THE FUNDAMENTAL MODULES OF THE CLASSICAL LIE ALGEBRAS

A Thesis

Submitted to the Faculty of Graduate Studies and Research

In Partial Fulfillment of the Requirements

for the Degree of

Master of Science

In

Mathematics

University of Regina

By

Gustavo Sergio Krimker Fernández

Regina, Saskatchewan

February 2012

UNIVERSITY OF REGINA

FACULTY OF GRADUATE STUDIES AND RESEARCH

SUPERVISORY AND EXAMINING COMMITTEE

Gustavo Sergio Krimker Fernandez, candidate for the degree of Master of Science in Mathematics**,** has presented a thesis titled, ***The Fundamental Modules of the Clasical Lie Algebras,*** in an oral examination held on February 17, 2012. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

External Examiner:	*Dr. Kaiming Zhao, Sir Wilfrid Laurier University
Supervisor:	Dr. Fernando Szechtman, Department of Mathematics & Statistics
Committee Member:	Dr. Allen Herman, Department of Mathematics & Statistics
Committee Member:	*Dr. Andrei Voldin, Department of Mathematics & Statistics
Committee Member:	Dr. Bruce Gilligan, Department of Mathematics & Statistics
Chair of Defense:	Dr. Miguel Sanchez, Faculty of Social Work

*Not present at defense

Abstract

The main objective of this Thesis is the construction of the fundamental modules of the classical Lie algebras. Weyl's Theorem shows that if L is a semisimple Lie algebra, then any finite dimensional $L-$module is a direct sum of irreducible $L-$modules. Since the classical algebras are semisimple, we just need the irreducible modules in order to obtain the others. On the other hand, the fundamental modules give us every irreducible $L-$ module and, therefore, every finite dimensional $L-$module.

Acknowledgments

First of all I thank my advisor Fernando Szechtman for introducing and teaching this subject to me. I am grateful to him for his time, support and guidance. I also wish to thank my other teachers in Regina, Doug Farenick, Allen Herman and Don Stanley, I thoroughly enjoyed their lectures. I would like to thank Bruce Gilligan and my colleagues Bahman Ahmadi, Ruhallah Ahmadi (Ruhi) and Yong Liu for many fruitful discussions, related and not related to mathematics. My thanks are due to Martin Argerami, Ara Steininger and Pamela Sparvier, they helped me whenever I needed. I would especially like to thank Natalio Guersenzvaig for encouraging me to continue my studies. I am also indebted to the Faculty of Graduate Studies and Research at the University of Regina in providing funding to finance my studies of mathematics. At last, I thank the School of Economics of the University of Buenos Aires for awarding a leave to perform this program.

Dedication

This thesis is dedicated to my wife María Inés.

Contents

Abstract **i**

Acknowledgments **ii**

Dedication **iii**

Table of Contents **iv**

1 Introduction **1**

2 Lie algebras **4**

2.1 Basic Concepts 4

2.2 Representations and Modules 8

2.3 Multilinear Algebra 11

2.4 The Lie algebra associated to a bilinear form 13

2.5 The Classical Lie algebras 15

2.6 The irreducible $\mathfrak{sl}(2)$-modules 17

2.7 Solvable Lie algebras 19

2.8 The Killing form 20

3 Semisimple Lie algebras **21**

3.1 Semisimple Lie algebras 21

3.2 Abstract root systems 22

3.3 The root space decomposition 24

3.4 Modules for semisimple Lie algebras 30

4 The fundamental modules of A_ℓ **37**

5 First fundamental modules of B_ℓ and D_ℓ **43**

5.1 Root space decomposition of B_ℓ 43

5.2 Root space decomposition of D_ℓ 48

5.3 Exterior powers of the natural module of D_ℓ 50

5.4 Exterior powers of the natural module of B_ℓ 52

6 The spin modules of B_ℓ and D_ℓ **55**

6.1 Clifford algebras 55

6.2 The Lie algebra of $C(V)$ 57

6.3 The spin module for B_ℓ 63

6.4 The spin modules for D_ℓ 67

7 The fundamental modules of C_ℓ **69**

7.1 The fundamental modules of C_ℓ 71

8 Final Remarks **76**

Chapter 1

Introduction

The main objective of this Thesis is the construction of the fundamental modules of the classical Lie algebras. These modules for any simple Lie algebra were first constructed by Cartan [Car13].

Let L be a finite dimensional semisimple Lie algebra. By Weyl's Theorem any finite dimensional $L-$module is a direct sum of irreducible $L-$modules. Since the classical algebras are semisimple, we just need the irreducible modules in order to obtain the others. On the other hand, from the fundamental modules we can construct every irreducible $L-$ module and, therefore, every finite dimensional $L-$module. Indeed, let $\lambda_1, \ldots, \lambda_\ell$ be the fundamental weights of L, with fundamental modules $V(\lambda_1), \ldots, V(\lambda_\ell)$ and maximal vectors $v_1, \ldots, v_\ell$. Let V be an arbitrary irreducible L-module. Then V has a unique highest weight λ. Since λ is integral and dominant, we have $\lambda = \sum_{1\leq i\leq \ell} m_i\lambda_i$, for unique non-negative integers $m_1, \ldots, m_\ell$. Consider the

L-module M which is the tensor product of m_1 copies of $V(\lambda_1)$, m_2 copies of $V(\lambda_2)$, $\ldots$, m_ℓ copies of $V(\lambda_\ell)$, i.e.

$$M = \underbrace{V(\lambda_1) \otimes \cdots \otimes V(\lambda_1)}_{m_1} \otimes \cdots \otimes \underbrace{V(\lambda_\ell) \otimes \cdots \otimes V(\lambda_\ell)}_{m_\ell}.$$

Then the submodule N of M generated by

$$v = \underbrace{v_1 \otimes \cdots \otimes v_1}_{m_1} \otimes \cdots \otimes \underbrace{v_\ell \otimes \cdots \otimes v_\ell}_{m_\ell}$$

is an irreducible L-module with highest weight λ, so $N \cong V$. Thus, the fundamental modules give us every irreducible finite dimensional $L-$module.

In this Thesis we present an explicit and basic construction of the fundamental modules of the classical Lie algebras. The exposition is reasonably self-contained. In fact, Chapter 2 reviews the required elementary concepts from Lie algebras and multilinear algebra, while Chapter 3 explores the structure of semisimple Lie algebras and their modules. This chapter gathers the most important tools from representation theory needed in later chapters.

The construction of the fundamental modules for the special linear algebra A_ℓ is given in Chapter 4. This involves the exterior powers of its natural module.

Exterior powers of the natural modules of the orthogonal algebras, D_ℓ and B_ℓ, also give the first fundamental modules of these algebras, as developed in Chapter 5. Here we also show that the last fundamental module of B_ℓ and the two last fundamental modules of D_ℓ cannot be obtained as exterior powers of their natural modules. These remaining modules, known as the spin modules, require another approach using the Clifford algebra instead of the exterior algebra.

Chapter 6 gives an overview of the Clifford algebra and the construction of the spin modules.

We end by constructing, in Chapter 7, the fundamental modules of the symplectic algebra C_ℓ.

Chapter 2

Lie algebras

Throughout this thesis all vector spaces are assumed to be over the complex numbers unless specifically stated otherwise.

Within the scope of this preliminary chapter we let V stand for an arbitrary vector space and refer to [Hum97] for all unproven statements, unless otherwise noted.

2.1 Basic Concepts

Definition 2.1.1. *A* ***Lie algebra*** *is a vector space L together with a multiplication $[\ ,\ ] : L \times L \to L$, called* ***bracket****, satisfying the following axioms:*

(L1) The bracket is bilinear,

(L2) $[xx] = 0$ *for all* $x \in L$,

(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$, *for all* $x, y, z \in L$.

Axiom *(L3)* is called the ***Jacobi identity***. Axiom *(L2)* can be replaced by

(L2)' $[xy] = -[yx]$ for all $x, y \in L$.

Example 2.1.2. *To any associative algebra A there corresponds a Lie algebra obtained by defining the bracket $[xy] = xy - yx$ on A. The Lie algebra corresponding to $End(V)$ is the **general linear algebra**, denoted by $\mathfrak{gl}(V)$. The Lie algebra corresponding to the full matrix algebra M_n is also known as the general linear algebra and denoted by $\mathfrak{gl}(n)$.*

For computational purposes it is useful to note that the basic matrices e_{ij} form a basis of $\mathfrak{gl}(n)$ and satisfy:

$$[e_{ij}e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}. \tag{2.1}$$

Example 2.1.3. *Any vector space can be seen as a Lie algebra with a trivial bracket. Such Lie algebra is said to be **abelian**.*

Definition 2.1.4. *Let A be an algebra, that is, a vector space with a bilinear multiplication $A \times A \to A$. A **derivation** of A is a linear map $\delta : A \to A$ satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$.*

For instance, if L is a Lie algebra and $x \in L$ then the map $ad\, x : L \to L$, given by $y \mapsto [xy]$, is a derivation of L.

Definition 2.1.5. *Let L_1 and L_2 be two Lie algebras. A map $\phi : L_1 \to L_2$ is called* *a **Lie homomorphism** if ϕ is linear and preserves the brackets, i.e.*

$$\phi([xy]) = [\phi(x)\phi(y)], \quad x, y \in L_1.$$

*A **Lie isomorphism** is a bijective Lie homomorphism.*

Example 2.1.6. *Suppose V has finite dimension n and let B be a basis of V. Then the map $M_B : \mathfrak{gl}(V) \to \mathfrak{gl}(n)$ that assigns to every endomorphism of V its matrix relative to B is a Lie isomorphism.*

Definition 2.1.7. *A subspace K of a Lie algebra L is called a **subalgebra** of L if $[xy] \in K$ for all $x, y \in K$.*

Example 2.1.8. *Let $\mathfrak{t}(n)$ be the subspace of $\mathfrak{gl}(n)$ of all upper triangular matrices. Then $\mathfrak{t}(n)$ is a subalgebra of $\mathfrak{gl}(n)$.*

Example 2.1.9. *Let A be an algebra and let $\mathrm{Der}(A)$ stand for the space of all derivations of A. Then $\mathrm{Der}(A)$ is a subalgebra of $\mathfrak{gl}(A)$.*

For instance, let $A = \mathbf{C}[X, X^{-1}]$ be the algebra of Laurent polynomials. The Lie algebra $\mathrm{Der}(A)$ is known as the Witt algebra and will be denoted by $\mathcal{W}$. A derivation of A is completely determined by its value on X. For $n \in \mathbf{Z}$ there is a unique derivation D_n satisfying $D_n(X) = X^{n+1}$, namely the derivation $m_{X^{n+1}} \circ d/dX$, where d/dX is the usual differentiation and $m_{X^{n+1}}$ is multiplication by X^{n+1}. Then $(D_n)_{n\in\mathbf{Z}}$ is a basis of $\mathcal{W}$ and

$$[D_m, D_n] = (n - m)D_{m+n}, \quad m, n \in \mathbf{Z}. \tag{2.2}$$

Definition 2.1.10. *A subspace I of a Lie algebra L is an **ideal** of L if $[xy] \in I$ for all $x \in L$ and $y \in I$.*

The kernel of a Lie homomorphism is always an ideal (the converse also holds). For instance, the trace map $\mathrm{tr} : \mathfrak{gl}(n) \to \mathbf{C}$ is a Lie homomorphism whose kernel is the ideal $\mathfrak{sl}(n)$, the **special linear algebra**, consisting of all $n \times n$ matrices having trace 0. Likewise, if $\dim V < \infty$ then $\mathfrak{sl}(V)$ is the ideal of $\mathfrak{gl}(V)$ consisting of traceless endomorphisms of V. Let $\mathfrak{n}(n)$ and $\mathfrak{d}(n)$ denote the subspaces of $\mathfrak{gl}(n)$ of all strictly upper triangular and diagonal matrices, respectively. Then the map $d : \mathfrak{t}(n) \to \mathfrak{d}(n)$ which assigns to every matrix in $\mathfrak{t}(n)$ its diagonal part is a Lie homomorphism with kernel $\mathfrak{n}(n)$. Thus $\mathfrak{n}(n)$ is an ideal of $\mathfrak{t}(n)$.

Definition 2.1.11. *A Lie algebra L is **simple** if $\dim L > 1$ and L has no ideals other than 0 and itself.*

Example 2.1.12. *The Lie algebra $\mathfrak{sl}(2)$ is simple.*

Proof. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then e, h, f form a basis of $\mathfrak{sl}(2)$ satisfying:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{2.3}$$

In particular, $ad\, h : L \to L$ is diagonalizable. A non-zero ideal I of L must have an eigenvector for $ad\, h$. Thus I must contain one of e, f, h, so $I = L$, by (2.3). □

Example 2.1.13. *The Witt algebra $\mathcal{W}$ is simple.*

Proof. This follows as above, using D_0 instead of h and (2.2) instead of (2.3). $\square$

Since the bracket is bilinear, it is easy to see that if I and J are two ideals of a Lie algebra L, then

$$I + J = \{x + y | x \in I, y \in J\}$$

is also an ideal. Similarly, $[IJ] = \{\sum [x_i y_i] | x_i \in I, y_i \in J\}$ is an ideal of L, by Jacobi's identity. For instance, $[\mathfrak{t}(n), \mathfrak{t}(n)] = \mathfrak{n}(n)$ and $[\mathfrak{gl}(n), \mathfrak{gl}(n)] = \mathfrak{sl}(n)$.

2.2 Representations and Modules

Definition 2.2.1. *Let L be a Lie algebra. A **representation** of L on V is a Lie homomorphism*

$$\phi : L \to \mathfrak{gl}(V).$$

*A **matrix representation** of L is a Lie homomorphism $\psi : L \to \mathfrak{gl}(n)$.*

A representation $\phi : L \to \mathfrak{gl}(V)$ is said to be finite dimensional if so is its underlying space V. In that case, if $n = \dim(V)$ and B is a basis of V, the matrix representation associated to ϕ is $\psi : L \overset{\phi}{\to} \mathfrak{gl}(V) \overset{M_B}{\to} \mathfrak{gl}(n)$.

Definition 2.2.2. *Let L be a Lie algebra. A **module** for L, or an L-**module**, is a vector space V together with an operation*

$$L \times V \to V, (x, v) \mapsto x \cdot v \quad (\text{or just } xv),$$

satisfying

(M1) $(ax + by)v = a(xv) + b(yv)$,

(M2) $x(av + bw) = a(xv) + b(xw)$,

(M3) $[xy]v = x(yv) - y(xv)$,

for all $x, y \in L$, $v, w \in V$ and $a, b \in \mathbf{C}$.

Given a representation $\phi : L \to \mathfrak{gl}(V)$ we can make V into an $L-$module by defining

$$xv = \phi(x)v, \quad \forall x \in L, v \in V.$$

Conversely, if V is an $L-$module, we obtain a representation of L on V by defining

$$\phi : L \to \mathfrak{gl}(V)$$

so that $\phi(x)$ is the linear map $v \mapsto xv$.

Example 2.2.3. *Let L be a Lie algebra. The adjoint map $ad : L \to \mathfrak{gl}(L)$, defined by $x \mapsto ad\, x$, is a Lie homomorphism, i.e., a representation of L on itself. This representation (resp. module) is called the* ***adjoint*** *representation (resp. module).*

For instance, let $L = \mathfrak{sl}(2)$ and $B = \{e, h, f\}$. Using (2.3) we see that the adjoint representation of L relative to B is

$$e \mapsto \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.4}$$

Example 2.2.4. *Suppose L is a Lie subalgebra of $\mathfrak{gl}(V)$ (resp. $\mathfrak{gl}(n)$). Then V (resp. the column space $\mathbf{C}^n$) is automatically an L-module. This module is called the* ***natural module*** *of L.*

Definition 2.2.5. *Let V and W be L-modules. An L-**homomorphism** is a linear map $f : V \to W$ satisfying $f(xv) = xf(v)$ for all $x \in L$ and $v \in V$.*

Definition 2.2.6. *Let V be an $L-$module. A **submodule** of V is a subspace W of V which is invariant under the action of L (i.e., $xW \subseteq W$ for all $x \in L$).*

For instance, the submodules of the adjoint module of a given Lie algebra L are just the ideals of L.

Definition 2.2.7. *A non-zero $L-$module V is said to be **irreducible** if its only submodules are 0 and V itself.*

For instance, the natural module of $\mathfrak{sl}(n)$ is easily seen to be irreducible, whereas the natural module of $\mathfrak{n}(n)$ has the following submodules and only these:

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle = \mathbf{C}^n,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbf{C}^n$ and $\langle v_1, \ldots, v_m \rangle$ stands for the subspace generated by the vectors $v_1, \ldots, v_m$.

