

A CLASSIFICATION OF HOMOGENEOUS KÄHLER MANIFOLDS
WITH DISCRETE ISOTROPY AND TOP NON VANISHING
HOMOLOGY IN CODIMENSION TWO

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Syedruhallah Ahmadi

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UNIVERSITY OF REGINA
FACULTY OF GRADUATE STUDIES AND RESEARCH
SUPERVISORY AND EXAMINING COMMITTEE

Seyedruhallah Ahmadi, candidate for the degree of Doctor of Philosophy in Mathematics, has presented a thesis titled, **A Classification of Homogeneous Kähler Manifolds with Discrete Isotropy and Top Non Vanishing Homology in Codimension Two**, in an oral examination held on Wednesday, July 24, 2013.. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

External Examiner: *Dr. Alan Huckleberry, Jacobs University

Supervisor: Dr. Bruce Gilligan, Department of Mathematics & Statistics

Committee Member: **Dr. Augustin-Liviu Mare,
Department of Mathematics & Statistics

Committee Member: Dr. Donald Stanley, Department of Mathematics & Statistics

Committee Member: **Dr. Fernando Szechtman,
Department of Mathematics & Statistics

Committee Member: Dr. Howard Hamilton, Department of Computer Science

Chair of Defense: Dr. R Nicholas Carleton, Department of Psychology

*via SKYPE

**Not present at defense

Abstract

Suppose G is a connected complex Lie group and Γ is a discrete subgroup such that $X := G/\Gamma$ is Kähler and the codimension of the top non-vanishing homology group of X with coefficients in \mathbb{Z}_2 is less than or equal to two. We show that G is solvable and a finite covering of X is biholomorphic to a product $C \times A$, where C is a Cousin group and A is $\{e\}$, \mathbb{C} , \mathbb{C}^* , or $\mathbb{C}^* \times \mathbb{C}^*$.

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To my family

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Chapter 1

Introduction

A Kähler manifold is a Hermitian manifold whose associated Hermitian form is closed. Compact Kähler manifolds have been extensively studied from a number of view points. For example, a compact complex surface admits a Kähler structure exactly when its first Betti number is even; this follows from the Enriques-Kodaira classification [GrHa78] and was also proved directly in [Buc99].

Dorfmeister and Nakajima classified Kähler manifolds on which the group of holomorphic isometries acts transitively [DN88]. They proved that any such manifold is holomorphically isomorphic to a bounded homogeneous domain, a flag manifold and \mathbb{C}^n modulo a lattice with additive structure or a

product of such manifolds.

The transitivity condition of isometries is a very strong condition. Borel-Remmert handled the compact case by showing that any connected compact homogeneous Kähler manifold X which is homogeneous under the automorphism group of X can be written as a direct product of a complex torus and a flag manifold (see Theorem 2.19).

The situation in the non-compact setting, however, is not so well studied nor understood. In order to consider problems that are tractable throughout this dissertation we consider a complex homogeneous manifold $X = G/H$, where G is a connected complex Lie group acting almost effectively on X and H is a closed complex subgroup of G , such that X has a Kähler structure. However, a classification of such homogeneous Kähler manifolds without any restriction is not realistic.

In this dissertation we consider homogeneous Kähler manifolds X with a topological invariant $d_X \leq 2$. We mention that d_X is dual to an invariant introduced by Abels [Abe76] and is defined as follows:

$$d_X = \min\{ r \mid H_{n-r}(X, \mathbb{Z}_2) \neq 0 \}$$

where by $H_{n-r}(X, \mathbb{Z}_2)$ we mean the singular homology of X with coefficients

in \mathbb{Z}_2 , and $\dim X = n$. Note that since X is a complex manifold, X is oriented.

The topological invariant d_X can be handled more easily in the algebraic setting, because algebraic groups have a finite number of connected components. In fact, Akhiezer classified G/H , where G is an algebraic group and H is an algebraic subgroup with $d_{G/H} = 1$ in [Akh77] and $d_{G/H} = 2$ in [Akh83]. On the other hand, a closed subgroup H of a connected Lie group G can have an infinite number of connected components, and this phenomenon can contribute to the topology of G/H and so influences $d_{G/H}$. For example, let $G = \mathbb{R}$ and $H = \mathbb{Z}$, and $G/H = \mathbb{S}^1$ which is compact, not because of any compactness of G , but because H is infinite discrete. This implies that the most delicate problems in working with the invariant d_X occur when the isotropy is discrete. The purpose of this dissertation is to classify Kähler G/H with $d_{G/H} = 2$ in the important special case when H is discrete. In fact we prove the following theorem in chapter 4.3. Note that the notion of a Cousin group is defined in Definition 2.3.

Theorem 1.1. *Let G be a connected complex Lie group and Γ a discrete subgroup of G such that $X := G/\Gamma$ is Kähler and $d_X \leq 2$. Then the group G is solvable and a finite covering of X is biholomorphic to a product $C \times A$, where*

C is a Cousin group and A is $\{e\}$, \mathbb{C}^* , \mathbb{C} , or $(\mathbb{C}^*)^2$. Moreover, $d_X = d_C + d_A$.

The dissertation is organized as follows.

In chapter 2 we give a brief review of particular known classification results of G/H some of which are used in our classification. In chapter 3 we introduce some tools to calculate d_X mentioned above. We also give some other tools that are used for our classification theorem. Note that in these chapters H might not be discrete unless otherwise stated.

Our main results are collected in chapter 4. While classifying homogeneous Kähler manifolds $X = G/\Gamma$ with discrete isotropy and $d_X \leq 2$, we prove that G is solvable. This is related to a variant of a question of Akhiezer in [Akh84] that is discussed in section 2.5.

Finally, in chapter 5 we present some projects which can be done in future. In the first project we introduce a method that will be used to give the full classification of homogeneous Kähler manifolds with top non-vanishing homology in codimension two, i.e., those Kähler manifolds which might not necessarily have discrete isotropy. The next project involves complex manifolds that are homogeneous under the holomorphic action of real Lie groups. Suppose $X = G/H$ where G is a real Lie group and H is a closed subgroup

of G such that G/H has a left invariant complex structure. We propose to investigate globalization of the G action when $d_X = 2$. The third project investigates extensions of the Akhiezer question in the Kähler setting. And the last one looks at finding some sufficient conditions in order that a holomorphically separable complex homogeneous manifold is Stein.

Chapter 2

Background

In this chapter we provide background material which we will need in Chapter 4 to prove the main theorem. In section 2.1 we define Kähler manifolds and present some of their properties and examples. In section 2.2 we give general information about Lie groups as transformation groups. Stein manifolds appear in the classification theorem as fundamental building blocks. We present some of their properties in section 2.3. In section 2.4 we introduce the classification of homogeneous manifold G/H when there is a special restriction on the complex Lie group G .

One motivation for Proposition 4.1 is the question of Akhiezer. We discuss this in section 2.5. Let $X = G/\Gamma$ be a homogeneous Kähler manifold

with discrete isotropy such that Γ is not contained in any proper parabolic subgroup of G . If there are no non-constant holomorphic functions on X , then lemma 2.2 states that the radical orbits of G are closed in X . A fact which we will use to prove Theorem 4.1.

2.1 Kähler manifolds

In this section we define Kähler manifolds and give a brief discussion concerning their structure. For more information we refer the reader to [HW01], [Can01] or [KN63].

A Riemannian metric on a complex manifold M is a covariant tensor field g of degree 2 which satisfies (1) $g(X, Y) \geq 0$, where $g(X, X) = 0$ if and only if $X = 0$ and (2) $g(X, Y) = g(Y, X)$, where X and Y are vector fields on M . An almost complex structure on a manifold M is a tensor field J which is, at every point m of M , an endomorphism of the tangent space $T_m M$ such that $J^2 = -1$, where 1 denotes the identity transformation of $T_m M$. The almost complex structure is complex when it is integrable. This means that there exist local complex coordinates on the manifold, i.e., an atlas of holomorphically compatible complex charts defining a complex structure on

the manifold. Note that this is known to be equivalent to the vanishing of the Nijenhuis tensor

$$N(X, Y) := [X, Y] - J[X, Y] + J[X, JY] + J[JX, Y],$$

see [NN57]. A Hermitian metric on a complex manifold is a Riemannian metric g which is invariant by the almost complex structure J , i.e.,

$$g(JX, JY) = g(X, Y) \text{ for all vector fields } X \text{ and } Y.$$

Note that any paracompact real manifold has a Riemannian metric. One can always find such metrics locally, and then use an appropriate partition of unity to extend to a global one. Given a Riemannian metric g on any almost complex manifold note that $h(X, Y) := g(X, Y) + g(JX, JY)$ defines a Hermitian metric on that manifold. Since h takes its values in \mathbb{C} , we can write $h(X, Y) = \mathcal{S}(X, Y) + i\mathcal{A}(X, Y)$. Note that \mathcal{S} is symmetric and \mathcal{A} is alternating, i.e., a 2-form on X called the associated Kähler form. If $d\mathcal{A} = 0$, then h is called a **Kähler metric**.

Definition 2.1. *A Kähler manifold is a complex manifold equipped with a Kähler metric.*

2.1.1 Properties of Kähler manifolds

- If M and N are Kähler then $M \times N$ is Kähler, with the product metric.
- For any compact Kähler manifold X , the odd Betti numbers are even (see page 117, [GrHa78]).

2.1.2 Examples of Kähler manifolds

- \mathbb{C}^n : A Hermitian metric on \mathbb{C}^n is given by $ds^2 = \sum_{j=1}^n dz_j \otimes d\bar{z}_j$.
- This metric induces a metric on the torus $T = \mathbb{C}^n/\Lambda$. Here $\Lambda \subset \mathbb{C}^n$ is a complete lattice acting on \mathbb{C}^n by translation, and the above metric is invariant under translations and thus pushes down to the quotient T .
- $\mathbb{C}\mathbb{P}_n$: Complex projective space is defined via the following equivalence relation. For $v, w \in \mathbb{C}^{n+1} \setminus \{0\}$, we let

$$v \sim w \iff v = \lambda w \text{ for some } \lambda \in \mathbb{C}^*.$$

Then we have the holomorphic \mathbb{C}^* -bundle

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow (\mathbb{C}^{n+1} \setminus \{0\})/\sim \cong \mathbb{C}\mathbb{P}_n,$$

where $\mathbb{C}\mathbb{P}_n$ is defined to be the base of this bundle. Let σ be a local section of π , i.e., $\pi \circ \sigma = \text{id}$ over an open subset V of $\mathbb{C}\mathbb{P}_n$. The

Fubini-Study metric on V is given by its associated $(1, 1)$ form

$$\omega := \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \|\sigma\|^2.$$

Let σ' be another local section of the map π defined over an open set W . If $V \cap W \neq \emptyset$, then $\sigma' = g \cdot \sigma$, where g is a nowhere vanishing holomorphic function on $V \cap W$. Thus we have

$$\begin{aligned} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|\sigma'\|^2 &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (\log \|\sigma\|^2 + \log g + \log \bar{g}) \\ &= \omega + \frac{\sqrt{-1}}{2\pi} (\partial \bar{\partial} \log g - \bar{\partial} \partial \log \bar{g}) \\ &= \omega \end{aligned}$$

Since this form is defined locally, and is independent of the section chosen, this construction yields a globally defined differential form of $\mathbb{C}\mathbb{P}_n$. One can also see that this form is positive and closed (see page 31, [GrHa78]), so it defines a Hermitian metric on $\mathbb{C}\mathbb{P}^n$. Thus, the Fubini-Study metric is Kähler.

- A complex submanifold of a Kähler manifold is Kähler by the induced Kähler structure. As a result, any Stein manifold (embedded in \mathbb{C}^n) or projective algebraic variety (embedded in $\mathbb{C}\mathbb{P}_n$) is of Kähler type.
- Any Riemann surface is Kähler. As noted above, any complex manifold

X admits Hermitian structures and the associated Hermitian form ω is closed for dimension reasons when $\dim_{\mathbb{C}} X = 1$.

2.1.3 Examples of non-Kähler manifolds

Compact Case:

Any connected, compact, complex, homogeneous manifold which is not the trivial product of a flag manifold and a torus is not a Kähler manifold (Theorem 2.19). We define a properly discontinuous action of the discrete group \mathbb{Z} on $\mathbb{C}^2 \setminus \{(0, 0)\}$ given by

$$(z, w) \sim 2^k(z, w)$$

for all $(z, w) \in \mathbb{C}^2$ and $k \in \mathbb{Z}$. The complex surface $(\mathbb{C}^2 \setminus \{(0, 0)\}) / \sim$ is compact and diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ and is called the Hopf surface. Since the Hopf surface is not the product of a flag manifold and a torus, it follows from the Theorem of Borel–Remmert [BR62] that the Hopf surface does not admit a Kähler metric.

Non-compact Case:

Consider the three-dimensional Heisenberg group

$$G := \left\{ \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{C} \right) \right\}$$

and consider the discrete subgroup

$$\Gamma := \left\{ \left(\begin{pmatrix} 1 & m & k + il \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : n, m, l, k \in \mathbb{Z} \right) \right\}.$$

The center of G is the subgroup

$$C := \left\{ \left(\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right) \right\}.$$

By a simple calculation one can see that the fibration $G/\Gamma \rightarrow G/C.\Gamma$ realizes

G/Γ as a torus bundle over $\mathbb{C}^* \times \mathbb{C}^*$.

Let G_0 be the real Lie subgroup

$$G_0 := \left\{ \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, z \in \mathbb{C} \right) \right\}$$

that contains Γ cocompactly. Let $\mathfrak{m} := \mathfrak{g}_0 \cap i\mathfrak{g}_0 = \mathfrak{g}'_0 \cap i\mathfrak{g}'_0$, and so $\mathfrak{g}'_0 \cap \mathfrak{m} \neq (0)$, where \mathfrak{g}_0 is the Lie algebra of G_0 , and \mathfrak{g}'_0 is its commutator group. It has been proven that if a nilmanifold G/Γ has the above form is Kähler, then $\mathfrak{m} \cap \mathfrak{g}'_0 = (0)$ (see Remark 4 (b) and Example 6 (b) in [OR87] for more details). Since this is not so here, G/Γ is not Kähler.

