WEAK EXPECTATION PROPERTIES OF C*-ALGEBRAS AND OPERATOR SYSTEMS

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Abstract

The purpose of this dissertation is two fold. Firstly, we prove a permanence result involving C*-algebras with the weak expectation property. More specifically, we show that if α is an amenable action of a discrete group G on a unital C*-algebra $\mathcal{A}$, then the crossed-product C*-algebra $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property if and only if $\mathcal{A}$ has this property. Secondly, the concept of a relatively weakly injective pair of operator systems is introduced and studied, motivated by relative weak injectivity in the C*-algebra category. E. Kirchberg [14] proved that the C*-algebra $C^*(\mathbb{F}_\infty)$ of the free group $\mathbb{F}_\infty$ on countably many generators characterizes relative weak injectivity for pairs of C*-algebras by means of the maximal tensor product. One of the main results in the latter part of this thesis is to show that $C^*(\mathbb{F}_\infty)$ also characterizes relative weak injectivity in the operator system category. A key tool is the theory of operator system tensor products [12, 13].

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Maa, Baba ar Choiti r jonno...

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Chapter 1

Introduction

In 1973, E. C. Lance studied C*-algebras which had the property that, when tensored with any other C*-algebra carried a unique norm. Such algebras are called *nuclear* C*-algebras [16]. A significant part of his work focused on finding conditions under which the maximal tensor product norm on a given C*-algebra and any other C*-algebra would extend as the maximal tensor product norm on the tensor product of a C*-algebra containing the the given one and the other C*-algebra. This phenomenon was characterized by the existence of certain weak expectations and was entitled the *weak expectation property.* Later on it was shown that nuclearity of a C*-algebra is equivalent to the *weak expectation property* in conjunction with another property called *exactness* [3].

In 1993, E. Kirchberg studied the weak expectation property and established a remarkable connection of this property to perhaps the single most outstanding open problem in operator algebras, the *Connes embedding problem* [14]. Kirchberg established that the Connes embedding problem is equivalent to determining whether the C*-algebra of the free group on countably many generators has the weak expectation property or not. He gave a tensorial characterization of C*-algebras with the weak expectation property and further studied given C*-algebra inclusions which admitted maximal tensor product norm extension as mentioned above. This is known as the concept of *relative weak injectivity*.

A quick survey of the mathematical literature reveals the enormous and penetrating study of the properties of nuclearity and exactness, thanks to decades of research by some of the best mathematicians of the century. The weak expectation property on the other hand is yet to receive a similar level of investigation.

This dissertation aims to make a contribution towards the understanding of the weak expectation property by exploring a permanence property of crossed product C*-algebras. Specifically, the following theorem is established in Chapter 3.

Theorem A : *If α is an amenable action of a discrete group G on a unital C*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property if and only if $\mathcal{A}$ does.*

Employing more direct and instructive techniques, distinct from those employed for proving Theorem A, a special case of the theorem above is also proved as stated below :

Theorem B : *If α is an action of an amenable discrete group G on a unital C^*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property if and only if $\mathcal{A}$ does.*

Further, with the recent advances in the theory of operator system tensor products, a formulation and study of the concept of relative weak injectivity is carried out in the operator system setting. The main results in this direction, established in Chapter 4 are given below.

Theorem C : *The following statements are equivalent for operator systems $\mathcal{S}$ and $\mathcal{T}$ for which $\mathcal{S} \subset \mathcal{T}$ ($\mathcal{S}$ unital subsystem of $\mathcal{T}$) :*

1. *$(\mathcal{S}, \mathcal{T})$ is a relatively weakly injective pair of operator systems;*
2. *$\mathcal{S} \otimes_c C^*(\mathbb{F}_\infty) \subset_{coi} \mathcal{T} \otimes_c C^*(\mathbb{F}_\infty)$;*
3. *For any unital completely positive map $\phi : S \to \mathcal{B}(\mathcal{H})$, there exist a unital completely positive map $\Phi : \mathcal{T} \to \phi(\mathcal{S})''$ such that $\Phi|_{\mathcal{S}} = \phi$;*
4. *$(C_u^*(\mathcal{S}), C_u^*(\mathcal{T}))$ is a relatively weakly injective pair of C^*-algebras.*

Existence of relatively weakly injective pairs of operator systems are also established in :

Theorem D : *If $\mathcal{S}$ is a separable unital operator subsystem of an operator system $\mathcal{T}$, then there exists a separable operator system $\mathcal{R}$ (unital subsystem of $\mathcal{T}$) such that $\mathcal{S} \subset \mathcal{R} \subset \mathcal{T}$ and $\mathcal{R}$ is relatively weakly injective in $\mathcal{T}$.*

This dissertation begins with a review of all preliminary concepts, required for the topics explored herein, in Chapter 2. Included in this chapter is an expository self-contained proof of E. Kirchberg's theorem :

Let $\mathbb{F}_\infty$ denote the free group on countably infinitely many generators and $C^*(\mathbb{F}_\infty)$ denote the full group C^*-algebra of $\mathbb{F}_\infty$. Then a C^*-algebra $\mathcal{A}$ satisfying the tensor product condition $\mathcal{A} \otimes_{max} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{min} C^*(\mathbb{F}_\infty)$ has the weak expectation property.

Chapter 3 studies crossed products of C*-algebras with the weak expectation property and Chapter 4 investigates the concept of relative weak injectivity in the operator system category. A concluding chapter indicates some future research avenues.

