SHIFT AND QUASI-SHIFT ENDOMORPHISMS ASSOCIATED WITH REPRESENTATIONS OF CUNTZ ALGEBRAS

A Thesis

Submitted to the Faculty of Graduate Studies and Research

In Partial Fulfillment of the Requirements

for the Degree of

Master of Science

In

Mathematics

University of Regina

By

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Regina, Saskatchewan

December 2012

UNIVERSITY OF REGINA

FACULTY OF GRADUATE STUDIES AND RESEARCH

SUPERVISORY AND EXAMINING COMMITTEE

Clifford Tyler Wood, candidate for the degree of Master of Science in Mathematics**,** has presented a thesis titled, ***Shift and Quasi-Shift Endomorphisms Associated with Representations of Cuntz Algebras,*** in an oral examination held on December 6, 2012. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

We present an overview of the main literature regarding *-endomorphisms of the von Neumann algebra $\mathscr{B}(H)$, the bounded linear operators on a separable, infinite-dimensional Hilbert space H. In doing so, we follow the established technique of exploiting the surjective correspondence with non-degenerate representations of Toeplitz-Cuntz algebras. Moreover, we introduce an unstudied class of endomorphism, which we call *quasi-shift* endomorphisms, and show that they are determined by the well-known class of *shift* endomorphisms. This is original research adapted from our recent pre-print *Quasi-shift Endomorphisms Associated with Representations of Cuntz Algebras* [13].

Acknowledgments

My sincere thanks goes to my thesis advisor, Dr. Remus Floricel. His breadth of knowledge is most impressive and his passion for mathematics is inspiring. I am truly grateful for the opportunity to work with him on this project and look forward to many future collaborations. Moreover, I wish to thank both Dr. Floricel and NSERC for providing financial support to pursue this project via an NSERC Discovery Grant.

I would also like to thank my colleagues, both faculty and fellow students, in Operator Algebras at the University of Regina. I have thoroughly enjoyed the our time together. In particular, I am grateful to Dr. Doug Farenick and Dr. Martín Argerami. I have been most fortunate for the opportunity to learn from such effective teachers and excellent researchers.

I am indebted to Dr. William Arveson and Dr. Marcelo Laca, whose work lead to the creation of this thesis, and wish to convey my respect and gratitude for their efforts.

Dedication

To my family, old and new, here and gone...

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction and Thesis Outline

The goal of this thesis is twofold. Firstly, to serve as a collective overview of significant results obtained in the study of *-preserving endomorphisms of $\mathscr{B}(H)$, the von Neumann algebra of bounded linear operators on a separable, infinite-dimensional Hilbert space H. The information contained within is sufficient to provide the reader with an introductory background to the study of *-endomorphisms of $\mathscr{B}(H)$, an important topic in many active areas of research [3],[15],[19],[16]. Secondly, to introduce and present original research regarding a class of *-endomorphism of $\mathscr{B}(H)$ that we believe to play a fundamental role in this subject area. In a recent pre-print [13] we consider this class of *-endomorphism defined on arbitrary von Neumann algebras.

To any unital *-endomorphism, $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$, we may associate a pair of von Neumann algebras,

$$\mathrm{Fix}(\alpha) \subseteq \mathrm{Tail}(\alpha),$$

called the *fixed-point* and *tail* algebras, respectively. As the name suggests, the fixed-point algebra is defined by

$$\mathrm{Fix}(\alpha) := \{X \in \mathscr{B}(H) \mid \alpha(X) = X\}, \tag{1.1}$$

and the tail algebra is given by the intersection of the range algebras $\alpha^k(\mathscr{B}(H))$,

$$\mathrm{Tail}(\alpha) := \bigcap_{k \geq 0} \alpha^k(\mathscr{B}(H)). \tag{1.2}$$

The class of unital *-endomorphisms known as *shift* endomorphisms is of particular importance in this thesis.

Definition 1.1.1. *A unital *-endomorphism, $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$, is said to be a* shift *if* $\mathrm{Tail}(\alpha) = \mathbb{C} \cdot 1$.

Our motive for creating this thesis was to introduce an unstudied class of unital *-endomorphism of $\mathscr{B}(H)$ which we refer to as *quasi-shift* endomorphisms.

Definition 1.1.2. *A unital *-endomorphism, $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$, is said to be a* quasi-shift *if* $\mathrm{Fix}(\alpha) = \mathrm{Tail}(\alpha)$.

Our terminology is chosen to reflect the fact that a quasi-shift endomorphism can be viewed as a generalization of a shift endomorphism. Namely, shift endomorphisms determine the intersection of the class of quasi-shift endomorphisms and the class of *ergodic* endomorphisms (i.e., unital *-endomorphisms such that $\mathrm{Fix}(\alpha) = \mathbb{C}\cdot 1$). Moreover, as a consequence of Theorem 4.1.4, quasi-shift endomorphisms are determined by shift endomorphisms.

Sufficient motivation for the study of quasi-shift endomorphisms can be obtained from the theory of Markov chains. The subalgebras $\mathrm{Fix}(\alpha)$ and $\mathrm{Tail}(\alpha)$ may be regarded as noncommutative analogues of the Poisson and tail boundaries, respectively [18],[3]. Markov chains for which these boundaries coincide are referred to as *steady chains* and have nice structural properties [18]. In fact, steady chains are completely characterized by a theorem of Kaimanovich. The main results of our research on quasi-shifts can be understood as non-commutative analogues of Kaimanovich's asymptotic characterization.

Theorem 1.1.3. [Kaimanovich,[18]] *A Markov chain (X, P, θ) on the state space (X, μ) with transition operator $P : L^\infty(X, \mu) \to L^\infty(X, \mu)$, initial distribution $\theta \prec \mu$ and one-dimensional distributions $\theta_n = \theta P^n, n \in \mathbb{N}$, is steady if and only if for all integers $n \geq 0, d > 0$, and all probability measures $\nu \prec \theta_n, \theta_{n+d}$ on X, we have*

$$\lim_{k\to\infty} ||\nu P^{k+d} - \nu P^k|| = 0.$$

Our analysis of quasi-shift endomorphisms mimics the pioneering work of Arveson, Laca, Bratteli and others regarding, most notably, shift endomorphisms. As such, it is fitting that a significant portion of this thesis be devoted to an exposition on the work of previous researchers regarding *-endomorphisms of $\mathscr{B}(H)$. In fact, this is most fortunate for the reader as, to our knowledge, no single resource exists which outlines the main literature regarding *-endomorphisms of the von Neumann algebra $\mathscr{B}(H)$. At the end of each chapter, we make a modest attempt to recognize and reference the contributions of these researchers. As such, we omit citations in the main body of the thesis.

The established technique for analysis of *-endomorphisms of $\mathscr{B}(H)$ is to exploit a correspondence with non-degenerate representations of Toeplitz-Cuntz algebras and, in particular, Cuntz algebras. In turn, a brief overview of these C^*-algebras and relevant subspaces constitutes the remainder of this introductory chapter.

We begin Chapter 2 by establishing the connection between *-endomorphisms of $\mathscr{B}(H)$ and non-degenerate representations of Toeplitz-Cuntz algebras (Corollary 2.1.3). In effect, we translate our study of endomorphisms into an equivalent study of representations. In fact, the majority of this thesis is written from the perspective of non-degenerate representations of Toeplitz-Cuntz algebras or of Cuntz algebras which, in turn, provided the motivation for the title.

The main body of the second chapter concerns analysis of two particular classes of non-degenerate representations of Toeplitz-Cuntz algebras referred to as *essential* and *singular* representations. Essential representations are shown to correspond to unital endomorphisms whereas singular representations correspond to endomorphisms which are far from unit-preserving called *completely non-unital*. Theorem 2.2.6 states that any non-degenerate representation of a Toeplitz-Cuntz algebra is given by some essential and singular representations, however, the latter is determined by the well-known Fock representation (see Section 2.3). In effect, we restrict our study to essential representations of Toeplitz-Cuntz algebras, or equivalently, to unital *-endomorphisms of $\mathscr{B}(H)$. Naturally, this provides a convenient transition into an exploration of shift endomorphisms. As our consideration of shift endomorphisms focuses on analysis of their conjugacy classes, we end Chapter 2 with a discussion of conjugacy and of cocycle conjugacy (outer conjugacy) for *-endomorphisms of $\mathscr{B}(H)$.

Chapter 3 is devoted to shift endomorphisms of $\mathscr{B}(H)$ and begins with a characterization by essential representations of Cuntz algebras (which are equivalent to essential representations of Toeplitz-Cuntz algebras). Theorem 3.2.7 is a significant result in the theory of *-endomorphisms of $\mathscr{B}(H)$ obtained by M. Laca and represents the main goal of this chapter. Crudely phrased, the theorem states that the conjugacy classes of shift endomorphisms correspond bijectively to classes of certain states of the canonical fixed-point subalgebra of a Cuntz algebra. As a result of this correspondence we provide an explicit example of a collection of non-conjugate shift endomorphisms indexed by an uncountable set. In effect, this halts further consideration of conjugacy for *-endomorphisms of $\mathscr{B}(H)$.

Our original contributions to this subject area are given in Chapter 4. In keeping with the theme of this thesis, we characterize the class of quasi-shift endomorphisms of $\mathscr{B}(H)$ by certain non-degenerate representations of Cuntz algebras which we call *steady* representations and, in Theorem 4.1.4, we give a direct integral decomposition of such representations. In Theorem 4.2.1, we give a Kaimanovich-type asymptotic characterization of quasi-shifts by states of $\mathscr{B}(H)$ for an arbitrary von Neumann algebra $M \subseteq \mathscr{B}(H)$. Moreover, we present a characterization of quasi-shift endomorphisms defined on the von Neumann algebra $\pi(\mathcal{O}_n)''$, where $\pi : \mathcal{O}_n \to \mathscr{B}(H)$ is a non-degenerate representation of the Cuntz algebra $\mathcal{O}_n$ on $n \geq 2$ generators (Theorem 4.2.4).

Chapter 5 contains a brief discussion on future work in this area.

1.2 Preliminaries and Relevant C^*-Algebras

At this point we would like to be explicit about the objects we study within this thesis and the recurring terminology we will use.

Let $\mathscr{B}(H)$ be the von Neumann algebra of bounded linear operators on a separable, infinite-dimensional Hilbert space H. We consider linear, multiplicative and involutive maps $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$, to which we have thus far referred to as *-endomorphisms of $\mathscr{B}(H)$.

Remark 1.2.1. *To avoid unnecessary repetition, in the remainder of this thesis we often use the phrase* "endomorphism" *in place of* "*-endomorphism of $\mathscr{B}(H)$".

At the risk of being over-simplistic, much of this thesis pertains to unital endomorphisms therefore we should also be explicit about this term. By *unital endomorphism*, we refer to an endomorphism, $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$, such that $\alpha(1) = 1$, where "1" represents the identity operator on the separable Hilbert space H. We caution the reader that this same notation is used to represent the identity element in a variety of spaces (similarly for the zero element) and we apologize for any confusion.

As mentioned in the introduction, much of this thesis concerns non-degenerate representations. For complete clarity, we define this term as well. A *non-degenerate representation* of a C^*-algebra $\mathfrak{U}$ on a Hilbert space K is a *-homomorphism (linear, multiplicative, involutive map) $\varphi : \mathfrak{U} \to \mathscr{B}(K)$ such that for any non-zero $\xi \in K$ there exists an element $u \in \mathfrak{U}$ satisfying $\varphi(u)\xi \neq 0$.

Remark 1.2.2. *All representations considered within this thesis are non-degenerate. To avoid unnecessary repetition, in the remainder of this thesis we often use the phrase* "representation" *in place of* "non-degenerate representation"*.*

In particular, we are interested in representations of Toeplitz-Cuntz and Cuntz algebras on the separable Hilbert space H. Therefore, we introduce some notation.

Notation 1.2.3. *The collection of representations of the Toeplitz-Cuntz algebra, $\mathcal{TO}_n$, on the separable Hilbert space H is denoted by* $\mathrm{Rep}(\mathcal{TO}_n, H)$*. Similarly for the Cuntz algebra, $\mathcal{O}_n$.*

1.2.1 Toeplitz-Cuntz and Cuntz Algebras

A collection of $2 \leq n \leq \infty, n \in \mathbb{N}$ isometries $\{s_i\}_{i=1}^n$, satisfying the relations

$$s_i^* s_j = \delta_{ij} \cdot 1, \qquad \sum_{i=1}^{n} s_i s_i^* < 1, \tag{1.3}$$

generates a universal C^*-algebra $\mathcal{TO}_n$ called the *Toeplitz-Cuntz algebra* [10]. For finitely many generators, the projection $1 - \sum_{i=1}^n s_i s_i^*$ generates a maximal ideal $\mathcal{J}_n$ which is isomorphic to the C*-algebra $\mathcal{K}$ of compact operators on a separable Hilbert space. The quotient $\mathcal{TO}_n/\mathcal{J}_n$ is the separable, purely-infinite and simple C^*-algebra $\mathcal{O}_n$ called the *Cuntz algebra* and we have the following short exact sequence

$$0 \to \mathcal{K} \to \mathcal{TO}_n \to \mathcal{O}_n \to 0.$$

One may define the Cuntz algebra $\mathcal{O}_n$ similarly to $\mathcal{TO}_n$ by using the relations

$$s_i^* s_j = \delta_{ij} \cdot 1, \qquad \sum_{i=1}^{n} s_i s_i^* = 1. \tag{1.4}$$

If $n = \infty$, the summation relation is replaced with $\sum_{i=1}^k s_i s_i^* \leq 1, \forall k \in \mathbb{N}$ and, in this case, $\mathcal{TO}_\infty$ is in fact the Cuntz algebra $\mathcal{O}_\infty$. For $2 \leq n \leq \infty$, J. Cuntz showed that $\mathcal{O}_n$ depends soley on the number of isometries chosen [10]. Explicitly, $\mathcal{O}_m \cong \mathcal{O}_n$ if and only if $m = n$. Contrary to the last comment, for simplicity we take the collection of isometries $\{s_i\}_{i=1}^n$ to be fixed in the remainder of this thesis.