2.2 Lie groups as transformation groups on complex manifolds

A complex Lie group is a set with the structure of both a complex manifold and a group such that the group operations are holomorphic. Since the sum of two solvable ideals is a solvable ideal, every complex Lie algebra contains a maximal solvable ideal called the radical. The corresponding connected subgroup is closed and in the case of complex Lie groups is complex. One sees this at the Lie algebra level, since if \mathfrak{r} is a solvable ideal in the complex Lie algebra \mathfrak{g} , then a calculation shows that $\mathfrak{r} + i\mathfrak{r}$ is also a solvable ideal in \mathfrak{g} . Since the radical is maximal with these properties, one has $\mathfrak{r} = \mathfrak{r} + i\mathfrak{r}$ and thus \mathfrak{r} is a complex ideal. In addition, the quotient of the complex Lie algebra \mathfrak{g} by the complex ideal \mathfrak{r} is a **complex** Lie algebra say \mathfrak{s} which is semisimple

(see theorem below). The following result was first proved by Levi for Lie algebras, but the corresponding statement for Lie groups is a consequence of his result.

Theorem 2.1 (Levi-Malcev theorem). *[Lev05] Let G be a simply connected, connected Lie group with radical R . Then there is a closed semi-simple subgroup S in G so that G is the semi-direct product $G = S \ltimes R$. The group S is unique up to conjugation. If G is complex, then so are R and S .*

If the Lie group is not simply connected, we may have to consider the universal covering group of G to apply the theorem. For example $GL_n(\mathbb{C}) = \mathbb{C}^* \cdot SL_n(\mathbb{C})$ as Lie groups, i.e., $G = R.S$, but $R \cap S \neq (e)$.

2.2.1 Lie groups as transformation groups

A holomorphic left G -action of a complex Lie group G on a complex manifold X is a holomorphic map $\phi : G \times X \rightarrow X$. For $g \in G$ and $x \in X$, we write $g.x$ instead of $\phi(g, x)$ for convenience. Note that $g : X \rightarrow X$ is holomorphic for any fixed $g \in G$. We say that G is acting as a Lie transformation group on X if ϕ in addition has the following two properties:

- 1) $gg'.x = g.(g'.x)$ for all $g, g' \in G$ and $x \in X$.

2) $e.x = x$ for all $x \in X$. Here by e we mean the identity element of G .

We call a G -action on X **effective** if it has an additional property that whenever $g.x = x$ for all $x \in X$, then $g = e$. By a G orbit $G.x$ of a point x we mean the set $G.x = \{g.x : g \in G\}$. For a subset $M \subseteq X$ the stabilizer H of M in G is a **subgroup** of G defined by

$$H = \text{Stab}_G(M) := \{ g \in G \mid g.M \subseteq M \}.$$

Furthermore if M is closed, then H is closed, and so is a Lie subgroup of G . If M is a closed complex analytic subvariety and the G -action is holomorphic with G a complex Lie group, then H is a **complex Lie subgroup** of G . Here M can also be a point, in which case we denote the stabilizer group by H_x . The map $gH_x \mapsto g.x$ establishes an identification between the left coset $G/H_x = \{gH_x : g \in G\}$ and the orbit $G.x$. In general if H is a closed complex subgroup of the complex Lie group G then we have the holomorphic principal bundle $H \rightarrow G \rightarrow G/H$, where G/H has a complex analytic quotient structure.

The ineffectivity of the holomorphic action of G on X is the normal closed complex Lie subgroup $J = \bigcap_{x \in X} H_x$. This will give us a natural action of G/J on X defined by $gJ.x := g.x$. The G -action is called **almost effective** if the ineffectivity of the G -action is discrete.

Note that from now on all our group actions are almost effective, unless otherwise stated. The reason for this assumption is the following. The identity component J^0 of J is a normal subgroup of G contained in H . One can replace G with $\hat{G} := G/J^0$ and H with $\hat{H} = H/J^0$ in the homogeneous manifold $X := G/H$ and again get the same manifold since $X = \hat{G}/\hat{H}$ but with an almost effective action of \hat{G} instead.

2.2.2 Facts about discrete isotropy

We begin with some facts about complex homogeneous manifolds of the form G/Γ , where Γ is a discrete subgroup of the complex Lie group G . These will be need later.

Suppose X is a connected complex manifold with a G -action of a complex Lie group G on X given by the holomorphic map $\phi(., .) : G \times X \rightarrow X$. For any $v \in \mathfrak{g}$, let $\langle v \rangle_{\mathbb{C}} = \{tv : t \in \mathbb{C}\}$ be the complex span of v and $\exp(\langle v \rangle_{\mathbb{C}})$ be the corresponding one parameter subgroup of G . For $x \in X$ the (complex) derivative at the point x defines a **fundamental vector field** by

$$\xi_v(x) := \left. \frac{d}{dt} \right|_{t=0} \phi(\exp tv, x).$$

The kernel of the map $v \rightarrow \xi_v$ is the Lie algebra of the isotropy subgroup of the G -action at the point x . If $X = G/\Gamma$ with Γ discrete, then the map

$v \mapsto \xi_v$ is injective. So in the setting of discrete isotropy a basis of \mathfrak{g} gives us a corresponding frame on X consisting of $n := \dim_{\mathbb{C}} X$ linearly independent holomorphic vector fields at every point, i.e., X is (complex) parallelizable.

Lemma 2.1 ([Wan54]). *Let X be a connected compact parallelizable complex manifold. Then $X = G/\Gamma$, where G is a connected complex Lie group and Γ is a discrete subgroup of G .*

Proof. Let $\dim X = n$. Since X is parallelizable, there are n linearly independent holomorphic vector fields X_1, \dots, X_n on X that generate the tangent space $T_x X$ at any point $x \in X$. Denote by \mathfrak{g} the Lie algebra of all holomorphic vector fields on X . For $v \in \mathfrak{g}$, there are holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(X)$ such that $v = \sum_{i=1}^n f_i X_i$. Every f_i is constant, since X is compact. So $\mathfrak{g} = \langle X_1, \dots, X_n \rangle_{\mathbb{C}}$ is an n -dimensional complex Lie algebra.

The holomorphic vector fields on any complex manifold X give rise to local holomorphic 1-parameter groups. In general, a similar statement about global 1-parameter groups does not hold. However, for X compact, every holomorphic vector field on X can be globally integrated to a holomorphic 1-parameter group of X . As a consequence, the connected, simply connected, complex Lie group G associated to the Lie algebra \mathfrak{g} acts holomorphically on the compact manifold X . The G -orbit of every point $x \in X$ is open because

the vector fields generating \mathfrak{g} also generate $T_x(X)$. Since X is connected, G is transitive on X . For dimension reasons the isotropy of the G -action on X is a discrete subgroup Γ of G . Hence X is biholomorphic to G/Γ . \square

2.3 Stein manifolds

Definition 2.2. *A complex manifold X is a Stein manifold if the following two conditions hold:*

1. *X is holomorphically separable, i.e., for two distinct points x and y in X , there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.*
2. *X is holomorphically convex, i.e., for any compact subset $K \subseteq X$, the holomorphic convex hull*

$$\hat{K} := \{x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$$

is again a compact subset of X .

Here we give some well known examples:

Example 2.1. 1. \mathbb{C}^n is Stein.

2. A domain $\Omega \subseteq \mathbb{C}^n$ is called a domain of holomorphy if there do not exist non-empty open sets $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \mathbb{C}^n$, where Ω_2 is connected, $\Omega_2 \not\subset \Omega$ and $\Omega_1 \subset \Omega \cap \Omega_2$ such that for every holomorphic function f on Ω there exists a holomorphic function g on Ω_2 with $f = g$ on Ω_1 . A connected, open set in \mathbb{C}^n is a domain of holomorphy if and only if it is holomorphically convex. Domains of holomorphy in \mathbb{C}^n are Stein.
3. Non-compact Riemann surfaces are Stein [BS49].
4. Every closed complex submanifold of a Stein manifold is Stein.
5. The Cartesian product of two Stein manifold is a Stein manifold.
6. Suppose $X \rightarrow Y$ is a covering map with X Stein. If the fiber of this covering map is finite, then Y is Stein [Ste56].

Theorem 2.2 (Embedding theorem). [Rem56] *Let X be a Stein manifold. Then there exists a proper biholomorphic map of X onto a closed complex submanifold of some \mathbb{C}^N .*

Remark 2.1. *Any Stein manifold is also a Kähler manifold. One can pull back a Kähler form on \mathbb{C}^N by using a proper biholomorphic map given by the embedding theorem cited above.*

Holomorphic bundles over Stein manifolds

If $X \rightarrow B$ is a holomorphic covering space and the base B is Stein, then X is Stein [Ste56]. This is also true for holomorphic vector bundles, but might fail for fiber bundles with fibers \mathbb{C}^n , $n > 1$, and with a nonlinear transition group; see for example Skoda [Sk77] for a \mathbb{C}^2 bundle over an open subset in \mathbb{C} whose total space is not Stein.

Here we present a version of the Grauert–Oka principle which we will use in the main theorem.

Theorem 2.3 (Grauert–Oka principle, Satz 6 in [Gr58]). *Let $X \xrightarrow{F} B$ be a holomorphic fiber bundle, where F is a complex manifold and B is a Stein complex manifold. Assume the group of the bundle is a (not necessarily connected) complex Lie group. If the bundle X is continuously homotopic to another bundle \tilde{X} over the Stein base B , then it is holomorphically homotopic to \tilde{X} . In particular, if the bundle \tilde{X} is topologically trivial, then it is holomorphically trivial.*

Theorem 2.4 (Satz 7 in [Gr58]). *Let X be an analytic fiber bundle over a non-compact Riemann surface Y with a connected complex Lie group as its structure group. Then X is analytically trivial.*

Remark 2.2. *We later use a classical result of Serre concerning the vanishing of certain homology groups of Stein manifolds [Ser53] (footnote, p. 59) stating that if X is an n -dimensional Stein manifold, then*

$$H_k(X, \mathbb{Z}_2) = 0$$

for all $k > n$, where by $H_k(X, \mathbb{Z}_2)$ we mean the k th homology group of X with coefficients in \mathbb{Z}_2 .

2.3.1 Lie's Flag theorem

A particular example of a Stein manifold as a homogeneous manifold is given by the orbits of a complex linear solvable Lie group G acting holomorphically on \mathbb{P}_n . By Lie's Flag theorem (see §4.1 in [Hum72]) there is a full flag

$$\{x\} \subsetneq L_1 \subsetneq L_2 \subsetneq \dots \subsetneq L_n = \mathbb{P}_n$$

that is stabilized by the G action on \mathbb{P}_n . Suppose $X := G.y$ is some orbit of the group G in \mathbb{P}_n and set $k := \min\{ m \mid X \cap L_m \neq \emptyset \}$. Without loss of generality we may assume that $y \in L_k$ and thus $X \subset L_k$, since L_k is G -invariant. Thus

$$X = X \cap L_k \subset L_k \setminus L_{k-1} \simeq \mathbb{C}^m.$$

As a consequence, X is holomorphically separable. In this setting Snow proved (Theorem 3.3, [Sno85]) that X is biholomorphic to $(\mathbb{C}^*)^s \times \mathbb{C}^t$ and thus that X is Stein.

2.4 Toward the classification of homogeneous manifolds

Let G be a complex Lie group and H a closed complex subgroup. Borel and Remmert [BR62] classified compact connected homogeneous Kähler manifolds (see also Theorem 2.19). They showed that any such manifold is the direct product of a complex torus and a flag manifold. It is not realistic to classify non-compact homogeneous Kähler manifolds G/H without any restriction.

In this section we present helpful results when there is a condition on G . We will use these results in chapter 4. Note that we are interested in applying the topological restriction $d_X \leq 2$ which we will introduce in section 3.4.

In the next result and throughout this thesis we need the concept of *an affine cone minus its vertex*. For the convenience of the reader we now recall what this is. Suppose $Q = S/P$ is a flag manifold, where S is a connected

complex semisimple Lie group and P is a parabolic subgroup. Then Q can be S equivariantly embedded into some projective space \mathbb{P}_N . Consider this projective space as the hyperplane at infinity in the projective space \mathbb{P}_{N+1} , i.e.,

$$\mathbb{P}_N = \{(0 : z_1 : \dots : z_{N+1}) \in \mathbb{P}_{N+1}\}$$

where we have used homogeneous coordinates. Now consider the complex manifold X consisting of all the complex lines joining the point $(1 : 0 : \dots : 0)$ to points in Q contained in the hyperplane at infinity, but minus the point $(1 : 0 : \dots : 0)$ itself and the points of Q . This is the affine cone minus its vertex over the flag manifold Q . It happens that $X = S/H$ is homogeneous under the induced S action and $S/H \rightarrow S/P$ is a \mathbb{C}^* bundle, where H is the kernel of a non-trivial character $\chi : P \rightarrow \mathbb{C}^*$. As an example, if $Q = \mathbb{P}_N$, then $X = \mathbb{C}^{N+1} - \{0\}$.

Theorem 2.5. [AG94] *Suppose G is a connected complex Lie group and H is a closed complex subgroup such that $X := G/H$ satisfies $\mathcal{O}(X) \neq \mathbb{C}$ and $d_X \leq 2$. Let $Y := G/J$ be the base of the holomorphic reduction (see section 3.3) of X and $F := J/H$ be its fiber.*

- a) *If $d_X = 1$, then F is compact and connected and Y is an affine cone minus its vertex.*

- b) If $d_X = 2$, then one of the following two cases occurs:

b₁) The fiber F is connected and satisfies $d_F = 1$ and the base Y is an affine cone minus its vertex,

b₂) The fiber F is compact and connected and $d_Y = 2$; moreover, Y is one of the following manifolds:

1) The complex line \mathbb{C} ;

2) The affine quadric Q_2 ;

3) $\mathbb{P}_2 \setminus Q$ where Q is a quadric curve;

4) A homogeneous holomorphic \mathbb{C}^* -bundle over an affine cone with its vertex removed. In this case Y is either an algebraic principal \mathbb{C}^* -bundle or is covered two-to-one by such.

2.4.1 G Abelian

Let G be an Abelian complex Lie group with a closed subgroup H . Thus, H is normal and G/H is an Abelian complex Lie group. In the following we define a special complex Lie group called a Cousin group, and then give a classification of complex Abelian Lie groups.

Cousin group

Cousin was the first to call attention to Abelian non-compact complex Lie groups, now called Cousin groups or toroidal groups, with the additional property that $\mathcal{O}(C) \cong \mathbb{C}$. These groups appear in the classification of Abelian complex Lie groups [Mor66].