Chapter 2

Preliminaries

This chapter briefly reviews the various well known facts found in mathematical literature needed for the development of the topics in this thesis. While most of these basics can be found in any introductory texts in C*-algebra theory as in the likes of [22], [20], [3], [19] etc., the contents of the last section draws on recent developments in the theory of tensor products of operator systems from [12], [13]. All vector spaces in this thesis are over the complex field $\mathbb{C}$ unless otherwise stated. All homomorphisms of C*-algebras are $*$-homomorphisms.

2.1 C*-algebras and operator systems

2.1.1 Normed algebras

An involutive Banach algebra $\mathfrak{B}$ is an algebra with involution which is also a Banach space and for all $b_1, b_2 \in \mathfrak{B}$,

$$\|b_1 b_2\| \leq \|b_1\| \|b_2\|.$$

Definition 2.1.1. *A C*-algebra is an involutive Banach algebra (say $\mathcal{A}$) such that for all $a \in \mathcal{A}$,*

$$\|a^* a\| = \|a\|^2.$$

Two important examples of C*-algebras are given below.

Example 2.1.2. Let X be a locally compact Hausdorff topological space. The space of all complex valued continuous functions on X vanishing at infinity, denoted by $C_0(X)$, is a *commutative* C*-algebra with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Example 2.1.3. Let $\mathcal{H}$ be a Hilbert space. The space of all continuous linear operators, denoted by $\mathcal{B}(\mathcal{H})$, is a C*-algebra with respect to the operator norm.

The importance of the above two examples are due to the theorems below.

Theorem 2.1.4. *(Gelfand) Every commutative C*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space X.*

Theorem 2.1.5. *(Gelfand-Naimark-Segal) Every C*-algebra is isomorphic to a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

For any $n \in \mathbb{N}$, the space of all $n \times n$ matrices over a C*-algebra $\mathcal{A}$, denoted by $\mathcal{M}_n(\mathcal{A})$, has a natural C*-algebra structure. Thus, associated to every C*-algebra is a natural family of matricial C*-algebras $\{\mathcal{M}_n(\mathcal{A})\}_{n\in\mathbb{N}}$.

Every C*-algebra $\mathcal{A}$ has a distinguished cone of positive elements. It is a well known fact that this cone, denoted by $\mathcal{A}^+$, is precisely the set $\{a^*a \mid a \in \mathcal{A}\}$. Indeed, there is a family of matricial cones of positive elements over a C*-algebra, namely $\{\mathcal{M}_n(\mathcal{A})^+\}_{n\in\mathbb{N}}$. These family of cones help to define a partial order on the set of self-adjoint elements at every matricial level. For self-adjoint $a, b \in \mathcal{A}$, define $a \geq b$ if $a - b \in \mathcal{A}^+$. The partial order at matricial levels are defined similarly.

2.1.2 Operator systems

Let $\mathcal{V}$ be a complex $*$-vector space [19, Chapter 13] with a family of distinguished cones over the self-adjoint $n \times n$ matrices over $\mathcal{V}$, usually denoted by $\{\mathcal{M}_n(\mathcal{V})^+\}_{n\in\mathbb{N}}$ and often referred to as the *positive* cones, satisfying the following properties

1. $\mathcal{M}_n(\mathcal{V})^+ \cap (-\mathcal{M}_n(\mathcal{V})^+) = \{0\}$ for all n.

2. for every $n \times m$ complex matrix λ, one has $\lambda^*\mathcal{M}_n(\mathcal{V})^+\lambda \subset \mathcal{M}_m(\mathcal{V})^+$.

The collection $\{\mathcal{M}_n(\mathcal{V})^+\}_{n\in\mathbb{N}}$ is called a *matrix order* on $\mathcal{V}$, and $\mathcal{V}$ itself is called a *matrix ordered* space.

Denote by $\mathcal{V}_h$, the set of all self-adjoint elements of $\mathcal{V}$. An *order unit* of $\mathcal{V}$ is an element $e \in \mathcal{V}_h$ such that, for every $x \in \mathcal{V}_h$, there exists $r \in \mathbb{R}^+$ so that $re + x \in \mathcal{M}_1(\mathcal{V})^+$. The order unit is called *Archimedean* if $re + x \in \mathcal{M}_1(\mathcal{V})^+$ for all $r > 0$ implies $x \in \mathcal{M}_1(\mathcal{V})^+$. Further, e is called an *Archimedean matrix order unit* if, for each n, $e_n = \mathrm{diag}(e, e, \ldots, e)$ is an Archimedean order unit for $\mathcal{M}_n(\mathcal{V})$.

Definition 2.1.6. *An operator system is a matrix ordered space with an archimedean matrix order unit.*

In [4], it was shown that, given any operator system $\mathcal{S}$, for any n, $\mathcal{M}_n(\mathcal{S})$ can be normed by the following recipe :

$$\|X\|_{\mathcal{M}_n(\mathcal{S})} = \inf\left\{ r : \begin{pmatrix} re_n & X \\ X^* & re_n \end{pmatrix} \in \mathcal{M}_{2n}(\mathcal{S})^+ \right\}$$

for $X \in \mathcal{M}_n(\mathcal{S})$.

Example 2.1.7. Let $\mathcal{S}_{\mathcal{H}}$ be a self adjoint unital subspace of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. $\mathcal{S}_{\mathcal{H}}$ is an operator system with positive cones $\{\mathcal{M}_n(\mathcal{S}_{\mathcal{H}})^+\}_{n\in\mathbb{N}}$ given by

$$\mathcal{M}_n(\mathcal{S}_{\mathcal{H}})^+ = \mathcal{M}_n(\mathcal{S}_{\mathcal{H}}) \cap \mathcal{M}_n(\mathcal{B}(\mathcal{H}))^+.$$

The importance of the example above is due to the fact that the converse is also true via appropriate morphisms (definition given below).