Definition 2.2.8. *Let V be an L-module and let S be a subset of V. We will write $\langle S \rangle_L$ for the L-**submodule generated** by S, that is, the smallest L-submodule of V, relative to inclusion, containing S. We will simply write $\langle s \rangle_L$ when $S = \{s\}$.*

It is clear that $\langle S\rangle_L$ is the span of S and all vectors of V of the form $x_m \cdots x_1 s$, where $x_i \in L$ and $s \in S$.

2.3 Multilinear Algebra

In this chapter our general references are [DF04] and [FH91].

Let L be a Lie algebra and let V be an L-module. The dual space V^* can be made into an L-module as follows:

$$(xf)(v) = -f(xv), \quad x \in L, f \in V^*, v \in V.$$

We will write $T(V)$ for the tensor algebra of V. Then $T(V)$ becomes an L-module via

$$x(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = xv_1 \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes xv_2 \otimes \cdots \otimes v_n + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes xv_n. \tag{2.5}$$

Indeed, given $x \in L$, there certainly exists a linear map $\phi(x) : T(V) \to T(V)$ given by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto xv_1 \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes xv_2 \otimes \cdots \otimes v_n + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes xv_n.$$

By definition $\phi(x)$ is derivation of $T(V)$. In order to compare the derivations $\phi([xy])$ and $[\phi(x), \phi(y)]$ it suffices to do that on the generating set V of $T(V)$. Obviously they agree on V, as this *is* an L-module. Since the map $x \mapsto \phi(x)$ is linear, (2.5) does make $T(V)$-into an L-module.

More generally, we can use (2.5) to turn $V_1 \otimes \cdots \otimes V_n$ into an L-module, provided so are $V_1, \ldots, V_n$. This follows by viewing $V_1 \otimes \cdots \otimes V_n$ as subspace of the $T(V)$, for $V = V_1 \oplus \cdots \oplus V_n$, in which case (2.5) shows that $V_1 \otimes \cdots \otimes V_n$ is an L-invariant subspace, and hence an L-module.

Let I be the ideal of $T(V)$ generated by all $v \otimes v$ with $v \in V$. Note that I is L-invariant. This follows from

$$xv \otimes v + v \otimes xv = (v + xv) \otimes (v + xv) - v \otimes v - xv \otimes xv.$$

The algebra $\Lambda(V) = T(V)/I$ is the exterior algebra of V. As the quotient of $T(V)$ by its submodule I, it follows that $\Lambda(V)$ is naturally an L-module. Given $v_1, \ldots, v_n \in V$ let $v_1 \wedge \cdots \wedge v_n$ denote the image of $v_1 \otimes \cdots \otimes v_n$ in $\Lambda(V)$. It is easy to see that the map $(v_1, \ldots, v_n) \mapsto v_1 \wedge \cdots \wedge v_n$ is multilinear and alternating. Moreover, we have

$$x(v_1 \wedge v_2 \wedge \cdots \wedge v_n) = xv_1 \wedge v_2 \wedge \cdots \wedge v_n + v_1 \wedge xv_2 \wedge \cdots \wedge v_n + \cdots + v_1 \wedge v_2 \wedge \cdots \wedge xv_n.$$

For $n \geq 0$ let $T^n(V)$ the L-submodule of $T(V)$ spanned by all $v_1 \otimes \cdots \otimes v_n$. The corresponding L-submodule of $\Lambda(V)$, namely $\Lambda^n(V) = (T^n(V) + I)/I$, is known as the n-th exterior power of V.

Suppose V has finite dimension m. Then

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^m(V),$$

where $\Lambda^0(V) \cong \mathbf{C}$, $\Lambda^1(V) \cong V$ and $\dim(\Lambda^i(V)) = \binom{m}{i}$ for $0 \leq i \leq m$. In fact, if

$w_1, \cdots, w_m$ form a basis of V then a basis for $\Lambda^i(V)$ is given by all $w_{j_1} \wedge \cdots \wedge w_{j_i}$ such that $1 \le j_1 < \cdots < j_i \le m$.

2.4 The Lie algebra associated to a bilinear form

All vector spaces, in particular all Lie algebras and representations, appearing in the remainder of this thesis, will be assumed to be finite dimensional.

Definition 2.4.1. *Let L be a Lie algebra and let V be an L-module. Given $v \in V$ we let $L(v) = \{x \in L \,|\, xv = 0\}$.*

Note that $L(v)$ is a subalgebra of L.

Example 2.4.2. *Let* $\mathrm{Bil}(V)$ *be the space of all bilinear forms $V \times V \to \mathbf{C}$. Then* $\mathrm{Bil}(V)$ *becomes a module for $L = \mathfrak{gl}(V)$ via*

$$(xf)(v, w) = -f(xv, w) - f(v, xw), \quad x \in L, f \in \mathrm{Bil}(V), v, w \in V.$$

Thus, for $f \in \mathrm{Bil}(V)$, we see that $L(f)$ is the subalgebra of all $x \in L$ satisfying

$$f(xv, w) + f(v, xw) = 0, \quad v, w \in V.$$

Let X' denote the transpose of $X \in M_n$. Then M_n is a module for $L = \mathfrak{gl}(n)$ via

$$X \cdot M = -X'M - MX, \quad X \in L, M \in M_n,$$

and for $M \in M_n$, we see that $L(M)$ is the subalgebra of all $X \in L$ satisfying

$$X'M + MX = 0. \tag{2.6}$$

Let $B = \{v_1, \ldots, v_n\}$ be a basis of V and let M be the matrix of $f \in \mathrm{Bil}(V)$ relative to B. The Lie isomorphism $M_B : \mathfrak{gl}(V) \to \mathfrak{gl}(n)$ carries $L(f)$ onto $L(M)$, i.e. an endomorphism x of V is in $L(f)$ if and only if its matrix $X = M_B(x)$ is in $L(M)$.

Note 2.4.3. *If $f \in \mathrm{Bil}(V)$ is non-degenerate then $L(f)$ is a subalgebra of $\mathfrak{sl}(V)$.*

Indeed, let M be the matrix of f relative to a basis B. Then M is invertible and condition (2.6) becomes $X' = -MXM^{-1}$. Taking traces yields $\mathrm{tr}\,(X) = -\mathrm{tr}\,(X)$.

Definition 2.4.4. *Let L be a Lie algebra with a subalgebra H and let V be an L-module. A linear functional $\lambda : H \to \mathbf{C}$ is said to be a* ***weight*** *for the action of H on V if the subspace*

$$V(H)_\lambda = V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in H\}$$

is non-zero, i.e., if H has a common eigenvector in V on which it acts via λ. In this case, the subspace V_λ is a ***weight space*** *for the action of H on V.*

Example 2.4.5. *Take $L = \mathfrak{gl}(n)$, $H = \mathfrak{d}(n)$ and $V = \mathbf{C}^n$. Then the weights for the action of H on V are the n coordinate functions $\varepsilon_1, \ldots, \varepsilon_n$, defined by*

$$\varepsilon_i(\mathrm{diag}(h_1, \ldots, h_n)) = h_i, \quad 1 \le i \le n. \tag{2.7}$$

The corresponding weight spaces are $\langle e_1 \rangle, \ldots, \langle e_n \rangle$.

Lemma 2.4.6. *(Invariance Lemma) Let L be a Lie algebra with an ideal I. Let V be an L-module and let $\lambda : I \to \mathbf{C}$ be a linear functional. Then $V_\lambda(I)$ is an L-submodule of V.*

2.5 The Classical Lie algebras

There are four families of subalgebras of $\mathfrak{gl}(V)$, or $\mathfrak{gl}(n)$, which are central to the theory of Lie algebras, known as the **classical Lie algebras**. They are the **special linear algebra** $\mathfrak{sl}(V)$, the **orthogonal algebras** $\mathfrak{o}(V) = L(f)$ with f non-degenerate and symmetric (divided into two families according to the parity of the dimension of V), and the **symplectic algebra** $\mathfrak{sp}(V) = L(f)$ with f non-degenerate and skew-symmetric. Since all invertible symmetric (resp. skew-symmetric) matrices of the same size are congruent to each other, the actual choice of the form f leaves the isomorphism type of orthogonal and symplectic algebras invariant.

For computational purposes it will be convenient to give a matrix interpretation of each of these families. We will also introduce an alternative common notation for the classical Lie algebras, namely $A_\ell, B_\ell, C_\ell, D_\ell$, where $\ell \geq 1$.

• The special Lie algebra $A_\ell = \mathfrak{sl}(\ell+1)$ is the Lie algebra of all $(\ell+1) \times (\ell+1)$ matrices having trace 0. Clearly $\dim(A_\ell) = \ell^2 + 2\ell$.

• The orthogonal algebra $B_\ell = \mathfrak{o}(2\ell+1)$ consists of all $X \in \mathfrak{gl}(2\ell+1)$ which satisfy condition (2.6) for

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}.$$

Let us write a matrix $X \in \mathfrak{gl}(2\ell+1)$ in blocks of the same size as those in M:

$$X = \begin{pmatrix} a & b & c \\ d & m & n \\ e & p & q \end{pmatrix}.$$

Then the condition $X \in B_\ell$, i.e., $X'M + MX = 0$, translates into

$$a = 0, \quad d = -c', \quad e = -b', \quad q = -m', \quad n = -n', \quad p = -p'.$$

In particular $\dim(B_\ell) = 2\ell^2 + \ell$.

- The orthogonal algebra $D_\ell = \mathfrak{o}(2\ell)$ consists of all $X \in \mathfrak{gl}(2\ell)$ which satisfy

condition (2.6) for

$$M = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}.$$

Then

$$X = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathfrak{gl}(2\ell)$$

is in D_ℓ if and only if $q = -m'$, $n = -n'$ and $p = -p'$. Thus $\dim(D_\ell) = 2\ell^2 - \ell$.

- The symplectic algebra $C_\ell = \mathfrak{sp}(2\ell)$ consists of all $X \in \mathfrak{gl}(2\ell, F)$ which satisfy

condition (2.6) for

$$M = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

Then

$$X = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathfrak{gl}(2\ell)$$

is in C_ℓ if and only if $q = -m'$, $n = n'$ and $p = p'$. In particular, $\dim(C_\ell) = 2\ell^2 + \ell$.

Example 2.5.1. *The natural module for each of the classical Lie algebras A_ℓ through D_ℓ is irreducible, with the only exception of D_1.*

Indeed, let L be a classical Lie algebra different from D_1 and let $V = \mathbf{C}^n$ be the corresponding column space. Let W be a non-zero submodule of V and let v be a non-zero vector in W. Acting on v by means of suitable elements of L we see that $e_1 \in W$. A similar argument now shows that all $e_2, \ldots, e_n$ are in W, as required.

It can be shown that any simple Lie algebra is isomorphic to one and only one of the following:

$$A_\ell\,(\ell \geq 1), B_\ell\,(\ell \geq 2), C_\ell\,(\ell \geq 3), D_\ell\,(\ell \geq 4), E_6, E_7, E_8, F_4, G_2,$$

where E_6, E_7, E_8, F_4, G_2 are the so-called exceptional Lie algebras. We will not make use of this result anywhere in this thesis.

2.6 The irreducible $\mathfrak{sl}(2)$-modules

Let $A = \mathbf{C}[X_1, \ldots, X_n]$ be the polynomial algebra in $n \geq 2$ variables $X_1, \ldots, X_n$. Each partial derivative $\partial/\partial X_j$ is a derivation of A. It follows automatically that each map $E_{ij} = m_{X_i} \circ \partial/\partial X_j$, where m_{X_i} is multiplication by X_i, is a derivation of A. Since any derivation of A is completely determined by its action on $X_1, \ldots, X_n$, it follows easily from (2.1) that the map $\phi : \mathfrak{gl}(n) \to \mathrm{Der}(A)$, given by $e_{ij} \mapsto E_{ij}$, is a representation of $\mathfrak{gl}(n)$ on A.

For $m \geq 0$ let A_m be the space of all homogeneous polynomials of degree m. We easily see that A_m is a $\mathfrak{gl}(n)$-invariant subspace of A of dimension $\binom{n+m-1}{m}$. We now view A_m as an $\mathfrak{sl}(n)$-module, by restriction. It is not difficult to see that A_m is an irreducible $\mathfrak{sl}(n)$-module. However, at this point we content ourselves with considering the case $n = 2$. Then A_m has dimension $m + 1$ and basis $B = \{X_1^m, X_1^{m-1}X_2, \dots, X_1X_2^{m-1}, X_2^m\}$. The matrices corresponding to e, h, f relative to B are respectively equal to

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & m \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 & \dots & 0 \\ 0 & m-2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 2-m & 0 \\ 0 & 0 & \dots & 0 & -m \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ m & 0 & 0 & \dots & 0 \\ 0 & m-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

It is now clear that A_m is an irreducible $\mathfrak{sl}(2)$-module.

Theorem 2.6.1. *Let m be a non-negative integer and let V be an $\mathfrak{sl}(2)$-module. Then the following conditions are equivalent:*

(1) V is isomorphic to A_m.

(2) V is irreducible and $\dim(V) = m + 1$.

(3) There is a common eigenvector $v \in V$ for the actions of h and e, with $hv = mv$ and $\dim(V) = m + 1$.

In what follows $V(m)$ will stand for the only irreducible $\mathfrak{sl}(2)$-module, up to isomorphism, having dimension $m + 1$.

2.7 Solvable Lie algebras

Definition 2.7.1. *Let L be a Lie algebra. The sequence of ideals of L recursively defined by*

$$L^{(0)} = L, \quad L^{(1)} = [L^{(0)}L^{(0)}], \quad L^{(2)} = [L^{(1)}L^{(1)}], \quad \ldots$$

and, in general

$$L^{(n)} = [L^{(n-1)}L^{(n-1)}], \quad n \geq 1,$$

*is called the **derived series** of L.*

Definition 2.7.2. *A Lie algebra L is called **solvable** if $L^{(n)} = 0$ for some n.*

Example 2.7.3. *The Lie algebra $\mathfrak{t}(n)$ is solvable.*

Theorem 2.7.4. *(Lie) Let L be a solvable Lie algebra and let V be a non-zero L-module. Then there exists a common eigenvector for the action of L on V.*

If L is a solvable Lie algebra and V is a non-zero L-module then Lie's Theorem ensures the existence of a linear functional $\lambda : L \to \mathbf{C}$ such that $V_\lambda \neq 0$.

Proposition 2.7.5. *If I, J are solvable ideals of L, then so is $I + J$.*

Corollary 2.7.6. *Let L be a Lie algebra. Then L has a unique solvable ideal, the* ***radical*** *of L and denoted by $RadL$, containing all solvable ideals of L.*

Example 2.7.7. *Consider the subalgebra L of $\mathfrak{gl}(3)$ of all matrices of the form*

$$\begin{pmatrix} a & b & x \\ c & -a & y \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $RadL$ consists of all of the above matrices satisfying $a = b = c = 0$.

2.8 The Killing form

The Killing form of a Lie algebra L is the symmetric bilinear form $\kappa : L \times L \to \mathbf{C}$

$$\kappa(x, y) = \mathrm{tr}\,((ad\,x)(ad\,y)), \quad x, y \in L.$$

For instance, if $L = \mathfrak{sl}(2)$ and $B = \{e, h, f\}$ then (2.4) shows that the matrix of κ relative to B is

$$M_B(\kappa) = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Theorem 2.8.1. *(Cartan) Let L be a Lie algebra. Then L is solvable if and only if $[LL]$ is in the radical of the Killing form of L.*

Chapter 3

Semisimple Lie algebras

In this preliminary chapter we explore the structure of semisimple Lie algebras and their modules. Again, we refer to [Hum97] for all unproven statements, unless otherwise noted.

3.1 Semisimple Lie algebras

Definition 3.1.1. *A Lie algebra L is called **semisimple** if $RadL = 0$.*

Example 3.1.2. *Every simple Lie algebra is semisimple.*

Theorem 3.1.3. *(Cartan) A Lie algebra is semisimple if and only if its Killing form is non-degenerate.*

Proposition 3.1.4. *Let L be a subalgebra of $\mathfrak{sl}(V)$. Assume that V is irreducible as L-module. Then L is semisimple.*

Proof. Let $R = RadL$. By Lie's theorem there is a linear functional $\lambda : R \to \mathbf{C}$ such that $V(R)_\lambda \neq 0$. By the Invariance Lemma we know that $V(R)_\lambda$ is L-invariant. But V is irreducible, so $V = V(R)_\lambda$, i.e. R acts on V via λ. But all operators in R have trace 0, so $\lambda = 0$, whence $R = 0$. □

Corollary 3.1.5. *The classical Lie algebras are all semisimple, except for D_1.*

Proof. By Note 2.4.3 all classical Lie algebras are subalgebras of a suitable $\mathfrak{sl}(V)$. By Example 2.5.1 we know that D_1 is the only classical Lie algebra whose natural module is not irreducible, so the result follows from Proposition 3.1.4 □

Theorem 3.1.6. *(Weyl) Let L be a semisimple Lie algebra and let V be an L-module. Then L is completely reducible, that is, every L-submodule of V has a complementary L-submodule.*

Theorem 3.1.7. *Let L be a semisimple Lie algebra. Let x be an element of L such that $\operatorname{ad} x : L \to L$ is diagonalizable. Then x acts diagonalizably on any L-module V.*

3.2 Abstract root systems

A root system is a triple $(E, (\ ,\ ), \Phi)$, where E is a non-zero finite dimensional real vector space, $(\ ,\ ) : E \times E \to \mathbf{R}$ is an inner product (positive definite symmetric bilinear form), and Φ is a subset of E (whose elements are called roots) satisfying:

(R1) $0 \notin \Phi$, Φ is finite, and Φ spans E.