The case $n = 1$ was studied by L. Coburn ten years before the work of Cuntz. In [8], it is shown that the structure of the resulting C^*-algebra is determined by three cases for the generator and, in each case, is isometrically *-isomorphic to a well-determined *-algebra. As such, we consider only Toeplitz-Cuntz and Cuntz algebras corresponding to two or more generating isometries.

1.2.2 The Generating Hilbert Space of $\mathcal{TO}_n$

The generating isometries $\{s_i\}_{i=1}^n, n \in \{2, 3, \ldots, \infty\}$ which define the Toeplitz-Cuntz algebra $\mathcal{TO}_n$ are an orthonormal basis for the n-dimensional Hilbert space

$$\mathcal{E} = \overline{\text{span}}^{||\cdot||}\{s_i\}_{i=1}^n \tag{1.5}$$

with inner product given by,

$$y^*x = \langle x, y\rangle 1, \quad \forall x, y \in \mathcal{E}. \tag{1.6}$$

One can deduce this claim from the proof of Theorem 2.1.1. The setting for the proof of Theorem 2.1.1 is that of $\mathscr{B}(H)$, however, the computations involved are identical to those required here. Here, we simply note that the space $\mathcal{E}$ generates the Toeplitz-Cuntz algebra as a C^*-algebra and is thus referred to as the *generating space of* $\mathcal{TO}_n$.

For our purposes, it is important to note that the representation theory of the Toeplitz-Cuntz algebra is fully captured by the representation theory of the generating space $\mathcal{E}$ (or, by extension, by the representation theory of any Hilbert space). More precisely, in [3] Arveson shows that if $\pi \in \text{Rep}(\mathcal{TO}_n, H)$ then the restriction $\pi\restriction_{\mathcal{E}}$ is an linear operator that satisfies the identity

$$\pi(y)^*\pi(x) = \langle x, y\rangle 1. \tag{1.7}$$

Conversely, any representation $\phi : \mathcal{E} \to \mathscr{B}(H)$, i.e. a mapping that satisfies (1.7), gives rise to a non-degenerate representation of $\mathcal{TO}_n$ on H such that $\pi \restriction_{\mathcal{E}} = \phi$. In turn, results obtained for $\pi(\mathcal{E})$ may be translated to $\pi(\mathcal{TO}_n)$ (or similarly to $\pi(\mathcal{O}_n)$).

The generating Hilbert space, $\mathcal{E}$, is a particular element of a countable family of Hilbert spaces contained by $\mathcal{TO}_n$ that we utilize in this thesis. We wish now to define this family. To this end, we begin by establishing some notation. Specifically, we will require a convenient means of indexing products of k elements chosen from the generating isometries $\{s_i\}_{i=1}^n$.

The indexing set for the product of k generating isometries is given by,

$$W_n^k = \{\mu = (\mu_1, \ldots, \mu_k) \mid \mu_i \in \{1, 2, \ldots, n\}\} \tag{1.8}$$

where we set $W_n^0 = \{0\}$. For $\mu = (\mu_1, \ldots, \mu_k) \in W_n^k$ we define s_μ as,

$$s_\mu := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} \qquad\qquad s_\mu^* := s_{\mu_k}^* s_{\mu_{k-1}}^* \cdots s_{\mu_1}^* \tag{1.9}$$

and $s_0 := 1$. With this, we represent the set of products of k isometries chosen from the generators $\{s_i\}_{i=1}^n$ of the Toeplitz-Cuntz algebra by

$$\mathcal{W}_n^k = \{s_\mu \mid \mu \in W_n^k\}. \tag{1.10}$$

For each $k \in \mathbb{N}$, the space

$$\mathcal{E}^k = \overline{\operatorname{span}}^{\|\cdot\|} \mathcal{W}_n^k \tag{1.11}$$

with the inner product (1.6) defined for elements of $\mathcal{E}^k$, is an n^k-dimensional Hilbert space with orthonormal basis $\mathcal{W}_n^k$. Here again, one may justify this claim by mimicking the relevant parts of the proof of Theorem 2.1.1 and exploiting the Kronecker delta relationship which defines the Toeplitz-Cuntz algebra. We note simply that $\mathcal{E}^k$ corresponds to (1.5) for $k = 1$ and moreover, for each $k \in \mathbb{N}$, (1.11) can be realized as a k-fold tensor product of the generating space, $\mathcal{E}$.

If U is a unitary operator on $\mathcal{E}$ there is a unique automorphism, $\gamma_U : \mathcal{TO}_n \to \mathcal{TO}_n$, corresponding to U such that

$$\gamma_U(x) = Ux, \quad \forall x \in \mathcal{E}. \tag{1.12}$$

This follows from the fact that $\{Us_i\}_{i=1}^n$ satisfies (1.3) and generates a C^*-algebra isomorphic to $\mathcal{TO}_n$. Such maps are called *quasi-free* automorphisms and if the unitary is of the form $U = \lambda \cdot 1$, $\lambda \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ then γ_U is called a *gauge automorphism*. The group

$$\gamma = (\gamma_\lambda)_{\lambda \in \mathbb{T}} \tag{1.13}$$

defined by gauge automorphisms is referred to as the *gauge group*. The gauge group may be used to define a fundamental subalgebra of the Cuntz algebra $\mathcal{O}_n$ which plays an important role in the study of shift endomorphisms.

1.2.3 The Fixed-point Subalgebra of $\mathcal{O}_n$

The *fixed-point algebra* of the action of the gauge group is defined by

$$\mathcal{F}_n := \{x \in \mathcal{O}_n \mid \gamma_\lambda(x) = x, \forall \lambda \in \mathbb{T}\}. \tag{1.14}$$

A more explicit definition for the fixed-point algebra may be developed using the indexing set given in the previous section. Denote by $\mathcal{F}_n^k$ the span of all words of the form $s_\mu s_\nu^*$ for $\mu, \nu \in W_n^k$. Then $\mathcal{F}_n^k \cong M_{n^k}(\mathbb{C})$ via

$$s_\mu s_\nu^* = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} s_{\nu_k}^* \cdots s_{\nu_2}^* s_{\nu_1}^* \mapsto e_{\mu_1\nu_1} \otimes \cdots \otimes e_{\mu_k\nu_k}$$

where $\{e_{ij}\}_{i,j=1}^n$ are the canonical matrix units for the matrix algebra $M_n(\mathbb{C})$. One can easily see that

$$\mathcal{F}_n = \overline{\bigcup_{k=0}^{\infty} \mathcal{F}_n^k}^{||\cdot||} \tag{1.15}$$

and from this formulation it is clear that $\mathcal{F}_n$ is a UHF-subalgebra of $\mathcal{O}_n$ of type n^∞. It should be noted that $\mathcal{F}_\infty$ is not uniformly-hyperfinite but rather an approximately-finite dimensional (AF) algebra. Moreover, the space $\mathcal{F}_\infty^k$ is isomorphic to the C^*-algebra $\mathcal{K}$ of compact operators on an infinite-dimensional, separable Hilbert space for each $k \in \mathbb{N}$ [10].

To supplement the last comment, let K be an infinite-dimensional, separable Hilbert space. For each $i \in \mathbb{N}$, let $\mathcal{K}_i$ be an isomorphic copy of the compact operators on K. If for each $j \in \mathbb{N}$, D_j denotes the subalgebra generated by

$$T_1 \otimes \cdots \otimes T_j \otimes 1 \cdots, \quad T_i \in \mathcal{K}_i, i \in \{1, 2, \ldots, j\},$$

then $\mathcal{F}_\infty$ may be realized as the limit of this sequence

$$D_0 + D_1 + \cdots + D_j$$

as $j \to \infty$. Moreover, a state ω of $\mathcal{F}_n$ is determined by the elements of this sequence and may be extended to a unique state $\hat{\omega}$ of $\mathcal{K}$ such that $\hat{\omega} \restriction_{D_j}$ is a normal state of D_j.

1.3 Comments & Citations

Our exposition is based primarily on two sources ([2] and [20]) and we rely heavily on [20]. M. Laca's paper, *Endomorphisms of $\mathcal{B}(\mathcal{H})$ and Cuntz algebras*, is an excellent resource for any beginner wishing to study *-endomorphisms of the von Neumann algebra $\mathscr{B}(H)$. As a result, the progression of topics in this thesis bears strong resemblance to that paper. In fact, [20] is based on Laca's research for his doctoral dissertation under the supervision of W. Arveson. In [20], Laca often refers to relevant work done by Arveson in [2] regarding semigroups of normal *-endomorphisms of $\mathscr{B}(H)$. We make note of a particular connection in Chapter 2.

A *-endomorphism of $\mathscr{B}(H)$ is *normal* if whenever $(X_i)_{i\in\mathcal{I}}$ is a bounded, increasing net of positive elements of $\mathscr{B}(H)$ then

$$\alpha(\sup_{i\in\mathcal{I}} X_i) = \sup_{i\in\mathcal{I}} \alpha(X_i).$$

In general, when studying endomorphisms defined on arbitrary von Neumann algebras we require normality, however, *-endomorphisms of $\mathscr{B}(H)$ are always normal ([24],Theorem V.5.1). Thus, we freely omit this condition in our exposition.

Another significant paper concerning *-endomorphisms of $\mathscr{B}(H)$ is [6]. In [6], O. Bratteli, P. Jorgensen, and G. Price consider only unital *-endomorphisms of $\mathscr{B}(H)$ and achieve similar results to that which Laca presents in [20]. Much of that paper focuses on developing techniques and concepts for differentiating between endomorphisms which possess invariant states and those which do not. This is, at least partially, motivated by R. Powers' result regarding conjugacy classes of shift endomorphisms having invariant pure states ([22], Theorem 2.3). Powers' paper [22] is another notable work concerning shift endomorphisms of $\mathscr{B}(H)$.

Nearly every statement made in Section 1.2.1 and Section 1.2.3 is nicely justified in K. Davidson's book [11] (pp. 144 - 155). Of particular note are Proposition V.4.2, Lemma V.5.2 and Theorem V.6.6. Naturally, Cuntz's original ground-breaking work, *Simple C^*-Algebras Generated by Isometries*, is also an excellent resource.

Chapter 2

Endomorphisms of $\mathscr{B}(H)$ and Representations of Toeplitz-Cuntz Algebras

This entire chapter, and much of the study of *-endomorphisms of $\mathscr{B}(H)$, relies on their correspondence with non-degenerate representations of Toeplitz-Cuntz algebras. Our first goal is to establish this connection. In the remainder of the chapter, we consider endomorphisms from the perspective of representations of Toeplitz-Cuntz algebras. Specifically, we focus our attention on two particular types of representations called *essential* and *singular*. In Section 2.2 (Theorem 2.2.6) we show that any representation decomposes into these two types and Section 2.3 is devoted to proving that singular representations correspond to the Fock representation. Lastly, we introduce general results for conjugacy and cocycle conjugacy of endomorphisms.

2.1 Realization of Endomorphisms by Representations of Toeplitz-Cuntz Algebras

As mentioned on multiple occasions, endomorphisms are determined by representations of Toeplitz-Cuntz algebras. The proof of this statement will follow easily from a result obtained by Arveson during his early work regarding semigroups of endomorphisms [2]. Here, the fact that an endomorphism is necessarily normal is implicit in the proof of the following theorem.

Theorem 2.1.1. *Let $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$ be an endomorphism. Then there exists an $n \in \mathbb{N} \cup \{\infty\}$, and a family of isometries $\{S_i\}_{i=1}^n$ on H satisfying $S_i^* S_j = \delta_{ij} \cdot 1$ such that*

$$\alpha(X) = \sum_{i=1}^{n} S_i X S_i^*, \quad \forall X \in \mathscr{B}(H). \tag{2.16}$$

Moreover, the linear space of operators

$$\mathcal{E}(\alpha) = \{A \in \mathscr{B}(H) \mid \alpha(X)A = AX, \, \forall X \in \mathscr{B}(H)\} \tag{2.17}$$

is a Hilbert space with orthonormal basis $\{S_i\}_{i=1}^n$, where the inner product is given by

$$B^* A = \langle A, B \rangle_\alpha 1, \quad \forall A, B \in \mathcal{E}(\alpha). \tag{2.18}$$

In the case $n = \infty$, the convergence of (2.16) is taken to be with respect to the strong operator topology.

Proof. We may view an arbitrary endomorphism $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$ as a representation of the von Neumann algebra $\mathscr{B}(H)$ on the separable Hilbert space H. Any such representation can be written as a countable direct sum of mutually disjoint irreducible representations. Since any irreducible representation is unitarily equivalent to the identity representation, the form $\alpha(X) = \sum_{i=1}^{n} S_i X S_i^*$ follows where $\{S_i\}_{i=1}^{n}$ are as in the statement of the theorem.

Next, we show that the space $\mathcal{E}(\alpha) = \{A \in \mathscr{B}(H) \mid \alpha(X)A = AX,\ \forall X \in \mathscr{B}(H)\}$ is a Hilbert space with respect to the inner product (2.18). Firstly, if $A, B \in \mathcal{E}(\alpha)$ then

$$B^*AX = B^*\alpha(X)A = XB^*A, \quad \forall X \in \mathscr{B}(H)$$

and therefore $B^*A = \lambda_{B^*A} \cdot 1$ where $\lambda_{B^*A} \in \mathbb{C}$. One can easily check that the map $\langle \cdot, \cdot \rangle_\alpha : \mathcal{E}(\alpha) \times \mathcal{E}(\alpha) \to \mathbb{C}$ satisfies the axioms of an inner product. To show completeness, we set $A = B$ in (2.18) which yields

$$||A||^2 = ||A^*A|| = ||\langle A, A\rangle_\alpha 1|| = \langle A, A\rangle_\alpha ||1|| = \langle A, A\rangle_\alpha.$$

This implies that the operator norm is equivalent to the Hilbert space norm and completeness is inherited from $\mathscr{B}(H)$.