Theorem 2.1.8. *(Choi-Effros) Every operator system is complete order isomorphic to a unital self-adjoint subspace of* $\mathcal{B}(\mathcal{H})$*, for some Hilbert space* $\mathcal{H}$*.*

2.1.3 Completely positive maps

Let $\mathcal{S}$ and $\mathcal{R}$ be operator systems. Let $\varphi : \mathcal{S} \to \mathcal{R}$ be a linear map. Call φ *positive* if $\varphi(\mathcal{S}^+) \subset \mathcal{R}^+$.

For each n, define $\varphi^{(n)} : \mathcal{M}_n(\mathcal{S}) \to \mathcal{M}_n(\mathcal{R})$ by $(s_{ij})_{i,j} \mapsto (\varphi(s_{ij}))_{i,j}$. Call φ *completely positive* (cp) if, $\varphi^{(n)}(\mathcal{M}_n(\mathcal{S})^+) \subset \mathcal{M}_n(\mathcal{R})^+$ for every n and *unital completely positive* (ucp) if the map is unital.

The map φ is called a *complete order isomorphism* if it is an algebraic isomorphism and both φ and φ^{-1} are completely positive. Further, it is called a *complete order injection* if it is a complete order isomorphism onto its range.

2.2 Tensor products and C*-algebras

2.2.1 Algebraic tensor products

Let V and W be vector spaces. Consider the set of all formal finite sums $\mathfrak{S}(V,W) = \{\sum_i \lambda_i(v_i \otimes w_i) : v_i \in V, w_i \in W, \lambda_i \text{ scalar}\}$. This set is a vector space with formal operations of addition (concatenation) and scalar multiplication. Let $\mathfrak{K}(V,W)$ denote the vector subspace of $\mathfrak{S}(V,W)$ spanned by all elements of the form :

- $(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w$
- $v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2$
- $\lambda(v \otimes w) - (\lambda v) \otimes w$
- $\lambda(v \otimes w) - v \otimes (\lambda w)$

Definition 2.2.1. *The algebraic tensor product of V and W, denoted by $V \otimes W$ is the vector space quotient $\mathfrak{S}(V, W)/\mathfrak{K}(V, W)$.*

It is a common practice to denote the equivalence class of the image of the element $v \otimes w \in \mathfrak{S}(V, W)$ in $V \otimes W$ by the same notation and we shall do the same. Further, such elements are referred to as the *elementary tensors* and they span $V \otimes W$.

2.2.2 Universal property of algebraic tensor products

The cartesian product of V and W, denoted $V \times W$, has a natural bilinear structure with pointwise operations. Define a *bilinear* map $\theta : V \times W \to V \otimes W$ by $(v, w) \mapsto v \otimes w$. Let Z be any other vector space and $\phi : V \times W \to Z$ be a bilinear map. Then there is a unique *linear* map $\tilde{\phi} : V \otimes W \to Z$ such that $\phi = \tilde{\phi} \circ \theta$. This is the *universal property* of tensor products, and further $V \otimes W$ is the unique vector space (up to isomorphism) with this property.

2.2.3 Involution and multiplication

Suppose that V and W are vector spaces with *involution*. Then the tensor product $V \otimes W$ has a unique involution such that

$$(v \otimes w)^* = v^* \otimes w^*.$$

If V and W are algebras, then so is their tensor product by means of

$$(v_1 \otimes w_1)(v_2 \otimes w_2) = v_1 v_2 \otimes w_1 w_2.$$

It can be shown that both of the above operations are well-defined.

2.2.4 Tensor product maps

Let V_1, V_2 and W_1, W_2 be vector spaces. Let $\phi_1 : V_1 \to W_1$ and $\phi_2 : V_2 \to W_2$ be linear maps. Then there is a unique linear map $\phi_1 \otimes \phi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ such that

$$\phi_1 \otimes \phi_2(v_1 \otimes v_2) = \phi_1(v_1) \otimes \phi_2(v_2)$$

for all $v_1 \in V_1$ and $v_2 \in V_2$.

Let Z be an algebra and $\varphi_1 : V_1 \to Z$ and $\varphi_2 : V_2 \to Z$ be linear maps. Then there is a unique linear map $\varphi_1 \cdot \varphi_2 : V_1 \otimes V_2 \to Z$ such that

$$\varphi_1 \cdot \varphi_2(v_1 \otimes v_2) = \varphi_1(v_1)\varphi_2(v_2).$$

In the case when V_1, V_2 and W_1, W_2 are involutive algebras and ϕ_1, ϕ_2 are $*$-homomorphisms, then the map $\phi_1 \otimes \phi_2$ as defined above is also a $*$-homomorphism.

For involutive algebras V_1, V_2 and Z, let $\pi_1 : V_1 \to Z$ and $\pi_2 : V_2 \to Z$ be $*$-homomorphisms with *commuting ranges*, then the map $\pi_1 \cdot \pi_2 : V_1 \otimes V_2 \to Z$ as defined above is also a $*$-homomorphism.

2.2.5 Tensor product inclusions

Let $V_1 \subset V_2$ and $W_1 \subset W_2$ be vector subspaces. Then there is a natural inclusion (as subspaces) given by

$$V_1 \otimes W_1 \subset V_2 \otimes W_2.$$

2.2.6 C*-algebra tensor products

Given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ is an involutive algebra with multiplication and involution defined as above.