(R2) If $\alpha \in \Phi$ then the only scalar multiples of α in Φ are α and $-\alpha$.

(R3) $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbf{Z}$ for all $\alpha, \beta \in \Phi$.

(R4) $\beta - [2(\beta, \alpha)/(\alpha, \alpha)]\alpha \in \Phi$ for all $\alpha, \beta \in \Phi$.

The Weyl group $W = W(\Phi)$ is the subgroup of the orthogonal group $O(E, (\ ,\ ))$ generated by the reflections r_α, where α runs through Φ, and

$$r_\alpha(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha, \quad x \in E.$$

The Weyl group can be identified with a subgroup of $Sym(\Phi)$ via the restriction map $w \mapsto w|_\Phi$. In particular, the Weyl group is finite.

A fundamental system is a basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Phi$ of E (whose elements are called fundamental roots) such that every $\alpha \in \Phi$ is an integral linear combination of the α_i with all coefficients ≥ 0 or all coefficients ≤ 0.

Theorem 3.2.1. *Every root system has a fundamental basis.*

Theorem 3.2.2. *Let Δ be a fundamental system. Then the Weyl group is generated by the fundamental reflections r_α, with $\alpha \in \Delta$.*

Let Δ be a fundamental system. We define a partial order $\leq$ on E by declaring $\mu \leq \lambda$ if there are non-negative integers $k_1, \ldots, k_\ell$ such that

$$\lambda - \mu = k_1\alpha_1 + \cdots + k_\ell\alpha_\ell. \tag{3.8}$$

We further define the sets of positive and negative roots to be

$$\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\} \text{ and } \Phi^- = -\Phi^+ = \{\alpha \in \Phi \mid \alpha < 0\}.$$

On occasion we will extend the above order on E by another one, say $\leq_{\mathbf{Q}}$, by allowing $k_1, \ldots, k_\ell$ to be non-negative rational numbers in the above definition.

Note that if $w \in W$ and Δ is a fundamental system then so is $w\Delta$.

Theorem 3.2.3. *The action of the Weyl group on the fundamental systems is simply transitive, i.e., given any fundamental systems Δ_1 and Δ_2 there exists a unique $w \in W$ such that $w\Delta_1 = \Delta_2$.*

Let Δ be a fundamental system. Then clearly so is $-\Delta$. By above, there exists a unique $w_0 \in W$ such that $w_0\Delta = -\Delta$. Since $w_0^2\Delta = \Delta$, we must have $w_0^2 = 1_E$, by uniqueness. Clearly, $w_0 = -1_E$ if and only if $-1_E \in W$, which need not be the case in general.

3.3 The root space decomposition

Definition 3.3.1. *A subalgebra H of a Lie algebra L is said to be **toral** if the map $\operatorname{ad} h : L \to L$ is diagonalizable for every h in H.*

For instance, the diagonal subalgebra of every classical Lie algebra is toral. It is easy to see that every toral subalgebra is abelian.

Proposition 3.3.2. *Let L be a non-zero semisimple Lie algebra. Then L has a non-zero toral subalgebra.*

Definition 3.3.3. *A **maximal toral subalgebra** of a Lie algebra L is a toral subalgebra not properly contained in any other toral subalgebra.*

Let L be a non-zero semisimple Lie algebra with toral subalgebra H. Since the operators $ad\, h$, $h \in H$, are diagonalizable and commute with each other, they can be simultaneously diagonalizable. Let $\Phi = \Phi(H)$ be the set of non-zero weights for the adjoint action of H on L. We have

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

where

$$L_\alpha = \{x \in L | [hx] = \alpha(h)x \text{ for all } h \in H\}, \quad \alpha \in H^*,$$

$$H \subseteq L_0,$$

$$\Phi = \{\alpha \in H^* \,|\, \alpha \neq 0 \text{ and } L_\alpha \neq 0\}.$$

Assume henceforth that H is maximal toral, which is equivalent to $H = L_0$. In this case, the weights are known as **roots**, and Φ is known as the **system of roots** associated to H. We refer to

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

as the **root space decomposition** of L relative to H. Here are a few basic properties.

It is known that Φ spans H^*. In particular Φ is non-empty. Moreover, $\Phi = -\Phi$. The only scalar multiples of a root are itself and its opposite.

Each L_α, $\alpha \in \Phi$, is one dimensional. In general, for $\alpha, \beta \in H^*$ we have

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta},$$

with equality if $\alpha, \beta, \alpha + \beta \in \Phi$.

Since Φ spans H^* there is a complex basis Δ of H^* made up of roots. Any other root is known to be a rational linear combination of roots in Δ. Hence the real span of Φ equals the real span of Δ. We denote this real space by E and its dimension by ℓ.

An important fact is that the restriction of the Killing form of L to H is non-degenerate. We thus obtain a linear isomorphism $H \to H^*$, $s \mapsto \kappa(s, -)$, whose inverse we denote by $t : H^* \to H$. Thus if $\theta \in H^*$ then t_θ is the only element of H such that

$$\kappa(t_\theta, h) = \theta(h), \quad h \in H.$$

Let $(\ ,\ )$ be the non-degenerate symmetric bilinear form on H^* transferred from H by means of t, i.e.

$$(\theta, \psi) = \kappa(t_\theta, t_\psi), \quad \theta, \psi \in H^*.$$

It is known to take rational values on $\Phi \times \Phi$ and hence real values on $E \times E$. Moreover, the restriction of $(\ ,\ )$ to E is known to be positive definite. This makes $(E, (\ ,\ ))$ into an Euclidean space.

We have

$$[L_\alpha, L_{-\alpha}] = \mathbf{C}t_\alpha, \quad \alpha \in \Phi,$$

where

$$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = (\alpha, \alpha) > 0.$$

Thus, if we set

$$h_\alpha = 2t_\alpha/\alpha(t_\alpha) = 2t_\alpha/(\alpha, \alpha), \quad \alpha \in \Phi$$

then h_α is an element of $[L_\alpha, L_{-\alpha}]$ satisfying

$$\alpha(h_\alpha) = 2\alpha(t_\alpha)/\alpha(t_\alpha) = 2.$$

Since $[L_\alpha, L_{-\alpha}]$ is one dimensional, we see that h_α is the unique element of this subspace where α takes the value 2. Let

$$S_\alpha = L_\alpha \oplus [L_\alpha, L_{-\alpha}] \oplus L_{-\alpha}.$$

Given $0 \neq x_\alpha \in L_\alpha$ there is a unique element $y_\alpha \in L_{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$. Since

$$[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha, \quad [h_\alpha, y_\alpha] = (-\alpha)(h_\alpha)y_\alpha = -2y_\alpha,$$

we see that $S_\alpha \cong \mathfrak{sl}(2)$.

For any $\theta \in H^*$ and $\alpha \in \Phi$ define

$$\langle \theta, \alpha \rangle = 2(\theta, \alpha)/(\alpha, \alpha).$$

This is linear in θ. We have the important relation:

$$\langle \theta, \alpha \rangle = 2(\theta, \alpha)/(\alpha, \alpha) = \kappa(t_\theta, 2t_\alpha/\alpha(t_\alpha)) = \theta(h_\alpha).$$

In particular this holds for all θ in E, which is the case we are interested in. In this case $\langle \theta, \alpha \rangle$ is real. Using the representation theory of $\mathfrak{sl}(2)$ one shows

$$\langle \beta, \alpha \rangle = \beta(h_\alpha) \in \mathbf{Z}, \quad \beta, \alpha \in \Phi.$$

It is also known that Φ is invariant under all r_α, $\alpha \in \Phi$. It follows that $(E, (\ ,\ ), \Phi)$ satisfies the axioms of an abstract root system. Note that in this context each reflection r_α, $\alpha \in \Phi$, is given by

$$r_\alpha(\theta) = \theta - 2(\theta, \alpha)/(\alpha, \alpha)\alpha = \theta - \langle \theta, \alpha \rangle \alpha = \theta - \theta(h_\alpha)\alpha, \quad \theta \in E.$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fundamental system. Then the elements x_α, y_α, $\alpha \in \Delta$, generate L as Lie algebra.

The $\ell \times \ell$ matrix C defined by

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle = \alpha_i(h_{\alpha_j})$$

is known as the **Cartan matrix**. It is invertible as a rational matrix.

Since Δ is a $\mathbf{C}$-basis of H^* the isomorphism $t : H^* \to H$ makes it clear that $\{t_{\alpha_1}, ..., t_{\alpha_\ell}\}$ is a $\mathbf{C}$-basis of H, and hence $\{h_{\alpha_1}, ..., h_{\alpha_\ell}\}$ is a $\mathbf{C}$-basis of H. The lineal functionals $\lambda_1, ..., \lambda_\ell$, forming the dual basis of $\{h_{\alpha_1}, ..., h_{\alpha_\ell}\}$, are known as the **fundamental weights**. The matrix expressing the fundamental roots in terms of fundamental weights is the transpose of the Cartan matrix. Indeed, if $\alpha_j = \sum_{1 \leq i \leq \ell} D_{ij}\lambda_i$ then applying both sides to h_{α_i} yields $C_{ji} = D_{ij}$. Thus each λ_i is a rational linear combination of the α_j. In particular, all λ_i are in E. The additive group Λ generated by the λ_i is called the **weight lattice**.

The Cartan matrix is invertible, when viewed as a matrix over $\mathbf{Q}$. The diagonal entries of C are always 2 and the off-diagonal entries are negative integers or 0. It follows (see [Car05], Proposition 10.18) that C^{-1} has non-negative coefficients. By the above, this implies that all fundamental weights λ_i satisfy $\lambda_i >_{\mathbf{Q}} 0$.

By definition of dual basis, if $\mu \in H^*$ then

$$\mu = \mu(h_{\alpha_1})\lambda_1 + \cdots + \mu(h_{\alpha_\ell})\lambda_\ell.$$

Hence $\mu \in \Lambda$ if and only if

$$\langle \mu, \alpha_i \rangle = \mu(h_{\alpha_i}) \in \mathbf{Z}, \quad 1 \leq i \leq \ell.$$

The elements of Λ are said to be **integral**. The elements μ of E satisfying $\mu(h_{\alpha_i}) \geq 0$ for all $1 \leq i \leq \ell$ are said to be **dominant**.

Note that the Weyl group preserves Λ. It is enough to verify this on the generators r_{α_i}, $1 \leq i \leq \ell$, of W and λ_j, $1 \leq j \leq \ell$, of Λ. We have

$$r_{\alpha_i}(\lambda_j) = \lambda_j - \lambda_j(h_{\alpha_i})\alpha_i = \lambda_j - \delta_{ij}\alpha_i \in \Lambda.$$

Given $\mu \in \Lambda$ we define its **dual** $\mu^* = -w_0(\mu) \in \Lambda$. Note that if μ is dominant then so is μ^*. In fact, we have the following more precise statement. For each $1 \leq i \leq \ell$ let $\sigma(i)$ be the only index such that $-w_0\alpha_i = \alpha_{\sigma(i)}$. We claim that $\lambda_i^* = \lambda_{\sigma(i)}$. Thus, the fundamental weights are permuted by duality. Clearly σ is a permutation of $\{1, \ldots, \ell\}$ of order 2. To verify the claim, observe that

$$\lambda_i^*(h_{\alpha_{\sigma(j)}}) = \frac{2(-w_0(\lambda_i), \alpha_{\sigma(j)})}{(\alpha_{\sigma(j)}, \alpha_{\sigma(j)})} = \frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

3.4 Modules for semisimple Lie algebras

Let L be a non-zero semisimple Lie algebra with maximal toral subalgebra H. We will maintain all of the notation and terminology introduced in §3.3.

Let V be a non-zero $L-$module. By Theorem 3.1.7 all operators $R(h), h \in H$, are diagonalizable. They also commute pairwise, since H is toral. Hence they are simultaneously diagonalizable. Thus V is the direct sum of the weight spaces for the action of H. Let Π be the finite set of all weights for this action. Thus

$$V = \bigoplus_{\mu \in \Pi} V_\mu.$$

Proposition 3.4.1. *If $\alpha \in \Phi$ and $\mu \in H^*$ then $L_\alpha V_\mu \subseteq V_{\mu+\alpha}$.*

Proof. Let $x \in L_\alpha, u \in V_\mu, h \in H$. Then,

$$hxu = xhu + [hx]u = x\mu(h)u + \alpha(h)xu = (\mu + \alpha)(h)xu.$$

□

Definition 3.4.2. *A **maximal vector** of V (relative to Δ) is a common eigenvector for the actions of H and all x_α, $\alpha \in \Delta$.*

It is easy to see that if $v \in V$ is a maximal vector then $x_\alpha v = 0$ for all $a \in \Delta$. In fact, if v is a maximal vector then $x_\alpha v = 0$ for all $\alpha \in \Phi^+$.

We now extend the order $\leq$ defined on E by means (3.8) to all of H^*.

Proposition 3.4.3. *V must have a maximal vector.*

Proof. Since Π is a finite subset of H^* it must have a maximal element, say λ. Since $\lambda \in \Pi$, $V_\lambda \neq 0$, so we can choose $v \in V_\lambda$ not zero. Then, by Proposition 3.4.1, $x_\alpha v \in V_{\alpha+\lambda} = 0$ for all $\alpha \in \Delta$ by the maximality of λ. □

Note that the subalgebra of L generated by H and all x_α, $\alpha \in \Delta$, is solvable. Hence Proposition 3.4.3 also follows from Lie's theorem.

Proposition 3.4.4. *Let V be an $L-$module and let $v \in V$ be a maximal vector. Let $U = \langle v \rangle_L$ the $L-$submodule of V generated by v and let U_1 be the span of v and all vectors of the form $y_{\beta_1} \ldots y_{\beta_n} v$, where $\beta_i \in \Delta$. Then $U = U_1$.*

Proof. Clearly $U_1 \subseteq U$ with equality if and only if U_1 is $L-$stable. Since L is generated as a Lie algebra by all $x_\alpha, y_\alpha, \alpha \in \Delta$, the $L-$stability of U_1 follows from the stability under all x_α and $y_\alpha \in \Delta$. The case of y_α is clear. The case of x_α follows by induction on the number of y's present before v. If this number is 0 then the vector is v and we know $x_\alpha v = 0$. Suppose the number is $m > 0$ and the result is true for all numbers less than m. We have

$$x_\alpha y_{\beta_1} \cdots y_{\beta_n} v = y_{\beta_1} x_\alpha \cdots y_{\beta_n} v + [x_\alpha y_{\beta_1}] y_{\beta_2} \cdots y_{\beta_n} v.$$

The first summand is in U_1 by inductive hypothesis and the fact that U_1 is y_{β_1}-invariant. The second term is 0 if $\beta_1 \neq \alpha$ since in that case the bracket belongs to $L_{\alpha-\beta_1}$, which is 0 because $\alpha - \beta_1$ cannot be a root. If $\beta_1 = \alpha$ then the bracket is h_α, in which case the second sum equals

$$(\lambda - (\beta_2 + \cdots + \beta_n))(h) y_{\beta_2} \cdots y_{\beta_n} v,$$

which is in U_1. This proves that $U = U_1$. □

Corollary 3.4.5. *Keep the above hypotheses and let λ be the weight of v. Then λ is the highest weight of U and* $\dim(U_\lambda) = 1$.