Definition 2.3 (Cousin group). *A Cousin group is a complex Lie group C which admits no non-constant holomorphic function.*

Let C be a connected complex Lie group with $\mathcal{O}(C) \cong \mathbb{C}$. Since $GL_{\mathbb{C}}(V)$ is holomorphically separable, there is no non-constant holomorphic homomorphism $C \rightarrow GL_{\mathbb{C}}(V)$ to the general linear group of a complex vector space V . Let $\mathfrak{V} = \mathfrak{C}$ be the Lie algebra of C . The kernel of the adjoint representation $\text{Ad} : C \rightarrow GL_{\mathbb{C}}(\mathfrak{C})$ is the center of C showing that **a connected complex Lie group C with $\mathcal{O}(C) \cong \mathbb{C}$ is Abelian.**

Since a Cousin group is Abelian, the exponential map $\exp : \mathfrak{C} \rightarrow C$ is a surjective homomorphism. The kernel Γ of the exponential map is discrete, so $C \cong \mathfrak{C}/\Gamma$. Since \mathfrak{C} is a vector space, it is isomorphic to \mathbb{C}^n for some n . This gives us the structure of a Cousin group $C = \mathbb{C}^n/\Gamma$ where Γ is a discrete additive subgroup of \mathbb{C}^n . **Note that we have the real torus $\text{Span}(\Gamma)_{\mathbb{R}}/\Gamma$**

inside any Cousin group.

Classification of connected, Abelian, complex Lie groups

We first give a classification of connected, Abelian, real Lie group G with Lie algebra \mathfrak{g} . Since $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$ we can consider the Lie algebra \mathfrak{g} as a vector space over \mathbb{R} . For an Abelian connected Lie group the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a surjective homomorphism with discrete kernel, thus a finitely generated free Abelian group (Theorem 3.6, page 25, [BT85]). Hence, $\text{Ker exp} = \langle v_1, \dots, v_p \rangle_{\mathbb{Z}}$ where $\{v_1, \dots, v_p\}$ is a linearly independent set of vectors in \mathfrak{g} . Let $V := \langle v_1, \dots, v_p \rangle_{\mathbb{R}}$ and choose a complementary vector subspace W with $\mathfrak{g} = V \oplus W$. Recall $\dim \mathfrak{g} = n$, $\dim V = p$. So $\dim W = n - p$. Then

$$\begin{aligned} G &\cong \mathfrak{g} / \text{Ker exp} \\ &\cong (V \oplus W) / \langle v_1, \dots, v_p \rangle_{\mathbb{R}} \\ &\cong V / \langle v_1, \dots, v_p \rangle_{\mathbb{Z}} \oplus W \\ &= (\mathbb{S}^1)^p \times \mathbb{R}^{n-p} \end{aligned}$$

Here $K := (\mathbb{S}^1)^p$ is the product of p copies of a circle \mathbb{S}^1 and is the unique maximal compact subgroup of G .

For an Abelian complex Lie group $G = \mathbb{C}^n / \Gamma$ one has the following classi-

fication due to Remmert and Morimoto, e.g., see [Mor66]. We now choose a complex vector space W that is complementary to $V := \langle v_1, \dots, v_p \rangle_{\mathbb{C}}$. Since G is holomorphically isomorphic to $(V/\Gamma) \times W$, we can reduce to the setting where Γ generates V as a complex vector space and $W \cong \mathbb{C}^t$.

Let $V_{\Gamma} := \langle \Gamma \rangle_{\mathbb{R}}$. Then the corresponding subgroup $K := V_{\Gamma}/\Gamma$ is the maximal compact subgroup of G . Set $W_{\Gamma} := V_{\Gamma} \cap iV_{\Gamma}$, the maximal complex vector subspace of V_{Γ} . The W_{Γ} -orbits in G are isomorphic to $W_{\Gamma}/W_{\Gamma} \cap \Gamma$ and give a complex foliation of G . The closure of the W_{Γ} -orbit through the identity of G is a subtorus L_1 of K and we can choose a complementary totally real subtorus L_2 such that $K = L_1 \times L_2$. There are uniquely defined vector subspaces \mathfrak{l}_i of V such that $L_i = \mathfrak{l}_i/\Gamma \cap \mathfrak{l}_i$ and we set $U_i := \mathfrak{l}_i + i\mathfrak{l}_i$. Assuming Γ generates V as a complex vector space, we have

$$G = V/\Gamma = U_1/(U_1 \cap \Gamma) \times U_2/(U_2 \cap \Gamma)$$

Note that $U_2/(U_2 \cap \Gamma) \cong (\mathbb{C}^*)^s$, since L_2 is totally real.

We claim $U_1/(U_1 \cap \Gamma)$ is a Cousin group, i.e., that $\mathcal{O}(U_1/(U_1 \cap \Gamma)) = \mathbb{C}$. For any $f \in \mathcal{O}(U_1/(U_1 \cap \Gamma))$ the restriction of $|f|$ to L_1 attains a maximum at some point $x \in L_1$, since the latter group is compact. The orbit map $\mathfrak{m} \mapsto G$ is holomorphic and thus the pullback of f to \mathfrak{m} is holomorphic and

its modulus attains a maximum at any preimage of x . By the Maximum Principle this pullback function is constant. Thus f is constant on the orbit of f through x . Since this orbit is dense in L_1 , the restriction of f to L_1 is constant. Finally, by using the Identity Principle it follows from the fact that $L_1 = \mathfrak{l}_1/\Gamma \cap \mathfrak{l}_1$ and $U_1 := \mathfrak{l}_1 + i\mathfrak{l}_1$ that f is constant on $U_1/(U_1 \cap \Gamma)$, i.e., $U_1/(U_1 \cap \Gamma)$ is a Cousin group.

Example 2.2. *The set $\{(1, 0), (0, 1), (i, i\alpha)\}$ is linearly independent over \mathbb{R} where $0 < \alpha < 1$ and $\alpha \notin \mathbb{Q}$. Let $V := \langle (1, 0), (0, 1), (i, i\alpha) \rangle_{\mathbb{R}}$. One can check that $K := V/\langle (1, 0), (0, 1), (i, i\alpha) \rangle_{\mathbb{Z}}$ is the maximal compact subgroup of $C := \mathbb{C}^2/\langle (1, 0), (0, 1), (i, i\alpha) \rangle_{\mathbb{Z}}$. We claim that $\mathcal{O}(C) = \mathbb{C}$. Here $\mathfrak{m} = \langle (1, \alpha) \rangle_{\mathbb{C}}$. The orbit of M in C is dense in K . Thus $L = K$ in this case and $L^{\mathbb{C}} = K^{\mathbb{C}} = C$. Suppose $f \in \mathcal{O}(C)$. Define $\sigma : \mathbb{C} \rightarrow \mathfrak{m}, z \mapsto z(1, \alpha)$. Consider the holomorphic map:*

$$\mathbb{C} \xrightarrow{\sigma} \mathfrak{m} \xrightarrow{\exp} M \xrightarrow{f} \mathbb{C}$$

Since M lies in K and $f(K)$ is compact, the holomorphic function $f \circ \exp \circ \sigma$ is a bounded, entire function, hence constant. But σ and \exp are not constant. Therefore $f|_M$ is constant and thus $f|_K$ is constant. However, K has real co-dimension one in C , it follows that f is constant on C .

2.4.2 G nilpotent

We need the following definition.

Definition 2.4. *A (principal) Cousin group tower of length one is a Cousin group. A (principal) Cousin group tower of length $n > 1$ is a (principal) holomorphic bundle with fiber a Cousin group and base a (principal) Cousin group of length $n - 1$.*

One has the following structure theorem.

Theorem 2.6. *[GH78] Let G be a connected, nilpotent, complex Lie group and let H be a closed complex subgroup of G . Then there exists a closed complex subgroup J of G , containing H , such that*

- $\mathcal{O}(G/H) = \pi^*(\mathcal{O}(G/J))$, where $\pi : G/H \rightarrow G/J$ is the bundle projection
- G/J is Stein
- the fiber J/H is connected
- $\mathcal{O}(J/H) = \mathbb{C}$
- J/H is a principal Cousin group tower (e.g., see Theorem IV.1.5 in [Oel85]).

Proof sketch.

One can reduce the situation to the discrete case by the following argument. The holomorphic reduction of G/H factors through the normalizer fibration since the base of the normalizer fibration is holomorphically separable (Lie's flag theorem). So it is enough to consider the normalizer fibration.

$$\begin{array}{ccc} G/H & & \\ \downarrow & \searrow^{N/H} & \\ G/J & \longrightarrow & G/N \end{array}$$

By assumption G is a connected nilpotent Lie group, and H^0 is a connected subgroup of G . Matsushima [Mat51] proved that for a nilpotent Lie group G , the normalizer $N := N_G(H^0)$ of a connected subgroup is connected. So $G/N = \mathbb{C}^n$, and $G/H = G/N \times N/H$ by Theorem 2.3. As a consequence, it is enough to study the fiber of the normalizer fibration which has discrete isotropy (to be discussed in section 3.1). From now on we assume $H = \Gamma$ is a discrete subgroup of G such that the smallest closed, connected, complex subgroup of G containing Γ is all of G . The center Z of G is a positive dimensional Γ -normal subgroup of G , i.e., $Z\Gamma$ is a closed subgroup of G (see [GH78]). We have the fibration

$$G/\Gamma \xrightarrow{Z\Gamma/\Gamma} G/Z\Gamma.$$

This fibration is applied to the base of the holomorphic reduction to prove by induction that G/Γ is holomorphically separable if and only if $G_{\mathbb{R}}/\Gamma$ is totally real in G/Γ if and only if G/Γ is Stein. For more explanation see [GH78]. \square

D. Akhiezer [Akh84] and K. Oeljeklaus [Oel85] showed the following result that applies to the fiber of the holomorphic reduction of any complex nilmanifold X and proves the remaining point in this structure result for complex nilmanifolds.

Theorem 2.7 (Theorem IV.1.5. [Oel85]). *Let $X = G/\Gamma$ be a complex nilmanifold with $\mathcal{O}(G/\Gamma) \cong \mathbb{C}$. Then X is a principal Cousin group tower.*

Proof sketch.

The proof is by induction on $\dim G$ and for $\dim G = 1$, it is clear that G/Γ is a torus. If G is Abelian, clearly G/Γ is a complex Lie group with $\mathcal{O}(G/\Gamma) = \mathbb{C}$ and thus G/Γ is a Cousin group. So we assume the center Z of G satisfies $0 < \dim Z < \dim G$. Its orbit $Z/Z \cap \Gamma$ is closed in G/Γ and is a connected Abelian complex Lie group. In the previous section we noted that $Z/Z \cap \Gamma$ is isomorphic to a product $\mathbb{C}^k \times (\mathbb{C}^*)^p \times C$, where C is a Cousin group. We need to show that $\dim C > 0$ in order to apply the induction.

As noted above there is a real connected subgroup $G_{\mathbb{R}}$ of G containing Γ cocompactly and since G/Γ is not Stein, $G_{\mathbb{R}}/\Gamma$ is not totally real. Thus the ideal $\mathfrak{m} := \mathfrak{g}_{\mathbb{R}} \cap i\mathfrak{g}_{\mathbb{R}} \neq 0$. Let \mathfrak{z} be the center of \mathfrak{g} , and set $\mathfrak{h} = \mathfrak{z} \cap \mathfrak{m}$. In a nilpotent Lie algebra every non-zero ideal intersects the center non-trivially, so $\mathfrak{h} \neq 0$ (see page 13 in [Hum72]). Let H be the Lie group corresponding to \mathfrak{h} . Let \mathfrak{k} denote the Lie algebra of the maximal compact subgroup K of $Z/Z \cap \Gamma$. Note that K contains a positive dimensional complex Lie subgroup, namely the orbit of H through e . Thus $\dim C > 0$. Let \widehat{C} denote the preimage of C in G and note that \widehat{C} is a central closed complex subgroup of G that has closed orbits in G/Γ . Then the holomorphic fibration $G/\Gamma \rightarrow G/\widehat{C}.\Gamma$ has the positive dimensional Cousin group C as fiber and the lower dimensional complex nilmanifold $G/\widehat{C}.\Gamma$ as base. Also because of the fact that \widehat{C} is central in G , this bundle is a principal bundle. Since necessarily $\mathcal{O}(G/\widehat{C}.\Gamma) = \mathbb{C}$, the result now follows by induction. \square

2.4.3 G solvable

Let G be a connected solvable complex Lie group and Γ be a discrete subgroup. Assume there exists a closed, connected (real) subgroup $G_{\mathbb{R}}$ of G containing Γ such that $G_{\mathbb{R}}/\Gamma$ is compact. The triple $(G, G_{\mathbb{R}}, \Gamma)$ will be called

a CRS manifold. The condition for the existence of such a real subgroup $G_{\mathbb{R}}$ in a given complex solvable Lie group G and discrete subgroup Γ is not known. However, if G is a nilpotent complex Lie group, then it contains such a real Lie subgroup [Mal49]. Note that the classification of G/Γ for a Kähler solvmanifold is given in [GO11] whenever $G_{\mathbb{R}}$ exists and $G_{\mathbb{R}}/\Gamma$ is of codimension two inside G/Γ . Cœuré and Loeb present the following example [CL85] which states that in the solvable case this real subgroup might exist but G/Γ might not even be Kähler.

Example 2.3. *Let $G_K = K \rtimes K^2$, where $K = \mathbb{Z}$ or \mathbb{C} with the group operation given by*

$$(z, b) \circ (z', b') := (z + z', e^{Az'}b + b')$$

where $z, z' \in K$, $b, b' \in K^2$ and A is the real logarithm of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $G_{\mathbb{C}}$ is a connected, simply connected three dimensional complex Lie group that contains the discrete subgroup $G_{\mathbb{Z}}$ such that its holomorphic reduction is given by

$$G_{\mathbb{C}}/G_{\mathbb{Z}} \rightarrow G_{\mathbb{C}}/G'_{\mathbb{C}} \cdot G_{\mathbb{Z}}$$

with the base biholomorphic to \mathbb{C}^* and the fiber to $\mathbb{C}^* \times \mathbb{C}^*$.

Huckleberry and E. Oeljeklaus showed that the base of the holomorphic reduction is Stein (Theorem, p. 58 in [HO86]). The following two theorems give valuable information for holomorphic reductions of Kähler solvmanifolds.

Theorem 2.8. [OR88] *Let X be a Kähler solvmanifold and let*

$$X \xrightarrow{F} Y$$

denote the holomorphic reduction of X . Then Y is a Stein manifold and F is a Cousin group. Moreover the first fundamental group $\pi_1(X)$ contains a nilpotent subgroup of finite index.

Theorem 2.9. [GO08] *Suppose G is a connected, solvable, complex Lie group and H is a closed complex subgroup of G with $X := G/H$ a Kähler manifold.*

Let

$$G/H \longrightarrow G/I$$

be the holomorphic reduction. Then there is a subgroup of finite index $\hat{I} \subset I$ such that the bundle

$$\hat{X} := G/\hat{I} \cap H \rightarrow G/\hat{I}$$

is a holomorphic $I^0/H \cap I^0$ -principal bundle and represents the holomorphic reduction of \hat{X} .