Definition 2.2.2. *A C*-tensor norm $\|\cdot\|_\alpha$ on $\mathcal{A} \otimes \mathcal{B}$ is a norm such that $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$, $\|x^*\|_\alpha = \|x\|_\alpha$ and $\|x^*x\|_\alpha = \|x\|_\alpha^2$ for all $x, y \in \mathcal{A} \otimes \mathcal{B}$.*

The algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ when endowed with a C*-tensor norm (say α as above) turns into a *pre C*-algebra*, the completion of which, is a C*-algebra tensor product of $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \otimes_\alpha \mathcal{B}$.

Two very important facts are in order :

1. A C*-norm <u>*always*</u> exists on $\mathcal{A} \otimes \mathcal{B}$.

2. $\mathcal{A} \otimes \mathcal{B}$ may have multiple C*-norms.

Definition 2.2.3. *A norm* $\|\cdot\|$ *on* $\mathcal{A} \otimes \mathcal{B}$ *is said to be a* *cross-norm* *when, for any* $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$, $\|a \otimes b\| = \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}}$.

Every C*-norm on $\mathcal{A} \otimes \mathcal{B}$ is a cross-norm [23].

2.3 The *min* and *max* C*-tensor products

Non-uniqueness of the C*-tensor norms was first discovered by Masamichi Takesaki in 1964 in [23]. We shall discuss two important canonical C*-tensor norms in some detail.

2.3.1 Tensor product of Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then their algebraic tensor product is a pre-Hilbert space with respect to the inner product

$$\langle \sum_i h_i \otimes k_i, \sum_j h'_j \otimes k'_j \rangle = \sum_{i,j} \langle h_i, h'_j \rangle \langle k_i, k'_j \rangle.$$

With slight abuse of notation we shall denote the Hilbert space completion of above by $\mathcal{H} \otimes \mathcal{K}$ and henceforth for all Hilbert space tensor product we shall do the same.

2.3.2 Tensor product of Hilbert space operators

Let $\mathcal{B}(\mathcal{X})$ denote the C*-algebra of all bounded linear operators on the Hilbert space $\mathcal{X}$. For operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, the operator $T \otimes S$ as defined below on the algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$ extends uniquely to a bounded linear operator on $\mathcal{H} \otimes \mathcal{K}$ which we shall also denote by $T \otimes S$. Thus $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and

$$T \otimes S(h \otimes k) = Th \otimes Sk,$$

for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$. Moreover it can be shown that

$$\|T \otimes S\| = \|T\| \|S\|.$$

2.3.3 The *min* tensor product

By virtue of the Gelfand-Naimark-Segal construction, every C*-algebra can be represented faithfully as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Definition 2.3.1. *Let $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ be faithful representations. The **min** norm on $\mathcal{A} \otimes \mathcal{B}$ is given by*

$$\| \sum_i a_i \otimes b_i \|_{min} = \| \sum_i \pi_{\mathcal{A}}(a_i) \otimes \pi_{\mathcal{B}}(b_i) \|_{\mathcal{B}(\mathcal{H} \otimes \mathcal{K})}.$$

The definition above is well defined as it can be shown that the *min* norm is independent of the faithful representation chosen [3, Proposition 3.3.11]. It is a fact that this is a genuine norm (and not just a semi-norm) on $\mathcal{A} \otimes \mathcal{B}$. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to the norm above is the *min* C*-tensor product of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \otimes_{\min} \mathcal{B}$. This is also known as the *spatial* tensor product in the literature.

2.3.4 Representations of the algebraic tensor product

A representation ρ of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called non-degenerate when

$$\overline{\operatorname{span}\{\rho(\mathcal{A})\mathcal{H}\}} = \mathcal{H}.$$

Similarly, one may have non-degeneracy of representations of involutive algebras (like the algebraic tensor product of two C*-algebras).

Non-degenerate representations of the algebraic tensor product of C*-algebras always exist. To see this, consider non-degenerate representations $\rho_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\rho_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}(\mathcal{K})$. Define representations of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H} \otimes \mathcal{K}$ by $\tilde{\rho}_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, $a \mapsto \rho_{\mathcal{A}}(a) \otimes I_{\mathcal{K}}$ and $\tilde{\rho}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, $b \mapsto I_{\mathcal{H}} \otimes \rho_{\mathcal{B}}(b)$, where $I_{(\cdot)}$ is the identity operator. Then $\tilde{\rho}_{\mathcal{A}}$ and $\tilde{\rho}_{\mathcal{B}}$ have commuting ranges. The representation given by $\rho : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, $\rho = \tilde{\rho}_{\mathcal{A}} \cdot \tilde{\rho}_{\mathcal{B}}$ is non-degenerate since $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ are.

The following is a well known result [3, Theorem 3.2.6]:

Theorem 2.3.2. *For C*-algebras $\mathcal{A}$ and $\mathcal{B}$, let $\pi : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be a non-degenerate representation. Then there exists non-degenerate representations $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}(\mathcal{H})$, with commuting ranges, such that $\pi = \pi_{\mathcal{A}} \cdot \pi_{\mathcal{B}}$.*

2.3.5 The *max* tensor product

The *max* tensor product is defined as

Definition 2.3.3. *For $x \in \mathcal{A} \otimes \mathcal{B}$, set*

$$\|x\|_{max} = \sup_{\pi} \|\pi(x)\|$$

where the supremum is taken over all non-degenerate representations of $\mathcal{A} \otimes \mathcal{B}$.