Proof. This follows from Proposition 3.4.4 and the fact that

$$y_{\beta_1} \ldots y_{\beta_n} U_\lambda \subseteq U_{\lambda-(\beta_1+\cdots+\beta_n)}, \quad \beta_i \in \Delta.$$

□

Proposition 3.4.6. *Keep the hypotheses of Proposition 3.4.4. Then U is irreducible.*

Proof. Let S be a non-zero submodule of U. By Weyl's Theorem, $U = S \oplus T$ for some L-submodule T of U. Let f be the projection of U onto S. Then f is an $L-$endomorphism of U. Let $\Pi(U)$ stand for the set of weights of U. Clearly f preserves each weight space U_μ of U, since

$$hf(v) = f(hv) = \mu(h) f(v), \quad \mu \in \Pi(U), v \in U_\mu.$$

In particular $f(V_\lambda) \subseteq V_\lambda$, but V_λ is one dimensional so $f(v) = av$ for some scalar $a \in \mathbf{C}$. It now follows from Proposition 3.4.4 that $f = a1_U$. Since S is not 0, we see that $a \neq 0$. But f is a projection, so $a = 1$, i.e. $U = S$. Thus U is irreducible. □

Corollary 3.4.7. *Every irreducible L-module V must be generated by a maximal vector v. Moreover, the weight of v is the highest in V.*

Corollary 3.4.8. *Let V_1 and V_2 be irreducible L-modules with respective highest weights λ_1 and λ_2. Then $V_1 \cong V_2$ if and only if $\lambda_1 = \lambda_2$.*

Proof. Suppose first $V_1 \cong V_2$. Then V_1 has highest weights λ_1 and λ_2, so $\lambda_1 = \lambda_2$. Suppose conversely that $\lambda_1 = \lambda_2$ and let v_1 and v_2 be maximal vectors in V_1 and V_2, respectively. Then $V_1 \oplus V_2$ is an L-module with maximal vector $v_1 + v_2$ of weight $\lambda_1 = \lambda_2$. Let U be the submodule of $V_1 \oplus V_2$ generated by $v_1 + v_2$. It is irreducible by Proposition 3.4.6. The projection maps of $V_1 \oplus V_2$ onto V_1 and V_2 are L-linear homomorphisms so their restrictions to U are so as well. These restrictions map $v_1 + v_2$ into v_1 and v_2, respectively, so they are bijective by the irreducibility of U, V_1 and V_2. Thus $V_1 \cong U \cong V_2$. □

Proposition 3.4.9. *All weights of V are integral, i.e., $\Pi \subset \Lambda$.*

Proof. Let $\mu \in \Pi$, $u \in V_\mu$ and $\alpha \in \Phi$. View V as an S_α-module. All eigenvalues of h_α acting on V must be integers by Weyl's theorem and the classification of irreducible $\mathfrak{sl}(2)$-modules. Since $h_\alpha u = \mu(h_\alpha)u$, it follows that $\mu(h_\alpha) \in \mathbf{Z}$. □

Proposition 3.4.10. *Suppose V is irreducible with highest weight λ. Then λ is dominant.*

Proof. Let $0 \neq v \in V_\lambda$. Given $\alpha \in \Delta$ we have $x_\alpha v = 0$ and $h_\alpha v = \lambda(h_\alpha)v$. As seen in §2.6, the dimension of the S_α-module generated by v is $\lambda(h_\alpha) + 1$, so $\lambda(h_\alpha) \geq 0$. □

Proposition 3.4.11. *The set Π of weights of V is invariant under the action of the Weyl group W of Φ.*

Proof. It suffices to show that Π is invariant under all reflections r_α, $\alpha \in \Phi$. Let $\mu \in \Pi$. Let q be the largest non-negative integer such that $\mu + q\alpha \in \Pi$ and let r be the largest non-negative integer such that $\mu - r\alpha \in \Pi$. Let M be the direct sum of all weight spaces $V_{\mu+i\alpha}$, where $-r \leq i \leq q$. By construction M is S_α-stable. Thus M is a non-zero finite dimensional S_α-module (not necessarily irreducible). The largest eigenvalue of h_α acting on M is $\mu(h_\alpha) + 2q$ and the smallest is $\mu(h_\alpha) - 2r$. By Weyl's theorem and the classification of irreducible $\mathfrak{sl}(2)$-modules, it follows that $\mu(h_\alpha) - 2r = -(\mu(h_\alpha) + 2q)$, i.e.

$$\mu(h_\alpha) = r - q,$$

and every integer of the same parity as $\mu(h_\alpha)$ between $\mu(h_\alpha) - 2r$ and $\mu(h_\alpha) + 2q$ is an eigenvalue of h_α on M. This implies that all summands of M are actually non-zero. In particular,

$$V_{r_\alpha(\mu)} = V_{\mu - \mu(h_\alpha)\alpha} = V_{\mu+(q-r)\alpha} \neq 0,$$

as claimed. $\square$

It is natural to ask whether every dominant integral $\lambda \in E$ is the highest weight of some irreducible L-module. The answer is positive, but the proof is hard.

It is easy to see that one can reduce the question to the case when λ is fundamental. Indeed, given a dominant integral $\lambda \in E$ there exist unique non-negative integers $k_1, \dots, k_\ell$ such that

$$\lambda = k_1\lambda_1 + \cdots + k_\ell\lambda_\ell.$$

Suppose $V_1, \dots, V_\ell$ are irreducible L-modules with highest weights $\lambda_1, \dots, \lambda_\ell$, respectively, and let $v_i \in V_i(\lambda_i)$ for $1 \leq i \leq \ell$. Let W_i be the k_i-th tensor power of V_i and let $U = W_1 \otimes \cdots \otimes W_\ell$. Consider the vector v of U given by $v = w_1 \otimes \cdots \otimes w_\ell$, where $w_i = v_i \otimes \cdots \otimes v_i$ (k_i times) for $1 \leq i \leq \ell$. We easily verify that v is a maximal vector of weight λ. By Proposition 3.4.6, the submodule of U generated by v is irreducible with highest weight λ.

Our goal is to give a concrete construction of the fundamental modules for each of the classical Lie algebras.

In what follows, if $\lambda \in E$ is integral dominant then $V(\lambda)$ stands for the unique irreducible L-module with highest weight λ up to isomorphism.

As an illustration, let us show how the highest weight modules $V(\lambda)$ behave with respect to duality.

Theorem 3.4.12. *Suppose $\lambda \in \Lambda$. Then $V(\lambda)^* \cong V(\lambda^*)$.*

Proof. Let $V = V(\lambda)$. Since V is irreducible, it is easy to see that so must be V^*. Thus, all we have to show is that λ^* is the highest weigh of V^*. For this purpose, we first find the weights of V^*. Let $\Pi = \Pi(V)$ be the set of weights of V. We claim that $\Pi(V^*) = -\Pi(V)$. To see this, let $\mu \in \Pi$. Given $f \in (V_\mu)^*$ we view f as an element of V^* by letting it be 0 on the sum of all other weight spaces of V. Then V^* is the direct sum of all $(V_\mu)^*$ as μ runs through Π. Moreover,

$$hf = -\mu(h)f, \quad \mu \in \Pi, f \in (V_\mu)^*.$$

Thus

$$(V_\mu)^* = (V^*)_{-\mu},$$

so $\Pi(V^*) = -\Pi(V)$, as claimed. Next we claim that $w_0\lambda$, which is a weight of V, is the smallest such a weight. Indeed, given $\mu \in \Pi$ we have $w_0\mu \in \Pi$, so $\lambda - w_0\mu$ is a sum of fundamental roots. Since w_0 sends Δ into $-\Delta$ it follows that $w_0\lambda - \mu$ is opposite to a sum of fundamental roots, i.e., $\mu \geq w_0\lambda$, as claimed. We deduce that $-w_0\lambda = \lambda^*$ is the highest weight of $-\Pi(V) = \Pi(V^*)$, as required. $\square$

Chapter 4

The fundamental modules of A_ℓ

In this chapter we will to construct all fundamental modules for the special linear algebra $L = \mathfrak{sl}(\ell+1) = A_\ell$. We will see that $V(\lambda_k) \cong \Lambda^k(V)$ for all $1 \le k \le \ell$, where $V = \mathbf{C}^{\ell+1}$ is the natural module of L.

Let H be the diagonal subalgebra of L, that is,

$$H = \{\mathrm{diag}(h_1, h_2, \ldots, h_{\ell+1}) | h_1 + \cdots + h_{\ell+1} = 0\}.$$

The diagonal matrices $e_{i,i} - e_{i+1,i+1}$, $1 \le i \le \ell$, form a basis of H. Moreover, these matrices together with all $e_{i,j}$, $1 \le i \ne j \le \ell+1$, form a basis of L.

For $1 \le i \le \ell+1$ let $\varepsilon_i : H \to \mathbf{C}$ be the i-th coordinate function, given by $h \mapsto h_i$. We claim that H is a maximal toral subalgebra of L with associated system of roots

$$\Phi = \{\varepsilon_i - \varepsilon_j \,|\, 1 \le i \ne j \le \ell+1\}$$

and corresponding weight spaces $L_{\varepsilon_i-\varepsilon_j} = \mathbf{C}e_{ij}$. Indeed, for $h \in H$, we have

$$[h, e_{ij}] = (h_i - h_j)e_{ij} = (\varepsilon_i - \varepsilon_j)(h)e_{ij}.$$

Thus $\mathbf{C}e_{ij} \subseteq L_{\varepsilon_i-\varepsilon_j}$ for $i \neq j$. Moreover, since H is abelian we have $H \subseteq L_0$. But $L = H \oplus \bigoplus_{i\neq j} \mathbf{C}e_{ij}$. It follows that $H = L_0$ and $\mathbf{C}e_{ij} = L_{\varepsilon_i-\varepsilon_j}$, as required.

For $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$ the only element of $[L_\alpha, L_{-\alpha}]$ on which α takes the value 2 is the diagonal matrix h_α, whose only non-zero diagonal entries are a 1 in position i and a -1 in position j.

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq \ell$. Then

$$\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$$

is a fundamental system for Φ. Indeed, given any $\varepsilon_i - \varepsilon_j$ with $i < j$, we can write

$$\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-1} - \varepsilon_j).$$

The corresponding Cartan matrix is tridiagonal, with 2's along the main diagonal and -1's on the first super and subdiagonals.

We take $x_{\alpha_i} = e_{i,i+1}$, in which case $y_{\alpha_i} = e_{i+1,i}$ indeed satisfies $[x_{\alpha_i}, y_{\alpha_i}] = h_{\alpha_i}$. The 2ℓ elements x_{α_i} and y_{α_i} clearly generate L.

From $\varepsilon_1 + \cdots + \varepsilon_{\ell+1} = 0$ and $\varepsilon_i - \varepsilon_{i+1} = \alpha_i$, $1 \leq i \leq \ell$, we see that all ε_i are in E. In particular, we can apply the elements of the Weyl group W to them. Since $r_{\varepsilon_i-\varepsilon_j}$ interchanges ε_i and ε_j and fixes all other ε_k, we see that W stabilizes $S =$

$\{\varepsilon_1, \ldots, \varepsilon_{\ell+1}\}$ and, in fact, the restriction map $W \to \mathrm{Sym}(S)$ is a group isomorphism, i.e. $W \cong S_{\ell+1}$.

Recall that the fundamental weights form the dual basis of $\{h_{\alpha_1}, h_{\alpha_2}, \ldots h_{\alpha_\ell}\}$. Given $1 \le k \le \ell$ we have

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k(h_{\alpha_i}) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Thus the fundamental weights are

$$\lambda_k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k, \quad 1 \le k \le \ell.$$

Let $V = \mathbf{C}^{\ell+1}$ be the natural module for L and let $e_1, \ldots, e_{\ell+1}$ be the standard basis of V. Since

$$x_{\alpha_i} e_1 = e_{i,i+1} e_1 = 0,\ 1 \le i \le \ell,\ \text{and } he_1 = h_1 e_1 = \varepsilon_1(h) e_1 = \lambda_1(h) e_1,\ h \in H,$$

we see that e_1 is a maximal vector of weight λ_1. Thus $\langle e_1 \rangle_L \cong V(\lambda_1)$. Since $e_i = e_{i,1} e_1 \in \langle e_1 \rangle_L$ for all $i > 1$ we see that $\langle e_1 \rangle_L = V$, whence $V \cong V(\lambda_1)$. Thus the first fundamental module is the natural module.

In fact $V(\lambda_k) \cong \Lambda^k(V)$ for all $1 \le k \le \ell$. To see this, suppose $2 \le k \le \ell$ and let $v = e_1 \wedge \cdots \wedge e_k \in \Lambda^k(V)$. We see that v is a maximal vector of weight λ_k, so $U = \langle v \rangle_L \cong V(\lambda_k)$. We will verify that $U = \Lambda^k(V)$ in two different ways.

• Let $1 \le m_1 < \cdots < m_k \le \ell + 1$. We wish to show that $e_{m_1} \wedge \cdots \wedge e_{m_k} \in U$. This is obvious if $j = m_j$ for all j. Otherwise, let j be the first index such that $j < m_j$. Then $e_{m_j,j}, \ldots, e_{m_k,k} \in L$ and $e_{m_j,j} \cdots e_{m_k,k} e_1 \wedge \cdots \wedge e_k = e_{m_1} \wedge \cdots \wedge e_{m_k} \in U$.

• Since $\varepsilon_1 + \cdots + \varepsilon_k$ is a weight of U, so are its W-conjugates

$$\varepsilon_{m_1} + \cdots + \varepsilon_{m_k}, \quad 1 \leq m_1 < \cdots < m_k \leq \ell + 1. \tag{4.9}$$

Since the only linear relation amongst $\varepsilon_1, \ldots, \varepsilon_{\ell+1}$, up to scaling, is $\varepsilon_1 + \cdots + \varepsilon_{\ell+1} = 0$, it follows that the W-conjugates (4.9) are all distinct. This implies, in particular, that $\dim(U) \geq \binom{\ell+1}{k} = \dim \Lambda^k(V)$, so $U = \Lambda^k(V)$.

We next wish to analyze the fundamental modules in terms of duality. Observe that the element w_0 of W sending Δ to $-\Delta$ is

$$w_0 = \cdots r_{\varepsilon_3 - \varepsilon_{\ell-1}} r_{\varepsilon_2 - \varepsilon_\ell} r_{\varepsilon_1 - \varepsilon_{\ell+1}},$$

since w_0 sends α_1 to $-\alpha_\ell$, $\alpha_2 \to -\alpha_{\ell-1}$, ... In particular, $w_0 = -1_E$ if and only if $\ell = 1$. As a permutation of S, written as the product of disjoint transpositions,

$$w_0 = \cdots (\varepsilon_3, \varepsilon_{\ell-1})(\varepsilon_2, \varepsilon_\ell)(\varepsilon_1, \varepsilon_{\ell+1}).$$

Observe next that

$$\lambda_1^* = \varepsilon_1^* = -\varepsilon_{\ell+1} = \varepsilon_1 + \cdots + \varepsilon_\ell = \lambda_\ell,$$

$$\lambda_2^* = (\varepsilon_1 + \varepsilon_2)^* = -\varepsilon_{\ell+1} - \varepsilon_\ell = \varepsilon_1 + \cdots + \varepsilon_{\ell-1} = \lambda_{\ell-1},$$

and, in general,

$$\lambda_k^* = \lambda_{\ell+1-k}, \quad 1 \leq k \leq \ell.$$

It follows that, for arbitrary non-negative integers $m_1, \ldots, m_\ell$, we have

$$(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_{\ell-1}\lambda_{\ell-1} + m_\ell\lambda_\ell)^* = m_\ell\lambda_1 + m_{\ell-1}\lambda_2 + \cdots + m_2\lambda_{\ell-1} + m_1\lambda_\ell.$$

Thus $V(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_{\ell-1}\lambda_{\ell-1} + m_\ell\lambda_\ell)$ is self-dual if and only if $m_1 = m_\ell$, $m_2 = m_{\ell-1}$, and so on. In particular, every $\mathfrak{sl}(2)$-module is self-dual.

The fundamental modules were seen above to be multiplicity free, that is, all weight spaces are one dimensional. This need not be the case at all for an arbitrary irreducible module $V(\lambda)$. For instance, let V be the $\mathfrak{sl}(3)$-module of homogeneous polynomials of degree 2 in the variables X_1, X_2, X_3 described in §2.6. If we first apply the automorphism $x \mapsto -x'$ of L and then let L act on V we obtain a new module that we will denote by U. Let $v = X_1^2 \otimes X_3^2 \in V \otimes U$. We readily see that v is a maximal vector of weight $\lambda = 2\lambda_1 + 2\lambda_2$. Thus $\langle v\rangle_L \cong V(\lambda)$. Now

$$y_{\alpha_2}y_{\alpha_1}v = e_{32}e_{21}v = 2e_{32}X_1X_2 \otimes X_3^2 = 2X_1X_3 \otimes X_3^2 - 4X_1X_2 \otimes X_2X_3$$

and

$$y_{\alpha_1}y_{\alpha_2}v = e_{21}e_{32}v = -2e_{21}X_1^2 \otimes X_2X_3 = -4X_1X_2 \otimes X_2X_3 + 2X_1^2 \otimes X_1X_3$$

are linearly independent vectors of $\langle v\rangle_L$ of weight

$$\lambda - (\alpha_1 + \alpha_2) = \lambda - (\lambda_1 + \lambda_2) = \lambda_1 + \lambda_2.$$

As an illustration, and to complete the irreducibility assertion made in §2.6, we consider next the case when V is the space of homogeneous polynomials of degree m in $\ell + 1$ variables $X_1, \ldots, X_{\ell+1}$, viewed as a module for $L = \mathfrak{sl}(\ell + 1)$. We will show that V is irreducible and isomorphic to $V(m\lambda_1)$. Indeed, it is clear that X_1^m is a maximal vector of weight $m\lambda_1$. Therefore, $\langle X_1^m \rangle_L \cong V(m\lambda_1)$. A typical monomial in V is of the form $X_1^{a_1} \cdots X_{\ell+1}^{a_{\ell+1}}$, where $a_1 + \cdots + a_{\ell+1} = m$. This monomial is in $\langle X_1^m \rangle_L$ since it is a non-zero scalar multiple of $e_{\ell+1,\ell}^{m-(a_1+\cdots+a_\ell)} \cdots e_{32}^{m-(a_1+a_2)} e_{21}^{m-a_1} X_1^m$. Thus $V = \langle X_1^m \rangle_L \cong V(m\lambda_1)$, as claimed.