2.4.4 G semi-simple

Let G be a semisimple complex Lie group and recall that G admits the structure of a linear algebraic group. A Borel subgroup of G is a maximal connected solvable subgroup of G . A parabolic subgroup is a subgroup of G containing a Borel subgroup. If P is a parabolic subgroup of G , then the manifold G/P is a flag manifold. For information on flag manifolds we refer the reader to [Wol69] and §3.1 in [Akh95].

Theorem 2.10 ([BO88] and Corollary 4.12 in [GMO13]). *Let G be a semi-simple complex Lie group and H a closed complex subgroup of G . Then the following conditions are equivalent:*

- (i) H is algebraic.
- (ii) the homogeneous space G/H is Kähler.

One important consequence of this theorem is that when X is a Kähler manifold under the transitive action of a semisimple group with discrete isotropy, then the isotropy group is automatically a finite set of points. One should note that since there is no known classification of discrete subgroups of complex Lie groups, even in low dimensional cases such as $SL(2, \mathbb{C})$, the

classification of semisimple homogeneous manifolds is impossible at the moment.

2.4.5 G linear algebraic

Theorem 2.11. *Let G be a connected complex linear algebraic group and H an algebraic subgroup of G . Set $X := G/H$.*

1. [Akh83] *Suppose $d_X = 2$. Then the space X is a twisted product $X = G \times_P F$. Here P is a parabolic subgroup of G and the manifold F is isomorphic to \mathbb{A}^1 , $\mathbb{C}^* \times \mathbb{C}^*$, or $\mathbb{P}_2 \setminus Q$ with Q a quadric, and the transitive operation $P \times F \rightarrow F$ is given by affine transformation, by group translations, or by projective transformations preserving Q .*
2. [Akh77] *If $d_{G/H} = 1$, then there exists a parabolic subgroup P of G containing H such that $G/H \rightarrow G/P$ is a \mathbb{C}^* -bundle over the flag manifold G/P .*

2.4.6 G reductive

A complex Lie group is reductive if it is the complexification of a totally real maximal compact subgroup. Then a finite covering of any reductive complex

Lie group has the form $S \times R$, where $R = (\mathbb{C}^*)^k$. In passing, we note that here $S \trianglelefteq G$. The following result had been proved earlier.

Theorem 2.12 ([Mat60], [Oni60]). *Let G be a reductive complex Lie group, H be a closed complex subgroup. Then G/H is Stein if and only if H is reductive.*

In passing we note that the Kähler setting is characterized by the following.

Theorem 2.13 (Theorem 5.1 in [GMO13]). *Suppose G is a reductive complex Lie group with Levi–Malcev decomposition $G = S.R$ and let H be a closed complex subgroup of G . Then G/H is Kähler if and only if $S.H$ is closed in G and $S \cap H$ is an algebraic subgroup of S .*

Theorem 2.14. *(Lemma in §2.4 in [BO73]) Assume G is a reductive complex Lie group. Let H be a closed subgroup with \overline{H} as its Zariski closure. Every H -invariant holomorphic function on G is necessarily \overline{H} -invariant.*

In other words, for G reductive the holomorphic reduction G/J of the complex homogeneous space G/H is also the holomorphic reduction for G/\overline{H} and so the holomorphic reduction map factors through G/\overline{H} as is given in

the following commutative diagram

$$\begin{array}{ccc}
 G/H & \longrightarrow & G/\overline{H} \\
 & \searrow & \swarrow \\
 & & G/J
 \end{array}$$

In fact J is algebraic (see Satz in §2.5 in [BO73]).

2.5 Akhiezer's question in Kähler setting

A Kähler manifold $X = G/\Gamma$ with discrete isotropy Γ and $d_X \leq 2$ is a solvmanifold (see proposition 4.1). This interesting phenomenon which we prove is related to a question of Akhiezer [Akh84]. The original question concerned the existence of analytic hypersurfaces in complex homogeneous manifolds. We consider the following variant of this question in the Kähler setting.

Modified Question of Akhiezer.

Suppose $X := G/H$ is a Kähler homogeneous manifold that satisfies

- (i) $\mathcal{O}(X) = \mathbb{C}$
- (ii) There is no proper parabolic subgroup of G which contains H

Is G then solvable?

Note that if G is known to be solvable, then it follows that X is a Cousin group by Theorem 2.8, since in that theorem the base of the holomorphic reduction of X will be a point. Hence X will be isomorphic to the fiber of its holomorphic reduction which is a Cousin group.

We will need the following.

Theorem 2.15. *(Corollary 6, p. 49 in [HO84]) Let G/H be a complex homogeneous manifold, and $N := N_G(H^0)$. If $\mathcal{O}(G/H) = \mathbb{C}$ and if $N \neq G$, then H is contained in a proper parabolic subgroup P in G .*

Next we prove that the radical orbits in some special setting are closed. We use this fact to prove proposition 4.1. Note that the following example shows the radical orbits might not be closed in general.

Example 2.4. *Consider $G := \mathbb{S}^1 \times \mathbb{R}$, where we consider the circle as complex numbers of modulus one. Let Γ be the discrete subgroup $\{(\pi i k \alpha, k) | k \in \mathbb{Z}\}$, where α is an irrational number. Note that $\Gamma \cdot \mathbb{R}$ is not closed in G . For a complex version of this embed the \mathbb{S}^1 into the diagonal matrices in $S = SL(2, \mathbb{C})$ which is biholomorphic to \mathbb{C}^* . Set $G = S \times \mathbb{C}$. One can see that $\Gamma \cdot R$ is not closed in G where R denotes the radical of G .*

Lemma 2.2. *Suppose G is a mixed connected complex Lie group, i.e., $G = S.R$ is a Levi–Malcev decomposition, where both S and R have positive dimension. Let $X := G/\Gamma$, where the discrete isotropy Γ is not contained in any proper parabolic subgroup of G . If there is no non-constant holomorphic function on X , then the radical orbits of G are closed in X .*

Proof. All R orbits are closed if and only if $R.\Gamma$ is closed. To prove that $R.\Gamma$ is closed we assume it is not and derive a contradiction. Let $H := \overline{R.\Gamma}$ where by bar we mean the topological closure. Auslander, using earlier results of Zassenhaus [Aus63] proved that H^0 is solvable. If H^0 is not a complex Lie group, let H_1 be the connected Lie subgroup of G associated to the complex Lie algebra $\mathfrak{h}_1 := \mathfrak{h} + i\mathfrak{h}$, where \mathfrak{h} denotes the Lie algebra of H^0 . Note that H_1 is solvable and complex. If H_1 is not closed, we continue this process which has to stop after a finite number of steps, because G has finite dimension.

We let I be the **'minimal' closed connected complex solvable** Lie group obtained in this manner. Define $J := N_G(I)$. Note that I is not the radical, since, by assumption, R does not have closed orbits and so $J \neq G$. Now consider the fibration $G/\Gamma \rightarrow G/J$. Since J is, by definition, the normalizer of a connected group we see that G/J is an orbit in some projective space (see section 3.1). Since $\mathcal{O}(G/\Gamma) \cong \mathbb{C}$, it then follows that

$\mathcal{O}(G/J) \cong \mathbb{C}$. Theorem 2.15 applies, so J is contained in a proper parabolic subgroup of G . If we prove that $\Gamma \subset J$ then Γ is contained in this proper parabolic subgroup which is a contradiction to our assumptions. To see this first note that for any $g \in H$ we have $gH^0g^{-1} = H^0$ and so we have $H^0 \subseteq gIg^{-1} \cap I$. But I is the minimal closed connected complex solvable Lie group that contains H^0 , so $gIg^{-1} = I$, hence $H \subseteq J$. But $\Gamma \subset H$. Thus $\Gamma \subset J$ and this contradiction proves that $R\Gamma$ is closed in G . \square

We first look at the case where $G = S \times R$, i.e., G is the trivial product group of a maximal semisimple subgroup S with the radical R of G .

Lemma 2.3 (Lemma 6 in [AG94]). *Suppose G is a complex Lie group whose Levi decomposition is a direct product $S \times R$. Let Γ be a discrete subgroup of G such that $S \cap \Gamma$ is finite. Then Γ is contained in a subgroup of the form $A \times R$, where A is an algebraic subgroup of S such that its identity component A^0 is solvable.*

We will use the following theorem in chapter 4.

Theorem 2.16 (p. 116 in [OR87] & Theorem 2 in [Gi04]). *Suppose G is a connected simply connected complex Lie group whose Levi-Malcev decompo-*

sition is a direct product $G = S \times R$. Suppose Γ is a discrete subgroup of G such that $X := G/\Gamma$ satisfies the conditions

(a) X is Kähler

(b) Γ is not contained in a proper parabolic subgroup of G and

(c) X does not possess non-constant holomorphic functions

Then $S = \{e\}$. That is, G/Γ is a Cousin group.

Depending on the analytic behavior of the radical orbits there is an answer to the Akhiezer's question. If the radical orbits do not possess any non-constant holomorphic functions then we have the following theorem which basically presents us to the situation of the Theorem 2.16.

Theorem 2.17 (Theorem 3 in [Gi04]). *Suppose G is a connected complex Lie group and Γ is a discrete subgroup of G such that G/Γ satisfies the condition (a), (b), and (c) of the Theorem 2.16. Let R denote the radical of G . Assume the typical radical orbit has no non-constant holomorphic function, that is, $\mathcal{O}(R.\Gamma/\Gamma) = \mathbb{C}$. Then $S = \{e\}$, that is, G is solvable.*

If the R orbits possess non-constant holomorphic functions, then the answer to Akhiezer's question in the general Kähler setting is not known. However, enough is known in certain cases that we are able to use these results

to prove the classification in our main theorem. The remaining special case that we need is when $G = S \ltimes R$ is a semidirect product and the radical R has complex dimension two. Then one has the following theorem which is extracted from the proof of Lemma 8 in [AG94].

Lemma 2.4. *Let G be a complex Lie group with Levi-Malcev decomposition $G = S \ltimes R$ with $\dim_{\mathbb{C}} R = 2$. Let Γ be a discrete subgroup of G such that $X = G/\Gamma$ is Kähler. Then Γ is contained in a proper subgroup of G of the form $A \ltimes R$, where A is a proper algebraic subgroup of S .*

Proof. Without loss of generality we may assume that G is simply connected. Suppose that the adjoint action of S on R is trivial. Then $G = S \times R$ and $S \cap \Gamma$ is finite by Theorem 2.10. The result follows by Lemma 2.3, i.e., $\Gamma \subseteq A \ltimes R$, where A is a proper algebraic subgroup of S .

Assume now that the adjoint action of S on R is not trivial. Then we claim that R is Abelian. If not, then R' is one-dimensional and S -invariant. Hence the S -action on R' is trivial and since S is semisimple, this action is completely reducible. So there is a one dimensional subspace of R complementary to R' that is also S -invariant. But this action is also trivial which gives a contradiction to the S -action on R being non-trivial. So R is Abelian and the S -action on R is irreducible.

We next note that G has the structure of a linear algebraic group. We see this in the following way. Consider a semisimple complex Lie group S with an irreducible two dimensional representation $R : S \rightarrow GL(2, \mathbb{C})$. Since S is perfect, its image is too and thus this image must be isomorphic to $SL(2, \mathbb{C})$ which is 3 dimensional. By the rank–nullity theorem the kernel of R has codimension 3 and is a normal subgroup of S . Thus S decomposes as a direct product $S = S_1 \cdot S_2$, where S_2 acts trivial and $S_1 \simeq SL(2, \mathbb{C})$ with the usual action on \mathbb{C}^2 . Any element $s \in S$ has the form $s_1 \cdot s_2$ with s_1 having different eigenvalues and s_2 acting trivially on R . So $G = S_2 \times (S_1 \ltimes \mathbb{C}^2)$. Since S_2 and $S_1 \ltimes \mathbb{C}^2$ are both linear algebraic, it follows that G has a linear algebraic structure.

We claim that Γ is contained in a proper algebraic subgroup of G . We assume not, i.e., that Γ is Zariski dense in G and derive a contradiction. Then $\pi(\Gamma)$ is Zariski dense in S , where $\pi : G \rightarrow S$ denotes the projection. It follows that $\pi(\Gamma)$ contains a free group with two generators that is also Zariski dense in S [Ti72]. We pick π –preimages of two of these generators in Γ and let $\tilde{\Gamma}$ be the subgroup of Γ generated by these elements. There is a torsion free subgroup $\tilde{\Gamma}_1$ of finite index in $\tilde{\Gamma}$ (Corollary 6.13, page 95, [Rag08]). Then $\pi(\tilde{\Gamma}_1)$ is also Zariski dense in S . For any $g \in G$ the intersection $\tilde{\Gamma}_1 \cap gSg^{-1}$ is

finite, since the gSg^{-1} -orbit in the Kähler manifold $G/\tilde{\Gamma}_1$ has finite isotropy [BO88]. Since $\tilde{\Gamma}_1$ is torsion free, we have $\tilde{\Gamma}_1 \cap gSg^{-1} = \{e\}$. Thus

$$\tilde{\Gamma}_1 \setminus \{e\} \subset G \setminus \bigcup_{g \in G} gSg^{-1} \subset G \setminus \pi^{-1}(S \setminus Z) = \pi^{-1}(Z),$$

where Z is a proper Zariski closed subset of S given by Lemma 7 in [AG94]. Then $\pi(\tilde{\Gamma}_1) \subset Z$, contradicting the fact that $\pi(\tilde{\Gamma}_1)$ is Zariski dense in S . Hence Γ is not Zariski dense in G , i.e., $A := \overline{\pi(\Gamma)}$ is a proper algebraic subgroup of S , so $\Gamma \subset A \times R$ as desired. \square

This has the following consequence which we use later.

Theorem 2.18. *Suppose G is a connected complex Lie group with Levi-Malcev decomposition $G = S \times R$ with $\dim_{\mathbb{C}} R = 2$. Let Γ be a discrete subgroup of G such that $X = G/\Gamma$ is Kähler, Γ is not contained in a proper parabolic subgroup of G and $\mathcal{O}(G/\Gamma) \simeq \mathbb{C}$. Then $S = \{e\}$, i.e., G is solvable.*

Proof. The radical orbits are closed by lemma 2.2. By lemma 2.4 the subgroup Γ is contained in a proper subgroup of G of the form $A \times R$, where A is an algebraic subgroup of S . Thus there are fibrations

$$G/\Gamma \longrightarrow G/R.\Gamma \longrightarrow S/A,$$

where the base $G/R.\Gamma = S/\Lambda$ with $\Lambda := S \cap R.\Gamma$. If A is reductive, then S/A is Stein and we get non-constant holomorphic functions on X as pullbacks

using the above fibrations. But this contradicts the assumption that $\mathcal{O}(X) \simeq \mathbb{C}$. If A is not contained in a proper reductive subgroup, then Theorem in §30.4 in [Hum75] applies and A is contained in a proper parabolic subgroup of S and so is Γ , thus contradicting the assumption that this is not the case. \square

We finish this chapter with a very well known theorem of Borel and Remmert concerning compact complex Kähler manifold. We will refer to this theorem through this thesis.