The norm above is well defined as the supremum is finite due to Theorem 2.3.2. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to the norm above is called the *max* C*-tensor product of $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \otimes_{\max} \mathcal{B}$.

The *max* norm is the largest possible norm on $\mathcal{A} \otimes \mathcal{B}$ while *min* is the smallest [22, Chapter IV]. In other words, for any other C*-norm $\|\cdot\|_\alpha$ on $\mathcal{A} \otimes \mathcal{B}$, it is always true that

$$\|x\|_{\min} \leq \|x\|_\alpha \leq \|x\|_{\max},$$

for all $x \in \mathcal{A} \otimes \mathcal{B}$.

Proposition 2.3.4. *(Universality) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C*-algebras. Let $\rho : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ be a homomorphism. Then there is a unique C*-homomorphism $\tilde{\rho} : \mathcal{A} \otimes_{max} \mathcal{B} \to \mathcal{C}$ extending ρ.*

2.4 Crossed product of C*-algebras

The crossed product construction in C*-algebra theory evolves around the action of a group on a C*-algebra via automorphisms. This procedure aims at constructing a new C*-algebra from a given C*-algebra $\mathcal{A}$, which encodes the group action on $\mathcal{A}$. All groups considered in this thesis are discrete.

2.4.1 Group Actions

Let G be a group. An action of G on $\mathcal{A}$ is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(\mathcal{A}),$$

where $\mathrm{Aut}(\mathcal{A})$ is the group of automorphisms of $\mathcal{A}$.

C*-algebras with a G-action on them are called G-C*-algebra. They are also referred to as C*-dynamical systems and denoted by $(\mathcal{A}, \alpha, G)$.

2.4.2 Covariant representations

Let $\mathcal{A}$ be a G-C*-algebra and the action be

$$\begin{aligned} \alpha : G &\to \mathrm{Aut}(\mathcal{A}) \\ g &\longmapsto \alpha_g \end{aligned}$$

Definition 2.4.1. *A covariant representation $(u_G, \pi_{\mathcal{A}}, \mathcal{H})$ of $\mathcal{A}$ consists of a unitary representation $g \longmapsto u_g$ of the group G on some Hilbert space $\mathcal{H}$ and a representation*

$\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *such that*

$$u_g \pi_{\mathcal{A}}(a) u_g^* = \pi(\alpha_g(a)),$$

for all $g \in G$ *and* $a \in \mathcal{A}$.

Covariant representations for G-C*-algebras always exist as we shall discuss shortly.

2.4.3 The universal crossed product

Let $C_c(G, \mathcal{A})$ denote the vector space of finitely supported functions on G taking values in $\mathcal{A}$. Denote by δ_g, the Dirac function at point $g \in G$. The set $\{a\delta_g\}_{a\in\mathcal{A}, g\in G}$ spans $C_c(G, \mathcal{A})$. A typical element $f \in C_c(G, \mathcal{A})$ looks like $f = \sum_i a_i \delta_{g_i}$, where the sum is finite.

For an action α of G on $\mathcal{A}$, define a product and involution on $C_c(G, \mathcal{A})$ by

$$f_1 f_2 = \sum_{i,j} a_i \alpha_{g_i}(b_j) \delta_{g_i g'_j} \quad \text{and} \quad f^* = \sum_l \alpha_{t_l^{-1}}(a_l^*) \delta_{t_l^{-1}},$$

where $f_1 = \sum_i a_i \delta_{g_i}$, $f_2 = \sum_j b_j \delta_{g'_j}$ and $f = \sum_l a_l \delta_{t_l}$.

The operations defined above turn $C_c(G, \mathcal{A})$ into an involutive algebra. Every non-degenerate representation of the involutive algebra $C_c(G, \mathcal{A})$ on a Hilbert space gives rise to a covariant representation of the G-C*-algebra $\mathcal{A}$ on the same Hilbert space. Conversely, every covariant representation of $\mathcal{A}$ is obtained from a non-degenerate representation of $C_c(G, \mathcal{A})$. To see this in the case when $\mathcal{A}$ is a *unital* C*-algebra, let $\rho_0 : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ be a non-degenerate representation. Since ρ_0 is non-degenerate, the span of $\{\rho_0(a\delta_g)\}_{a\in\mathcal{A}, g\in G}$ is dense in $\mathcal{H}$. As a consequence, $\rho_0(1_{\mathcal{A}}\delta_e) =$

$I_{\mathcal{H}}$, where $1_{\mathcal{A}} \in \mathcal{A}$ is the unit and $e \in G$ is the identity element. So one has, for $g \in G$,

$$\rho_0(1_{\mathcal{A}}\delta_g)^* \rho_0(1_{\mathcal{A}}\delta_g) = \rho_0(1_{\mathcal{A}}\delta_e).$$

Above equality shows that for every $g \in G$, $\rho_0(1_{\mathcal{A}}\delta_g)$ is a unitary operator. Define a unitary representation u_0 by

$$g \longmapsto \rho_0(1_{\mathcal{A}}\delta_g)$$

and a representation π_0 of $\mathcal{A}$ by

$$a \longmapsto \rho_0(a\delta_e).$$

A simple computation shows that $u_0(g)\pi_0(a)u_0(g)^* = \pi_0(\alpha_g(a))$. Thus, $(u_0, \pi_0, \mathcal{H})$ is a covariant representation corresponding to ρ_0.