Now, we show that the isometries $\{S_i\}_{i=1}^{n}$ appearing in (2.16) are an orthonormal basis for the space $\mathcal{E}(\alpha)$. By (2.18) we have

$$\langle S_j, S_i\rangle_\alpha 1 = S_i^* S_j = \delta_{ij} \cdot 1$$

implying that the isometries are pairwise orthogonal. Moreover,

$$||S_i||^2 = \langle S_i, S_i\rangle_\alpha = \delta_{ii} = 1, \quad \forall i \in \{1, \ldots, n\}$$

shows that each is a unit vector. Lastly, we show that the sequence $\{S_i\}_{i=1}^n$ is maximal. To this end, we show that ${\{S_i\}_{i=1}^n}^\perp = \{0\}$. If $B \in {\{S_i\}_{i=1}^n}^\perp$, then $S_i^* B = 0$, for each $i \in \{1, \ldots, n\}$. Therefore,

$$BX = \alpha(X)B = \sum_{i=1}^n S_i X S_i^* B = 0, \quad \forall X \in \mathscr{B}(H).$$

If we take $X = 1$ then it follows that $B = 0$, which completes the proof. $\square$

Definition 2.1.2. *The* index *of an endomorphism $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$ refers to the dimension of the Hilbert space $\mathcal{E}(\alpha)$ given in (2.17). The collection of endomorphisms with index $n \in \{2, 3, \ldots, \infty\}$ is denoted by* $\mathrm{End}_n(\mathscr{B}(H))$.

Corollary 2.1.3. *Let $\alpha \in \mathrm{End}_n(\mathscr{B}(H))$. Then there exists a $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$ such that*

$$\alpha(X) = \sum_{i=1}^n \pi(s_i) X \pi(s_i)^*, \quad \forall X \in \mathscr{B}(H). \tag{2.19}$$

Moreover, the restriction of the representation $\pi : \mathcal{TO}_n \to \mathscr{B}(H)$ to $\mathcal{E}$ is a unitary operator onto the Hilbert space $\mathcal{E}(\alpha)$.

Proof. Let $\alpha \in \mathrm{End}_n(\mathscr{B}(H))$ and $\{S_i\}_{i=1}^n$ be as in the statement of Theorem 2.1.1. Viewing $C^*(S_1, \ldots, S_n)$ as a subalgebra of $\mathscr{B}(H)$, let $\pi : \mathcal{TO}_n \to C^*(S_1, \ldots, S_n)$ be the canonical isomorphism where $\pi(s_i) = S_i, \forall i \in \{1, \ldots, n\}$. Hence, by (2.16)

$$\alpha(X) = \sum_{i=1}^n \pi(s_i) X \pi(s_i)^*, \quad \forall X \in \mathscr{B}(H)$$

as desired.

Finally, the last statement of the theorem is clear after noting that the map $\pi\restriction_{\mathcal{E}}$ is an isomorphism which follows immediately from the fact that both $\mathcal{E}$ and $\mathcal{E}(\alpha)$ are n-dimensional Hilbert spaces. $\square$

Notation 2.1.4. *In light of Corollary 2.1.3, we represent the fact that an endomorphism $\alpha \in \mathrm{End}_n(\mathscr{B}(H))$ has the form (2.19) determined by $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$ with the notation, α_π.*

Any *-endomorphism of $\mathscr{B}(H)$ of index n is determined by a non-degenerate representation of $\mathcal{TO}_n$, however, as the following proposition states this representation is not unique. Nevertheless, the index of an endomorphism is an invariant and, as such, is taken to be fixed and having a value in the set $\{2, 3, \ldots, \infty\}$ unless otherwise stated.

Proposition 2.1.5. *Let $\alpha_\pi \in \mathrm{End}_m(\mathscr{B}(H))$ and $\beta_\sigma \in \mathrm{End}_n(\mathscr{B}(H))$. Then $\alpha_\pi = \beta_\sigma$ if and only if $m = n$ and $\pi = \sigma \circ \gamma_U$ where γ_U is a quasi-free automorphism.*

Proof. We begin by proving the forward implication. By the last comment of Corollary 2.1.3,

$$\mathcal{E}(\alpha_\pi) = \{A \in \mathscr{B}(H) \mid \alpha_\pi(X)A = AX, \, \forall X \in \mathscr{B}(H)\}$$

is isomorphic (as Hilbert spaces) to the the generating space $\mathcal{E}_m$ (1.5) of the Toeplitz-Cuntz algebra $\mathcal{TO}_m$; similarly for the endomorphism β_σ and $\mathcal{TO}_n$. Thus, if $\alpha_\pi = \beta_\sigma$

then $\mathcal{E}_m \cong \mathcal{E}(\alpha_\pi) = \mathcal{E}(\beta_\sigma) \cong \mathcal{E}_n$ implies $m = n$. Since the endomorphisms have equal index, we note that $\pi, \sigma \in \mathrm{Rep}(\mathcal{TO}_n, H)$ and let $\mathcal{E}$ represent the generating Hilbert space of $\mathcal{TO}_n$ as usual. Both π and σ are unitary operators on $\mathcal{E}$ thus the map $U = \sigma^{-1} \circ \pi$ is also a well-defined unitary operator on $\mathcal{E}$. One can easily see that $\pi = \sigma \circ \gamma_U$ on $\mathcal{E}$ and hence also on $\mathcal{TO}_n$. This establishes the forward implication.

We now prove the reverse implication. Suppose now that $m = n$ and $\pi = \sigma \circ \gamma_U$ for some quasi-free automorphism, γ_U. Then, $\pi = \sigma$ on $\mathcal{E}$ and we have that $\alpha_\pi(1) = \beta_\sigma(1)$. By Theorem 2.1.1 and the last line of Corollary 2.1.3,

$$(\alpha_\pi(X) - \beta_\sigma(X))A = A(X - X) = 0, \quad \forall A \in \mathcal{E}(\alpha) = \mathcal{E}(\beta), \forall X \in \mathscr{B}(H)$$

implies that $\alpha_\pi(X) = \beta_\sigma(X)$ on $\pi(\mathcal{E})H = \sigma(\mathcal{E})H$ (i.e., on the range of the projection $\alpha_\pi(1) = \beta_\pi(1)$). Thus,

$$(\alpha_\pi(X) - \beta_\sigma(X))\xi = (\alpha_\pi(X) - \beta_\sigma(X))\alpha_\pi(1)\xi = 0, \quad \forall X \in \mathscr{B}(H), \forall \xi \in H$$

which concludes the proof. □

Our study of *-endomorphisms of $\mathscr{B}(H)$ will, from this point onward, be replaced by the equivalent study of non-degenerate representations of Toeplitz-Cuntz algebras.

2.2 Essential and Singular Representations of Toeplitz-Cuntz Algebras

In this section, we consider two types of representations of Toeplitz-Cuntz algebras corresponding to unital endomorphisms and to endomorphisms which are far from unit-preserving.

Definition 2.2.1. *An endomorphism* $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$ *is called* completely non-unital *if* $\alpha^k(1) \to 0$ *as* $k \to \infty$ *in the strong operator topology.*

Naturally, we may consider the condition that defines a completely non-unital endomorphism in the context of representations of Toeplitz-Cuntz algebras.

Observation 2.2.2. *For any* $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$*, the iterates* $\alpha_\pi^k(1), k \in \mathbb{N}$ *determine a decreasing sequence of projections onto the closed vector spaces* $\overline{\pi(\mathcal{E}^k)H}$*. This is clear after noting that whenever* $\xi \in H$*, then*

$$\alpha^k(1)\xi = \sum_{s \in \mathcal{W}_n^k} \pi(s)\pi(s^*)\xi \in \overline{\pi(\mathcal{E}^k)H}$$

and for any $r \in \mathcal{W}_n^k$*,*

$$\alpha^k(1)\pi(r)\xi = \sum_{s \in \mathcal{W}_n^k} \pi(s)\pi(s)^*\pi(r)\xi = \pi(r)\xi.$$

As we have a decreasing sequence of closed vector spaces, the intersection

$$H_\infty = \bigcap_{k \geq 1}^{\infty} \overline{\pi(\mathcal{E}^k)H}$$

is thus a closed vector space. In turn, the sequence of projections $\{\alpha_\pi^k(1)\}_{k=1}^\infty$ *converges to* P_{H_∞}*, the projection corresponding to* H_∞.

Using Observation 2.2.2 one can easily prove the following proposition.

Proposition 2.2.3. *Let* $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$*. Then* α_π *is unital if and only if* $\overline{\pi(\mathcal{E})H} = H$*, and* α_π *is completely non-unital if and only if* $\bigcap_{k\geq 1}^{\infty} \overline{\pi(\mathcal{E}^k)H} = \{0\}$.

This characterization of unital and completely non-unital endomorphisms motivates some terminology at the level of representations.

Definition 2.2.4. *A representation* $\pi : \mathcal{TO}_n \to \mathscr{B}(H)$ *is said to be* essential *if* $\overline{\pi(\mathcal{E})H} = H$*. A representation is called* singular *if* $\bigcap_{k\geq 1}^{\infty} \overline{\pi(\mathcal{E}^k)H} = \{0\}$.

Observation 2.2.5. *For* $n < \infty$*, Proposition 2.2.3 suggests that* $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$ *is essential if and only if*

$$\pi(1 - \sum_{i=1}^{n} s_i s_i^*) = 0$$

or, equivalently, if and only if $\ker \pi = \mathcal{J}_n$*, where* $\mathcal{J}_n$ *is the ideal generated by the projection* $1 - \sum_{i=1}^n s_i s_i^*$*. It follows that every representation of the Cuntz algebra* $\mathcal{O}_n = \mathcal{TO}_n/\mathcal{J}_n$ *is essential for finite index,* n*. However, if* $n = \infty$ *then there exists representations of* $\mathcal{O}_\infty$ *that are not essential.*

In his work with semigroups of endomorphisms, Arveson obtained a decomposition for a representation of a continuous product system into singular and essential subrepresentations. The following theorem is an adaptation of Proposition 1.14 of [2] for the case of a single endomorphism. In effect, this theorem implies that any endomorphism is determined by a unital and completely non-unital component. With the correspondence given by Corollary 2.1.3, our study of endomorphisms may be replaced with an equivalent study of essential and singular representations of Toeplitz-Cuntz algebras.

Theorem 2.2.6. *Let $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$. Then there exists a unique decomposition*

$$\pi = \pi_E \oplus \pi_S$$

such that π_E is essential and π_S is singular. Moreover, this decomposition is central.

Proof. We aim to define the subrepresentations π_E and π_S. To this end, we take $H_\infty = \bigcap_{k\geq 1}^{\infty} \overline{\pi(\mathcal{E}^k)H}$ as in Observation 2.2.2 and define

$$\pi \restriction_E := \pi \restriction_{H_\infty} \qquad \text{and} \qquad \pi_S := \pi \restriction_{H_\infty^\perp} .$$

Then,

$$H_\infty \supseteq \overline{\pi(\mathcal{E}^k)H_\infty} = \overline{\pi(\mathcal{E}^k)P_{H_\infty}H} = P_{H_\infty}\overline{\pi(\mathcal{E}^k)H} \supseteq P_{H_\infty}H_\infty = H_\infty$$

shows that π_E is essential. Similarly,

$$\bigcap_{k\geq 0} \overline{\pi_S(\mathcal{E}^k)H_\infty^\perp} = \bigcap_{k\geq 0} \overline{\pi(\mathcal{E}^k)H_\infty^\perp} \supseteq H_\infty^\perp \cap H_\infty = \{0\}$$

confirms singularity for π_S.

To prove the decomposition is central, we firstly prove that H_∞ is invariant under $\pi(\mathcal{E})$. This follows from Observation 2.2.2 and the following computation:

$$\pi(\mathcal{E})\overline{\pi(\mathcal{E}^k)H} \subseteq \overline{\pi(\mathcal{E})\pi(\mathcal{E}^k)H} = \overline{\pi(\mathcal{E}^{k+1})H} \subseteq \overline{\pi(\mathcal{E}^k)H}, \quad \forall k \in \mathbb{N}.$$

Similarly, H_∞ is invariant under $\pi(\mathcal{E})^*$ as

$$\pi(\mathcal{E})^*\overline{\pi(\mathcal{E}^{k+1})H} \subseteq \overline{\pi(\mathcal{E}^k)H}, \quad \forall k \in \mathbb{N}.$$

Since the vector space H_∞ is invariant under both $\pi(\mathcal{E})$ and $\pi(\mathcal{E})^*$, it is invariant under $\pi(\mathcal{TO}_n)$ and $P_{H_\infty} \in \pi(\mathcal{TO}_n)'$. Moreover, we have that $\alpha^k(1) \in (\pi(\mathcal{E}) \cup \pi(\mathcal{E})^*)''$ for each $k \in \mathbb{N}$. By Observation 2.2.2, it follows that $P_{H_\infty} \in (\pi(\mathcal{E}) \cup \pi(\mathcal{E})^*)''$, which finishes the proof. $\square$

We end this section by exploiting the Gelfand-Naimark-Segal construction to characterize essential and singular representations (or, equivalently unital and completely non-unital endomorphisms) in terms of states of Toeplitz-Cuntz algebras. In particular, essential and singular representations of the Toeplitz Cuntz algebra $\mathcal{TO}_n$ can be determined by the asymptotic behavior of the canonical backward shift on the dual space $\mathcal{TO}_n^*$ of $\mathcal{TO}_n$.