Theorem 2.19. *[BR62] Let X be a connected compact complex Kähler manifold such that the group of all holomorphic transformations of X is transitive. Then X is the product of a complex torus and a flag manifold.*

Chapter 3

Basic Tools

Fibration Methods

One of our standard tools throughout is the use of fibrations, in particular, those that arise as homogeneous fibrations, i.e.,

$$G/H \xrightarrow{J/H} G/J.$$

where G is a complex Lie group, and J and H are closed complex subgroups with H contained in J . Let H^0 be the identity component of H , and \tilde{H} be the largest subgroup of H normal in J . The group of the bundle is J/\tilde{H} acting in J/H as left translations (see the Theorem on page 30, [Ste57]). If this action is trivial the bundle splits as the trivial product $G/J \times J/H$.

Here we give a brief explanation of some fibrations that we will use in later chapters. For further discussion we refer the reader to the references [HO84], [Hu90] and [HO81].

3.1 Normalizer fibration

The normalizer of a subgroup of a group has been a useful tool in group theory for a long time. Tits was the first who analyzed the structure of compact complex homogeneous manifolds using the normalizer subgroup [Ti62]. Given a complex homogeneous manifold $X = G/H$, where G is a (connected) complex Lie group and H is a closed complex subgroup, let $N := N_G(H^0)$ be the normalizer in G of the identity component H^0 of H . Then $H \subset N$ and N is a closed complex subgroup of G . One then has the fibration $G/H \rightarrow G/N$. This fibration is useful because its base G/N is an orbit in some projective space. To see this consider the adjoint representation $\text{Ad} : G \rightarrow GL(\mathfrak{g})$, where \mathfrak{g} is the Lie algebra of G . Let \mathfrak{h} be the Lie algebra of H . Since \mathfrak{h} is a vector subspace of dimension k in the vector space \mathfrak{g} of dimension n , we may consider \mathfrak{h} as a point in Grassmann manifold $\text{Gr}(k, n)$, and N as its isotropy group. As a conclusion we can always consider the base of a normalizer

fibration G/N as an orbit in some Grassmann manifold. By Plücker embedding theorem this Grassmann manifold can be equivariantly embedded in a complex projective space \mathbb{P}_m .

Since H^0 is normal in N , it acts trivially on the fiber $F = N/H$, and we can write the fiber as $F = \hat{N}/\Gamma$, where $\Gamma := H/H^0$ is a discrete subgroup of the complex Lie group $\hat{N} := N/H^0$. Thus, the fiber of the normalizer fibration is always complex parallelizable. If the fiber of the normalizer fibration is also compact and Kähler, then the lemma below shows that it is a torus.

Lemma 3.1 (Corollary 2 in [Wan54]). *Let X be a compact complex parallelizable Kähler manifold. Then X is a torus, i.e. $X = \mathbb{C}^n/\Gamma$, where Γ is a co-compact, discrete additive subgroup.*

Proof. By Lemma 2.1 we have $X = G/\Gamma$. To prove the lemma it is enough to prove that the Lie algebra \mathfrak{g} is Abelian and so $G \cong \mathbb{C}^n$. Let $\omega_1, \dots, \omega_n$ be a basis of right-invariant holomorphic 1-forms on G . Because these forms are right G -invariant they are right Γ -invariant so we can consider them as forms on X . Since X is compact and Kähler, holomorphic forms on X are closed. Hence for the basis X_1, \dots, X_n of invariant vector fields on X one

has

$$0 = d\omega_i(X_j, X_k) = \frac{1}{2}\{X_j(\omega_i(X_k)) - X_k(\omega_i(X_j))\} - \omega_i([X_j, X_k]) = -\omega_i([X_j, X_k])$$

for all i, j , and k (see page 36, [KN63]). This implies $[X_j, X_k] = 0$ for all j and k , and therefore the Lie algebra \mathfrak{g} is Abelian. \square

The base of the normalizer fibration of a compact homogeneous manifold is a flag manifold. If a homogeneous complex manifold is not compact, it might happen that the base is compact. The next lemma is the Borel Fixed Point Theorem and is stated here for future reference.

Lemma 3.2. *If the base G/N of the normalizer fibration is compact, then it is a flag manifold, i.e., homogeneous under the effective action of a semisimple group.*

3.2 Commutator fibration

The base of the normalizer fibration is an orbit in some projective space. In this setting G is represented as a complex linear group with \mathfrak{g} as its Lie algebra. Let \overline{G} be the algebraic closure of G and note that the corresponding \overline{G} -orbit $\overline{G}/\overline{H}$ contains the G -orbit G/H . By abuse of language we are using

\overline{H} to denote the isotropy of the \overline{G} orbit and not the algebraic closure of H . Since \overline{G} is an algebraic group acting algebraically on the projective space, its commutator subgroup \overline{G}' is also algebraic and has closed orbits. This gives rise to the following diagram

$$\begin{array}{ccc} G/H & \longrightarrow & \overline{G}/\overline{H} \\ \downarrow & & \downarrow \\ G/I & \longrightarrow & \overline{G}/\overline{G}' \cdot \overline{H} \end{array}$$

Let G' be the commutator group of G with \mathfrak{g}' as its Lie algebra. By Chevalley's theorem (Theorem 13, page 173, [Che51]), one has $\mathfrak{g}' = \overline{\mathfrak{g}'}$ and thus $G' = \overline{G}'$. Now we claim that $I = G'.H = G \cap \overline{G}' \cdot \overline{H}$. Let $g = g_1.g_2 \in G \cap \overline{G}' \cdot \overline{H}$, where $g_1 \in \overline{G}' = G'$ and $g_2 \in \overline{H}$. But since $g_2 \in G \cap \overline{H} = H$ we conclude $g = g_1.g_2 \in G'.H$ and hence $G \cap \overline{G}' \cdot \overline{H} \subseteq G'.H$. The opposite inclusion is also true and we get the equality as desired. As a consequence, $G/I = G/G'.H$ is an orbit in the affine algebraic Abelian group $\overline{G}/\overline{G}' \cdot \overline{H}$. Since the latter group is Stein, we see that $G/G'.H$ is also Stein and we have the commutator group fibration

$$G/H \longrightarrow G/G'.H$$

If G is also solvable, then the commutator group is unipotent. So the fiber of the above fibration is algebraically isomorphic to some affine space \mathbb{C}^n .

3.3 Holomorphic reduction

Let $X := G/H$ be a homogeneous complex manifold, where G is a (connected) complex Lie group, and H a closed complex subgroup. Define an equivalence relation \sim on X by

$$x \sim y \iff f(x) = f(y) \quad \forall f \in \mathcal{O}(X).$$

We have the natural map

$$\pi : X \rightarrow X/\sim = G/J$$

where

$$J := \{ g \in G \mid f(gH) = f(eH) \quad \forall f \in \mathcal{O}(G/H) \}.$$

Then J is a closed complex subgroup of G and contains H . The map π is called the holomorphic reduction of X . Note that for an arbitrary complex manifolds (non-homogeneous), X/\sim might not be locally compact and so might not have the structure of a complex space. Also, the base of a holomorphic reduction need not to be Stein and one may not have $\mathcal{O}(J/H) \neq \mathbb{C}$.

Example 3.1. In $G := SL(2, \mathbb{C})$ we let

$$H := \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$$

The algebraic closure of H is

$$\overline{H} = \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : w \in \mathbb{C} \right\}$$

We have the intermediate fibration

$$G/H \rightarrow G/\overline{H} \rightarrow G/J$$

where J is the subgroup of G that appears in the definition of the holomorphic reduction of G/\overline{H} . Since G is reductive, the theorem of Barth and Otte (Theorem 2.14) applies and every H -invariant holomorphic function on G is necessarily \overline{H} -invariant. Note that $\overline{H} \subseteq J$. Hence the holomorphic reduction of G/\overline{H} and G/H are the same. In this case $G/\overline{H} \cong \mathbb{C}^2 - \{(0,0)\}$ which is holomorphically separable, i.e., $J = \overline{H}$.

So we see that the base of the holomorphic reduction is not necessarily Stein. The fiber $J/H = \mathbb{C}^*$ is Stein and so not necessarily a Cousin group.

The holomorphic reduction π has the following properties:

- The homogeneous complex manifold G/J is holomorphically separable
- $\pi^* \mathcal{O}(G/J) \simeq \mathcal{O}(G/H)$, i.e., all holomorphic functions on G/H arise as the pullbacks of holomorphic functions on G/J via the holomorphic map π .

3.4 Topological invariant d_X

By d_X we mean the co-dimension of the top (singular) homology group of X with coefficients in \mathbb{Z}_2 that does not vanish.

Definition 3.1. *Let X be an oriented manifold with $\dim_{\mathbb{R}} X = n$. Define*

$$d_X := \min\{ k : H_{n-k}(X, \mathbb{Z}_2) \neq 0 \}.$$

Note that $d_X = 0$, exactly if the manifold X is compact. Otherwise, d_X is positive. The integer d_X is dual to an invariant introduced by H. Abels [Abe76]. The following theorem is a good tool to calculate d_X , when X has **connected isotropy**. The reader should note that this does not apply in the setting of discrete isotropy and one needs the fibration lemma given in the next section in order to handle that setting.

Theorem 3.1 (Covariant fibration [Mos55]). *Let G be a connected real Lie group and H a connected closed Lie subgroup. Further, suppose K is a maximal compact subgroup of G and L is a maximal compact subgroup of H contained in K . Then G/H can be retracted by a strong retraction to the compact space K/L , i.e., G/H admits the structure of a real vector bundle over K/L . Thus*

$$d_{G/H} = \dim(G/H) - \dim(K/L)$$

Since $G = K \times \mathbb{R}^s$ and $H = L \times \mathbb{R}^t$ where $L \subseteq K$ and $G/H \xrightarrow{\mathbb{R}^{s-t}} K/L$, we have $d_{G/H} = s - t$

For example since $X = \mathbb{C}^*$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ we see that $d_X = 1$. Here we bring a more interesting example which we use later.

Example 3.2 (Lemmas 1 and 2, [Akh83]). *Suppose G is an algebraic group and H is an algebraic subgroup with unipotent radicals U and V respectively. Then a direct computation of dimensions shows us that*

$$d_{G/H} = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} H + \dim_{\mathbb{C}} U - \dim_{\mathbb{C}} V.$$

As a consequence, if J is an algebraic subgroup such that $H \subset J \subset G$, then

$$d_{G/H} = d_{G/J} + d_{J/H}.$$

Example 3.3. *We consider what happens for $S := SL(2, \mathbb{C})$ and various algebraic subgroups of this S . Note that $K = SU(2)$ here.*

Subexample (a): *Let $H = B$ be a Borel subgroup of S . Then S/H is compact and $d_{S/H} = 0$.*

Subexample (b): *Let H be a maximal unipotent subgroup of a Borel subgroup of S . Such H is isomorphic to \mathbb{C} and $L = \{e\}$. The minimal compact orbit in S/H is $K/L = SU(2)/\{e\}$ and S/H fibers as a holomorphic \mathbb{C}^* -bundle over \mathbb{P}_1 . It follows that $d_{S/H} = 1$.*

Subexample (c): Now let $H = \text{diag}(\alpha, \alpha^{-1}) \cong \mathbb{C}^*$. Here $L \cong S^1$ and $V \cong \mathbb{R}$, where the equivalences are topological. Hence, the minimal compact orbit in S/H is $K/L = SU(2)/S^1$ and S/H fibers as an affine \mathbb{C} -bundle over \mathbb{P}_1 , i.e., S/H is the affine quadric. Thus, $d_{S/H} = 2$. The normalizer $N(H)$ of H gives a 2:1 covering $S/H \rightarrow S/N(H)$, where the base is $\mathbb{P}_2 \setminus Q$ with Q a quadric curve. Thus $d_{S/N(H)} = d_{\mathbb{P}_2 \setminus Q} = 2$.

Subexample (d): Let $H = \{e\}$, then $d_S = 3$.

Example 3.4. Let G be the Heisenberg group consisting of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{C}$. For any integer k from 0 to 6, we can make discrete subgroups Γ such that $d_{G/\Gamma} = k$. For example if we define Γ to be a subgroup of G with $x, z \in \mathbb{Z}$ and $y \in \mathbb{Z} + i\mathbb{Z}$ then $d_{G/\Gamma} = 2$ since $G/\Gamma \xrightarrow{T} \mathbb{C}^* \times \mathbb{C}^*$, where T is a one dimensional complex torus.

3.4.1 The fibration lemma

If H is a connected subgroup of a connected Lie group G such that $I \subset H \subset G$, where I is another connected subgroup of G , by Iwasawa decomposition we have the homeomorphisms $G = K \times \mathbb{R}^s$, $H = L \times \mathbb{R}^t$, and $I = M \times \mathbb{R}^u$ where we can choose maximal compact subgroups K, L and M of G, H and I respectively, such that $M \subseteq L \subseteq K$. By Theorem 3.1 we have the simple calculation

$$d_{G/I} = s - u = s - t + t - u = d_{G/H} + d_{H/I}.$$

This was noted in example 3.2 in the setting of algebraic groups.

Since we are dealing with isotropy subgroups that are not connected, we need the next observations to handle these settings. Interesting such groups exist, e.g., see the proof of Proposition 4.2.

Lemma 3.3. *Given a locally trivial fiber bundle $X \xrightarrow{F} B$ with F, X, B connected smooth manifolds the following were proved using spectral sequences in §2 in [AG94]:*

1. *If the bundle is orientable (e.g., if $\pi_1(B) = 0$), then $d_X = d_F + d_B$.*
2. *If B has the homotopy type of a q -dimensional CW complex, then $d_X \geq d_F + (\dim B - q)$.*

3. If B is homotopy equivalent to a compact manifold, then $d_X \geq d_F + d_B$.

For example, this happens if B is homogeneous with connected isotropy or is a solvmanifold [Aus63].

Chapter 4

Main Results

In this chapter we give the classification theorem for homogeneous Kähler manifolds with discrete isotropy and with top non-vanishing homology in codimension at most two.