Conversely, let $(u_G, \pi_{\mathcal{A}}, \mathcal{H})$ be a covariant representation of the C*-dynamical system $(\mathcal{A}, \alpha, G)$. The linear map given by

$$\begin{aligned} \rho : C_c(G, \mathcal{A}) &\rightarrow \mathcal{B}(\mathcal{H}) \\ \sum_i a_i \delta_{g_i} &\longmapsto \sum_i \pi_{\mathcal{A}}(a_i) u_G(g_i) \end{aligned}$$

is a representation of $C_c(G, \mathcal{A})$ on $\mathcal{H}$.

Definition 2.4.2. *The universal crossed product of the C*-dynamical system* $(\mathcal{A}, \alpha, G)$ *is the completion of* $C_c(G, \mathcal{A})$ *with respect to the norm*

$$\|f\| = \sup_{\rho} \|\rho(f)\|,$$

where the supremum runs over all non-degenerate homomorphisms $\rho : C_c(G, \mathcal{A}) \rightarrow \mathcal{B}(\mathcal{H})$. The universal crossed product is denoted by $\mathcal{A} \rtimes_\alpha G$.

The following universal property justifies the nomenclature above.

Proposition 2.4.3. *(Universal property) For every covariant representation $(u, \pi, \mathcal{H})$ of a G-C^*-algebra $\mathcal{A}$ there is a homomorphism*

$$\begin{aligned} \sigma : \mathcal{A} \rtimes_\alpha G &\rightarrow \mathcal{B}(\mathcal{H}) \\ \sum_i a_i \delta_{g_i} &\longmapsto \sum_i \pi(a_i) u_{g_i} \end{aligned}$$

for all $\sum_i a_i \delta_{g_i} \in C_c(G, \mathcal{A})$.

2.4.4 The reduced crossed product

As mentioned earlier, the following construction shows that covariant representations of G-C*-algebras always exist.

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a faithfully represented G-C*-algebra. Define a new representation $\pi_\mathcal{A}$ of $\mathcal{A}$ on $\mathcal{H} \otimes l^2(G) \cong \bigoplus_{g \in G} \mathcal{H}$ by

$$a \longmapsto \bigoplus_{g \in G} \alpha_{g^{-1}}(a) \in \mathcal{B}(\bigoplus_{g \in G} \mathcal{H}).$$

Let λ denote the *left regular representation* of G. The unitary representation of G on $\mathcal{H} \otimes l^2(G)$ given by $g \longmapsto 1_\mathcal{A} \otimes \lambda_g$ and the representation $\pi_\mathcal{A}$ constitutes a covariant representation of $\mathcal{A}$, that is,

$$(1_\mathcal{A} \otimes \lambda_g) \pi_\mathcal{A}(a) (1_\mathcal{A} \otimes \lambda_g^*) = \pi_\mathcal{A}(\alpha_g(a)).$$

This representation is called a *regular representation.* The regular representation of $\mathcal{A}$ gives rise to a representation of $C_c(G, \mathcal{A})$ on $\mathcal{H} \otimes l^2(G)$, which is known as the regular representation of $C_c(G, \mathcal{A})$.

Definition 2.4.4. *The reduced crossed product of a C*-dynamical system* $(\mathcal{A}, \alpha, G)$ *is the norm closure of the image of the regular representation of* $C_c(G, \mathcal{A})$ *in* $\mathcal{B}(\mathcal{H} \otimes l^2(G))$. *The reduced crossed product is denoted by* $\mathcal{A} \rtimes_{\alpha,r} G$.

Proposition 2.4.5. *[3, Proposition 4.1.5] The reduced crossed product* $\mathcal{A} \rtimes_{\alpha,r} G$ *is independent of the choice of the faithful representation* $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$.

2.5 The weak expectation property of C*-algebras

The weak expectation property of C*-algebras was introduced by E.C. Lance [16] to study nuclearity of certain group C*-algebras. In the following, the symbol '$\subset$' shall denote subalgebra and/or C*-subalgebra as shall be clear from the context.

2.5.1 C*-tensor product inclusions

Let $\mathcal{A}_1$ be a C*-subalgebra of $\mathcal{A}_2$, denoted by $\mathcal{A}_1 \subset \mathcal{A}_2$. For any C*-algebra $\mathcal{B}$, the algebraic tensor product inclusion holds, that is

$$\mathcal{A}_1 \otimes \mathcal{B} \subset \mathcal{A}_2 \otimes \mathcal{B}.$$

It is a natural question whether the C*-tensor products preserve the inclusion analogous to the purely algebraic case. In other words, if α is a C*-norm, then is it true that

$$\mathcal{A}_1 \otimes_\alpha \mathcal{B} \subset \mathcal{A}_2 \otimes_\alpha \mathcal{B} \ ?$$

While there is little to no hope of answering this question in its utmost generality, due to the lack of precise description of every possible C*-tensor norm, the two important cases of $\alpha = \min$ and $\alpha = \max$ do have a concrete answer.

2.5.2 The *min* tensor product inclusion

Given C*-algebras $\mathcal{A}_1 \subset \mathcal{A}_2$ and $\mathcal{B}$ as above, let $\pi_2 : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$ and $\rho : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ be faithful representations of $\mathcal{A}_2$ and $\mathcal{B}$. The C*-algebra $\mathcal{A}_2 \otimes_{\min} \mathcal{B}$ is the one generated by the set of operators $\{\pi_2(a) \otimes \rho(b)\}_{a \in \mathcal{A}_2, b \in \mathcal{B}}$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Let $\pi_1 = \pi_2|_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$. So, π_1 is a faithful representation of $\mathcal{A}_1$. The C*-algebra $\mathcal{A}_1 \otimes_{\min} \mathcal{B}$ is the C*-subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ generated by $\{\pi_1(a') \otimes \rho(b)\}_{a' \in \mathcal{A}_1, b \in \mathcal{B}}$. Since $\pi_1(a') = \pi_2(a')$ for all $a' \in \mathcal{A}_1 \subset \mathcal{A}_2$, evidently $\mathcal{A}_1 \otimes_{\min} \mathcal{B}$ is a C*-subalgebra of $\mathcal{A}_2 \otimes_{\min} \mathcal{B}$. Therefore, the *min* tensor product inclusion is always true in general.