Definition 2.2.7. *The* backward shift *on $\mathcal{TO}_n^*$ is the linear map $\alpha^* : \mathcal{TO}_n^* \to \mathcal{TO}_n^*$ defined by*

$$\alpha^*\omega(x) = \sum_{i=1}^{n} \omega(s_i x s_i^*), \quad \forall \omega \in \mathcal{TO}_n^*, \forall x \in \mathcal{O}_n. \tag{2.20}$$

Observation 2.2.8. *(1) The map $\alpha^* : \mathcal{TO}_n^* \to \mathcal{TO}_n^*$ is a contractive operator whose iterates are given by*

$$\alpha^{*k}\omega(x) = \sum_{s\in\mathcal{W}_n^k} \omega(sxs^*), \quad \forall\omega \in \mathcal{TO}_n^*, \forall x \in \mathcal{O}_n \tag{2.21}$$

for each $k \in \mathbb{N}$.

(2) For $n < \infty$, α^* is the adjoint of the canonical endomorphism of $\mathcal{TO}_n$

$$\alpha : x \mapsto \sum_{i=1}^{n} s_i x s_i^*.$$

(3) One can use (2.20) to define a backward shift on the dual on the Cuntz algebra $\mathcal{O}_n$, as well as on the dual of the algebra $\mathcal{F}_n$. We will use the notation α^* in these cases as well.

(4) The name "backward shift" is motivated by the fact that $\alpha^*\omega = \omega_2 \otimes \omega_3 \otimes \cdots$, when $\omega = \omega_1 \otimes \omega_2 \otimes \cdots$ is any product state of $\mathcal{F}_n$, $n \in \mathbb{N}$.

Theorem 2.2.9. *Let $\omega : \mathcal{TO}_n \to \mathbb{C}$ be a state with corresponding GNS representation $\pi : \mathcal{TO}_n \to \mathscr{B}(H)$. Then π is essential if and only if $||\alpha^{*k}\omega|| = 1$ for all $k \geq 1$, and π is singular if and only if $||\alpha^{*k}\omega|| \to 0$ as $k \to \infty$.*

Proof. Let Ω be a cyclic vector corresponding to a state ω of $\mathcal{TO}_n$ with GNS representation $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$. Then,

$$\begin{aligned} ||\alpha^{*k}\omega|| &= \alpha^{*k}\omega(1) \\ &= \sum_{s\in\mathcal{W}_n^k} \langle \pi(s)\pi(s)^*\Omega, \Omega\rangle \\ &= \langle \sum_{s\in\mathcal{W}_n^k} \pi(s)\pi(s)^*\Omega, \Omega\rangle \\ &= \langle \alpha^k(1)\Omega, \Omega\rangle. \end{aligned}$$

By Observation 2.2.2, $\alpha^k(1) \to P_{H_\infty}$ as $k \to \infty$. Thus by the computation above, $||\alpha^{*k}\omega||$ converges to $\langle P_{H_\infty}\Omega, \Omega\rangle = ||P_{H_\infty}\Omega||^2$ as $k \to \infty$. Therefore, $||\alpha^{*k}\omega|| = 1$ for each $k \geq 0$ if and only if $||P_{H_\infty}\Omega|| = 1$. Similarly, $||\alpha^{*k}\omega|| \to 0$ if and only if $P_{H_\infty}\Omega = 0$. Since the decomposition of π is central, it follows that $P_{H_\infty}\Omega$ is cyclic for π_E and $P_{H_\infty^\perp}\Omega$ is cyclic for π_S. Therefore, π is essential if and only if $P_{H_\infty^\perp}\Omega = 0$, and π is singular if and only if $P_{H_\infty}\Omega = 0$. □

2.3 Singular Representations of Toeplitz-Cuntz Algebras

In this section, we show that a singular representation of a Toeplitz-Cuntz algebra corresponds to a multiple of the left regular representation of the Toeplitz-Cuntz algebra on the full Fock space up to unitary equivalence. To this end, let $\mathcal{E}$ be the n-dimensional Hilbert space (1.5) for any $2 \leq n \leq \infty$. Let $F_\mathcal{E}$ denote the full *Fock* *space* over $\mathcal{E}$,

$$F_{\mathcal{E}} = \bigoplus_{k=0}^{\infty} \mathcal{E}^{\otimes k}$$

where $\mathcal{E}^{\otimes k} = \underbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{k \text{ times}}$, $k \in \mathbb{N}$ and $\mathcal{E}^{\otimes 0}$ is the one dimensional space spanned by a unit vector Ω. For any $e \in \mathcal{E}$, we define the left creation operators $\ell_k(e)$ by

$$\begin{aligned} \ell_k(e) : \mathcal{E}^{\otimes k} &\rightarrow \mathcal{E}^{\otimes k+1} \quad k \geq 1 \\ \ell_k(e)(e_1 \otimes \cdots \otimes e_k) &:= e \otimes e_1 \otimes \cdots \otimes e_k \\ \ell_k(e)\Omega &:= e. \end{aligned}$$

The left creation operators give rise to the mapping $\ell : \mathcal{E} \rightarrow \mathscr{B}(F_{\mathcal{E}})$,

$$\ell(e) = \bigoplus_{k=1}^{\infty} \ell_k(e). \tag{2.22}$$

One can easily see that ℓ is a representation of the Hilbert space $\mathcal{E}$ on $F_{\mathcal{E}}$ (see also [12]), and thus it satisfies (1.7). In particular, it generates a unique non-degenerate representation of the Toeplitz-Cuntz algebra $\mathcal{TO}_n$ on $F_{\mathcal{E}}$, denoted by the same letter.

Definition 2.3.1. *The non-degenerate representation $\ell : \mathcal{TO}_n \rightarrow \mathscr{B}(F_{\mathcal{E}})$ will be referred as the* Fock representation*.*

Proposition 2.3.2. *The Fock representation $\ell : \mathcal{TO}_n \rightarrow \mathscr{B}(F_{\mathcal{E}})$ is singular.*

Proof. The subspaces $\overline{\ell(\mathcal{E}^k)F_{\mathcal{E}}} = \bigoplus_{i=k}^{\infty} \mathcal{E}^{\otimes i}$ converge to $\{0\}$ as $k \rightarrow \infty$ and thus $\bigcap_{k\geq 0}^{\infty} \overline{\ell(\mathcal{E}^k)F_{\mathcal{E}}} = \{0\}$. $\square$

Theorem 2.3.3. *Let $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$ and $\mathcal{E}$ be the generating Hilbert space of $\mathcal{TO}_n$. Then π is singular if and only if it is unitarily equivalent to a multiple of the Fock representation with multiplicity equal to* $\dim[\pi(\mathcal{E})H]^\perp$.

Proof. We begin by proving the forward implication. Thus, we proceed by constructing a unitary which implements the desired equivalence.

Let $\pi \in \mathrm{Rep}(\mathcal{TO}_n, H)$ be singular with endomorphism $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$. Let N_k be the subspace corresponding to the projection $\alpha_\pi^k(1) - \alpha_\pi^{k+1}(1)$ for each $k \geq 0$,

$$N_k = (\alpha_\pi^k(1) - \alpha_\pi^{k+1}(1))H = \overline{\pi(\mathcal{E}^k)N_0}, \qquad N_0 = \overline{\mathrm{span}}(\pi(\mathcal{E})H)^\perp. \tag{2.23}$$

The subspaces N_k are mutually orthogonal as the projections $\alpha_\pi^k(1) - \alpha_\pi^{k+1}(1)$, $k \geq 0$, are mutually orthogonal. Moreover, since $\alpha_\pi^k(1) \to 0$ the summation

$$(1 - \alpha_\pi(1)) + (\alpha_\pi(1) - \alpha_\pi^2(1)) + (\alpha_\pi^2(1) - \alpha_\pi^3(1)) + \dots$$

converges strongly to the identity map. Thus, the Hilbert space H decomposes as a direct sum,

$$H = \bigoplus_{k=0}^{\infty} N_k. \tag{2.24}$$

Let ℓ be the Fock representation and for each $k \geq 0$, let $U_k : \mathcal{E}^{\otimes k} \otimes N_0 \to N_k$ be defined by

$$U_k(e_1 \otimes \dots \otimes e_k \otimes \xi) := \pi(e_1 \cdots e_k)\xi.$$

Each U_k extends to a unitary operator onto N_k since for any $\xi, \eta \in H$,

$$\begin{aligned}
\langle \pi(e_1 \cdots e_k)\xi, \pi(f_1 \cdots f_k)\eta \rangle &= \langle \pi(f_k^* \cdots f_1^* e_1 \cdots e_k)\xi, \eta \rangle \\
&= (\prod_{i=1}^{k} \langle e_i, f_i \rangle) \langle \xi, \eta \rangle \\
&= \langle e_1 \otimes \cdots \otimes e_k \otimes \xi, f_1 \otimes \cdots \otimes f_k \otimes \eta \rangle.
\end{aligned}$$

Taking the direct sum gives a unitary operator

$$U : F_{\mathcal{E}} \otimes N_0 = \bigoplus_{k=1}^{\infty} \mathcal{E}^{\otimes k} \otimes N_0 \to \bigoplus_{k=0}^{\infty} N_k = H.$$

Now, we show that the unitary operator $U : F_{\mathcal{E}} \otimes N_0 \to H$ implements the desired equivalence. For any $x, e_1, \ldots, e_k \in \mathcal{E}$ and $\xi \in N_0$,

$$\begin{aligned}
\pi(x)U(e_1 \otimes \cdots \otimes e_k \otimes \xi) &= \pi(x)\pi(e_1 \cdots e_k)\xi \\
&= U(x \otimes e_1 \otimes \cdots e_k \otimes \xi) \\
&= U(\ell(x)(e_1 \otimes \cdots \otimes e_k) \otimes 1 \cdot \xi) \\
&= U(\ell \otimes 1)(x)(e_1 \otimes \cdots \otimes e_k \otimes \xi)
\end{aligned}$$

therefore $\pi(\mathcal{E}) = U(\ell \otimes 1)U^{-1}(\mathcal{E})$. Hence, the representations are equivalent on $\mathcal{TO}_n$ which establishes the forward implication.

The converse follows by applying Proposition 2.3.2, and the proof is complete. $\square$

2.4 Conjugacy and Cocycle Conjugacy for Endomorphisms

Much of the existing literature regarding *-endomorphisms of $\mathscr{B}(H)$ pertains to classification of types of endomorphisms. Naturally, the first notion of equivalence for endomorphisms that is considered is conjugacy. We end this chapter with a brief exposition on conjugacy, and a weaker notion of equivalence called cocycle conjugacy, for arbitrary *-endomorphisms of $\mathscr{B}(H)$.

Definition 2.4.1. *Two endomorphisms $\alpha, \beta : \mathscr{B}(H) \to \mathscr{B}(H)$ are* conjugate *if there is an automorphism, θ, of $\mathscr{B}(H)$ such that $\alpha = \theta^{-1} \circ \beta \circ \theta$. Two endomorphisms $\alpha, \beta : \mathscr{B}(H) \to \mathscr{B}(H)$ are said to be* cocycle-conjugate *if there is an automorphism $\psi : \mathscr{B}(H) \to \mathscr{B}(H)$ such that α is conjugate to $\beta \circ \psi$.*

As expected, we wish to translate conjugacy into the context of representations of Toeplitz-Cuntz algebras. For brevity, we introduce the following definition.

Definition 2.4.2. *Two representations $\pi, \sigma \in \mathrm{Rep}(\mathcal{TO}_n, H)$ are* quasi-free equivalent *if there exists a quasi-free automorphism γ_U such that $\pi \circ \gamma_U$ is unitarily equivalent to σ.*

Theorem 2.4.3. *Let α_π and β_σ be endomorphisms corresponding to $\pi \in \mathrm{Rep}(\mathcal{TO}_m, H)$ and $\sigma \in \mathrm{Rep}(\mathcal{TO}_n, H)$, respectively. Then α_π and β_σ are conjugate if and only if $m = n$ and π is quasi-free equivalent to σ. Moreover, α_π and β_σ are cocycle-conjugate if and only if $m = n$ and $\dim[1 - \alpha_\pi(1)] = \dim[1 - \beta_\sigma(1)]$.*

Proof. We begin by proving the first statement. The definition of conjugacy is equivalent to the existence of a unitary operator $U : H \to H$ such that

$$\alpha(X) = U^*\beta(UXU^*)U, \quad \forall X \in \mathscr{B}(H).$$

By Corollary 2.1.3 this implies,

$$\sum_{i=1}^{n} \sigma(s_i)X\sigma(s_i)^* = \sum_{i=1}^{n}(U\pi(s_i)U^*)X(U\pi(s_i)U^*)^*, \ \forall X \in \mathscr{B}(H).$$

Thus, α and β are conjugate if and only if σ and $U\pi(\cdot)U^*$ determine the same endomorphism. All that remains is to apply Proposition 2.1.5 and the proof of the first statement is complete.

Now, we prove the second statement. Since both α_π and β_σ act on H, cocycle-conjugacy implies that $\alpha_\pi = \beta_\sigma \circ \psi$ for some automorphism $\psi : \mathscr{B}(H) \to \mathscr{B}(H)$. Since automorphisms of $\mathscr{B}(H)$ are unit-preserving, we have that

$$1 - \alpha_\pi(1) = 1 - (\beta_\sigma \circ \psi)(1) = 1 - \beta_\sigma(1)$$

and therefore $\dim[1 - \alpha_\pi(1)] = \dim[1 - \beta_\sigma(1)]$. Moreover, we have that $\alpha_\pi(1) = \beta_\sigma(1)$ and hence $m = n$.