In section 4.1 we prove that all such manifolds are solvmanifolds. This partially answers the modified question of Akhiezer discussed in section 2.5.

In section 4.2 we prove that any homogeneous Kähler manifold with discrete isotropy which is a torus bundle over $\mathbb{C}^* \times \mathbb{C}^*$ splits as a product.

We use this in section 4.3 to prove the main theorem. We prove that any homogeneous Kähler manifold X with $d_X = 2$ and with discrete isotropy is a product of a Cousin group and one of $\{e\}, \mathbb{C}^*, \mathbb{C}$, or $(\mathbb{C}^*)^2$.

4.1 Reduction to solvable case

We now prove one of the key ingredients that will allow us to prove our classification in the case of discrete isotropy. First we need the following lemmas.

Lemma 4.1. *Let G be a connected complex Lie group that acts holomorphically, transitively, and effectively on any of the following complex manifolds:*

1. $C \times \mathbb{C}$,
2. $C \times \mathbb{C}^*$, or
3. $C \times (\mathbb{C}^*)^2$

where C is a Cousin group. Then G is solvable.

Proof. We first claim that the G -action on $C \times A$ induces a transitive G -action on A , where A is \mathbb{C} , \mathbb{C}^* , or $\mathbb{C}^* \times \mathbb{C}^*$. To see this let $p : C \times A \rightarrow A$ be the projection map to the second factor. Since $\mathcal{O}(C) \cong \mathbb{C}$ and A is holomorphically separable, $p(g.(C \times \{z\}))$ is always a point in A , for all $g \in G$ and $z \in A$. Since the G -action is holomorphic and p is also holomorphic, this induces the required action on A .

Let $G = S \ltimes R$ be a Levi-Malcev decomposition of the complex Lie group G (Theorem 2.1). For any complex Lie group H , if $\phi : S \rightarrow H$ is a Lie homomorphism, then $\phi(S)$ is semi-simple (see section 5.2., [Hum72]), a fact which we use later.

If $A = \mathbb{C}$, it follows that the G -action on \mathbb{C} is given by a homomorphism $\psi : G \rightarrow \text{Aut}(\mathbb{C})$. But $\text{Aut}(\mathbb{C}) = \mathbb{C}^* \ltimes \mathbb{C}$ is solvable, i.e., $\psi(S) = \{e\}$ by a fact mentioned above.

Let $A = \mathbb{C}^*$. Note that $\text{Aut}(\mathbb{C}^*)^0 \cong \mathbb{C}^*$, which is solvable. Hence $\psi(S) = \{e\}$ for any homomorphism $\psi : G \rightarrow \text{Aut}(\mathbb{C}^*)$.

Let $A = (\mathbb{C}^*)^2$ and assume that the G -action on $(\mathbb{C}^*)^2$ is given by a representation $\phi : G \rightarrow \text{Aut}(\mathbb{C}^*)^2$. First, $\phi(S)$ has no 1-dimensional orbit in A . Such an orbit would be a \mathbb{P}_1 . Since $(\mathbb{C}^*)^2$ is holomorphically separable, this is not possible. If S has a two dimensional orbit, then the S -action is transitive, i.e., $(\mathbb{C}^*)^2 = S/H$ for some algebraic subgroup H of S . Consider part of the exact homology sequence for the fibration $S \xrightarrow{H} S/H$ as follows

$$\cdots \longrightarrow \pi_1(S) \longrightarrow \pi_1((\mathbb{C}^*)^2) \longrightarrow \pi_0(H) \rightarrow \cdots .$$

Both $\pi_1(S)$ and $\pi_0(H)$ are finite, but $\pi_1((\mathbb{C}^*)^2) \cong \mathbb{Z}^2$ is infinite giving us a contradiction. Hence S acts ineffectively on $A = (\mathbb{C}^*)^2$ too.

Under the G action on $C \times A$, the semisimple group S stabilizes $C \times \{z_0\}$, which is given by a representation $\mu : G \rightarrow \text{Aut}^0(C) \cong C$. This means $\mu(S) = \{e\}$. So S is acting ineffectively, i.e., $S = \{e\}$. Hence G is solvable in all cases. \square

Lemma 4.2. *Let G be a connected complex Lie group and Γ a discrete subgroup of G such that $X = G/\Gamma$ is a homogeneous Kähler manifold. Assume that Γ is not contained in any proper parabolic subgroup of G , $\mathcal{O}(G/\Gamma) = \mathbb{C}$, and $d_X \leq 2$. Then G is solvable.*

Proof. According to Lemma 2.2 the R orbits are closed and we have the fibration:

$$G/\Gamma \longrightarrow G/R.\Gamma = S/\Lambda$$

where $\Lambda = S \cap R.\Gamma$ is Zariski dense and discrete in S . There are two cases. One case is when the R orbits have no non-constant holomorphic functions. Using Theorem 2.17 we get the desired conclusion. The other case is when $\mathcal{O}(R.\Gamma/\Gamma) \neq \mathbb{C}$. Let J be the closed complex subgroup of $R.\Gamma$ that contains Γ so that we have the holomorphic reduction $R.\Gamma/\Gamma \rightarrow R.\Gamma/J$. (Note that this exists by the discussion in §3.3.) This holomorphic reduction gives the intermediate fibration

$$G/\Gamma \rightarrow G/J \rightarrow G/R.\Gamma$$

Note that since $J^0 \triangleleft G$ (Theorem 2.15, page 39) we can make the following quotients:

$$\widehat{G} = G/J^0 \quad \widehat{R} = R/J^0 \quad \widehat{J} = J/J^0$$

which gives $G/J = \widehat{G}/\widehat{J}$ and $\widehat{G} = S.\widehat{R}$, where S is a Levi factor for G . Applying the result stated in section 3.3 on page 57 and the topological condition $d_{G/\Gamma} \leq 2$ we have

$$2 \geq d_{G/\Gamma} \geq d_{R.\Gamma/J} \geq \dim_{\mathbb{C}} R.\Gamma/J.$$

Hence $\dim_{\mathbb{C}} \widehat{R} = 1$, or 2. If $\dim_{\mathbb{C}} \widehat{R} = 1$, then S acts trivially on \widehat{R} and Theorem 2.17 on page 42 applies and \widehat{G} is solvable. Since $\widehat{G} = S.\widehat{R}$ we have $S = \{e\}$. But $G = S.R$, hence G is also solvable. If $\dim_{\mathbb{C}} \widehat{R} = 2$, then either $\widehat{G} = S \times \widehat{R}$ which again is the case dealt with in Theorem 2.17, or \widehat{G} is not a trivial product of S and \widehat{R} . This with Theorem 2.18 on page 45 proves G is solvable. \square

Lemma 4.3. *Let C be a connected Abelian Lie group and A a closed, connected subgroup. Then $d_A \leq d_C$.*

Proof. The maximal compact subgroups in this setting are unique and $C = K \times V$ and $A = L \times W$ with $L \subseteq K$ and with the vector space group W

contained in the vector space group V . It now follows from the definition of d that $d_A = \dim_{\mathbb{R}} W \leq \dim_{\mathbb{R}} V = d_C$. \square

Lemma 4.4. *Let G be a connected complex Lie group, and Γ a discrete subgroup. Let $X = G/\Gamma$ be a Kähler manifold. For any semisimple Lie subgroup S of G , the S orbit $S/S \cap \Gamma$ is closed.*

Proof. Let S be a complex semi-simple Lie subgroup of G . Pick $x \in X$ so that the S orbit through x is of minimal dimension. Such an S orbit is necessarily closed inside X . Otherwise, it has a boundary of a smaller dimension (Theorem 3.6, [GMO11]). Note that remark 2.2 in the same paper guarantees that we are in the situation of the theorem. Since $\dim S.x = \dim S - \dim(S \cap \Gamma)$, and $\dim(S \cap \Gamma) = 0$, we conclude that all S orbits are minimal, i.e. of the same dimension and closed. \square

Lemma 4.5. *Let Γ be a finite subgroup of a semisimple Lie group S . Then $d_{S/\Gamma} \geq 3$.*

Proof. This follows since

$$d_{S/\Gamma} = d_S = \operatorname{codim}_{\mathbb{R}} K \geq 3,$$

where $S = K^{\mathbb{C}}$ and K is any maximal compact subgroup of S . \square

Lemma 4.6. *Suppose $G = SL(2, \mathbb{C}) \times R$ and Γ is a discrete subgroup G with $X = G/\Gamma$ Kähler and $d_X \leq 2$. Then Γ is not contained in a proper parabolic subgroup P of G*

Proof. We assume Γ is contained in a proper parabolic subgroup P of G and derive a contradiction. Such a group has the form $P = B \times R$, where B is isomorphic to a Borel subgroup of $SL(2, \mathbb{C})$. Consider the holomorphic reduction

$$P/\Gamma \xrightarrow{J/\Gamma} P/J.$$

and note that because P is solvable, J/Γ is a Cousin group [OR88] and P/J is a Stein manifold [HO86]. Note that $\mathcal{O}(P/\Gamma) \neq \mathbb{C}$. Otherwise, P/Γ itself would be a Cousin group and P would be Abelian, a contradiction. Thus, P/J is a positive dimensional Stein manifold with $d_{P/J} \leq 2$. Hence $P/J \cong \mathbb{C}, \mathbb{C}^*, \mathbb{C}^* \times \mathbb{C}^*$, or the complex Klein bottle.

Consider the following diagram with its induced $S = SL(2, \mathbb{C})$ -orbit fibrations

$$\begin{array}{ccccc} G/\Gamma & \xrightarrow{F=J/\Gamma} & G/J & \xrightarrow{P/J} & G/P = \mathbb{P}_1 \\ \cup & & \cup & & \parallel \\ S/S \cap \Gamma & \xrightarrow{A_1} & S/S \cap J & \xrightarrow{A_2} & S/B = \mathbb{P}_1 \end{array} \quad (4.1)$$

Note that since $\dim P/J > 0$ and P/J is not compact, it follows that $d_F \leq 1$ by the Fibration Lemma. Since A_1 is a closed Abelian subgroup of F , Lemma 4.3 implies $d_{A_1} \leq 1$. Also Lemma 4.5 for $S := SL(2, \mathbb{C})$ implies that $d_{S/S \cap \Gamma} = 3$. We will consider all the possibilities for $S \cap J$ and use the fibration $S/S \cap \Gamma \rightarrow S/S \cap J$ in order to derive a contradiction.

- First observe that G/J is Kähler in all cases:

- (a) The one dimensional space P/J is equivariantly embeddable in \mathbb{P}_1 . So G/J is an open subset in a homogeneous \mathbb{P}_1 -bundle over \mathbb{P}_1 which is Kähler by a theorem of Kodaira [Kod54]. Thus G/J is Kähler.

- (b) If $P/J \cong \mathbb{C}^* \times \mathbb{C}^*$, then Lemma 3.3 gives $d_{G/J} = 2$, and $d_F = 0$. In fact F is a compact complex torus (see Lemma 3.1). Since F is a torus, by means of integration over this compact fiber [Bla56] we can push the Kähler metric down to conclude G/J is Kähler.

Now since G/J is Kähler, any S -orbit in G/J is Kähler and, as a consequence, has algebraic isotropy $S \cap J$ (Theorem 2.10) The algebraic subgroups of a Borel subgroup B in $S = SL(2, \mathbb{C})$ are, up to isomorphism, finite subgroups, $A = \text{diag}(\alpha, \alpha^{-1})$, $N(A)$, a maximal unipotent subgroup U of B , and B itself.

1. $\dim(S \cap J) = 2$. Then $S \cap J = B$, so $d_{S/B} = 0$ (see example 3.3 (a)).

By Lemma 3.3 we have $d_{S/S \cap \Gamma} = d_{A_1} + d_{S/B} \leq 1 + 0 = 1 < 3$.

2. $\dim(S \cap J) = 1$. We consider the fibration 4.1 mentioned above. There are two possibilities:

(i) $S \cap J \cong \mathbb{C}^*$. Then $S/S \cap J$ is an affine quadric or the complement of a quadric in \mathbb{P}_2 . In both cases $A_2 \cong \mathbb{C}$. Note that $P/J \neq \mathbb{C}^*$. So $P/J \cong \mathbb{C}$ or $\mathbb{C}^* \times \mathbb{C}^*$. By fibration lemma (Lemma 3.3) we have $d_F = 0$. Hence F is compact. By Lemma 4.3 we see that $d_{A_1} = 0$. Thus A_1 is a torus. But this also means $S \cap \Gamma$ is infinite which is in contradiction with the fact that discrete isotropy group of Kähler manifolds transitive under a semisimple group must be finite (see the remark after Theorem 2.10 on page 35).

(ii) $S \cap J \cong \mathbb{C}$. Then $S/S \cap J$ is a finite quotient of $\mathbb{C}^2 \setminus \{(0, 0)\}$ and so $A_2 \cong \mathbb{C}^*$. Then $P/J \cong \mathbb{C}$, $\mathbb{C}^* \times \mathbb{C}^*$, or \mathbb{C}^* . For the first two cases F is compact and we get the same contradiction as in (i). If $P/J \cong \mathbb{C}^*$, then $d_F = 1$ by fibration lemma. Hence $d_{A_1} \leq 1$ and $A_1 \cong \mathbb{C}^*$, or a torus. In both cases $S \cap \Gamma$ is infinite which is in contradiction with the fact that $S \cap \Gamma$ is finite, as explained in the previous case.

3. $\dim(S \cap J) = 0$ and then $S \cap J$ is finite. In this case $\dim(S/S \cap J) = \dim S - \dim(S \cap J) = 3$, and $\dim G/J = 3$. It then follows that $S/S \cap J$ is open in G/J . Note that the only case that $\dim G/J = 3$ is when it is $\mathbb{C}^* \times \mathbb{C}^*$ bundle over S/P , i.e., $d_{G/J} = 2$. Lemma 4.4 says this S-orbit is also closed. Thus, $G/J = S/S \cap J$ which means $d_{S/S \cap J} = 2$. Since $d_{A_1} \leq d_F = 0$, we see that $d_{S/S \cap J} < 3$, which is a contradiction.

We conclude that the assumption that there is a proper parabolic subgroup of G which contains Γ yields a contradiction and the lemma follows. \square

We now combine the above to prove the main result of this section, namely G is solvable in our setting.

Proposition 4.1. *Let G be a connected complex Lie group and Γ a discrete subgroup of G such that $X = G/\Gamma$ is a homogeneous Kähler manifold. If $d_X \leq 2$, then G is solvable.*

Proof. First suppose $X = S/\Gamma$ is a Kähler manifold, where S is a semisimple complex Lie group. By Theorem 2.10 we see that Γ is algebraic and so finite. But then $d_X \geq 3$ by lemma 4.5. This is a contradiction to the assumption $d_X \leq 2$. We conclude that G cannot be semisimple and so must have a positive dimensional radical, i.e., $\dim R > 0$.

