ |A_1| = (n-t-2)! \left( 1 + (t+2)(n-t-2) \right) . \]

We will now show that \( |A_1| \geq |A_0| \) for all values of \( n \) such that \( t+3 \leq n \leq 2t \).
(The requirement that \( t+3 \leq n \) is necessary since for \( n \leq t+1 \), \( A_1 \) is not defined, and for \( n = t+2 \), any permutation with \( t+1 \) elements of \([t+2]\) fixed will have to have all \( t+2 \) elements fixed.)

Let \( n = 2t + \ell \). Substituting \( 2t + \ell \) for \( n \) in Equations 6.3.1 and 6.3.2, we see that \( |A_1| \geq |A_0| \) if and only if
\[ 1 + (t+2)(t + \ell - 2) \geq (t + \ell)(t + \ell - 1) . \]
This is equivalent to the requirement that

\[(1 - \ell)(t + \ell - 2) \geq 1. \tag{6.3.3}\]

Since \(n \geq t + 3\), it follows that \(t + \ell \geq 3\). Then it is easy to see that Equation 6.3.3 does not hold if \(\ell \geq 1\), that is, if \(n \geq 2t + 1\). It does hold for all \(\ell < 1\). In fact, for all \(\ell < 0\) and for \(\ell = 0\) when \(t > 3\), we have a strict inequality.

Thus, when \(n \leq 2t\) and \(t > 3\),

\[|A_1| > |A_0|\]

which proves that \(n \geq 2t + 1\) is the best possible bound.
Chapter 7

Conclusion

In this thesis we have adapted techniques used in the proof of the Complete Erdős-Ko-Rado Theorem [1] for use with $t$-intersecting families of permutations. Using a newly defined compression operation on families of permutations that are closed under the $x$-fixing operation of Cameron and Ku [2], we have proved a number of results analogous to existing results for left-shifted set systems. The main result is the following theorem.

**Theorem 7.0.1.** Let $\mathcal{A}$ be a fixed $t$-intersecting family of permutations from $S_n$. If $n \geq 2t + 1$, then

$$|\mathcal{A}| \leq (n - t)!.$$  

Further, if $|\mathcal{A}| = (n - t)!$, then $\mathcal{A}$ is trivially $t$-intersecting.

The bound on $n$ given in this theorem is a new result and we have shown that it is
optimal. Our theorem is limited to fixed families due to problems with the application of the $x$-fixing operation to arbitrary families of permutations. However, the families of permutations, $\mathcal{A}_i$, defined in Section 6.3 are fixed so these families and equivalent families are covered by the theorem. Based on similarities with the results for set systems, it seems reasonable to conjecture that the maximum size of a $t$-intersecting family of permutations for all values of $n$ and $t$ is achieved by one of these families. Proving this conjecture would also prove the following conjecture.

**Conjecture 7.0.2.** Let $\mathcal{A}$ be a $t$-intersecting family of permutations from $S_n$. If $n \geq 2t + 1$, then

$$|\mathcal{A}| \leq (n - t)!.$$ 

Further, if $|\mathcal{A}| = (n - t)!$, then $\mathcal{A}$ is trivially $t$-intersecting.

Another approach to extending our theorem to all families of permutations would be to further modify the $x$-fixing operation in order to maintain the size of the family. This approach would have the advantage of proving that only trivially $t$-intersecting families of permutations are of the maximum size when $n \geq 2t + 1$.

### 7.1 Directions for future research

The question of the size and structure of $t$-intersecting families of permutations of maximum size when $n$ is small relative to $t$ remains open. For set systems, the
question of the size and structure of maximum sized $t$-intersecting $k$-set systems for all values of $n$, $k$ and $t$ was answered by Ahlswede and Khachatrian in [1]. Since we have been able to adapt part of their proof for use with permutations, further adaption of their proof seems to be a promising approach. However, since it requires families of permutations that are fixed, any result obtained would be limited to fixed families. Table A.1 shows a comparison of the sizes of $A_0$, $A_1$, $A_2$, $A_3$ and $A_4$ for various values of $t$ and $n < 2t$.

Other questions that have arisen during preparation of this thesis include:

1. Can the $x$-fixing operation be modified so that the size of any maximal $t$-intersecting family of subsets will not change on application of the $x$-fixing operation? As we saw in Section 4.3, only families that contain two permutations, $\sigma$ and $\pi$, such that $\sigma_x = \pi_x$ and that do not contain $\sigma_x$ or the permutations formed by transposing $\sigma(x)$ and $\pi(x)$ in $\sigma$ and $\pi$ remain a problem. We conjecture that these families are not the largest possible families. This is known to be true for small values of $n$ but remains to be proved for all $n$.

2. Is it possible to define some new operation on families of permutations that will result in a family of permutations with the properties attained using the $x$-fixing and compression operations but without the problems associated with the $x$-fixing operation?
3. Is there some optimal order for performing the \((i, j)\)-compression operations on a family of permutations and, if so, can we determine the number of operations that will be sufficient to give a compressed family in all cases? This has been done for set systems, but the situation is more complicated for permutations since a third element, \(\sigma^{-1}(i)\), is affected by the \((i, j)\)-compression of a permutation \(\sigma\).

4. Do the families of permutations that contain \(t + i\) of \(t + 2i\) given mappings (i.e., families equivalent to \(A_i\) for some \(i \in \mathbb{N}\)) attain the maximum possible size for all values of \(n\) and \(t\)? If so, are they the only families that attain the maximum size? For set systems, this question was answered by the Complete Erdős-Ko-Rado Theorem. Could there be some other approach to this question that takes advantage of the group structure of \(S_n\)?
Bibliography


Appendix A

Largest $A_i$ for $n \leq 2t$

Examples of formulas for the size of $A_i$:

1. $|A_0| = (n - t)!$

2. $|A_1| = (n-t-2)! \left[ 1 + (t+2)(n-t-2) \right]$

3. $|A_2| = (n-t-4)! \left[ 1 + (t+4)(n-t-4) + \frac{(t+4)(t+3)}{2} ((n-t-3) + (n-t-4)^2) \right]$

Table A.1 shows which of the $A_i$ with $i \in \{0, \ldots, 4\}$ is the largest for various values of $t$ and $n \leq 2t$. Each row gives the results for the value of $t$ shown on the left. The columns represent values of $n$ relative to $t$ with the value decreasing from left to right. For instance, the first column is $n = 2t$, the second is $n = 2t - 1$ and so on. The numbers in the body of the table are the value of $i$ for the largest $A_i$. The blank area in the upper right corner is values of $n$ and $t$ where $t > n$. Calculations were done using Maple V.
Table A.1: Largest $A_i$ for various values of $t$ and $n = 2t-x$

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