For the converse, let S be a partial isometry on H having initial projection $1 - \alpha_\pi(1)$ and final projection $1 - \beta_\sigma(1)$. Define a unitary operator on H by

$$W := \sum_{i=1}^{n} \sigma(s_i)\pi(s_i)^* + S.$$

Then,

$$\begin{aligned}
\beta_\sigma(X)W &= \sum_{j=1}^{n} \sigma(s_j)X\sigma(s_j)^*[\sum_{i=1}^{n} \sigma(s_i)\pi(s_i)^* + S] \\
&= \sum_{j=1}^{n} \sigma(s_j)X\pi(s_j)^* + \sum_{j=1}^{n} \sigma(s_j)X\sigma(s_j)^* S \\
&= \sum_{j=1}^{n} \sigma(s_j)X\pi(s_j)^* \\
&= \sum_{j=1}^{n} \sigma(s_j)X\pi(s_j)^* + S\sum_{j=1}^{n} \pi(s_j)X\pi(s_j)^* \\
&= [\sum_{i=1}^{n} \sigma(s_i)\pi(s_i)^* + S]\sum_{j=1}^{n} \pi(s_j)X\pi(s_j)^* \\
&= W\alpha_\pi(X),
\end{aligned}$$

for all $X \in \mathscr{B}(H)$, which implies that α_π and β_σ are cocycle conjugate. This completes the proof. □

Recall that Theorem 2.2.6 gave a decomposition for representations of Toeplitz-Cuntz algebras into essential and singular subrepresentations. In turn, this implies that an endomorphism is given by a unital and completely non-unital component. We may capture this result in regards to conjugacy.

Theorem 2.4.4. *Two endomorphisms are conjugate if and only if they have conjugate unital and non-unital components.*

Proof. The forward implication is obtained by restricting the isomorphism implementing conjugacy to each component.

For the converse, let $\alpha_\pi \in \mathrm{End}_m(\mathscr{B}(H)), \beta_\sigma \in \mathrm{End}_n(\mathscr{B}(H))$ where $\pi = \pi_E + \pi_S$ and $\sigma = \sigma_E + \sigma_S$ as in Theorem 2.2.6. Suppose that π_E is quasi-free equivalent to σ_E and that π_S is quasi-free equivalent to σ_S. It follows that $m = n$ and π_E is unitarily equivalent to $\pi_S \circ \gamma_U$ for some quasi-free automorphism γ_U. As both π_S and σ_S are singular (and hence both unitarily equivalent to a multiple of the Fock representation) we must also have that π_S is unitarily equivalent to $\sigma_S \circ \gamma_U$. Thus, π is quasi-free equivalent to $\sigma \circ \gamma_U$. Applying Proposition 2.4.3 concludes the proof. $\square$

2.5 Comments & Citations

In Section 2.1, Theorem 2.1.1 was used to prove the correspondence between endomorphisms and representation of Toeplitz-Cuntz algebras. This theorem appears as Proposition 2.1 in [2]. Combinations of Theorem 2.1.1 and Corollary 2.1.3 appear in [6] and [20] as Theorem 3.1 and Theorem 2.1 respectively. The statement of uniqueness for an endomorphism (Proposition 2.1.5) is taken from [[20], Proposition 2.2].

The index of an endomorphism given in Definition 2.1.2 is referred to as the *Powers index* in [6]. This terminology is a result of R. Powers' observation that the commutant $\alpha(\mathscr{B}(H))'$ of a unital endomorphism $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$ is a factor of type $\mathrm{I}_n, n \in \{2, 3, \ldots, \infty\}$. In [22], Powers refers to the index as the *multiplicity* of α.

The main body of this chapter was devoted to characterizing unital and completely non-unital endomorphisms. In doing so, we follow the flow of [20] in which Theorems 2.2.6, 2.3.3 and 2.2.9 all appear. As we already noted, Theorem 2.2.6 is a modification of Proposition 1.14 of [2] where Arveson obtains a decomposition of a *product system* into an essential and singular component. Arveson introduced product systems as a means of studying cocycle conjugacy in the realm of E-semigroups of endomorphisms of type I_∞ factors. A fundamental problem in the theory of E_0-semigroups is their classification up to cocycle conjugacy [3]. To avoid digression, we simply wish to note that cocycle conjugacy for semigroups of endomorphisms is a topic of great interest. The cocycle conjugacy portion of Theorem 2.4.3 (which appears as Propositions 2.2 and 2.3 of [20]) is a generalization of R. Powers' result ([22], Theorem 2.4) which considers only shift endomorphisms.

We wish to note the characterizations of unital and completely non-unital endomorphisms given in this chapter.

	Unital	**Completely Non-Unital**
Endomorphisms of $\mathscr{B}(H)$	$\alpha(1) = 1$	$\alpha^k(1) \to 0$ as $k \to \infty$
Representations of $\mathcal{TO}_n$	$\overline{\pi(\mathcal{E})H} = H$	$\bigcap_{k\geq 1}^{\infty} \overline{\pi(\mathcal{E}^k)H} = \{0\}$
States of $\mathcal{TO}_n$	$\|\|\alpha^{*k}\omega\|\| = 1,\ \forall k \geq 0$	$\|\|\alpha^{*k}\omega\|\| \to 0$ as $k \to \infty$

Chapter 3

Shift Endomorphisms

In the previous chapter, we proved that endomorphisms are determined by essential and singular representations of Toeplitz-Cuntz algebras. Moreover, Theorem 2.3.3 implies that unital endomorphisms have a richer structure than completely non-unital endomorphisms. Thus, the remainder of this thesis concerns unital endomorphisms or, equivalently, essential representations of Toeplitz-Cuntz algebras. For any $n \in \{2, 3, \ldots, \infty\}$, essential representations of $\mathcal{TO}_n$ correspond to essential representations of $\mathcal{O}_n$ as noted in Observation 2.2.5. In this chapter, we focus on the class of shift endomorphisms in regards to conjugacy. Our main goal is to establish a bijection between the conjugacy classes of shift endomorphisms of index n and certain pure states of the the fixed-point subalgebra $\mathcal{F}_n$ of the Cuntz algebra $\mathcal{O}_n$. We end this chapter with an example of non-conjugate shift endomorphisms.

3.1 Characterization of Shift Endomorphisms

In keeping with the theme of Chapter 2, one may obtain a characterization of ergodic and shift endomorphisms by representations of Cuntz algebras.

Notation 3.1.1. *The subsets of ergodic and shift endomorphisms of* $\mathrm{End}_n(\mathscr{B}(H))$ *are denoted by* $\mathrm{Erg}_n(\mathscr{B}(H))$ *and* $\mathrm{Shift}_n(\mathscr{B}(H))$*, respectively.*

Theorem 3.1.2. *Let* $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$ *where* $\pi : \mathcal{O}_n \to \mathscr{B}(H)$ *is an essential representation. Then,*

(1) $\alpha_\pi \in \mathrm{Erg}_n(\mathscr{B}(H))$ *if and only if* π *is irreducible, and*

(2) $\alpha_\pi \in \mathrm{Shift}_n(\mathscr{B}(H))$ *if and only if* $\pi \restriction_{\mathcal{F}_n}$ *is irreducible.*

This theorem will follow easily from a result due to M. Laca.

Proposition 3.1.3. *Let* $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$ *where* $\pi : \mathcal{O}_n \to \mathscr{B}(H)$ *is an essential representation. Then,*

$$\mathrm{Fix}(\alpha_\pi) = \pi(\mathcal{O}_n)' \tag{3.25}$$

$$\mathrm{Tail}(\alpha_\pi) = \pi(\mathcal{F}_n)' \tag{3.26}$$

Proof. Equation (3.26) is proven similar to equation (3.25) on the spaces $\pi(\mathcal{F}_n^k)$ and using the fact that $\mathcal{F}_n$ is the inductive limit of the spaces $\mathcal{F}_n^k$.

Equation (3.25) follows easily from a basic containment argument. If $X \in \pi(\mathcal{O}_n)'$ then

$$\begin{aligned}\alpha_\pi(X) &= \sum_{i=1}^{n} \pi(s_i) X \pi(s_i)^* \\ &= X \sum_{i=1}^{n} \pi(s_i)\pi(s_i)^* \\ &= X.\end{aligned}$$

Therefore $X \in \mathrm{Fix}(\alpha_\pi)$ and $\pi(\mathcal{O}_n)' \subseteq \mathrm{Fix}(\alpha_\pi)$.

Conversely, if $X \in \mathrm{Fix}(\alpha_\pi)$ then

$$\alpha_\pi(X) = \sum_{i=1}^{n} \pi(s_i) X \pi(s_i)^* = X$$

Multiplying on the left by $\pi(s_j)^*$ gives

$$X\pi(s_j)^* = \pi(s_j)^* X, \ j = 1, \ldots, n.$$

Similarly, multiplying by $\pi(s_j)$ on the right gives

$$\pi(s_j)X = X\pi(s_j), \ j = 1, \ldots, n.$$

Since X commutes with the generators of $\pi(\mathcal{O}_n)$ it commutes with any element of $\pi(\mathcal{O}_n)$. Therefore, $\mathrm{Fix}(\alpha_\pi) \subseteq \pi(\mathcal{O}_n)'$, and the proof is complete. $\square$

Proof of Theorem 3.1.2. The proof follows immediately from Proposition 3.1.3 and the definition of irreducibility. $\square$

Observation 3.1.4. *Using the well-known correspondence of pure states and irreducible representations, one can extend Theorem 3.1.3 to characterize ergodic and shift endomorphisms in terms of states of Cuntz algebras.*

We may also characterize shift endomorphisms in terms of states of the von Neumann algebra $\mathscr{B}(H)$. In Chapter 4, we give a similar condition for quasi-shift endomorphisms (Theorem 4.2.1).

Proposition 3.1.5. *Let $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$ where $\pi : \mathcal{O}_n \to \mathscr{B}(H)$ is an essential representation. Then $\alpha_\pi \in \mathrm{Shift}_n(\mathscr{B}(H))$ if and only if for any normal states ω_1 and ω_2 of $\mathscr{B}(H)$ we have*

$$\lim_{k\to\infty} ||(\omega_1 - \omega_2) \circ \alpha_\pi^k|| = 0. \tag{3.27}$$

Proof. We begin by proving the forward implication. Here, we note that for any normal linear functional ω of $\mathscr{B}(H)$,

$$||\omega \circ \alpha_\pi^k|| = ||\omega \restriction_{\alpha_\pi^k(\mathscr{B}(H))} ||, \quad \text{for each } k \in \mathbb{N}. \tag{3.28}$$

Let $k \in \mathbb{N}$ be fixed. Indeed, for every $X \in \mathscr{B}(H)$, we have

$$|\omega(\alpha_\pi^k(X))| \leq ||\omega \restriction_{\alpha_\pi^k(\mathscr{B}(H))} || \cdot ||\alpha_\pi^k(X)||.$$

On the other hand, if $X \in \alpha_\pi^k(\mathscr{B}(H))$ has norm one and

$$|\omega(X)| = ||\omega \restriction_{\alpha_\pi^k(\mathscr{B}(H))} ||,$$

then there exists an $X_0 \in \mathscr{B}(H)$ such that $X = \alpha_\pi^k(X_0)$ and $||X_0|| = ||X|| = 1$ and hence,

$$\begin{aligned} ||\omega \restriction_{\alpha_\pi^k(\mathscr{B}(H))} || &= |\omega(X)| \\ &= |\omega \circ \alpha_\pi^k(X_0)| \\ &\leq ||\omega \circ \alpha_\pi^k||. \end{aligned}$$

Therefore, we have (3.28) and

$$\lim_{k\to\infty} ||\omega \circ \alpha_\pi^k|| = \lim_{k\to\infty} ||\omega \restriction_{\alpha_\pi^k(\mathscr{B}(H))} || = ||\omega \restriction_{\mathrm{Tail}(\alpha_\pi)} ||.$$

Now, suppose $\alpha_\pi \in \mathrm{Shift}_n(\mathscr{B}(H))$ and that ω_1 and ω_2 are normal states of $\mathscr{B}(H)$. If we define $\omega' := \omega_1 - \omega_2$ then $\omega' \restriction_{\mathrm{Tail}(\alpha_\pi)} = 0$ and

$$\lim_{k\to\infty} ||(\omega_1 - \omega_2) \circ \alpha_\pi^k|| = ||\omega' \restriction_{\mathrm{Tail}(\alpha_\pi)} || = 0.$$

as required.

For the converse, we aim to show that $\bigcap_{k\geq 0} \alpha_\pi^k(\mathscr{B}(H)) = \mathbb{C} \cdot 1$ for which it suffices to show that for every self-adjoint normal linear functional λ such that $\lambda(1) = 0$ we have $\lambda(\mathrm{Tail}(\alpha_\pi)) = \{0\}$. By the Jordan decomposition ([21], Theorem 3.2.5), every self-adjoint normal linear functional that annihilates the identity operator must be a scalar multiple of the difference of two normal states. Hence, if ω' is such a functional then there exists an ω_1 and ω_2 such that $\omega' = \omega_1 - \omega_2$. Now, since

$$\alpha_\pi^k(\mathrm{Tail}(\alpha_\pi)) = \mathrm{Tail}(\alpha_\pi),$$

for each $X \in \mathrm{Tail}(\alpha_\pi)$ there exists operators $X_k \in \mathrm{Tail}(\alpha_\pi)$ such that

$$\alpha_\pi^k(X_k) = X \quad \text{and} \quad ||X_k|| = ||\alpha_\pi^k(X_k)|| = ||X||$$

for all $k \in \mathbb{N}$. Thus,

$$|\omega'(X)| = |(\omega_1 \circ \alpha_\pi^k - \omega_2 \circ \alpha_\pi^k)(X_k)| \leq ||\omega_1 \circ \alpha_\pi^k - \omega_2 \circ \alpha_\pi^k|| \cdot ||X||, \quad \forall k \in \mathbb{N}.$$

Therefore, if $\lim_{k\to\infty} ||(\omega_1 - \omega_2) \circ \alpha_\pi^k|| = 0$ then $\omega'(X) = 0, \forall X \in \mathrm{Tail}(\alpha_\pi)$ which concludes the proof. $\square$

3.2 Identification of Conjugacy Classes

The goal of this section is to establish Theorem 3.2.7. We prove that the conjugacy classes of $\mathrm{Shift}_n(\mathscr{B}(H))$ are in bijective correspondence with the quasi-free equivalence classes of essential quasi-invariant pure states of $\mathcal{F}_n$. To this end, we begin by defining terminology.