We prove that G is solvable by induction on the dimension of G . Clearly, if $\dim_{\mathbb{C}} G < 3$, then the group G is solvable. So we assume that the result holds for any connected complex Lie group whose dimension is strictly less than n and assume G is a complex Lie group with $\dim_{\mathbb{C}} G = n$.

We first claim that Γ is not contained in a proper parabolic subgroup of G . In order to prove this we assume that Γ is contained in a proper parabolic subgroup of G and derive a contradiction. Let P be a maximal such subgroup. Since P/Γ is Kähler $d_{P/\Gamma} \leq 2$, and $\dim P/\Gamma < n$, the induction hypothesis implies P is solvable.

Recall that a maximal proper parabolic subgroup of a complex semisimple Lie group S is solvable if and only if it is isomorphic to the Borel subgroup B in $S \cong SL(2, \mathbb{C})$. Then $P = B \times R \subseteq G = S \times R$. We are now in the situation of lemma 4.6 which gives us the desired contradiction. We conclude that Γ is not contained in a proper parabolic subgroup of G .

We now have to consider two cases as follows. First assume that X has no non-constant holomorphic functions, i.e., $\mathcal{O}(X) = \mathbb{C}$. Then lemma 4.2 shows that G is solvable.

Next assume X has non-constant holomorphic functions, i.e., $\mathcal{O}(X) \neq \mathbb{C}$.

Under these assumptions the classification given in the Main Theorem in [AG94] (see also Theorem 2.5 on page 23) applies and the base G/J of the holomorphic reduction

$$G/\Gamma \xrightarrow{J/\Gamma} G/J$$

is one of the following:

1. an affine cone minus its vertex,
2. the affine line \mathbb{C} ,
3. the affine quadric Q_2 ,
4. $\mathbb{P}_2 \setminus Q$ where Q is a quadric curve, or
5. a homogeneous holomorphic \mathbb{C}^* -bundle over an affine cone with its vertex removed.

Note that in case 2 the fiber is a torus. By Lemma 4.1, G is solvable.

For 3 & 4 we get fibrations

$$G/\Gamma \rightarrow G/J \rightarrow G/P \cong \mathbb{P}_1$$

with P a proper parabolic subgroup of G . Now $\Gamma \subseteq P$ gives a contradiction.

For 1 & 5 first note that a cone minus its vertex is a \mathbb{C}^* bundle over a flag manifold G/P (see section 2.4), where P is a parabolic subgroup in G .

In both cases we get the fibration

$$G/\Gamma \xrightarrow{J/\Gamma} G/J \longrightarrow G/P.$$

We cannot have $G \neq P$, i.e., P a proper parabolic subgroup, because then $\Gamma \subset P \subset G$ which is a contradiction, as noted above. If $G = P$, i.e., if the base of the second fibration is a point, then G/J or a 2:1 covering of it is biholomorphic to \mathbb{C}^* or $(\mathbb{C}^*)^2$. Note that if G/J or its 2:1 covering is $(\mathbb{C}^*)^2$, the fiber J/Γ is a torus. By Lemma 4.1 it follows that G is solvable. If $G/J = \mathbb{C}^*$, then the fiber J/Γ is Kähler and there are two separate cases to consider. In 5 this fiber will be compact and we have a torus bundle over $\mathbb{C}^* \times \mathbb{C}^*$. The result then follows from Lemma 4.1. In 1 if $d_X = 2$, then $d_{J/\Gamma} = 1$ by the fibration lemma. Since J has dimension strictly less than the dimension of G , it follows by the induction hypothesis that J is solvable. But then G is solvable too, since $G/J = \mathbb{C}^*$, so completing the proof. \square

4.2 Triviality of torus bundle over $\mathbb{C}^* \times \mathbb{C}^*$

In subsection 2.1.3 we consider a discrete subgroup Γ of the 3-dimensional Heisenberg group G such that the holomorphic reduction of G/Γ is a non-trivial holomorphic torus bundle over $\mathbb{C}^* \times \mathbb{C}^*$. The reason the bundle is

not even homeomorphic to a product $T \times (\mathbb{C}^*)^2$ is because the fundamental group of the latter is Abelian, while the fundamental group $\pi_1(G/\Gamma) \cong \Gamma$ is nilpotent and not Abelian. Note that this example is not Kähler. In sharp contrast to this we consider in this section homogeneous Kähler solvmanifolds with discrete isotropy which fiber as torus bundles over $\mathbb{C}^* \times \mathbb{C}^*$ and prove that such X are biholomorphic to the product of the torus and $(\mathbb{C}^*)^2$. The structure of the holomorphic reductions of Kähler solvmanifolds has been analyzed in [OR88] and [GO08], see also Theorem 2.8 and Theorem 2.9.

Proposition 4.2. *Suppose G is a connected, simply connected solvable complex Lie group and Γ is a discrete subgroup such that G/Γ is Kähler and has holomorphic reduction*

$$X = G/\Gamma \xrightarrow{T} G/J \cong \mathbb{C}^* \times \mathbb{C}^* = Y,$$

where $T = J/\Gamma$ is a compact complex torus. Then a finite covering of G/Γ is biholomorphic to the product $T \times \mathbb{C}^* \times \mathbb{C}^*$.

Proof. There are two cases to consider, namely either J^0 is normal in G or not. If J^0 is normal in G , then the group G/J^0 is a simply connected 2-dimensional complex Lie group. Up to isomorphism there are two possibilities, the Abelian case and the solvable case.

First assume that the group G/J^0 is Abelian and let $\alpha : G \rightarrow G/J^0$ be the projection homomorphism. Since Y contains the real subgroup $\mathbb{S}^1 \times \mathbb{S}^1$ the pullback $G_0 := \alpha^{-1}(\mathbb{S}^1 \times \mathbb{S}^1)$ is a real subgroup of G . We then have the fibration

$$\begin{array}{ccc} G/\Gamma & \xrightarrow{T} & \mathbb{C}^* \times \mathbb{C}^* \\ \cup & & \cup \\ G_0/\Gamma & \xrightarrow{T} & \mathbb{S}^1 \times \mathbb{S}^1 \end{array}$$

with compact fibers T . Thus G_0/Γ is compact. The triple (G, G_0, Γ) is a Kähler CRS manifold (see subsection 2.4.3 for definition). Thus the bundle splits as a product (Theorem 6.14, 2(iii) page 189, [GO11]).

Next we assume that G/J^0 is isomorphic to B , the Borel subgroup of $SL(2, \mathbb{C})$, and we let \mathfrak{b} denote its Lie algebra. Let \mathfrak{g} be the Lie algebra of G and $\mathfrak{g}/\mathfrak{j}$ be the Lie algebra of G/J^0 . Let $\pi : G \rightarrow G/J^0$ be the quotient map and $d\pi : \mathfrak{g} \rightarrow \mathfrak{b} := \mathfrak{g}/\mathfrak{j}$ its differential.

By definition the nilradical \mathfrak{n} is the largest nilpotent ideal in \mathfrak{g} . Since the sum of two nilpotent ideal is nilpotent (Proposition 6 on p.25 in [Jac62]), \mathfrak{n} contains every nilpotent ideal in \mathfrak{g} . In particular, \mathfrak{g}' and \mathfrak{j} are in \mathfrak{n} . Also $d\pi(\mathfrak{n})$ is a nilpotent ideal in $\mathfrak{g}/\mathfrak{j} = \mathfrak{b}$. Hence, it is contained in the nilradical \mathfrak{n}_b of \mathfrak{b} . Thus, we have $\mathfrak{j} \subseteq \mathfrak{n} \subseteq d\pi^{-1}(\mathfrak{n}_b)$. Note that $\dim \mathfrak{j} = n - 2$ and

$\dim d\pi^{-1}(\mathfrak{n}_b) = n - 1$. But $\mathfrak{n} \neq \mathfrak{j}$ since $\mathfrak{g}/\mathfrak{j}$ is not Abelian (the quotient by the nilradical is always Abelian). So $\mathfrak{n} = d\pi^{-1}(\mathfrak{n}_b)$. Let N be its Lie group. Define $\Gamma_N := \Gamma \cap N$. Since N is nilpotent, the exponential map $\exp : \mathfrak{n} \rightarrow N$ is surjective. Let $\gamma \in \Gamma_N$. Then there is $y \in \mathfrak{n}$ such that $\gamma = \exp(y)$. Let $\mathfrak{u} = \langle y \rangle_{\mathbb{C}}$ be the Lie algebra generated by y . Let U be its Lie group. Then $\mathfrak{n} = \mathfrak{u} \oplus \mathfrak{j}$ and so $N = U \times J^0$. Note that $[y, \mathfrak{j}] = 0$, since Γ centralizes J^0 , see the proof of Theorem 1 in [GO08]. So N is Abelian. Define $\Gamma_U := \Gamma \cap U \cong \mathbb{Z}$ and $\Gamma_J := \Gamma \cap J^0$ which is a full lattice in J^0 . It is immediate that

$$N/\Gamma_N = U/\Gamma_U \times J^0/\Gamma_J.$$

Since $\Gamma/\Gamma_N \simeq \mathbb{Z}$, we choose $\gamma \in \Gamma$ so that γ projects to a generator of Γ/Γ_N . Pick any $w \in \mathfrak{g} \setminus \mathfrak{n}$ and define $A := \{\exp(sw) : s \in \mathbb{C}\}$. Note that A is complementary to N in G , and thus $G = A \ltimes N$. Let $\gamma_A \in A$ and $\gamma_N \in N$ so that $\gamma = \gamma_A \gamma_N$. Then since γ and γ_N both centralize J^0 , we see that γ_A centralizes J^0 . Since $\gamma_A \in A$, we have $\exp(h) = \gamma_A$ with $h = sw$ for some $s \in \mathbb{C}$. Note that

$$[h, \mathfrak{j}] = 0, \tag{4.2}$$

a fact which we use later.

Since $\mathfrak{a} + \mathfrak{u}$ is isomorphic to $\mathfrak{b} = \mathfrak{g}/\mathfrak{j}$ as a vector space, we may take $e_+ \in \mathfrak{u}$

so that

$$[d\pi(h), d\pi(e_+)] = 2d\pi(e_+).$$

Let $\{e_1, \dots, e_{n-2}\}$ be a basis for \mathfrak{j} . It follows that there are structure constants a_i so that

$$[h, e_+] = 2e_+ + \sum_{i=1}^{n-2} a_i e_i.$$

The remaining structure constants are 0, due to the fact $[h, \mathfrak{j}] = 0$, see (4.2). Conversely, any choice of the a_i defines a solvable Lie algebra \mathfrak{g} and the corresponding simply-connected group $G = A \ltimes N$.

We compute the action of $\gamma_A \in A$ on N by conjugation. For this note that the adjoint representation restricted to \mathfrak{n} , i.e., the map $\text{ad}_h : \mathfrak{n} \rightarrow \mathfrak{n}$, is expressed by the matrix

$$M := [\text{ad}_h] = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2} & 0 & \cdots & 0 \end{pmatrix}.$$

So the action of A on N is through the one parameter group of linear transformations $t \rightarrow e^{tM}$ for $t \in \mathbb{C}$. For $k \geq 1$

$$(tM)^k = \frac{1}{2}(2t)^k M,$$

and it follows that

$$e^{tM} = \frac{1}{2}(e^{2t} - 1)M + \text{Id}.$$

Now the projection of the element γ_A acts trivially on the base $Y = G/J$, so $t = \pi ik$ where $k \in \mathbb{Z}$. Hence γ_A acts trivially on U . Also γ_N acts trivially on N , since N is Abelian. Thus γ acts trivially on N and as a consequence, although G is a non-Abelian solvable group the manifold $X = G/\Gamma$ is just the quotient of \mathbb{C}^n by a discrete additive subgroup of rank $2n-2$. Its holomorphic reduction is the original torus bundle which, since we are now dealing with the Abelian case, is trivial.

Assume J^0 is not normal in G and set $N := N_G(J^0)$. Let

$$G/J \xrightarrow{N/J} G/N$$

be the normalizer fibration. Since the base G/N of the normalizer fibration is an orbit in some projective space Lie's flag theorem applies (see section 3.1) and G/N is holomorphically separable and thus Stein [Sno85]. Since we also have $d_{G/N} \leq 2$ we see that $G/N \cong \mathbb{C}$, \mathbb{C}^* or $\mathbb{C}^* \times \mathbb{C}^*$. Assume $G/N \cong \mathbb{C}$. Since $d_X \leq 2$ the fibration lemma (Lemma 3.3) applies and $d_{N/J} = 0$, i.e., N/J is biholomorphic to a torus T . By Grauert theorem $G/J \cong T \times \mathbb{C}$. However, we assume that $G/J = \mathbb{C}^* \times \mathbb{C}^*$ and this gives us a contradiction.