2.5.3 The *max* tensor product inclusion

In general, the *max* tensor product inclusion does ***not*** have an affirmative solution. That is to say, it is possible to find C*-algebras $\mathcal{A}_1 \subset \mathcal{A}_2$ and $\mathcal{B}$ such that, $\mathcal{A}_1 \otimes_{\max} \mathcal{B}$

is not a C*-subalgebra of $\mathcal{A}_2 \otimes_{\max} \mathcal{B}$.

To be more precise, the norm induced by the restriction of $\|\cdot\|_{\mathcal{A}_2 \otimes_{\max} \mathcal{B}}$ on $\mathcal{A}_1 \otimes \mathcal{B}$ is not equal to $\|\cdot\|_{\mathcal{A}_1 \otimes_{\max} \mathcal{B}}$. In other words, if $\iota : \mathcal{A}_1 \hookrightarrow \mathcal{A}_2$ denotes the inclusion map, then the homomorphism (see Proposition 2.3.4)

$$\widetilde{\iota \otimes \mathrm{id}_{\mathcal{B}}} : \mathcal{A}_1 \otimes_{\max} \mathcal{B} \to \mathcal{A}_2 \otimes_{\max} \mathcal{B}$$

has a non-trivial kernel. The following example illustrates the statement above.

Example 2.5.1. Let $\mathbb{F}_2$ denote the free group on two generators. Let λ and β denote the *left* and the *right regular representations* of $\mathbb{F}_2$ on $l^2(\mathbb{F}_2)$ respectively [20]. Consider the C*-algebra inclusion $\mathrm{C}^*_\lambda(\mathbb{F}_2) \subset \mathcal{B}(l^2(\mathbb{F}_2))$, where $\mathrm{C}^*_\lambda(\mathbb{F}_2)$ is the reduced group C*-algebra of $\mathbb{F}_2$. For the inclusion $\iota : \mathrm{C}^*_\lambda(\mathbb{F}_2) \hookrightarrow \mathcal{B}(l^2(\mathbb{F}_2))$, it is true that

$$\ker \widetilde{\iota \otimes \mathrm{id}_{\mathcal{B}}} \neq \{0\},$$

where $\mathcal{B} = \mathrm{C}^*_\beta(\mathbb{F}_2)$ is the C*-subalgebra of $\mathcal{B}(l^2(\mathbb{F}_2))$ generated by the right regular representation β of $\mathbb{F}_2$. The reader is referred to [3, Proposition 3.6.9] for the proof.

Given the example above, it is natural to ask if there are any conditions under which the *max* tensor product inclusion might hold. A couple of closely related necessary and sufficient conditions were found by E.C. Lance and E. Kirchberg which we discuss below.

2.5.4 Weak expectations

Definition 2.5.2. *(Weak expectations) Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a non-degenerate C*-algebra. A unital completely positive map*

$$\varphi : \mathcal{B}(\mathcal{H}) \to \overline{\mathcal{A}}^{SOT}$$

is called a weak expectation when $\varphi(a) = a$ for all $a \in \mathcal{A}$. Here $\overline{\mathcal{A}}^{SOT}$ denotes the strong operator topology closure of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

Definition 2.5.3. *(Weak expectation property of representations) Let $\mathcal{A}$ be a C*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful non-degenerate representation of $\mathcal{A}$ on $\mathcal{H}$. The representation π is said to have the weak expectation property if there exists a weak expectation for the C*-algebra $\pi(\mathcal{A})$.*

Definition 2.5.4. *(Weak expectation property of C*-algebras) A C*-algebra $\mathcal{A}$ is said to have the* ***weak expectation property*** *if <u>every</u> faithful non-degenerate representation π of $\mathcal{A}$ has the weak expectation property.*

The weak expectation property of C*-algebras is the key to the *max* tensor product inclusion problem as can be seen from the following result.

Theorem 2.5.5. *(E. C. Lance, [16]) Let $\mathcal{A}$ be a C*-algebra. For any C*-algebra $\mathcal{B}$ containing $\mathcal{A}$ as a C*-subalgebra, the following statements are equivalent :*

1. *$\mathcal{A}$ has the weak expectation property.*

2. For any C^**-algebra* $\mathcal{C}$, $\mathcal{A} \otimes_{max} \mathcal{C} \subset \mathcal{B} \otimes_{max} \mathcal{C}$.

The proof of the well known theorem above is easy but fairly long and can be found in [16, Theorem 3.3] or [3, Proposition 3.6.2, Corollary 3.6.8].

Remark 2.5.6. The definition of the weak expectation property of C*-algebras involves all faithful non-degenerate representations. Therefore it may also be rephrased in terms of the *universal representation* of the C*-algebra as follows.