Definition 3.2.1. *A state ω of $\mathcal{F}_n$ is said to be* essential *if it admits an extension to a state $\widetilde{\omega}$ of $\mathcal{O}_n$ with an essential GNS representation.*

Observation 3.2.2. *It follows from Theorem 2.2.9 that a state ω of $\mathcal{F}_n$ is essential if and only if $||\alpha^{*k}\omega|| = 1, \forall k \geq 0$, i.e. if and only if $\alpha^{*k}\omega$ is a state for all $k \geq 0$. Moreover, we deduce from Observation 2.2.5 that any state of $\mathcal{F}_n$ is essential for $n < \infty$. Definition 3.2.1 is therefore relevant only for states of $\mathcal{F}_\infty$.*

We will be considering states which are quasi-equivalent to their image under the backward shift. We refer the reader to [17] for a full analysis of quasi-equivalence of states. Here, we simply note that two states ω and ϕ of a C^*-algebra $\mathfrak{U}$ are *quasi-equivalent* if their GNS-representations π_ω and π_ϕ are quasi-equivalent, i.e., there exists a *-isomorphism φ from $\pi_\omega(\mathfrak{U})''$ to $\pi_\phi(\mathfrak{U})''$ such that $\varphi(\pi_\omega(u)) = \pi_\phi(u)$ for all $u \in \mathfrak{U}$.

Definition 3.2.3. *An essential state ω of $\mathcal{F}_n$ is said to be* quasi-invariant *if it is quasi-equivalent to*

$$\alpha^*\omega = \sum_{i=1}^{n} \omega(s_i x s_i^*), \quad x \in \mathcal{F}_n.$$

Unfortunately, the backward shift α^* does not preserve purity; the state $\alpha^*\omega$ may not be pure even if ω is pure. However, by considering a different state we may preserve purity and characterize quasi-invariance of states in terms of unitary equivalence.

Lemma 3.2.4. *Let ω be a state of $\mathcal{F}_n$ and define the state $\omega'(x) : \mathcal{F}_n \to \mathbb{C}$ by*

$$\omega'(x) := \omega(s_1^* x s_1), \quad \forall x \in \mathcal{F}_n.$$

Then ω' is pure if and only if ω is pure. Moreover, ω is quasi-invariant if and only if it its GNS representation is unitarily equivalent to the GNS representation of ω'.

Proof. See Lemma 4.2 of [20]. □

The following theorem is key to establishing the bijective correspondence of Theorem 3.2.7 by showing that shift endomorphisms can be characterized by quasi-invariance of states.

Theorem 3.2.5. *The GNS representation $\pi : \mathcal{F}_n \to \mathscr{B}(H)$ associated with an essential pure state ω extends to a representation $\widetilde{\pi} : \mathcal{O}_n \to \mathscr{B}(H)$ if and only if ω is quasi-invariant. Moreover, $\widetilde{\pi}$ is unique up to gauge automorphism.*

Proof. We begin by proving the forward implication. Let $\widetilde{\pi}$ be an extension to $\mathcal{O}_n$ of the GNS representation $\pi : \mathcal{F}_n \to \mathscr{B}(H)$ associated with an essential pure state ω of $\mathcal{F}_n$ with cyclic vector Ω. One can easily see that

$$\alpha^*\omega(x) = \sum_{i=1}^{n} \langle \pi(x)\pi(s_i^*)\Omega, \pi(s_i^*)\Omega \rangle, \quad x \in \mathcal{F}_n.$$

Since ω is pure, it follows from Proposition 10.3.14 of [17] that $\alpha^*\omega$ is quasi-equivalent to ω, as desired.

Now we prove the converse. Let ω be an essential quasi-invariant pure state of $\mathcal{F}_n$ and let $\pi : \mathcal{F}_n \to \mathscr{B}(H)$ be its GNS representation with cyclic vector Ω. We aim to construct the extension $\widetilde{\pi} : \mathcal{O}_n \to \mathscr{B}(H)$. To this end, we construct a collection of isometries $\{S_i\}_{i=1}^n$ satisfying (1.4). The canonical isomorphism from $\mathcal{O}_n$ to $C^*(S_1, \ldots, S_n)$ then gives a suitable representation which restricts to π.

To construct the isometries, we use the state defined in Lemma 3.2.4. Since ω is quasi-invariant, π it is unitarily equivalent to the GNS representation of ω', so there is a unit vector $\Omega' \in H$ such that $\omega'(x) = \omega(s_1^* x s_1) = \langle \pi(x)\Omega', \Omega' \rangle$ for all $x \in \mathcal{F}_n$. Define S_1 on vectors of the form $\pi(x)\Omega$ by

$$S_1 : \pi(x)\Omega \mapsto \pi(s_1 x s_1^*)\Omega', \quad x \in \mathcal{F}_n.$$

For any $x, y \in \mathcal{F}_n$,

$$\begin{aligned}
\langle S_1\pi(x)\Omega, S_1\pi(y)\Omega \rangle &= \langle \pi(s_1 x s_1^*)\Omega', \pi(s_1 y s_1^*)\Omega' \rangle \\
&= \langle \pi(s_1 y^* s_1^* s_1 x s_1^*)\Omega', \Omega' \rangle \\
&= \omega'(s_1 y^* x s_1^*) \\
&= \omega(y^* x) \\
&= \langle \pi(x)\Omega, \pi(y)\Omega \rangle
\end{aligned}$$

Since $\pi(\mathcal{F}_n)\Omega$ is dense in H, the map S_1 extends to an isometry on H which we will also denote by S_1. Moreover, with a little computation (which we omit for simplicity) one can show that $\pi(s_1 s_1^*) = S_1 S_1^*$.

If we define $S_i := \pi(s_i s_1^*)S_1$ for all $i = 1, 2, \ldots, n$ then,

$$\begin{aligned}
S_i^* S_i &= S_1^* \pi(s_1 s_i^*)\pi(s_i s_1^*)S_1 \\
&= S_1^* \pi(s_1 s_1^*) S_1 \\
&= S_1^*(S_1 S_1^*) S_1 \\
&= 1
\end{aligned}$$

shows that each S_i is an isometry. Moreover, since ω is essential we have that

$$\begin{aligned}\sum_{i=1}^{n} S_i S_i^* &= \sum_{i=1}^{n} \pi(s_i s_1^*) S_1 S_1^* \pi(s_1 s_i^*) \\ &= \sum_{i=1}^{n} \pi(s_i s_1^* s_1 s_1^* s_1 s_i^*) \\ &= \sum_{i=1}^{n} \pi(s_i s_i^*) \\ &= 1\end{aligned}$$

Therefore, the isometries $\{S_i\}_{i=1}^n$ satisfy (1.4). Viewing $C^*(S_1, \ldots, S_n)$ as a subalgebra of $\mathscr{B}(H)$, let $\widetilde{\pi} : \mathcal{O}_n \to C^*(S_1, \ldots, S_n)$ be the canonical isomorphism (essential representation of $\mathcal{O}_n$ on H) such that $\widetilde{\pi}(s_i) = S_i$ for each $i \in \{1, \ldots, n\}$.

We now aim to show that $\widetilde{\pi} : \mathcal{O}_n \to \mathscr{B}(H)$ is an extension of $\pi : \mathcal{F}_n \to \mathscr{B}(H)$. To prove that $\widetilde{\pi} \restriction_{\mathcal{F}_n} = \pi$ it suffices to show that the representations are equivalent on elements of the form rs^* where $r, s \in \mathcal{W}_n^k$ for all $k \in \mathbb{N}$. This is done by induction on the value, k.

For $k = 0$, we trivially have that $\widetilde{\pi}(rs^*) = 1 = \pi(rs^*)$ for $r, s \in \mathcal{W}_n^0 = \{1\}$.

Assume now that the representations are equivalent up to k. Then,

$$\begin{aligned}\widetilde{\pi}(s_i r s^* s_j^*) &= \widetilde{\pi}(s_i)\widetilde{\pi}(rs^*)\widetilde{\pi}(s_j^*) \\ &= \pi(s_i s_1^*) S_1 \pi(rs^*) S_1^* \pi(s_1 s_j^*) \\ &= \pi(s_i s_1^*) \pi(s_1 r s^* s_1^*) S_1 S_1^* \pi(s_1 s_j^*) \\ &= \pi(s_i s_1^* s_1 r s^* s_1^*) \pi(s_1 s_1^*) \pi(s_1 s_j^*) \\ &= \pi(s_i r s^* s_j^*), \quad \forall i, j \in \{1, \ldots, n\}\end{aligned}$$

and hence, the representations agree for $k+1$. By induction, and by the fact that $\mathcal{F}_n$ is generated by elements of the form rs^* where $r, s \in \mathcal{W}_n^k$ for each $k \in \mathbb{N}$, we have that $\widetilde{\pi}(x) = \pi(x)$ for all $x \in \mathcal{F}_n$.

All that remains is to show that the extension $\widetilde{\pi} : \mathcal{O}_n \to \mathscr{B}(H)$ is unique. If $\widetilde{\pi}_1$ extends π, then

$$\widetilde{\pi}_1(s_j) = \pi(s_j s_1^*)\widetilde{\pi}_1(s_1), \quad j = 1, 2, \ldots, n$$

and $\widetilde{\pi}_1(s_1)\pi(x)\Omega = \pi(s_1 x s_1^*)\widetilde{\pi}_1(s_1)\Omega$. Since the isometries $\{s_i\}_{i=1}^n$ generate $\mathcal{O}_n$ and $\pi(\mathcal{F}_n)\Omega$ is dense in H, the extension $\widetilde{\pi}_1$ is completely determined by $\Omega' = \pi(s_1)\Omega$. For $x \in \mathcal{F}_n$,

$$\langle \pi(x)\Omega', \Omega' \rangle = \langle \pi(s_1^* x s_1)\Omega, \Omega \rangle = \omega'(x),$$

and thus Ω' is determined uniquely up to scalar multiple of modulus one because π is irreducible. If $\widetilde{\pi}_2$ is another extension, then $\widetilde{\pi}_2(s_1)\Omega = \lambda\Omega'$ for some $\lambda \in \mathbb{T}$ and this implies $\widetilde{\pi}_1 = \widetilde{\pi}_2 \circ \gamma_\lambda$, which concludes the proof. □

Corollary 3.2.6. *If π_1 and π_2 are representations of $\mathcal{O}_n$ such that $\pi_1 \restriction_{\mathcal{F}_n}$ and $\pi_2 \restriction_{\mathcal{F}_n}$ are irreducible and quasi-free equivalent then π_1 and π_2 are quasi-free equivalent.*

Proof. If π_1 and π_2 are as in the statement of the theorem then there exists a unitary operator W on H and a quasi-free automorphism such that

$$\pi_1(\mathcal{F}_n) = \mathrm{Ad}_W \circ \pi_2 \circ \gamma_U(\mathcal{F}_n).$$

Thus π_1 and π_2 both are extensions to $\mathcal{O}_n$ of the same irreducible representation of

$\mathcal{F}_n$. By the uniqueness portion of Theorem 3.2.5 they differ by a gauge automorphism γ_λ and hence $\pi_1(\mathcal{O}_n) = \mathrm{Ad}_W \circ \pi_2 \circ \gamma_{\lambda U}(\mathcal{O}_n)$, which concludes the proof. $\square$

Theorem 3.2.7. *The extension procedure of Theorem 3.2.5 establishes a bijection between the conjugacy classes of* $\mathrm{Shift}_n(\mathscr{B}(H))$ *and the quasi-free equivalence classes of essential quasi-invariant pure states of* $\mathcal{F}_n$.

Proof. Given an essential quasi-invariant pure state of $\mathcal{F}_n$, Theorem 3.2.5 provides a representation π of $\mathcal{O}_n$. Since ω is pure it follows that $\pi \restriction_{\mathcal{F}_n}$ is irreducible, so by Theorem 3.1.2 we have $\alpha_\pi \in \mathrm{Shift}_n(\mathscr{B}(H))$. Finally, Corollary 3.2.6 and Theorem 2.4.3 ensure that quasi-free equivalent states define conjugate endomorphisms.

To establish the forward correspondence, we note that for any shift endomorphism the restriction to $\mathcal{F}_n$ of any vector state in any representation of $\mathcal{O}_n$ implementing the shift must be quasi-invariant (Proposition 10.3.14 of [17]). By Theorem 2.4.3 conjugate shift endomorphisms α_π and β_σ define quasi-free equivalent representations. In the case of shift endomorphisms, these restrict to irreducible representations on $\mathcal{F}_n$ and hence the corresponding vector states are quasi-equivalent. $\square$

3.2.1 An Example of Non-Conjugate Shift Endomorphisms

In this section we consider a family of product states of $\mathcal{F}_n$ indexed by an uncountable set which, by the correspondence given in Theorem 3.2.7, give rise to non-conjugate shift endomorphisms.

Definition 3.2.8. *A state $\omega : \mathcal{F}_n \to \mathbb{C}$ with GNS representation π is called a* factor state *if $\pi(\mathcal{F}_n)''$ is a factor.*

We aim to characterize quasi-invariance of factor states of $\mathcal{F}_n$, in turn, we require a characterization of quasi-equivalence in this case.