Observe finally that V, like all fundamental modules, is also multiplicity free. However, unlike the case of all $V(\lambda_i)$ now the weights of V do not form a single W-orbit, unless $m = 0$ (in which case V is trivial) or $m = 1$ (in which case V is $V(\lambda_1)$). Indeed, note first of all that $X_1^{a_1} \cdots X_{\ell+1}^{a_{\ell+1}}$ is a weight vector of weight $(a_1 - a_2)\lambda_1 + \cdots + (a_\ell - a_{\ell+1})\lambda_\ell$. If $X_1^{b_1} \cdots X_{\ell+1}^{b_{\ell+1}}$ is a monomial in V of the same weight then $a_i = b_i$ for all $1 \leq i \leq \ell + 1$ by the linear independence of $\lambda_1, \ldots, \lambda_\ell$ and the invertibility of the $(\ell+1) \times (\ell+1)$ matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.$$

This shows that V is multiplicity free. Assume next $m \geq 1$. Then $m\lambda_1$ has exactly $\ell + 1$ conjugate weights under W, namely $m\varepsilon_1, \ldots, m\varepsilon_{\ell+1}$, attained at the monomials $X_1^m, \ldots, X_{\ell+1}^m$. These form a basis of V only when $m = 1$.

Chapter 5

First fundamental modules of B_ℓ and D_ℓ

Since D_1 is abelian, we will assume $\ell \geq 2$ when referring to D_ℓ. Here we analyze the first ℓ exterior powers of D_ℓ and use this information to obtain corresponding results for B_ℓ. In particular, we will obtain the first $\ell-2$ fundamental modules of D_ℓ and the first $\ell-1$ fundamental modules of B_ℓ. The remaining fundamental modules for B_ℓ and D_ℓ, known as the spin modules, will be considered in Chapter 6.

Throughout the entire exposition we will view D_ℓ as the subalgebra of B_ℓ whose blocks b, c, d, e are 0, in the notation of §2.5.

5.1 Root space decomposition of B_ℓ

Let H be the subspace of all diagonal matrices of B_ℓ, i.e., all matrices

$$h = \mathrm{diag}(0, h_1, h_2, \ldots, h_\ell, -h_1, -h_2, \ldots, -h_\ell), \quad h_i \in \mathbf{C}.$$

It will be convenient to number the rows and columns of matrices in B_ℓ from 0 to 2ℓ. Using this notation, a basis of B_ℓ can be obtained by adjoining to a basis of H, say all $e_{i,i} - e_{i+\ell,i+\ell}$ with $1 \le i \le \ell$, the following matrices: $e_{0,\ell+i} - e_{i,0}$ and $e_{0,i} - e_{i+\ell,0}$ for $1 \le i \le \ell$, $e_{i,j} - e_{j+\ell,i+\ell}$ where $1 \le i \ne j \le \ell$, as well as $e_{i,j+\ell} - e_{j,i+\ell}$ and $e_{i+\ell,j} - e_{j+\ell,i}$ with $1 \le i < j \le \ell$.

For $1 \le i \le \ell$ let $\varepsilon_i : H \to \mathbf{C}$ be the linear functional defined by $\varepsilon_i(h) = h_i$. We claim that H is a maximal toral subalgebra of B_ℓ with associated system of roots

$$\Phi = \{\pm\varepsilon_i, 1 \le i \le \ell, \pm(\varepsilon_i + \varepsilon_j), 1 \le i < j \le \ell, \varepsilon_i - \varepsilon_j, 1 \le i \ne j \le \ell)\}.$$

Indeed, let $h \in H$. Then

$$[h, e_{i,0} - e_{0,i+\ell}] = h_i e_{i,0} + h_{i+\ell} e_{0,i+\ell} = h_i e_{i,0} - h_i e_{0,i+\ell} = \varepsilon_i(h)(e_{i,0} - e_{0,i+\ell}),$$

$$[h, e_{0,i} - e_{i+\ell,0}] = -h_i e_{0,i} - h_{i+\ell} e_{i+\ell,0} = -h_i e_{0,i} + h_i e_{i+\ell,0} = -\varepsilon_i(h)(e_{0,i} - e_{i+\ell,0}),$$

$$\begin{aligned}
[h, e_{i,j} - e_{j+\ell,i+\ell}] &= (h_i - h_j)e_{i,j} - (h_{j+\ell} - h_{i+\ell})e_{j+\ell,i+\ell} \\
&= (h_i - h_j)e_{i,j} - (-h_j + h_i)e_{j+\ell,i+\ell} \\
&= (h_i - h_j)(e_{i,j} - e_{j+\ell,i+\ell}) \\
&= (\varepsilon_i - \varepsilon_j)(h)(e_{i,j} - e_{j+\ell,i+\ell}),
\end{aligned}$$

$$\begin{aligned}
[h, e_{i,j+\ell} - e_{j,i+\ell}] &= (h_i - h_{j+\ell})e_{i,\ell+j} - (h_j - h_{i+\ell})e_{j,i+\ell} \\
&= (h_i + h_j)(e_{i,j+\ell} - e_{j,i+\ell}) \\
&= (\varepsilon_i + \varepsilon_j)(h)(e_{i,j+\ell} - e_{j,i+\ell}),
\end{aligned}$$

$$
\begin{aligned}
[h, e_{j+\ell,i} - e_{i+\ell,j}] &= (h_{j+\ell} - h_i)e_{j+\ell,i} - (h_{i+\ell} - h_j)e_{i+\ell,j} \\
&= (-h_j - h_i)(e_{j+\ell,i} - e_{i+\ell,j}) \\
&= -(h_i + h_j)(e_{j+\ell,i} - e_{i+\ell,j}) \\
&= -(\varepsilon_i + \varepsilon_j)(h)(e_{j+\ell,i} - e_{i+\ell,j}).
\end{aligned}
$$

The total number of roots found so far is $\ell + \ell + \ell^2 - \ell + 2\,\frac{\ell(\ell-1)}{2} = 2\ell^2$. Since H is abelian, $\dim(H) = \ell$, and $\dim(B_\ell) = 2\ell^2 + \ell$, the claim follows.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \alpha_\ell = \varepsilon_\ell$. Then

$$
\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell\}
$$

is a fundamental system. Indeed, for a root of the form ε_i,

$$
\begin{aligned}
\varepsilon_i &= (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{\ell-1} - \varepsilon_\ell) + \varepsilon_\ell \\
&= \alpha_i + \alpha_{i+1} + \cdots + \alpha_\ell,
\end{aligned}
$$

if $i < j$,

$$
\begin{aligned}
\varepsilon_i - \varepsilon_j &= (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-1} - \varepsilon_j) \\
&= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j,
\end{aligned}
$$

and for a root of the form $\varepsilon_i + \varepsilon_j$,

$$
\begin{aligned}
\varepsilon_i + \varepsilon_j &= (\varepsilon_i - \varepsilon_{i+1}) + \cdots + (\varepsilon_{j-1} - \varepsilon_j) + (\varepsilon_j - \varepsilon_{j+1}) \cdots + (\varepsilon_{\ell-1} - \varepsilon_\ell) + \varepsilon_\ell \\
&\quad + (\varepsilon_{j-1} - \varepsilon_j) + (\varepsilon_j - \varepsilon_{j+1}) \cdots + (\varepsilon_{\ell-1} - \varepsilon_\ell) + \varepsilon_\ell \\
&= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{\ell-1} + \alpha_\ell).
\end{aligned}
$$

Thus every root is a linear combination of elements of Δ with the same sign.

We temporarily let $L = B_\ell$. For $1 \le i \le \ell - 1$ we set

$$x_{\alpha_i} = e_{i,i+1} - e_{i+1+\ell,i+\ell} \in L_{\alpha_i} \text{ and } y_{\alpha_i} = e_{i+1,i} - e_{i+\ell,i+1+\ell} \in L_{-\alpha_i}.$$

This is correct since

$$[x_{\alpha_i} y_{\alpha_i}] = e_{i,i} - e_{i+1,i+1} + e_{i+1+\ell,i+1+\ell} - e_{i+\ell,i+\ell} = h_{\alpha_i},$$

where

$$\alpha_i(h_{\alpha_i}) = 2.$$

We further set $x_{\alpha_\ell} = e_{0,2\ell} - e_{\ell,0} \in L_{\alpha_\ell}$, in which case $y_{\alpha_\ell} = 2(e_{2\ell,0} - e_{0,\ell}) \in L_{-\alpha_\ell}$. This works since

$$[x_{\alpha_\ell} y_{\alpha_\ell}] = 2(-e_{2\ell,2\ell} + e_{\ell,\ell}) = h_{\alpha_\ell},$$

where

$$\alpha_\ell(h_{\alpha_\ell}) = 2.$$

The elements $h_{\alpha_1}, \ldots h_{\alpha_\ell}$ defined above form a basis of H. We can explicitly describe them as follows:

$$h_{\alpha_1} = diag\ (0, \underbrace{1, -1, 0, \ldots, 0}_{\ell}, \underbrace{-1, 1, 0, \ldots, 0}_{\ell}),$$

$$h_{\alpha_2} = diag\ (0, \underbrace{0, 1, -1, 0, \ldots, 0}_{\ell}, \underbrace{0, -1, 1, 0, \ldots, 0}_{\ell}),$$

$$\vdots$$

$$h_{a_{\ell-1}} = diag\ (0, \underbrace{0, \ldots, 0, 1, -1}_{\ell}, \underbrace{0, \ldots, 0, -1, 1}_{\ell}),$$

$$h_{a_\ell} = diag\ (0, \underbrace{0, \ldots, 0, 2}_{\ell}, \underbrace{0, \ldots, 0, -2}_{\ell}).$$

Next we find the dual basis of $\{h_{\alpha_1}, \ldots, h_{a_\ell}\}$.

Let $\lambda_k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$, where $1 \leq k \leq \ell - 1$. Then for $1 \leq i \leq \ell$ we have $\lambda_k(h_{\alpha_i}) = \delta_{ki}$. Thus $\lambda_1, \ldots, \lambda_{\ell-1}$ are the first $\ell - 1$ fundamental weights. Moreover,

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_\ell(h_{\alpha_i}) = \begin{cases} 0 & \text{if } i < \ell, \\ 2 & \text{if } i = \ell. \end{cases}$$

Thus the remaining fundamental weight is

$$\lambda_\ell = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_\ell).$$

Note that the reflection $r_{\varepsilon_i - \varepsilon_j}$ interchanges ε_i and ε_j and fixes all other roots ε_k. Moreover, the reflection r_{ε_i} sends ε_i to $-\varepsilon_i$ and fixes all other roots ε_k. It follows that the Weyl group W is isomorphic to $C_2^\ell \rtimes S_\ell$.

The element w_0 of W sending Δ to $-\Delta$ is

$$w_0 = r_{\varepsilon_\ell} \cdots r_{\varepsilon_1} = -1_E.$$

It follows that every B_ℓ-module is self-dual.

5.2 Root space decomposition of D_ℓ

The maximal toral subalgebra H of B_ℓ previously described is a maximal toral subalgebra of D_ℓ. The associated system of roots is

$$\Phi_0 = \{\pm(\varepsilon_i + \varepsilon_j), 1 \le i < j \le \ell,\ \varepsilon_i - \varepsilon_j, 1 \le i \ne j \le \ell\}.$$

Let $\beta_1 = \varepsilon_1 - \varepsilon_2, \beta_2 = \varepsilon_2 - \varepsilon_3, \dots, \beta_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \beta_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$. Then the set $\Delta_0 = \{\beta_1, \beta_2, \dots, \beta_{\ell-1}, \beta_\ell\}$ is a fundamental system. Indeed, if $i < j$ then

$$\varepsilon_i - \varepsilon_j = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1},$$

$$\varepsilon_i + \varepsilon_j = (\beta_i + \beta_{i+1} + \cdots + \beta_{\ell-2}) + (\beta_j + \beta_{j+1} + \cdots + \beta_{\ell-1} + \beta_\ell).$$

Thus every root is a linear combination of elements of Δ_0 with the same sign.

The elements $h_{\beta_1}, \dots, h_{\beta_\ell}$ corresponding to Δ_0 are as follows. The first $\ell - 1$ are precisely those for B_ℓ, namely $h_{\beta_i} = e_{i,i} - e_{i+1,i+1} + e_{i+1+\ell,i+1+\ell} - e_{i+\ell,i+\ell}$ for $1 \le i < \ell$, while h_{β_ℓ} is exactly the element $h_{\varepsilon_{\ell-1}+\varepsilon_\ell}$ of B_ℓ, namely

$$h_{\beta_\ell} = e_{\ell-1,\ell-1} + e_{\ell,\ell} - e_{2\ell-1,2\ell-1} - e_{2\ell,2\ell}.$$

We further let

$$x_{\beta_\ell} = e_{\ell,2\ell-1} - e_{\ell-1,2\ell},\ y_{\beta_\ell} = e_{2\ell-1,\ell} - e_{2\ell,\ell-1}.$$

The fundamental weights $\mu_1, \ldots, \mu_\ell$ corresponding to $h_{\beta_1}, \ldots, h_{\beta_\ell}$ are as follows. The first $\ell - 2$ of them are precisely those for B_ℓ, namely $\mu_k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$ for $1 \leq k \leq \ell - 2$. Now

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\ell-1}(h_{\beta_i}) = \begin{cases} 0 & \text{if } i \leq \ell - 2, \\ 1 & \text{if } i = \ell - 1 \text{ or } i = \ell. \end{cases}$$

It follows that

$$\mu_{\ell-1} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\ell-1} - \varepsilon_\ell)$$

and

$$\mu_\ell = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\ell-1} + \varepsilon_\ell)$$

are the last two fundamental weights.

The reflection $r_{\varepsilon_i - \varepsilon_j}$ interchanges ε_i and ε_j and fixes all other ε_k. The reflection $r_{\varepsilon_i + \varepsilon_j}$ sends ε_i to $-\varepsilon_j$, ε_j to $-\varepsilon_i$, and fixes all other ε_k. It follows that the Weyl group W_0 of D_ℓ is isomorphic to $T \rtimes S_\ell$, where T is the subgroup of C_2^ℓ of all $(a_1, \ldots, a_\ell)$, where $a_i = \pm 1$ and the number i such that $a_i = -1$ is even.

Whether -1_E is in W_0 or not depends on whether ℓ is even or not. Indeed, suppose $\ell = 2m$. Then

$$w_0 = r_{\varepsilon_m - \varepsilon_{m+1}} \cdots r_{\varepsilon_2 - \varepsilon_{\ell-1}} r_{\varepsilon_1 - \varepsilon_\ell} r_{\varepsilon_m + \varepsilon_{m+1}} \cdots r_{\varepsilon_2 + \varepsilon_{\ell-1}} r_{\varepsilon_1 + \varepsilon_\ell} = -1_E,$$

so every D_ℓ-module is self-dual. Suppose next $\ell = 2m + 1$. Then

$$w_0 = r_{\varepsilon_m - \varepsilon_{m+1}} \cdots r_{\varepsilon_2 - \varepsilon_{\ell-2}} r_{\varepsilon_1 - \varepsilon_{\ell-1}} r_{\varepsilon_m + \varepsilon_{m+1}} \cdots r_{\varepsilon_2 + \varepsilon_{\ell-2}} r_{\varepsilon_1 + \varepsilon_{\ell-1}}.$$

This sends ε_i to $-\varepsilon_i$, for $1 \le i < \ell$, but fixes ε_ℓ, so $w_0 \neq -1_E$. Note also that $\mu_k^* = \mu_k$, for $1 \le k \le \ell - 2$, $\mu_{\ell-1}^* = \mu_\ell$ and $\mu_\ell^* = \mu_{\ell-1}$.