Assume $G/N \cong \mathbb{C}^* \times \mathbb{C}^*$. By Chevalley's theorem (Theorem 13, page 173, [Che51]) the commutator group G' acts algebraically. Hence the G' -orbits are closed and one gets the commutator fibration

$$G/N \xrightarrow{\mathcal{C}} G/G'.N.$$

Since G is solvable, it follows that G' is unipotent and the G' -orbits are cells, i.e., $G'.N/N \cong \mathbb{C}$. By the Fibration Lemma the base of the commutator fibration is a torus. But it is proved in [HO81] that the base of a commutator fibration is always Stein which is a contradiction. Now the only case remaining is when $G/N \cong \mathbb{C}^*$. By the fibration lemma (Lemma 3.3) we get $d_{N/J} = 1$ and hence $N/J = \mathbb{C}^*$

Since G is simply connected, G admits a Hochschild-Mostow hull ([HM64]), i.e., there exists a solvable linear algebraic group

$$G_a = (\mathbb{C}^*)^k \times G$$

that contains G as a Zariski dense, topologically closed, normal complex subgroup. By passing to a subgroup of finite index we may, without loss of generality, assume the Zariski closure $G_a(\Gamma)$ of Γ in G_a is connected. Gilligan and Oeljeklaus [GO08] proved that $G_a(\Gamma) \supseteq J^0$, and that $G_a(\Gamma)$ is nilpotent. Let $\pi : \widehat{G}_a(\Gamma) \rightarrow G_a(\Gamma)$ be the universal covering and set $\widehat{\Gamma} := \pi^{-1}(\Gamma)$. Since

$\widehat{G}_a(\Gamma)$ is a simply connected, nilpotent, complex Lie group, the exponential map from the Lie algebra $\mathfrak{g}_a(\Gamma)$ to $\widehat{G}_a(\Gamma)$ is bijective. For any subset of $\widehat{G}_a(\Gamma)$ and, in particular for the subgroup $\widehat{\Gamma}$, the smallest closed, connected, complex (resp. real) subgroup \widehat{G}_1 (resp. \widehat{G}_0) of $\widehat{G}_a(\Gamma)$ containing $\widehat{\Gamma}$ is well-defined. By construction $\widehat{G}_0/\widehat{\Gamma}$ is compact – e.g., see Theorem 2.1 (2) \iff (4) in Raghunathan ([Rag08]). Set $G_1 := \pi(\widehat{G}_1)$ and $G_0 := \pi(\widehat{G}_0)$. We consider the CRS manifold (G_1, G_0, Γ) , see §2.4.3. Note that the homogeneous CR–manifold G_0/Γ fibers as a T –bundle over $S^1 \times S^1$. In order to understand the complex structure on the base $S^1 \times S^1$ of this fibration consider the following diagram

$$\begin{array}{ccccc}
\widehat{G}_0/\widehat{\Gamma} & \subset & \widehat{G}_1/\widehat{\Gamma} & \subseteq & \widehat{G}_a(\Gamma)/\widehat{\Gamma} \\
\parallel & & \parallel & & \parallel \\
G_0/\Gamma & \subset & G_1/\Gamma & \subseteq & G_a(\Gamma)/\Gamma \\
T \downarrow & & T \downarrow & & \downarrow T \\
S^1 \times S^1 = G_0/(G_0 \cap J^0.\Gamma) & \subset & G_1/J^0.\Gamma & \subseteq & G_a/J^0.\Gamma
\end{array}$$

As observed in the proof of Part I of Theorem 1 in [GO08] the manifold $G_a/J^0.\Gamma$ is a holomorphically separable solvmanifold and thus is Stein [HO86]. So $G_1/J^0.\Gamma$ is also Stein and thus $G_0/(G_0 \cap J^0.\Gamma)$ is totally real in $G_1/J^0.\Gamma$. The corresponding complex orbit $G_1/J^0.\Gamma$ is then biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

It now follows by Theorem 6.14 in [GO08] that a finite covering of G_1/Γ splits as a product of a torus with $\mathbb{C}^* \times \mathbb{C}^*$ and, in particular, that a subgroup of finite index in Γ is Abelian.

Now set $A := \{ \exp t\xi \mid t \in \mathbb{C} \}$, where $\xi \in \mathfrak{g} \setminus \mathfrak{n}$ and \mathfrak{n} is the Lie algebra of N^0 . Then $G = A \ltimes N^0$ and any $\gamma \in \Gamma$ can be written as $\gamma = \gamma_A \cdot \gamma_N$ with $\gamma_A \in A$ and $\gamma_N \in N$. The fiber $G/\Gamma \rightarrow G/N$ is the N^0 -orbit of the neutral point and Γ acts on it by conjugation. Since N/Γ is Kähler and has two ends, it follows by Proposition 1 in [GOR89] that (a finite covering of) N/Γ is biholomorphic to a product of the torus and \mathbb{C}^* . (By abuse of language we still call the subgroup having finite index Γ .) Now the bundle $G/\Gamma \rightarrow G/N$ is associated to the bundle

$$\mathbb{C} = G/N^0 \longrightarrow G/N = \mathbb{C}^*$$

and thus $G/\Gamma = \mathbb{C} \times_{\rho} (T \times \mathbb{C}^*)$, where $\rho : N/N^0 \rightarrow \text{Aut}(T \times \mathbb{C}^*)$ is the adjoint representation. Since Γ is Abelian, this implies γ_A acts trivially on $\Gamma_N := \Gamma \cap N^0$. Now suppose J has complex dimension k . Then γ_A is acting as a linear map on $N^0 = \mathbb{C} \ltimes J^0 = \mathbb{C}^{k+1}$ and commutes with the additive subgroup Γ_N that has rank $2k + 1$ and spans N^0 as a linear space. This implies γ_A acts trivially on N^0 and, as a consequence, the triviality of a finite covering of the bundle, as required. \square

4.3 Main theorem

In the following we classify Kähler G/Γ when Γ is discrete and $d_X \leq 2$. Note that $d_X = 0$ means X is compact and this is the classical result of Borel–Remmert (Theorem 2.19) and the case $d_X = 1$ corresponds to X having more than one end and this was handled in [GOR89] (Theorem, p.164); the proof will show this equivalence.

Theorem 4.1. *Let G be a connected complex Lie group and Γ a discrete subgroup of G such that $X := G/\Gamma$ is Kähler and $d_X \leq 2$. Then the group G is solvable and a finite covering of X is a direct product $C \times A$, where C is a Cousin group and A is $\{e\}, \mathbb{C}^*, \mathbb{C}$, or $(\mathbb{C}^*)^2$. Moreover, $d_X = d_C + d_A$.*

Proof. By proposition 4.1 we only need to consider the case that G is solvable. If there is no non-constant holomorphic function on X , i.e., if $\mathcal{O}(X) \cong \mathbb{C}$, then X is a Cousin group (Theorem 2.8 on page 34). The case when X is compact ($d_X = 0$) is Theorem 3.1 on page 49.

So we assume that $\mathcal{O}(X) \neq \mathbb{C}$, i.e., we have non-constant holomorphic functions on X . Let

$$G/\Gamma \xrightarrow{C=J/\Gamma} G/J$$

be the holomorphic reduction of G/Γ . It is known that the base is Stein

(see the Theorem on p. 58 in [HO86]), a finite covering of the bundle is principal (Theorem 2.9 on page 34), and the fiber is biholomorphic to a Cousin group (Theorem 2.8 on page 34). Since G/J is Stein, by the fibration lemma (Lemma 3.3 on page 57) one can see that

$$\dim_{\mathbb{C}} G/J \leq d_{G/J} \leq d_X \leq 2.$$

Let $d_X = 1$. Then $d_{G/J} = 1$. Since G/J is Stein, G/J is biholomorphic to \mathbb{C}^* . Since Theorem 2.9 on page 34 states that a finite cover of a Kähler solvmanifold has a holomorphic reduction that is a Cousin group principal bundle, this means up to some finite cover the structure group of the bundle is a connected complex Lie group. So the Grauert–Röhrl theorem (Theorem 2.4) then states that this finite covering is trivial.

Let $d_X = 2$. Again by fibration lemma it follows that $G/J \cong \mathbb{C}, \mathbb{C}^*, \mathbb{C}^* \times \mathbb{C}^*$ or a complex Klein bottle [AG94]. The case of \mathbb{C}^* is handled as above. A torus bundle over \mathbb{C} is trivial by Grauert–Oka principle (Theorem 2.3). To be more specific note that since \mathbb{C} is continuously contractible to a point, we conclude that the bundle is a topologically trivial bundle and hence is holomorphically trivial by Theorem 2.3. Finally, it is enough to discuss the case $\mathbb{C}^* \times \mathbb{C}^*$ since we discussed the first two cases already and a Klein bottle is a 2-1 cover of $\mathbb{C}^* \times \mathbb{C}^*$. But this is proposition 4.2. The proof of the

theorem is now complete as we covered all the possibilities. \square

Remark 4.1. *We list the possibilities that can occur in the theorem.*

1. *Suppose $d_X = 0$. Then X is a torus. This corresponds to X compact.*

2. *Suppose $d_X = 1$. Then one of the following holds:*

(i) *X is a Cousin group of hypersurface type*

(ii) *a finite covering of X is $T \times \mathbb{C}^*$ with T a torus*

This corresponds to X having two ends, see [GOR89].

3. *For $d_X = 2$, then one of the following holds:*

(i) *X is a Cousin group*

(ii) *a finite covering of X is a Cousin group of hypersurface type times \mathbb{C}^* , i.e., of the form $C \times \mathbb{C}^*$, where C is a Cousin group with $d_C = 1$.*

(iii) *X is a product $T \times \mathbb{C}$*

(iv) *a finite covering of X is a product $T \times (\mathbb{C}^*)^2$*

Remark 4.2. *Let G be the product of the 3-dimensional Heisenberg group and \mathbb{C} . As noted in Example 6 (a) in [OR88] there is a discrete subgroup Γ of G such that G/Γ is Kähler and $d_{G/\Gamma} = 3$. No finite covering of its*

holomorphic reduction splits as a product. So $d = 2$ is optimal for the Main Theorem.

Remark 4.3. *The example discussed in subsection 2.1.3 has $d = 2$, but is not Kähler and its holomorphic reduction does not split holomorphically as a product.*

Chapter 5

Future Work

5.1 Complete classification

Let G be a complex Lie group and H a closed subgroup such that the homogeneous manifold $X = G/H$ is Kähler with $d_X \leq 2$. In this dissertation we give a classification of such manifolds when the isotropy H is discrete. However a classification in the general case is missing. We intend to solve the general case where H is not necessarily discrete. In chapter 3 we introduced the normalizer and commutator fibrations. By fibration lemma (Lemma 3.3) it is immediate that the base of the normalizer fibration

$$X := G/H \rightarrow G/N$$

has $d_{G/N} \leq 2$.

Chevalley showed (Theorem I3, page 173, [Che51]) that $\mathfrak{g}' = \overline{\mathfrak{g}'}$ and thus $G' = \overline{G'}$ is acting as an **algebraic group** on a projective space. Hence there is a closed G' orbit. But we know that $G' \trianglelefteq G$. Thus all G' orbits are closed and we can consider the commutator fibration

$$G/N \xrightarrow{G'.N/H} G/G'.N$$

The base of the commutator fibration is a positive dimensional Abelian Stein Lie group. One has to analyze all the possibilities of the bundle over this Stein base.

5.2 Globalization of holomorphic action

Let X be a connected, compact, complex manifold. It was proved by Bochner–Montgomery [BM47] proved that the identity component $G = Aut^0(X)$ of the automorphism group of X is a complex Lie group acting holomorphically on X . Assume H is a connected real Lie group acting holomorphically on X . Any H orbit lies inside the corresponding G orbit. In fact if H acts transitively on X , i.e., if $H.x = X$, then $X = G.x$. So a holomorphic transitive action of a real Lie group automatically extends to the complex Lie group G .

Gilligan and Heinzner extended this result to any connected homogeneous complex manifold X with $d_X = 1$ which can be thought of as "very close to compact" homogeneous manifolds [GH98].

The situation fails for bigger d_X . An obvious example of this is an open disk in \mathbb{C} which has $d = 2$.

For a more interesting example consider the action of $H = SL(3, \mathbb{R})$ on $\mathbb{C}P_2$. There are two orbits which are $\mathbb{R}P_2$ and its complement $X = \mathbb{C}P_2 \setminus \mathbb{R}P_2$ which has $d_X = 2$. No complex Lie group acts transitively and holomorphically on X [HS81] Analyzing the real group action similar to the methods used in [GH98] and using the classification of $X = G/H$ where G is **linear algebraic** homogeneous manifold with $d_X = 2$ [Akh83] and looking at the real orbits inside the homogeneous manifolds with the methods used in [HS81] we have strong evidence that we can prove unless some special cases like the example above happen, any homogeneous manifold $X = G/H$ which is homogeneous under the holomorphic and transitive action of a **Lie group** G with $d_X = 2$ is also homogeneous under the holomorphic and transitive action of the globalization $G^{\mathbb{C}}$ of G .

One of the essential tools we can use is an analogue of the normalizer fibration which was defined for a complex manifold X homogeneous under

the transitive holomorphic action of a real Lie group G in the Nancy Band [HO84]. The following observations concerning this fibration are essential for our purposes:

1. Given $X = G/H$ a homogeneous complex manifold, there exists a closed subgroup J of G containing H with $J \subset N(H^0)$ (but not necessarily equal!) such that G/J is G -equivariantly an orbit in some projective space.
2. The orbit G/J is open in the corresponding $G^{\mathbb{C}}$ -orbit, where $G^{\mathbb{C}}$ denotes the smallest connected complex subgroup of the automorphism group of the projective space that contains G .
3. The fiber J/H is complex parallelizable.
4. The bundle $G/H \rightarrow G/J$ is a locally trivial **holomorphic** fiber bundle with structure group the **complex** Lie group J/H^0 .

5.3 The Akhiezer Question

Almost thirty years ago, Akhiezer [Akh84] posed a question concerning the existence of analytic hypersurfaces in complex homogeneous manifolds. The

following variant of his question turns out to be essential for our proof of Theorem 4.1 stated above in order to show that the group G is solvable. As far as we know, there is no general answer to this question. But there is enough known in special cases to be useful for our present purposes.

5.3.1 Modified Question of Akhiezer

Suppose G/H is a Kähler homogeneous manifold that satisfies

- (a) $\mathcal{O}(G/H) \simeq \mathbb{C}$
- (b) there is no proper parabolic subgroup of G that contains H

Then G/H is an Abelian complex Lie group; in particular, the group G is solvable.

In the future we intend to consider other settings where this problem might be solvable.

5.4 Stein sufficiency conditions

Suppose G is a connected complex Lie group and H is a closed complex subgroup such that $X := G/H$ is holomorphically separable. There are obvious examples where such X are not Stein, e.g., $X = \mathbb{C}^n \setminus \{0\}$ for $n > 1$.

One setting where one might hope that holomorphic separability implies Stein for complex homogeneous manifolds would be if the isotropy subgroup is discrete. There is, however, a counterexample to this in [Oel92]. In this counterexample one has a homogeneous complex manifold $Y := G/\Gamma$ with $d_{G/\Gamma} = 4 < 5 = \dim_{\mathbb{C}} G/\Gamma$. This violates the homological condition of Serre (see footnote, p. 79 in [Ser53]) for Steinness of a complex manifold.

This suggests that one should impose a further condition. Suppose $X = G/\Gamma$ is holomorphically separable and satisfies the condition

$$d_{G/\Gamma} \geq \dim_{\mathbb{C}} G/\Gamma.$$

Again, one constructs counterexamples by considering $X := \mathbb{C}^k \times Y$, where Y is the complex homogeneous manifold in the previous paragraph. As k increases by one, the invariant d increases by two and the dimension by one.

Some further type of condition is needed that would prevent such examples. We will study this question in the setting where $G = S \times R$ is a mixed group and R is a simply connected nilpotent complex Lie group with the property that the smallest connected complex Lie subgroup of G that contains $\Gamma \cap R$ is assumed to be R . This can be expanded to include cases where G is algebraic and $\Gamma \cap R$ is Zariski dense in R .

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