Proposition 3.2.9. *Essential factor states $\omega_1, \omega_2 : \mathcal{F}_n \to \mathbb{C}$ are quasi-equivalent if and only if $\lim_{k\to\infty} ||\alpha^{*k}\omega_1 - \alpha^{*k}\omega_2|| = 0$.*

Proof. In the finite dimensional case, the map α^* is the adjoint of the endomorphism α and

$$||\alpha^*(\omega_1 - \omega_2)|| = ||(\omega_1 - \omega_2) \restriction_{\alpha^j(\mathcal{F}_n)} ||.$$

Theorem 2.7 of [23] then gives the desired correspondence.

In the case where $\mathcal{E}$ is infinite dimensional, let α be the endomorphism induced by the unilateral shift $T_1 \otimes \cdots \otimes T_j \mapsto 1 \otimes T_1 \otimes \cdots \otimes T_j$ where each T_i is a compact operator. It is known that factor states ω_1 and ω_2 of a C^*-algebra $\mathcal{U}$ are quasi-equivalent if and only if $\frac{1}{2}(\omega_1 + \omega_2)$ is a factor state ([7], Proposition 2.4.27), moreover this happens if and only if $\frac{1}{2}(\hat{\omega_1} + \hat{\omega_2})$ is a factor state (where $\hat{\omega}$ is as in Section 1.2.3). Now, applying Theorem 2.6.11 of [7], $\frac{1}{2}(\hat{\omega_1} + \hat{\omega_2})$ is a factor state if and only if $||(\hat{\omega}_1 - \hat{\omega}_2) \restriction_{\alpha^j(\mathcal{K})} || \to 0$ as $j \to \infty$. Assuming now that the states are essential, $\alpha^*\omega = (\hat{\omega} \circ \alpha) \restriction_{\mathcal{F}_\infty}$ and thus

$$||(\hat{\omega}_1 - \hat{\omega}_2) \circ \alpha^j|| = ||\alpha^{*j}(\omega_1 - \omega_2)||$$

which completes the proof. $\square$

Let E be a separable, infinite-dimensional Hilbert space with a sequence of unit vectors $\{\xi_k\}_{k=1}^{\infty}$. For each ξ_k, let $\omega_k(x) = \langle \xi_k, x\xi_k \rangle$ be a vector state of $\mathcal{K}(E)$, the algebra of compact operators on E. Let $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ denote the pure factor state given by the tensor product of the vector states ω_k.

Let $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ and $\omega' = \bigotimes_{k=1}^{\infty} \omega'_k$ be two such product states. Quasi-equivalence corresponds to unitary equivalence for states, therefore it follows from Proposition 3.2.9 that ω and ω' are unitarily equivalent if and only if $\lim_{k\to\infty} ||\alpha^{*k}\omega - \alpha^{*k}\omega'|| = 0$. By the work of Guichardet [14] this condition is equivalent to the convergence of the series $\sum_{k=1}^{\infty}(1 - |\langle \xi_k, \xi'_k \rangle|)$. Noting that, for a quasi-free automorphism γ_U,

$$(\omega \circ \gamma_U)(x) = \bigotimes_{k=1}^{\infty} \langle U\xi_k, xU\xi_k \rangle, \quad x \in \mathcal{K}(E)$$

it follows that ω and ω' are quasi-free equivalent if and only if there exists a unitary operator U on E such that the series $\sum_{k=1}^{\infty}(1 - |\langle \xi_k, U\xi'_k \rangle|)$ converges. Since the dual map α^* is a unilateral shift on $\mathcal{F}_n$ [Observation 2.2.8 (4)] we have that the factor state ω of $\mathcal{F}_n$ is quasi-invariant (i.e., ω is quasi-equivalent to $\alpha^*\omega$) if and only if $\sum_{k=1}^{\infty}(1 - |\langle \xi_k, \xi_{k-1} \rangle|) < \infty$.

We may further the characterization of this class of quasi-invariant states by considering the subspaces $\mathbb{C} \cdot \xi_k$, $k \in \mathbb{N}$. Specifically, if θ_k denotes the angle determined by $\mathbb{C} \cdot \xi_k$ and $\mathbb{C} \cdot \xi_{k-1}$ then $|\langle \xi_k, \xi_{k-1} \rangle| = \cos\theta_k$. It follows that the series

$$\sum_{k=1}^{\infty}(1 - |\langle \xi_k, \xi_{k-1} \rangle|) = \sum_{k=1}^{\infty}(1 - \cos\theta_k)$$

converges if and only $\sum_{k=1}^{\infty}\theta_k^2$ converges. We have thus determined that the factor state ω of $\mathcal{F}_n$ is quasi-invariant if and only the series $\sum_{k=1}^{\infty}\theta_k^2$ converges. We use this to construct an example of an uncountable collection of non-conjugate shift endomorphisms.

Example 3.2.10. *Let $s \in (\frac{1}{4}, \frac{1}{2}]$ and let v_1 and v_2 be orthogonal unit vectors in E. For the sequence of vectors $\xi_k = \cos(k^{-s})v_1 + \sin(k^{-s})v_2$, $k \in \mathbb{N}$ the angle θ_{k+1} is given by*

$$\theta_{k+1} = k^{-s} - (k+1)^{-s} = k^{-s}[1 - (1 + \frac{1}{k})^{-s}] = O(k^{-(s+1)}).$$

In this case, the series $\sum_{k=1}^{\infty}\theta_k^2$ converges and, by our previous argument, the product state determined by the sequence $\{\xi_k\}_{k=1}^{\infty}$ is quasi-invariant.

Now, consider another value $s' \in (\frac{1}{4}, \frac{1}{2}]$ and a similar statement as made above. Moreover, assume that $s' \neq s$. If the unitary operator U on E does not fix the limit $\lim_k \xi_k$ then the series $\sum_{k=1}^{\infty}(1 - |\langle \xi_k, U\xi_k' \rangle|)$ diverges. Suppose U fixes the one-dimensional subspace corresponding to

$$v_1 = \lim_k \xi_k = \lim_k \xi_k'.$$

The angle between the subspaces $\mathbb{C} \cdot \xi_k$ and $\mathbb{C} \cdot U\xi_k'$ is at least $k^{-s} - k^{-s'}$, and the series $\sum_{k=1}^{\infty}(k^{-s} - k^{-s'})^2$ diverges. Therefore, different values of the parameter s give rise to non-conjugate shift endomorphisms.

3.3 Comments & Citations

The characterization of shift and ergodic shift endomorphisms in terms of irreducible representations (Theorem 3.1.2) appears as Theorem 3.3 of [6] and simply as an observation in [20]. Both are based on Proposition 3.1.3 which is Proposition 3.1 of [20].

The proof of Proposition 3.1.5 is similar to the technique we use for Theorem 4.2.1 and is an adaptation of Proposition 2.9.2 of [3]. We note that

$$\lim_{k\to\infty} ||\omega \restriction_{\alpha_\pi^k(\mathscr{B}(H))} || = ||\omega \restriction_{\mathrm{Tail}(\alpha_\pi)} ||$$

may be deduced directly.

Observation 3.3.1. *If X_k is a decreasing family of weak*-closed subspaces of the dual X^* of a Banach space X, then for every weak*-continuous linear functional ω on X^*, the sequence of norms $\{\|\omega \restriction_{X_k} \|\}_{k\geq 1}$ is decreasing, and*

$$\lim_{k\to\infty} \|\omega \restriction_{X_k} \| = \|\omega \restriction_{\bigcap_k X_k} \|.$$

The bulk of material which constitutes Section 3.2 is taken directly from [20]; specifically, Lemma 4.2, Theorem 4.3, Corollary 4.4 and Theorem 4.5 of [20] give the bijective correspondence between shifts and states of $\mathcal{F}_n$. Moreover, Example 3.2.10 is given in [20]. A similar formulation is given by Example 5.5 of [6], and in that paper, Theorem 3.2.7 is obtained using similar arguments for index $n < \infty$.

We wish to note the characterizations of ergodic and shift endomorphisms given in this chapter. Let ω be a state of $\mathcal{O}_n$ with GNS representation $\pi \in \mathrm{Rep}(\mathcal{O}_n, H)$.

$$\alpha_\pi \in \mathrm{Erg}_n(\mathscr{B}(H)) \Leftrightarrow \pi \text{ is irreducible } \Leftrightarrow \omega \text{ is pure}$$

$$\alpha_\pi \in \mathrm{Shift}_n(\mathscr{B}(H)) \Leftrightarrow \pi \restriction_{\mathcal{F}_n} \text{ is irreducible } \Leftrightarrow \omega \restriction_{\mathcal{F}_n} \text{ is pure}$$

Chapter 4

Quasi-shift Endomorphisms

In a recent pre-print [13], we considered quasi-shift endomorphisms of arbitrary von Neumann algebras. In this thesis, we present original research regarding quasi-shift endomorphisms of von Neumann algebras, in particular the algebras $\mathscr{B}(H)$ and $\pi(\mathcal{O}_n)''$ for $\pi \in \mathrm{Rep}(\mathcal{O}_n, H), n \in \{2, 3, \ldots \infty\}$. In each case, we freely translate existing definitions and notation to suit our purposes without explicit mention.

4.1 Quasi-shift Endomorphisms and Steady Representations

In this section, we characterize quasi-shift endomorphisms of $\mathscr{B}(H)$ in terms of representations of Cuntz algebras and provide a direct integral decomposition for such representations. In light of the correspondence between endomorphisms and representations of Cuntz algebras this decomposition determines the form of a quasi-shift endomorphism.

Definition 4.1.1. *An essential representation* $\pi \in \mathrm{Rep}(\mathcal{O}_n, H)$ *is said to be* steady *if* $\pi(\mathcal{F}_n)$ *is weakly* dense in* $\pi(\mathcal{O}_n)$*.*

Proposition 4.1.2. *Let* $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$ *where* $\pi \in \mathrm{Rep}(\mathcal{O}_n, H)$ *is essential. Then* π *is a steady representation if and only if* α_π *is a quasi-shift of* $\mathscr{B}(H)$*. In this case, the following conditions are equivalent:*

1. π *is irreducible;*

2. $\alpha_\pi \in \mathrm{Shift}_n(\mathscr{B}(H))$*;*

3. $\alpha_\pi \in \mathrm{Erg}_n(\mathscr{B}(H))$*.*

Proof. The proof follows easily from Proposition 3.1.3 and Theorem 3.1.2. □

Corollary 4.1.3. *Let* $\pi \in \mathrm{Rep}(\mathcal{O}_n, H)$*. Then* π *is a steady representation if and only if* $\pi(\mathcal{E}^k) \subset \pi(\mathcal{F}_n)''$*, for every* $k \in \mathbb{N}$*.*

We may obtain a decomposition of a steady representation analogous to the decomposition of a representation of a Toeplitz-Cuntz algebra given in Theorem 2.2.6.

Theorem 4.1.4. *Let* $\pi \in \mathrm{Rep}(\mathcal{O}_n, H)$ *be a steady representation and*

$$\pi = \int_X^{\oplus} \pi_x \, d\mu(x)$$

be an arbitrary direct integral decomposition of π *into irreducible representations* $\pi_x : \mathcal{O}_n \to H_x$*, where* (X, μ) *is a standard measure space. Then* π_x *is a steady representation for* μ*-almost every* $x \in X$*.*

Proof. We argue by contradiction. Assume that there is a Borel set $B \subseteq X$, $\mu(B) > 0$, such that π_x is not a steady representation of $\mathcal{O}_n$, for every $x \in B$. In particular, $\pi_x(\mathcal{F}_n)$ is not irreducible in $\mathscr{B}(H_x)$, for every $x \in B$. Using a standard Borel selection argument, one can find a measurable family of projections $\{P_x\}_{x\in B}$, $P_x \in \mathscr{B}(H_x)$, $0 \neq P_x \neq 1_{H_x}$, such that P_x commutes with $\pi_x(\mathcal{F}_n)$ for every $x \in B$. Define $P_x = 0$ for $x \in X \setminus B$, and consider the projection $P \in \mathscr{B}(H)$

$$P = \int_X^{\oplus} P_x \, d\mu(x).$$

It is clear that P commutes with $\pi(\mathcal{F}_n)$. Moreover, since $\pi(\mathcal{F}_n)$ is weakly dense in $\pi(\mathcal{O}_n)$, it follows that $P \in \pi(\mathcal{O}_n)'$. Since

$$\int_X^{\oplus} (\pi_x(a)P_x - P_x\pi_x(a)) \, d\mu(x) = \pi(a)P - P\pi(a) = 0, \ a \in \mathcal{O}_n,$$

we conclude that $P_x \in \pi_x(\mathcal{O}_n)'$ for almost every x. Now since $\pi_x(\mathcal{O}_n)$ is irreducible for all x, the latter implies that $P_x \in \{0, 1_{H_x}\}$ for almost every x, which contradicts the choice of P_x for $x \in B$. $\square$

In particular, Theorem 4.1.4 implies that the representations $\pi_x : \mathcal{O}_n \to \mathscr{B}(H_x)$ give rise to shift endomorphisms $\alpha_{\pi_x} \in \mathrm{End}(\mathscr{B}(H_x))$. In this respect, quasi-shift endomorphisms are determined by shift endomorphisms.

4.2 Asymptotic Characterizations of Quasi-shift Endomorphisms

We now prove a noncommutative analogue at the level of endomorphisms of Kaimanovich's characterization of steady Markov chains (Theorem 1.1.3).

Theorem 4.2.1. *Let $M \subseteq \mathscr{B}(H)$ be a von Neumann algebra and $\alpha \in \mathrm{End}(M)$. The following statements are equivalent:*

(1) α is a quasi-shift;

(2) For every unit vector $\xi \in H$,

$$\lim_{n\to\infty} \|(\omega_\xi \circ \alpha - \omega_\xi) \upharpoonright_{\alpha^n(M)} \| = 0$$

where $\omega_\xi(X) := \langle X\xi, \xi\rangle, X \in \mathscr{B}(H)$;

(3) For every normal state ω of M,

$$\lim_{n\to\infty} \|\omega \circ \alpha^{n+1} - \omega \circ \alpha^n\| = 0. \tag{4.29}$$

Proof. The equivalence (1) $\Leftrightarrow$ (2) is a direct consequence of Observation 3.3.1.