5.3 Exterior powers of the natural module of D_ℓ

Let $V = \mathbf{C}^{2\ell+1}$ be the natural module for B_ℓ and let $e_0, e_1, \ldots, e_{2\ell}$ be the canonical basis of V. Let V_0 be the span of $e_1, \ldots, e_{2\ell}$. Then V_0 is a D_ℓ-submodule of V, clearly isomorphic to the natural module of D_ℓ.

We now turn our attention to the decomposition of the exterior powers $\Lambda^k(V_0)$, $1 \le k \le \ell$, into irreducible D_ℓ-submodules and their relationship to the fundamental modules of D_ℓ.

We note first that V_0 is irreducible. Moreover, V_0 is isomorphic to $V(\mu_1)$, if $\ell > 2$, and to $V(\mu_1 + \mu_2)$, if $\ell = 2$. Indeed,

$$\Pi(V_0) = \{\varepsilon_1, \ldots, \varepsilon_\ell, -\varepsilon_1, \ldots, -\varepsilon_\ell\},$$

where each of these weights has multiplicity one in V_0. The highest weight amongst these is ε_1, attained at e_1. Since $\Pi(V_0)$ is a W_0-orbit, it follows that

$$V_0 = \langle e_1 \rangle_{D_\ell} \cong V(\varepsilon_1),$$

where $\varepsilon_1 = \mu_1$ if $\ell > 2$ and $\varepsilon_1 = \mu_1 + \mu_2$ if $\ell = 2$. More generally, we have the following

Theorem 5.3.1. *We have* $\Lambda^k(V_0) \cong V(\mu_k)$ *if* $1 \le k \le \ell - 2$ *and* $\Lambda^k(V_0) \cong V(\mu_\ell + \mu_{\ell-1})$ *if* $k = \ell - 1$.

Proof. By above, we may assume that $1 < k \leq \ell - 1$. We also know from above that $\Pi(V_0) = \{\varepsilon_1, \ldots, \varepsilon_\ell, -\varepsilon_1, \ldots, -\varepsilon_\ell\}$, where each of these weights has multiplicity one in V_0. It follows that the weights of $\Lambda^k(V_0)$ are sums of k distinct weights of V_0. It is readily seen that $\varepsilon_1 + \cdots + \varepsilon_k$ is higher than all other weights of $\Lambda^k(V_0)$. Moreover, $\varepsilon_1 + \cdots + \varepsilon_k$ has multiplicity one in $\Lambda^k(V_0)$ and is attained at $v = e_1 \wedge \cdots \wedge e_k$. It follows that $\langle v \rangle_{D_\ell} \cong V(\varepsilon_1 + \cdots + \varepsilon_k)$, where $\varepsilon_1 + \cdots + \varepsilon_k = \mu_k$ if $k < \ell - 1$ and $\varepsilon_1 + \cdots + \varepsilon_k = \mu_\ell + \mu_{\ell-1}$ if $k = \ell - 1$. It remains to verify that $\Lambda^k(V_0) = \langle v \rangle_{D_\ell}$.

Let $\lambda \in E$ be dominant integral satisfying $\varepsilon_1 + \cdots + \varepsilon_k > \lambda$. It is not difficult to see that $\lambda = \mu_{k-2}, \mu_{k-4}, \ldots,$ where $\mu_0 = 0$ (see an entirely analogous argument for C_ℓ in Lemma 7.1.1). It suffices to see that $\Lambda^k(V_0)$ has no maximal vectors of any of these weights. To this end, for each $1 \leq i \leq [k/2]$ let $S(i)$ be the set of all sequences $(J_1, \ldots, J_i)$ such that $k - 2i < J_1 < \cdots < J_i \leq \ell$. To each $J \in S(i)$ we assign the element $E_J \in \Lambda^k(V_0)$ defined by

$$E_J = e_1 \wedge \cdots \wedge e_{k-2i} \wedge e_{J(1)} \wedge \cdots \wedge e_{J(i)} \wedge e_{J(1)+\ell} \cdots \wedge e_{J(i)+\ell}.$$

A basis for the weight space $(\Lambda^k(V_0))_{\mu_{k-2i}}$ is given by all E_J with $J \in S(i)$. By skillfully using $x_{\beta_1}, \ldots, x_{\beta_\ell}$ we see that no linear combination of the E_J is maximal. □

Note 5.3.2. *We have $\Lambda^\ell(V_0) \cong V(2\mu_\ell) \oplus V(2\mu_{\ell-1})$. More precisely, let U_1 and U_2be the D_ℓ-submodules of $\Lambda^\ell(V_0)$ generated by $e_1 \wedge \cdots \wedge e_{\ell-1} \wedge e_\ell$ and $e_1 \wedge \cdots \wedge e_{\ell-1} \wedge e_{2\ell}$, respectively. Then $\Lambda^\ell(V_0) = U_1 \oplus U_2$, with $U_1 \cong V(2\mu_\ell)$ and $U_2 \cong V(2\mu_{\ell-1})$*

5.4 Exterior powers of the natural module of B_ℓ

Here we analyze the B_ℓ-modules $\Lambda^k(V)$, $1 \leq k \leq \ell$, and their relationship to the natural module of B_ℓ.

Note first that the natural module V is irreducible, with $V \cong V(\lambda_1)$ if $\ell > 1$ and $V \cong V(2\lambda_1)$ if $\ell = 1$. Indeed, the $2\ell+1$ weights of V are $0, \varepsilon_1, \ldots, \varepsilon_\ell, -\varepsilon_1, \ldots, -\varepsilon_\ell$, attained at $e_0, e_1, \ldots, e_\ell, e_{\ell+1}, \ldots, e_{2\ell}$. Clearly ε_1 is higher than all other weights. It follows that $\langle e_1 \rangle_{B_\ell} \cong V(\varepsilon_1)$. Since the weights $\varepsilon_1, \ldots, \varepsilon_\ell, -\varepsilon_1, \ldots, -\varepsilon_\ell$ form a W-orbit, $e_1, \ldots, e_{2\ell}$ are all in $\langle e_1 \rangle_{B_\ell}$. But $(e_{0,1} - e_{\ell+1,0})e_1 = e_0$, so $\langle e_1 \rangle_{B_\ell} = V$.

Suppose next that $\ell \geq 2$. It is clear that $\mathbf{C}e_0$ is a D_ℓ-submodule of V, with $V = V_0 \oplus \mathbf{C}e_0$. More generally, for $k \geq 1$ let X (resp. Y) be the subspace of $\Lambda^k(V)$ spanned by all $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ (resp. $e_0 \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}$) such that $1 \leq i_1 < \cdots < i_k \leq 2\ell$ (resp. $1 \leq i_1 < \cdots < i_{k-1} \leq 2\ell$). Then clearly X and Y are D_ℓ-invariant submodules of $\Lambda^k(V) = X \oplus Y$, with $X \cong_{D_\ell} \Lambda^k(V_0)$ and $Y \cong_{D_\ell} \Lambda^{k-1}(V_0)$.

Theorem 5.4.1. *Suppose $2 \leq k \leq \ell$. Then $\Lambda^k(V)$ is irreducible, with $\Lambda^k(V) \cong V(\lambda_k)$ if $2 \leq k < \ell$, and $\Lambda^k(V) \cong V(2\lambda_\ell)$ if $k = \ell$.*

Proof. Let $v = e_1 \wedge \cdots \wedge e_k \in \Lambda^k(V)$. It is clear that v is a maximal vector of weight $\varepsilon_1 + \cdots + \varepsilon_k$. Thus $\langle v \rangle_{B_\ell} \cong V(\lambda_k)$ if $1 \leq k < \ell$ and $\langle v \rangle_{B_\ell} \cong V(2\lambda_\ell)$ if $k = \ell$.

We claim that $\langle v \rangle_{B_\ell}$ contains X. This is clear if $k < \ell$, since $v \in X$, which was previously shown to be generated as D_ℓ-module by v. If $k = \ell$ then by Note 5.3.2 X

is generated as D_ℓ-module by v and $w = e_1 \wedge \cdots \wedge e_{\ell-1} \wedge e_{2\ell}$. Since

$$(e_{0,\ell} - e_{2\ell,0})(e_{0,\ell} - e_{2\ell,0})v = -w,$$

the claim follows. Now from

$$(e_{0,1} - e_{\ell+1,0})v = e_0 \wedge e_2 \wedge \cdots \wedge e_k$$

and the previously shown irreducibility of Y as a D_ℓ-module, we deduce that $\langle v \rangle_{B_\ell}$ contains Y. Since $\Lambda^k(V) = X \oplus Y$, it follows that $\langle v \rangle_{B_\ell} = \Lambda^k(V)$. $\square$

Combining previous results we can see how $V(\lambda_k)$ decomposes as a D_ℓ-module. It is convenient to set $\mu_0 = 0$.

Corollary 5.4.2. *If* $1 \leq k \leq \ell - 2$ *then*

$$V(\lambda_k) \cong_{D_\ell} V(\mu_k) \oplus V(\mu_{k-1})$$

and

$$V(\lambda_{\ell-1}) \cong_{D_\ell} V(\mu_\ell + \mu_{\ell-1}) \oplus V(\mu_{\ell-2}),\ V(2\lambda_\ell) \cong_{D_\ell} V(2\mu_\ell) \oplus V(2\mu_{\ell-1}) \oplus V(\mu_\ell + \mu_{\ell-1}).$$

Regarding the spin modules, we will see in the next chapter that

$$V(\lambda_\ell) \cong_{D_\ell} V(\mu_\ell) \oplus V(\mu_{\ell-1}).$$

Note finally that the second half of exterior powers $\Lambda^k(V)$, $\ell < k \leq 2\ell$, do not furnish any new modules. Indeed, since B_ℓ is a subalgebra of $A_{2\ell}$, the exterior powers

$\Lambda^1(V), \ldots, \Lambda^{2\ell}(V)$ appear in pairs of dual B_ℓ-modules, as indicated in Chapter 4. But every B_ℓ-module is self-dual, so

$$\Lambda^{2\ell}(V) \cong \Lambda^1(V),\ \Lambda^{2\ell-1}(V) \cong \Lambda^2(V), \ldots$$

Exactly the same phenomenon occurs for the D_ℓ-modules $\Lambda^k(V_0)$. Indeed, D_ℓ is a subalgebra of $\mathfrak{sl}(V_0)$, so the exterior powers $\Lambda^1(V_0), \ldots, \Lambda^{2\ell}(V_0)$ appear in pairs of dual D_ℓ-modules. But each of them is self-dual, which follows from Theorem 5.3.1 and Note 5.3.2, together with our earlier description of duality amongst the fundamental modules of D_ℓ.

Chapter 6

The spin modules of B_ℓ and D_ℓ

In Chapter 5 we saw that $V(\lambda_\ell)$, $V(\mu_\ell)$ and $V(\mu_{\ell-1})$ cannot be obtained as exterior powers of the natural modules of B_ℓ and D_ℓ. The construction of these last modules requires another approach using the Clifford algebra instead of the exterior algebra.

6.1 Clifford algebras

There are several ways to define the Clifford algebra (see, for instance, [Lam05],[Car05], [Art57], [GW00]) [GM91]).

Recall a quadratic space is a pair (V, f) where V is a finite dimensional complex vector space and $f : V \times V \to \mathbf{C}$ is a symmetric bilinear form.

Definition 6.1.1. *Let (V, f) be a quadratic space. A unital associative algebra $\mathcal{A}$ is said to be **compatible with** (V, f) if V is a subspace of $\mathcal{A}$ and*

$$v^2 = f(v) \cdot 1, \quad \forall v \in V.$$

Definition 6.1.2. *Let (V, f) be a quadratic space. A unital associative algebra C is said to be a **Clifford algebra** for (V, f) if it is compatible with (V, f) and satisfies the following universal property: given any unital associative algebra $\mathcal{A}$ compatible with (V, f), there exists a unique homomorphism of algebras $\phi : C \to \mathcal{A}$ such that $\phi(v) = v$ for any $v \in V$.*

In order to construct a Clifford algebra, consider a vector space V of finite dimension n and let $f : V \times V \to \mathbf{C}$ be a symmetric bilinear form. For brevity we will write (v, w) instead of $f(v, w)$. Let $\mathcal{T}(V)$ be the tensor algebra of V and $\mathcal{J}$ be the two sided ideal of $\mathcal{T}(V)$ generated by the elements

$$v \otimes v - (v, v) \cdot 1, \quad \forall v \in V$$

We can identify V with its image in $C(V) = \mathcal{T}(V)/\mathcal{J}$, in which case $C(V)$ is a Clifford algebra for (V, f) (see [Lam05] for details).

Note that

$$\begin{aligned}(v + w) \otimes (v + w) - (v + w, v + w)1 &= v \otimes v - (v, v)1 + w \otimes w - (w, w)1 \\ &\quad + (v \otimes w + w \otimes v - 2(v, w)1)\end{aligned}$$

implies $v \otimes w + w \otimes v - 2(v, w)1 \in \mathcal{J}$ for all $v, w \in V$.

We fix a basis $v_1, \ldots, v_n$ of V. Then, as an associative algebra with 1, $C(V)$ is generated [Car05] by $v_1, \ldots, v_n$ subject to the relations:

$$v_i v_i = (v_i, v_i)1, \tag{6.10}$$

$$v_j v_i = -v_i v_j + 2(v_i, v_j)1.$$

It is also known [Lam05] that $\dim C(V) = 2^n$ and the set of elements

$$v_1^{\tau_1} v_2^{\tau_2} \ldots v_n^{\tau_n}, \quad \tau_i = 0, 1, \tag{6.11}$$

is a basis of $C(V)$. There is also a decomposition

$$C(V) = C(V)^+ \oplus C(V)^-,$$

where $C(V)^+$ is spanned by products of an even number of elements of V and $C(V)^-$ is spanned by products of an odd number of elements of V. We have $\dim C(V)^+ = \dim C(V)^- = 2^{n-1}$.

6.2 The Lie algebra of $C(V)$

As usual, the associative algebra $C(V)$ can be made into a Lie algebra $[C(V)]$ by defining $[xy] = xy - yx$.

Let L be the subspace of $[C(V)]$ spanned by the elements $[v, w]$ for all $v, w \in V$.

Thus L is spanned by all elements $[v_i, v_j]$, where $i < j$. Note that

$$[v_iv_j] = v_iv_j - v_jv_i \tag{6.12}$$

$$= v_iv_j + v_iv_j - 2(v_i, v_j) \cdot 1$$

$$= 2v_iv_j - 2(v_i, v_j) \cdot 1$$

Since these elements are linearly independent, we deduce

$$\dim L = \frac{n(n-1)}{2}.$$

Lemma 6.2.1. *Let $x, y, z \in V$. Then $[[xy]z] = 4(y,z)x - 4(x,z)y$.*

Proof. We have

$$[[xy]z] = [xy - yx, z] = xyz - zxy - yxz + zyx.$$

Applying the relations (6.10) we infer

$$[[xy]z] = -xzy + 2(y,z)x + xzy - 2(x,z)y + yzx - 2(x,z)y - yzx + 2(y,z)x$$

$$= 4(y,z)x - 4(x,z)y.$$

□

Lemma 6.2.2. *Let $x, y, z, w \in V$. Then*

$$[[xy][zw]] = 4(y,z)[xw] - 4(y,w)[xz] + 4(x,w)[yz] - 4(x,z)[yw].$$

Proof. Since $ad\,[xy]$ is a derivation of $[C(V)]$, we have

$$[[xy][zw]] = [[[xy]z]w] + [z[[xy]w]] = [[[xy]z]w] - [[[xy]w]z].$$

Hence, by Lemma 6.2.1,

$$\begin{aligned}[[xy][zw]] &= [4(y,z)x - 4(x,z)y, w] - [4(y,w)x - 4(x,w)y, z] \\ &= 4(y,z)[xw] - 4(x,z)[yw] - 4(y,w)[xz] + 4(x,w)[yz].\end{aligned}$$

□

Corollary 6.2.3. *L is a Lie subalgebra of $[C(V)]$.*

Proof. This follows immediately for Lemma 6.2.2. □

By means of the adjoint representation, $C(V)$ is a $[C(V)]-$module and, in particular, an $L-$module. Applying Lemma 6.2.1, we see that V is an $L-$submodule of $C(V)$.

Lemma 6.2.4. *If f is non degenerate then V is a faithful $L-$module.*

Proof. Suppose $x \in L$ satisfies $xV = 0$. We claim that $x = 0$. Indeed, we have $x = \sum_{1 \leq i < j \leq n} a_{ij}[v_i v_j]$ for some $a_{ij} \in \mathbf{C}$. Since f is non-degenerate there exists $w \in V$

such that $(v_1, w) = 1$ and $(v_j, w) = 0$ for $j > 1$. By Lemma 6.2.1 we have

$$\begin{aligned} 0 = [xw] &= \sum_{1 \le i < j \le n} a_{ij}[[v_i v_j]w] \\ &= 4 \sum_{1 \le i < j \le n} a_{ij}(v_j, w)v_i - 4 \sum_{1 \le i < j \le n} a_{ij}(v_i, w)v_j \\ &= -4 \sum_{1 \le i < j \le n} a_{ij}(v_i, w)v_j. \end{aligned}$$

It follows that $a_{12} = \cdots = a_{1n} = 0$. Successively repeating this process, starting with the list $v_2, \ldots, v_n$ until we reach the list v_{n-1}, v_n, we see that all $a_{ij} = 0$.

□

Lemma 6.2.5. *Let $x \in L$ and $v, w \in V$. Then*

$$([xv], w) + (v, [xw]) = 0. \tag{6.13}$$

Proof. It suffices to verify this for elements of L of the form $x = [yz]$ with $y, z \in V$. By Lemma 6.2.1:

$$\begin{aligned} ([xv], w) + (v, [xw]) &= ([[yz]v], w) + (v, [[yz]w]) \\ &= (4(z, v)y - 4(y, v)z, w) + (v, 4(z, w)y - 4(y, w)z) \\ &= 4(z, v)(y, w) - 4(y, v)(z, w) + 4(z, w)(v, y) - 4(y, w)(v, z) \\ &= 0. \end{aligned}$$

□

Let $R : L \to \mathfrak{gl}(V)$ be the representation of L associated to V, i.e. $R(x)v = [xv]$ for $x \in L$ and $v \in V$. Let $L(f)$ be the subalgebra of $\mathfrak{gl}(V)$ associated to f, as explained in §2.4. By Lemma 6.2.5

$$f(R(x)v, w) + f(v, R(x)w) = 0, \quad x \in L, v \in V,$$

which means that $R(L)$ is a subalgebra of $L(f)$. Suppose next that f is non-degenerate, in which case $L(f)$ is the orthogonal algebra. Then $R : L \to L(f)$ is a monomorphism, by Lemma 6.2.4. Let us compare the dimensions of L and $L(f)$.

$$\dim(L) = \frac{2\ell(2\ell-1)}{2} = 2\ell^2 - \ell = \dim(D_\ell) = \dim(L(f)), \quad n = 2\ell,$$
$$\dim(L) = \frac{(2\ell+1)(2\ell)}{2} = 2\ell^2 + \ell = \dim(B_\ell) = \dim(L(f)), \quad n = 2\ell+1.$$

We have proven

Theorem 6.2.6. *If f is non-degenerate then $R : L \to L(f)$ is an isomorphism, i.e. $L \cong D_\ell$ if $n = 2\ell$ and $L \cong B_\ell$ if $n = 2\ell+1$.*

Let us give an explicit matrix description of this. We suppose henceforth that $n = 2\ell + 1$ (this case will suffice since D_ℓ is naturally embedded in B_ℓ). Let $B = \{v_0, v_1, \dots, v_\ell, v_{-1}, v_{-2}, \dots v_{-\ell}\}$ be a basis of V and let $f : V \times V \to f$ be the non-degenerate bilinear form defined by

$$(v_0, v_0) = 1,$$
$$(v_i, v_{-i}) = 1, \quad 1 \le i \le \ell,$$
$$(v_i, v_j) = 0, \quad i \neq -j. \tag{6.14}$$

The matrix of f relative to B is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}.$$

Let $\psi : L \overset{R}{\to} L(f) \overset{M_B}{\to} L(M) = B_\ell$ be the matrix representation associated to R and the basis B. By above, ψ is an isomorphism. We next describe the elements of B_ℓ corresponding to the basis elements $[v_i v_j]$ of L, where $v_i, v_j \in B$. The answer could not be more satisfactory. Using Lemma 6.2.1 and the definition (6.14) of f we easily verify the following:

$$[v_i v_j] \mapsto 4(e_{i,j+\ell} - e_{j,i+\ell}),\ [v_{-i} v_{-j}] \mapsto 4(e_{i+\ell,j} - e_{j+\ell,i}), \quad 1 \le i < j \le \ell,$$

$$[v_i v_{-j}] \mapsto 4(e_{i,j} - e_{j+\ell,i+\ell}), \quad 1 \le i, j \le \ell,$$

$$[v_0 v_i] \mapsto 4(e_{0,i+\ell} - e_{i,0}),\ [v_0 v_{-i}] \mapsto 4(e_{0,i} - e_{i+\ell,0}), \quad 1 \le i \le \ell. \tag{6.15}$$

In particular, for $h_1, \ldots, h_\ell \in \mathbf{C}$, the element

$$h = \frac{1}{4} \sum_{1 \le i \le \ell} h_i [v_i v_{-i}] \in L \tag{6.16}$$

corresponds to the diagonal matrix $h = \mathrm{diag}(0, h_1, h_2, \ldots, h_\ell, h_{-1}, h_{-2}, \ldots h_{-\ell})$ of the maximal toral subalgebra H of B_ℓ previously considered. It follows that the elements (6.16) form a maximal toral subalgebra of L.

6.3 The spin module for B_ℓ

For $1 \le i \le \ell$ let $u_{-i} = v_0 v_{-i}$ and $u_i = v_0 v_i$. Let U be the subspace of $C(V)$ spanned by all elements of the form

$$u_{-j_1} u_{-j_2} \dots u_{-j_k} u_1 u_2 \dots u_\ell, \tag{6.17}$$

where $1 \le j_1 < j_2 < \cdots < j_k \le \ell$ and $0 \le k \le \ell$. Clearly $U \subseteq C(V)^+$. Using the relations (6.10) together with the fact that v_0 is orthogonal to all other vectors from B, as well as the fact that the vectors (6.11) are linearly independent, we see that the vectors (6.17) are linearly independent. Thus $\dim(U) = 2^\ell$.

Lemma 6.3.1. *We have the following relations in U:*

1. $u_i^2 = u_{-i}^2 = 0$ *for all* $1 \le i \le \ell$*;*
2. $u_i u_j = -u_j u_i$ *if* $1 \le i, j \le \ell$*;*
3. $u_{-i} u_{-j} = -u_{-j} u_{-i}$ *if* $1 \le i, j \le \ell$*;*
4. $u_{-i} u_j = -u_j u_{-i}$ *if* $1 \le i \ne j \le \ell$*;*
5. $u_i u_{-i} + u_{-i} u_i = -2 \cdot 1$ *for all* $1 \le i \le \ell$*.*

Proof. This follows from (6.10) and (6.14). □

Proposition 6.3.2. *U is a left ideal of $C(V)^+$.*

Proof. Note that $C(V)^+$ is generated as an algebra by the elements u_i and u_{-i} for $1 \le i \le \ell$. Thus, we just need to show that $u_i U \subseteq U$ and $u_{-i} U \subseteq U$. We consider a typical spanning vector (6.17) of U. We wish to show that u_i and u_{-i} send this vector back into U.

Suppose first, $i \in \{j_1, \dots, j_k\}$, i.e. $i = j_t$ for some t such that $1 \le t \le k$. Applying the relations given in Lemma 6.3.1, we have

$$\begin{aligned}
u_i u_{-j_1} \dots u_{-j_k} u_1 \dots u_\ell &= u_{j_t} u_{-j_1} \dots u_{-j_k} u_1 \dots u_\ell \\
&= (-1)^{t-1} u_{-j_1} \dots u_{j_t} u_{-j_t} \dots u_{-j_k} u_1 \dots u_\ell \\
&= (-1)^{t-1} u_{-j_1} \dots (-2 \cdot 1 - u_{-j_t} u_{j_t}) \dots u_{-j_k} u_1 \dots u_\ell \\
&= (-1)^t 2 u_{-j_1} \dots \widehat{u_{j_t}} \dots u_{-j_k} u_1 \dots u_\ell \\
&\quad + (-1)^t u_{-j_1} \dots u_{-j_t} u_{j_t} \dots u_{-j_k} u_1 \dots u_\ell \\
&= (-1)^t 2 u_{-j_1} \dots \widehat{u_{j_t}} \dots u_{-j_k} u_1 \dots u_\ell
\end{aligned}$$

since $u_{j_t}^2 = 0$ and we can move u_{j_t} next to itself at the cost of a sign per move. By the very same reason, if $i \notin \{j_1, \dots, j_k\}$ then

$$u_i u_{-j_1} \dots u_{-j_k} u_1 \dots u_\ell = 0.$$

Therefore,

$$u_i u_{-j_1} \dots u_{-j_k} u_1 \dots u_\ell = \begin{cases} (-1)^t 2 u_{-j_1} \dots \widehat{u_{-i}} \dots u_{-j_k} u_1 \dots u_\ell & \text{if } i = j_t \in \{j_1 \dots j_k\}, \\ 0 & \text{if } i \notin \{j_1 \dots j_k\}, \end{cases} \tag{6.18}$$

where $\widehat{u}$ means the term u is omitted. Similarly we obtain

$$u_{-i}u_{-j_1}\dots u_{-j_k}u_1\dots u_\ell = \begin{cases} 0 & \text{if } i \in \{j_1 \dots j_k\}, \\ (-1)^t u_{-j_1}\dots u_{-i}\dots u_{-j_k}u_1\dots u_\ell & \text{if } i \notin \{j_1\dots j_k\}, \end{cases} \tag{6.19}$$

where t is the number of indices $j_1, \dots, j_k$ which are less than i. This shows that $C(V)^+U \subseteq U$ □

We have just seen that U is a $C(V)^+-$module under left multiplication. Thus U is automatically a $[C(V)^+]-$module under the same action, and hence an $L-$module by restriction. We can view U as a B_ℓ-module via the isomorphism $\psi : L \to B_\ell$ described above.

Theorem 6.3.3. *The B_ℓ-module U is irreducible and isomorphic to $V(\lambda_\ell)$.*

Proof. Consider the vector $u_1 \cdots u_\ell$. We claim that $u_1 \cdots u_\ell$ is a weight vector of weight λ_ℓ. Indeed, let $1 \leq i \leq \ell$. Then

$$\begin{aligned}[v_i v_{-i}] &= v_i v_{-i} - v_{-i} v_i \\ &= v_i v_0 v_0 v_{-i} - v_{-i} v_0 v_0 v_i \\ &= -u_i u_{-i} + u_{-i} u_i.\end{aligned}$$

Therefore,

$$\begin{aligned}[v_i v_{-i}]u_1 \cdots u_\ell &= (-u_i u_{-i} + u_{-i} u_i) u_1 \cdots u_\ell \\ &= -u_i u_{-i} u_1 \cdots u_\ell \\ &= (u_{-i} u_i + 2 \cdot 1) u_1 \cdots u_\ell \\ &= 2 u_1 \cdots u_\ell.\end{aligned}$$

Let $h = \operatorname{diag}(0, h_1, \ldots, h_\ell, h_{-1}, \ldots, h_{-\ell}) \in H$. Then

$$\begin{aligned}h u_1 \cdots u_\ell &= \left(\frac{1}{4} \sum_{1 \le i \le \ell} h_i [v_i v_{-i}] \right) u_1 \cdots u_\ell \\ &= \frac{1}{2}(h_1 + \cdots + h_\ell) u_1 \cdots u_\ell \\ &= \lambda_\ell(h) u_1 \cdots u_\ell.\end{aligned}$$

The W-conjugates of $\lambda_\ell = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_\ell)$ are the 2^ℓ weights

$$\frac{1}{2} \sum_{i=1}^{\ell} \sigma_i \varepsilon_i,$$

where $\sigma_i = \pm 1$. Note that λ_ℓ is strictly higher than all other weights of these form, according the order defined in §3.4. Since $\dim(U) = 2^\ell$, we conclude that $U = \langle u_1 \cdots u_\ell \rangle_{B_\ell}$, where $u_1 \cdots u_\ell$ is a maximal vector of weight λ_ℓ, as required. □

Proposition 6.3.4. *Each vector $u_{-j_1} u_{-j_2} \ldots u_{-j_k} u_1 u_2 \ldots u_\ell$ considered in (6.17) is a weight vector of weight $\frac{1}{2} \sum_{1 \le i \le \ell} \sigma_i \varepsilon_i$, where $\sigma_i = -1$ if $i \in \{j_1, \ldots, j_k\}$ and $\sigma_i = 1$ otherwise.*

Proof. It follows from (6.15) that $[v_0 v_{-i}] \in L_{-\varepsilon_i}$ for all $1 \leq i \leq \ell$. Since

$$[v_0 v_{-i}] = v_0 v_{-i} - v_{-i} v_0 = u_{-i} + u_{-i} = 2u_{-i}, \quad 1 \leq i \leq \ell, \tag{6.20}$$

we deduce that

$$[v_0 v_{-j_1}] \cdots [v_0 v_{-j_k}] u_1 \cdots u_\ell = 2^k u_{-j_1} \cdots u_{-j_k} u_1 \cdots u_\ell$$

is a weight vector of weight

$$\lambda_\ell - (\varepsilon_{j_1} + \cdots + \varepsilon_{j_k}) = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_\ell) - (\varepsilon_{j_1} + \cdots + \varepsilon_{j_k}) = \frac{1}{2} \sum_{1 \leq i \leq \ell} \sigma_i \varepsilon_i.$$

□

6.4 The spin modules for D_ℓ

Theorem 6.4.1. *Consider U as a D_ℓ-module by restricting the action of B_ℓ. Then $U \cong U_0 \oplus U_1$, where U_0, U_1 are the irreducible D_ℓ-submodules respectively generated by $u_1 \cdots u_\ell$ and $u_{-\ell} u_1 \cdots u_\ell$. Moreover, $U_0 \cong V(\mu_\ell)$ and $U_1 \cong V(\mu_{\ell-1})$. Furthermore, U_0 is spanned by all vectors (6.17) with k even and U_1 by those with k odd. In particular, $\dim(U_0) = 2^{\ell-1} = \dim(U_1)$.*

Proof. We maintain the notation previously introduced for B_ℓ. Since $\Delta_0 \subset \Phi^+$ and $u_1 \cdots u_\ell$ is a maximal vector relative to Δ, it must also be so relative to Δ_0. Thus U_0 is an irreducible D_ℓ-module of highest weight μ_ℓ. Our description of W_0 ensures that all weights $\frac{1}{2}\sum_{1\le i\le \ell}\sigma_i\varepsilon_i$ of U, where the number i such that $\sigma_i = -1$ is even, are weights of U_0. It follows that U_0 contains all vectors (6.17) with k even.

We claim that $u_{-\ell}u_1\cdots u_\ell$ is also a maximal weight relative to Δ_0, necessarily of weight

$$\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{\ell-1}-\varepsilon_\ell),$$

by Proposition 6.3.4. Indeed, from (6.15) and (6.20) we deduce that $u_{-\ell}u_1\cdots u_\ell$ is a non-zero scalar multiple of $y_{\alpha_\ell}u_1\cdots u_\ell$. Now $[x_{\alpha_i}y_{\alpha_\ell}] = 0$ since $\alpha_i-\alpha_\ell\notin\Phi$ for $1\le i<\ell$. Therefore,

$$x_{\alpha_i}y_{\alpha_\ell}u_1\cdots u_\ell = y_{\alpha_\ell}x_{\alpha_i}u_1\cdots u_\ell = 0,\quad 1\le i<\ell.$$

The description of x_{β_ℓ} given in Section 5.2 together with (6.15) show that $x_{\beta_\ell}u_{-\ell}u_1\cdots u_\ell$ is a scalar multiple of $[v_\ell, v_{\ell-1}]u_{-\ell}u_1\cdots u_\ell$. On the other hand, we easily see that $[v_\ell, v_{\ell-1}] = -u_\ell u_{\ell-1}+u_{\ell-1}u_\ell$. Moreover,

$$u_{\ell-1}u_{-\ell}u_1\cdots u_\ell = 0,\quad u_{\ell-1}u_\ell u_{-\ell}u_1\cdots u_\ell = -2u_{\ell-1}u_1\cdots u_\ell = 0$$

as seen in the proof of Proposition 6.3.2. It follows that $[v_\ell, v_{\ell-1}]u_{-\ell}u_1\cdots u_\ell = 0$, whence $x_{\beta_\ell}u_{-\ell}u_1\cdots u_\ell = 0$. This proves our claim. The remaining assertions for U_1 follow exactly as those for U_0. This completes the proof. □

From our earlier work we also deduce that the spin modules for D_ℓ are self-dual when ℓ is even, and dual to each other when ℓ is odd.