(1) $\Rightarrow$ (2). Assume that α is a quasi-shift. It follows that the von Neumann algebras $\alpha^n(M)$ decrease to the fixed-point algebra $\mathrm{Fix}(\alpha)$ as $n \to \infty$. Hence by Observation 3.3.1,

$$\lim_{n\to\infty} \|(\omega_\xi \circ \alpha - \omega_\xi) \upharpoonright_{\alpha^n(M)} \| = \lim_{n\to\infty} \|(\omega_\xi \circ \alpha - \omega_\xi) \upharpoonright_{\mathrm{Fix}(\alpha)} \|,$$

which is zero because $\omega_\xi \circ \alpha(X) = \omega_\xi(X)$, for every $X \in \mathrm{Fix}(\alpha)$.

(2) $\Rightarrow$ (1). Let $X \in \mathrm{Tail}(\alpha)$ satisfying $\|X\| \leq 1$. We claim that X is fixed by α. Indeed, if $Y = \alpha(X) - X$, then for any unit vector $\xi \in H$, we have

$$\begin{aligned} |\omega_\xi(Y)| = |(\omega_\xi(\alpha(X)) - \omega_\xi(X)| &\leq \|(\omega_\xi \circ \alpha - \omega_\xi) \restriction_{\mathrm{Tail}(\alpha)} \| \\ &= \lim_{n\to\infty} \|(\omega_\xi \circ \alpha - \omega_\xi) \restriction_{\alpha^n(M)} \| \\ &= 0. \end{aligned}$$

Since ξ is arbitrary, $Y = 0$ follows.

(2) $\Leftrightarrow$ (3). This equivalence is immediately obtained from the following facts:

(i) $\|\omega \circ \alpha\| = \|\omega \restriction_{\alpha(M)} \|$, for ever $\omega \in M_*$, and

(ii) every normal state of M is an infinite convex combination of vector states ω_ξ, with $\xi \in H$, $\|\xi\| = 1$.

This concludes the proof. $\square$

Observation 4.2.2. *Every endomorphism α of a von Neumann algebra M can be realized as the minimal dilation of a unital normal completely positive map ϕ of a von Neumann algebra N [3]. More precisely, if $p \in M$ is a projection such that*

$$\alpha(p) \geq p \textit{ and } \sup_k \alpha^k(p) = 1,$$

then the restriction ϕ of α to the corner $N = pMp$, $\phi(x) = p\alpha(x)p, \forall x \in N$, is a unital normal completely positive map of the von Neumann algebra N of which the minimal *dilation is the endomorphism α. Since $\sup_k \alpha^k(p) = 1$, one can easily see that the asymptotic condition in the above theorem is conserved throughout this construction. Consequently, α is a quasi-shift of M if and only if the sequence $\{\omega \circ (\phi^{k+1} - \phi^k)\}_{k\in\mathbb{N}}$ norm-converges to zero, for every $\omega \in N_*$. Examples of completely positive maps that satisfy this property were discussed in [4].*

Lastly, we give an asymptotic condition which determines when an endomorphism $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$ restricts to a quasi-shift of the von Neumann algebra $\pi(\mathcal{O}_n)''$ for $\mathrm{Rep}(\mathcal{O}_n, H)$ in terms of the backward shift $\alpha^* : \mathcal{O}_n^* \to \mathcal{O}_n^*$. To simplify the proof of the theorem, we note the following Lemma.

Lemma 4.2.3. *Let $\alpha : \mathscr{B}(H) \to \mathscr{B}(H)$ be an endomorphism with linear space $\mathcal{E}(\alpha)$. Then for every $U \in \mathcal{E}(\alpha^{m_1})$, $V \in \mathcal{E}(\alpha^{m_2})$, $m_1, m_2 \in \mathbb{N}$ and for all sufficiently large $k > 0$, we have*

$$\alpha^k(A)UV^* = UV^*\alpha^{k-m_1+m_2}(A), \quad X \in \mathscr{B}(H). \tag{4.30}$$

Proof. Direct computation. $\square$

Theorem 4.2.4. *Let ω be state of $\mathcal{O}_n$ with essential GNS representation $\pi \in \mathrm{Rep}(\mathcal{O}_n, H)$. The following conditions are equivalent:*

(1) The restriction $\alpha_\pi \restriction_M$ of the endomorphism $\alpha_\pi \in \mathrm{End}_n(\mathscr{B}(H))$ to the von Neumann algebra $M = \pi(\mathcal{O}_n)''$ is a quasi-shift.

(2) For every $u \in \mathcal{E}^{m_1}$, $v \in \mathcal{E}^{m_2}$, $m_1, m_2 \in \mathbb{N}$, one has

$$\lim_{k\to\infty} \|\alpha^{*(k+1)}(\omega_{u,v}) - \alpha^{*k}(\omega_{u,v})\| = 0, \tag{4.31}$$

where $\omega_{u,v}(x) := \omega(uxv^*)$, $x \in \mathcal{O}_n$.

Proof. Let $\xi \in H$ be the unit cyclic vector in the GNS construction for ω, and let $\mathcal{E}(\alpha_\pi) = \pi(\mathcal{E})$ be the linear space (2.17) for α_π. Since the C^*-algebra generated by the Hilbert space $\mathcal{E}(\alpha_\pi)$ is isomorphic to $\mathcal{O}_n$, Condition (2) in the statement of the theorem is equivalent to the following (2'): For every $m_1, m_2 \in \mathbb{N}$ and every pair of operators $U \in \mathcal{E}(\alpha_\pi^{m_1})$, $V \in \mathcal{E}(\alpha_\pi^{m_2})$, the linear functional $\omega_{V^*\xi,U^*\xi}$ defined as $\omega_{V^*\xi,U^*\xi}(X) := \langle XV^*\xi, U^*\xi\rangle$, $X \in \mathscr{B}(H)$, satisfies

$$\lim_{k\to\infty} \|(\omega_{V^*\xi,U^*\xi} \circ \alpha_\pi - \omega_{V^*\xi,U^*\xi}) \restriction_{\alpha_\pi^k(M)} \| = 0. \tag{4.32}$$

Implication (1) $\Rightarrow$ (2$'$) follows immediately from Theorem 4.2.1.

We now prove the implication (2$'$) $\Rightarrow$ (1). By Theorem 4.2.1, it is enough to show that there is a dense vector subspace $H_0 \subset H$ with the property that for every pair of vectors η, $\zeta \in H_0$ and every $k \in \mathbb{N}$, one has

$$\lim_{k\to\infty} \|(\omega_{\eta,\zeta} \circ \alpha_\pi^{k+1} - \omega_{\eta,\zeta} \circ \alpha_\pi^k) \restriction_M \| = 0 \tag{4.33}$$

where $\omega_{\zeta,\eta}$ denotes the linear functional defined on $\mathscr{B}(H)$ by

$$\omega_{\eta,\zeta}(X) := \langle X\eta, \zeta\rangle,\ X \in \mathscr{B}(H).$$

For this purpose, we take H_0 be the linear space of all finite sums of vectors of the form $UV^*\xi$, where $U, V \in \mathcal{E}_\infty(\alpha_\pi) := \bigcup_{k\in\mathbb{N}} \mathcal{E}(\alpha_\pi^k)$. Note that H_0 is dense in H because

$$M = \overline{\operatorname{span}}^{\text{weak}^*}\{UV^* \,|\, U, V \in \mathcal{E}_\infty(\alpha_\pi)\}$$

and ξ is a cyclic vector for M. Here we have made use of the following fact: While the set of all products

$$\mathcal{P} = \{UV^* \,|\, U, V \in \mathcal{E}_\infty(\alpha_\pi)\}$$

is not closed under multiplication, we do have

$$\{Z_1 Z_2 \,|\, Z_i \in \mathcal{P}\} \subseteq \mathcal{P} + \mathcal{P}. \tag{4.34}$$

Since $\omega_{\eta,\zeta}$ is sesquilinear in η, ζ, every functional of the form $\omega_{\eta,\zeta}$ with $\eta, \zeta \in H_0$ can be decomposed into a finite sum of functionals of the form $\omega_{X\xi,Y\xi}$, where $X, Y \in \mathcal{P}$. In particular, it suffices to show that (4.33) holds for all vectors of the form $\eta = X\xi$, $\zeta = Y\xi$, where $X, Y \in \mathcal{P}$. To that end we firstly show that (4.33) holds in the case where

$$\eta = UV^*\xi,\ \zeta = \xi, \tag{4.35}$$

where $U \in \mathcal{E}(\alpha_\pi^{m_1})$, $V \in \mathcal{E}(\alpha_\pi^{m_2})$ for some positive integers $m_1\, m_2 \in \mathbb{N}$. Indeed, for large $k \in \mathbb{N}$ and for every $A \in \mathscr{B}(H)$, one has

$$\begin{aligned}
\omega_{\eta,\zeta} \circ \alpha_\pi^{k+1}(A) &= \langle \alpha_\pi^{k+1}(A) UV^*\xi, \xi\rangle \\
&= \langle U\alpha_\pi^{k+1-m_1}(A) V^*\xi, \xi\rangle \\
&= \langle \alpha_\pi^{k+1-m_1}(A) V^*\xi, U^*\xi\rangle
\end{aligned}$$

and similarly

$$\omega_{\eta,\zeta} \circ \alpha_\pi^k(A) = \langle \alpha_\pi^{k-m_1}(A) V^* \xi, U^* \xi \rangle.$$

Therefore for large k, we have

$$\| \left(\omega_{\eta,\zeta} \circ \alpha_\pi^{k+1} - \omega_{\eta,\zeta} \circ \alpha_\pi^k \right) \restriction_M \| = \| \left(\omega_{V^*\xi, U^*\xi} \circ \alpha_\pi^{k+1-m_1} - \omega_{V^*\xi, U^*\xi} \circ \alpha_\pi^{k-m_1} \right) \restriction_M \|$$

The term on the right has the form of (4.33) (with k replaced with $k - m_1$), so it tends to zero as $k \to \infty$, proving (4.33) in the special case (4.35). Now consider the general case in which $\eta = X\xi$, $\zeta = Y\xi$ with X, $Y \in \mathcal{P}$. Writing $Y = UV^*$ with $U \in \mathcal{E}(\alpha_\pi^{m_1})$, $V \in \mathcal{E}(\alpha_\pi^{m_2})$, Lemma 4.2.3 implies that for sufficiently large k we have

$$\begin{aligned} \omega_{X\xi, Y\xi}(\alpha_\pi^k(A)) &= \langle Y^* \alpha_\pi^k(A) X\xi, \xi \rangle = \langle \alpha_\pi^{k-m_1+m_2}(A) Y^* X \xi, \xi \rangle, \text{ and} \\ \omega_{X\xi, Y\xi}(\alpha_\pi^{k+1}(A)) &= \langle Y^* \alpha_\pi^{k+1}(A) X\xi, \xi \rangle = \langle \alpha_\pi^{k+1-m_1+m_2}(A) Y^* X \xi, \xi \rangle. \end{aligned}$$

Since Y^*X decomposes into a sum $Z_1 + Z_2$ with $Z_i \in \mathcal{P}$, the preceding formulas imply that the difference

$$\omega_{X\xi, Y\xi}(\alpha_\pi^{k+1}(A)) - \omega_{X\xi, Y\xi}(\alpha_\pi^k(A))$$

decomposes into a sum of two differences of expressions having the general form of (4.35) above, and now we can argue as before to conclude that

$$\lim_{k \to \infty} \| \left(\omega_{X\xi, Y\xi} \circ \alpha_\pi^{k+1} - \omega_{X\xi, Y\xi} \circ \alpha_\pi^k \right) \restriction_M \| = 0$$

as required. $\square$

Observation 4.2.5. *Using the notation of the previous theorem, we deduce from Proposition 3.1.3 that* $\mathrm{Fix}(\alpha \restriction_M)$ *coincides with the center of* M*. In particular, if* $\alpha \restriction_M$ *is a quasi-shift, then* $\alpha \restriction_M$ *is a shift of* M *if and only if* ω *is a factor state of* $\mathcal{O}_n$*. Moreover, if* ω *is a pure state, then* α *is a quasi-shift of* $\mathscr{B}(H)$ *iff* α *is a shift of* $\mathscr{B}(H)$*. In particular, the above asymptotic condition fully characterizes all shifts of* $\mathscr{B}(H)$*, provided that* ω *is a pure state of* $\mathcal{O}_n$*.*

Chapter 5

Future Work

In this thesis, we introduced and studied the class of quasi-shift endomorphisms and gave some asymptotic characterizations. We would like to construct a large class of examples of quasi-shift endomorphisms of von Neumann algebras. As of now, we have considered how to obtain such examples in connection with Observation 4.2.2, however, the details of which remain unpolished. Moreover, we would like to further our asymptotic characterizations of quasi-shifts, in particular to von Neumann algebras which are not related to representations of Cuntz algebras.

Our primary goal for future research in this area is to extend our consideration of quasi-shift endomorphisms of von Neumann algebras to E_0-semigroups. Naturally, this is much more sophisticated and many technical details arise. In particular, we require a replacement for the notable role that Cuntz algebras played in the discrete case. For E_0-semigroups, Cuntz algebras are replaced by the so-called *spectral* C^*-*algebras* of exponential product systems. These algebras were introduced and studied by W. Arveson in [3]. Our future work on this topic will thus rely on the pioneering work of W. Arveson as was the case for single *-endomorphisms of the von Neumann algebra $\mathscr{B}(H)$.

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