Chapter 7

The fundamental modules of C_ℓ

Let $V = \mathbf{C}^{2\ell}$ and let $B = \{e_1, \ldots, e_\ell, e_{\ell+1}, \ldots, e_{2\ell}\}$ be the canonical basis of V. $f : V \times V \to \mathbf{C}$ be the non-degenerate skew-symmetric bilinear form whose matrix relative to B is

$$M = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

Let $L = L(f)$, the symplectic Lie algebra $\mathfrak{sp}(2\ell)$. We recall at this point that the isomorphism $M_B : \mathfrak{gl}(V) \to \mathfrak{gl}(2\ell)$ sends $L(f)$ onto $L(M) = C_\ell$. It will be convenient in this chapter to deal with linear operators rather than matrices.

Let H be the subalgebra of all $h \in L$ that diagonally on B. For $1 \leq i \leq \ell$ let $\varepsilon_i : H \to \mathbf{C}$ be the linear functional defined by $\varepsilon_i(h) = h_i$. Then H is a maximal toral subalgebra of L with associated system of roots

$$\Phi = \{\pm(\varepsilon_i + \varepsilon_j), 1 \leq i \leq j \leq \ell,\, \varepsilon_i - \varepsilon_j, 1 \leq i \neq j \leq \ell)\}.$$

The corresponding root spaces are

$$L_{\varepsilon_i+\varepsilon_j} = \mathbf{C}(e_{i,j+\ell} + e_{j,i+\ell}),\ L_{-(\varepsilon_i+\varepsilon_j)} = \mathbf{C}(e_{i+\ell,j} + e_{j+\ell,i}),\ L_{\varepsilon_i-\varepsilon_j} = \mathbf{C}(e_{i,j} - e_{j+\ell,i+\ell}).$$

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \alpha_\lambda = 2e_\ell$. Then

$$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$$

is a fundamental system. The corresponding elements $h_{\alpha_1}, \dots, h_{\alpha_\ell}$ of H are

$$h_{\alpha_1} = e_{1,1} - e_{2,2} - e_{\ell+1,\ell+1} + e_{\ell+2,\ell+2}, \dots, h_{\alpha_{\ell-1}} = e_{\ell-1,\ell-1} - e_{\ell,\ell} - e_{2\ell-1,2\ell-1} + e_{2\ell,2\ell}$$

and

$$h_{\alpha_\ell} = e_{\ell,\ell} - e_{2\ell,2\ell}.$$

Indeed, if we set

$$x_{\alpha_1} = e_{1,2} - e_{2+\ell,1+\ell}, \dots, x_{\alpha_{\ell-1}} = e_{\ell-1,\ell} - e_{2\ell,2\ell-1}, x_{\alpha_\ell} = e_{\ell,2\ell}$$

and

$$y_{\alpha_1} = e_{2,1} - e_{1+\ell,2+\ell}, \dots, y_{\alpha_{\ell-1}} = e_{\ell,\ell-1} - e_{2\ell-1,2\ell}, y_{\alpha_\ell} = e_{2\ell,\ell}$$

Then

$$[x_\alpha, y_\alpha] = h_\alpha \text{ and } \alpha(h_\alpha) = 2, \quad \alpha \in \Delta.$$

The corresponding fundamental weights are

$$\lambda_1 = \varepsilon_1, \lambda_2 = \varepsilon_1 + \varepsilon_2, \dots, \lambda_\ell = \varepsilon_1 + \dots + \varepsilon_\ell.$$

The reflection $r_{\varepsilon_i-\varepsilon_j}$ interchanges ε_i and ε_j and fixes all other ε_k. Moreover, the reflection $r_{2\varepsilon_i}$ sends ε_i to $-\varepsilon_i$ and fixes all other ε_k. It follows that the Weyl group W is isomorphic to $C_2^\ell \rtimes S_\ell$.

The element w_0 of W sending Δ to $-\Delta$ is

$$w_0 = r_{2\varepsilon_\ell} \cdots r_{2\varepsilon_1} = -1_E.$$

It follows that every L-module is self-dual.

7.1 The fundamental modules of C_ℓ

Note that V is irreducible and isomorphic to $V(\lambda_1)$. Indeed,

$$\Pi(V) = \{\varepsilon_1, \ldots, \varepsilon_\ell, -\varepsilon_1, \ldots, -\varepsilon_\ell\},$$

where each of these weights has multiplicity one in V. The highest weight amongst these is ε_1, attained at e_1. Since $\Pi(V)$ is a W-orbit, it follows that

$$V = \langle e_1 \rangle_L \cong V(\varepsilon_1) = V(\lambda_1).$$

Suppose next $2 \le k \le \ell$. We describe next the decomposition of $\Lambda^k(V)$ in irreducible submodules. We require the following

Lemma 7.1.1. *Suppose $2 \le k \le \ell$. Let $\lambda \in E$ be dominant and integral, and suppose that $\lambda < \lambda_k$. Then $\lambda = \lambda_{k-2}, \lambda_{k-4}, \ldots$, where λ_0 is understood to be 0.*

Proof. Suppose $\beta \in E$ satisfies $\beta >_{\mathbf{Q}} 0$. Then

$$\beta = a_1\alpha_1 + \cdots + a_{\ell-1}\alpha_{\ell-1} + a_\ell\alpha_\ell,$$

where $a_1, \ldots, a_{\ell-1}, a_\ell$ are non-negative rational numbers, not all equal to 0. We may write $\beta \in E$ in the form

$$\begin{aligned}\beta &= a_1(\varepsilon_1 - \varepsilon_2) + a_2(\varepsilon_2 - \varepsilon_3) + \cdots + a_{\ell-1}(\varepsilon_{\ell-1} - \varepsilon_\ell) + 2a_\ell\varepsilon_\ell \\ &= a_1\varepsilon_1 + (a_2 - a_1)\varepsilon_2 + \cdots + (a_{\ell-1} - a_{\ell-2})\varepsilon_{\ell-1} + (2a_\ell - a_{\ell-1})\varepsilon_\ell.\end{aligned}$$

It follows that if $\beta \in E$ satisfies $\beta >_{\mathbf{Q}} 0$ then, when writing β as a rational linear combination of $\varepsilon_1, \ldots, \varepsilon_\ell$, the first non-zero coefficient must be positive.

Now let λ be as stated. We have $\lambda = m_1\lambda_1 + \cdots + m_\ell\lambda_\ell$, where $m_1, \ldots, m_\ell$ are non-negative integers. We also know that $\lambda_k > \lambda$. As indicated in Chapter 3 all fundamental weights are rationally positive. Therefore, if $m_i > 0$ for $i \geq k$ then $\lambda_k >_{\mathbf{Q}} \lambda \geq_{\mathbf{Q}} \lambda_i$, which yields the contradiction $-(\varepsilon_{k+1} + \cdots + \varepsilon_i) >_{\mathbf{Q}} 0$. If $m_i > 0$ and $m_j > 0$ for $1 \leq i \neq j < k$ then $\lambda_k >_{\mathbf{Q}} \lambda \geq_{\mathbf{Q}} \lambda_i + \lambda_j$, which yields the contradiction $\varepsilon_{i+1} + \cdots + \varepsilon_k - (\varepsilon_1 + \cdots + \varepsilon_j) >_{\mathbf{Q}} 0$. Thus $m_i > 0$ for at most one index i, necessarily less than k. Arguing as above, we see that $m_i > 0$ forces $m_i = 1$. Thus $\lambda = \lambda_i$, where $0 \leq i < k$. But then $\lambda_k > \lambda_i$ forces $i \equiv k \mod 2$. □

As a way of illustration, we will consider first the decomposition of $\Lambda^2(V)$. We claim that $\Lambda^2(V)$ is not irreducible and

$$\Lambda^2(V) = \langle e_1 \wedge e_2 \rangle_L \oplus \langle e_1 \wedge e_{\ell+1} + \cdots + e_\ell \wedge e_{2\ell} \rangle_L, \tag{7.21}$$

where

$$\langle e_1 \wedge e_2 \rangle_L \cong V(\lambda_2) \text{ and } \langle e_1 \wedge e_{\ell+1} + \cdots + e_\ell \wedge e_{2\ell} \rangle_L \cong V(0). \tag{7.22}$$

Moreover, $\langle e_1 \wedge e_2 \rangle_L$ is the kernel of the canonical L-epimorphism $\theta : \Lambda^2(V) \to \mathbf{C}$

$$v \wedge w \mapsto f(v, w), \quad v, w \in V.$$

Indeed, the weights of $\Lambda^2(V)$ are sums of 2 distinct of $\Pi(V)$. The highest amongst these is $\lambda_2 = \varepsilon_1 + \varepsilon_2$, which has multiplicity one in $\Lambda^2(V)$ and is attained at $e_1 \wedge e_2$. By Lemma 7.1.1 the only dominant weight of $\Lambda^2(V)$ smaller that λ_2 is 0. The 0-weight space of $\Lambda^2(V)$ has basis $e_1 \wedge e_{\ell+1}, \ldots, e_\ell \wedge e_{2\ell}$. Up to scaling, the only linear combination of these vectors which produces a maximal vector is $e_1 \wedge e_{\ell+1} + \cdots + e_\ell \wedge e_{2\ell}$. This yields (7.21) and (7.22). A routine calculation shows that θ is an L-epimorphism. It obviously contains $e_1 \wedge e_2$, and hence $\langle e_1 \wedge e_2 \rangle_L$, in its kernel. Since the only other submodule of $\Lambda^2(V)$ is $\langle e_1 \wedge e_{\ell+1} + \cdots + e_\ell \wedge e_{2\ell} \rangle_L$, which is obviously not in $\ker \theta$, we infer $\ker \theta = \langle e_1 \wedge e_2 \rangle_L$.

Suppose next that $2 \leq k \leq \ell$. For $0 \leq i \leq [k/2]$ let $S(i)$ be the set of all sequences $(J_1, \ldots, J_i)$ such that $k - 2i < J_1 < \cdots < J_i \leq \ell$. To each $J \in S(i)$ we assign the element $E_J \in \Lambda^k(V)$ defined by

$$E_J = e_1 \wedge \cdots \wedge e_{k-2i} \wedge e_{J(1)} \wedge \cdots \wedge e_{J(i)} \wedge e_{J(1)+\ell} \cdots \wedge e_{J(i)+\ell}.$$

We further define $E_{\lambda_{k-2i}} \in \Lambda^k(V)$ by

$$E_{\lambda_{k-2i}} = \sum_{J \in S(i)} E_J.$$

Thus, for instance,

$$E_{\lambda_k} = e_1 \wedge \cdots \wedge e_k,$$

$$E_{\lambda_{k-2}} = e_1 \wedge \cdots \wedge e_{k-2} \wedge \left(\sum_{k-2<r\leq \ell} e_r \wedge e_{r+\ell} \right),$$

$$E_{\lambda_{k-4}} = e_1 \wedge \cdots \wedge e_{k-4} \wedge \left(\sum_{k-4<r<s\leq \ell} e_r \wedge e_s \wedge e_{r+\ell} \wedge e_{s+\ell} \right).$$

Theorem 7.1.2. *Suppose $2 \leq k \leq \ell$. Then*

$$\Lambda^k(V) \cong V(\lambda_k) \oplus V(\lambda_{k-2}) \oplus V(\lambda_{k-4}) \oplus \cdots,$$

where the last summand is $V(0)$ if k is even and $V(\lambda_1)$ if k is odd. Moreover,

$$\dim V(\lambda_k) = \binom{2\ell}{k} - \binom{2\ell}{k-2},$$

$$V(\lambda_{k-2i}) \cong \langle E_{\lambda_{k-2i}} \rangle_L, \quad 0 \leq i \leq [k/2],$$

and $\langle E_{\lambda_k} \rangle_L$ is the kernel of the epimorphism of L-modules $\theta_k : \Lambda^k(V) \to \Lambda^{k-2}(V)$

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{r<s} (-1)^{r+s-1} f(v_r, v_s) v_1 \wedge \cdots \wedge \widehat{v_r} \wedge \cdots \wedge \widehat{v_s} \wedge \cdots \wedge v_k.$$

Proof. By above $\Pi(V) = \{\varepsilon_1, \ldots, \varepsilon_\ell, -\varepsilon_1, \ldots, -\varepsilon_\ell\}$, where each of these weights has multiplicity one in V. It follows that the weights of $\Lambda^k(V)$ are sums of k distinct weights of V. It is readily seen that λ_k is higher than all other weights of $\Lambda^k(V)$. Moreover, λ_k has multiplicity one in $\Lambda^k(V)$ and is attained at E_{λ_k}. It follows that $\langle E_{\lambda_k} \rangle_L \cong V(\lambda_k)$.

By Lemma 7.1.1 the dominant weights of $\Lambda^k(V)$ smaller than λ_k are $\lambda_{k-2}, \lambda_{k-4}, \ldots$. Let $1 \leq i \leq [k/2]$. A basis for the weight space $(\Lambda^k(V))_{\lambda_{k-2i}}$ is given by all E_J with

$J \in S(i)$. Up to scaling, the only linear combination of the E_J which produces a maximal vector is $E_{\lambda_{k-2i}}$. It follows that

$$\Lambda^k(V) = \langle E_{\lambda_k} \rangle_L \oplus \langle E_{\lambda_{k-2}} \rangle_L \oplus \langle E_{\lambda_{k-4}} \rangle_L \oplus \cdots ,$$

where

$$\langle E_{\lambda_{k-2i}} \rangle_L \cong V(\lambda_{k-2i}), \quad 0 \leq i \leq [k/2].$$

A routine calculation shows that θ_k is an L-homomorphism. By definition $\theta(E_{\lambda_k}) = 0$ so $\langle E_{\lambda_k} \rangle_L \subseteq \ker \theta_k$. This also follows from the fact that $\lambda_k > \lambda_{k-2}$, the highest weight of $\Lambda^{k-2}(V)$. Let $1 \leq i \leq [k/2]$. By definition, the composite L-homomorphism $\theta_{k-(i-1)} \circ \cdots \circ \theta_k : \Lambda^k(V) \to \Lambda^{k-2i}(V)$ sends $E_{\lambda_{k-2i}}$ into a non-zero integral multiple of $e_1 \wedge \cdots \wedge e_{k-2i}$. In particular, $E_{\lambda_{k-2i}}$ is not in $\ker \theta_k$. Since the only irreducible submodules of $\Lambda^k(V)$ are those generated by $E_{\lambda_k}, E_{\lambda_{k-2}}, \ldots$, we obtain $\ker \theta_k = \langle E_{\lambda_k} \rangle_L$. Thus the image of θ_k is isomorphic to $V(\lambda_{k-2}) \oplus V(\lambda_{k-4}) \oplus \cdots \cong \Lambda^{k-2}(V)$. In particular, $\Lambda^{k-2}(V)$ has the same dimension as the image of θ_k, so θ_k is surjective. This yields the stated dimension for $\ker \theta_k = \langle E_{\lambda_k} \rangle_L \cong \Lambda^k(V)$. □

Chapter 8

Final Remarks

One of reasons that decided me to write about this topic was to leave a self-contained and elementary record of the fundamental modules for the classical Lie algebras, available to all those who are beginning their studies in the representation theory of Lie algebras and wish to utilize an accessible reference on the subject.

Secondly, while the classification of the irreducible modules for semisimple Lie algebras is well understood, the problem of classifying the indecomposable modules for general Lie algebras is wide open. For Lie algebras of the form $\mathfrak{s} \ltimes V$, where $\mathfrak{s}$ is semisimple and V is a non-trivial irreducible $\mathfrak{s}$-module, the irreducible $\mathfrak{s}$-modules play the role of building blocks for the indecomposable $\mathfrak{s} \ltimes V$-modules. The latter are formed by putting the former together in rather complicated way. Thus, a detailed knowledge of the irreducible modules for semisimple Lie algebras is necessary when attempting to classify indecomposable modules for more complicated Lie algebras.

Bibliography

[Art57] E. Artin. Geometric algebra. 1957.

[Car13] É. Cartan. Les groupes prejectifs qui ne laissent invariante aucune multiplicité plane. *Bull. Soc. Math. France*, 41(1):53–96, 1913.

[Car05] R.W. Carter. *Lie algebras of finite and affine type*. Cambridge University Press, 2005.

[DF04] D.S. Dummit and R.M. Foote. Abstract algebra. 2004.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer, 1991.

[GM91] J.E. Gilbert and M.A.M. Murray. *Clifford algebras and Dirac operators in harmonic analysis*, volume 26. Cambridge University Press, 1991.

[GW00] R. Goodman and N.R. Wallach. *Representations and invariants of the classical groups*, volume 68. Cambridge University Press, 2000.

[Hum97] J.E. Humphreys. *Introduction to Lie algebras and representation theory.* Springer, 1997.

[Lam05] T.Y. Lam. *Introduction to quadratic forms over fields.* American Mathematical Society, 2005.