Let $\mathcal{A}$ be a C*-algebra with the weak expectation property and let $\pi_u : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u)$ denote the universal representation of $\mathcal{A}$. The universal representation is a faithful non-degenerate representation of $\mathcal{A}$ and thus by hypothesis has the weak expectation property. Conversely, let π_u have the weak expectation property. Denote by $\Phi_u : \mathcal{B}(\mathcal{H}_u) \to \mathcal{A}^{**}$ the weak expectation of π_u, where $\mathcal{A}^{**} = \overline{\pi_u(\mathcal{A})}^{\mathrm{SOT}}$ is the universal enveloping von Neumann algebra of $\mathcal{A}$. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be any faithful non-degenerate representation. By universal properties of $\mathcal{A}^{**}$, let $\tilde{\pi} : A^{**} \to \overline{\pi(\mathcal{A})}^{\mathrm{SOT}}$ be the unique normal unital homomorphism (owing to the non-degeneracy of π) extending π. Further let $\rho : \pi(\mathcal{A}) \to \mathcal{B}(\mathcal{H}_u)$ be the homomorphism $\pi(a) \longmapsto \pi_u(a)$ for all $a \in \mathcal{A}$. Since $\mathcal{B}(\mathcal{H}_u)$ is injective, the homomorphism ρ extends to a unital completely positive map $\Phi_\rho : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_u)$. It is easy to see that the unital completely positive map

$$\tilde{\pi} \circ \Phi_u \circ \Phi_\rho : \mathcal{B}(\mathcal{H}) \to \overline{\pi(\mathcal{A})}^{\mathrm{SOT}}$$

is a weak expectation of π. Thus, the universal representation of $\mathcal{A}$ having the weak expectation property is equivalent to $\mathcal{A}$ having the weak expectation property.

Remark 2.5.7. From the statement of Theorem 2.5.5, one notes that the question of *max* tensor product inclusion involving a C*-algebra $\mathcal{A}_1$ with the weak expectation property, among the pair $\mathcal{A}_1 \subset \mathcal{A}_2$, somewhat ignores the C*-algebra $\mathcal{A}_2$ containing it. A related property, known as *relative weak injectivity*, which we discuss in a later section, addresses the issue of *max* tensor product inclusion of a given pair of C*-algebras $\mathcal{A}_1 \subset \mathcal{A}_2$, where the C*-subalgebra $\mathcal{A}_1$ may not have the weak expectation property.

2.5.5 Kirchberg's tensor product characterization

E. Kirchberg gave the first tensorial characterization of the weak expectation property in his seminal work on exactness of group C*-algebras [14]. In this section we extract and present a partially detailed proof (we prove only one direction of equivalence) of this important fact as the full proof requires a very deep and difficult result of Kirchberg besides a long background preparation which we omit for the sake of the length of this exposition.

Theorem 2.5.8. *(Tensor product criterion, [14]) Let $\mathbb{F}_\infty$ denote the free group on countably infinitely many generators and $C^*(\mathbb{F}_\infty)$ denote the full group C*-algebra of $\mathbb{F}_\infty$. Then the following statements are equivalent :*

1. *$\mathcal{A}$ has the weak expectation property.*

2. $\mathcal{A} \otimes_{max} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{min} C^*(\mathbb{F}_\infty)$.

Proof. (2) $\Rightarrow$ (1). Let the C*-algebra $\mathcal{A}$ be faithfully and non-degenerately represented on $\mathcal{H}$, denoted by $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. The goal is to show that π has the weak expectation property, that is to say $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$ has a weak expectation.

Denote the commutant of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H})$ by $\pi(\mathcal{A})'$. Further, denote the group of unitaries of $\pi(\mathcal{A})'$ by $\mathcal{U}(\pi(\mathcal{A})')$. Let the cardinality of $\mathcal{U}(\pi(\mathcal{A})')$ be $\mathcal{N}$. Consider the free group $\mathbb{F}_{\mathcal{N}}$ on $\mathcal{N}$ generators, and $\mathrm{C}^*(\mathbb{F}_{\mathcal{N}})$, the full group C*-algebra of $\mathbb{F}_{\mathcal{N}}$.

Let $\iota : \pi(\mathcal{A}) \hookrightarrow \mathcal{B}(\mathcal{H})$ be the inclusion homomorphism and $\pi_{\mathcal{N}} : \mathrm{C}^*(\mathbb{F}_{\mathcal{N}}) \to \pi(\mathcal{A})'$ be the canonical surjective homomorphism mapping the universal unitaries in $\mathrm{C}^*(\mathbb{F}_{\mathcal{N}})$ onto $\mathcal{U}(\pi(\mathcal{A})')$. Then the map

$$\iota \cdot \pi_{\mathcal{N}} : \pi(\mathcal{A}) \otimes \mathrm{C}^*(\mathbb{F}_{\mathcal{N}}) \to \mathcal{B}(\mathcal{H})$$

is a homomorphism since ι and $\pi_{\mathcal{N}}$ have commuting ranges. By the universal property of the *max* tensor product, $\iota \cdot \pi_{\mathcal{N}}$ extends uniquely as a homomorphism

$$\rho : \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_{\mathcal{N}}) \to \mathcal{B}(\mathcal{H}).$$

From definition of ρ it is clear that $\rho(a \otimes 1) = \iota(a) = \pi(a)$ for all $a \in \mathcal{A}$ and $\rho(1 \otimes \mathrm{C}^*(\mathbb{F}_{\mathcal{N}})) = \pi(\mathcal{A})'$.

Let $\theta : \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_{\mathcal{N}}) \to \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_{\mathcal{N}})$ be the canonical homomorphism. Let $x \in \ker \theta$. Next, choose a countable subset $X \subset \mathcal{N}$ such that x belongs to the closure of the set $\pi(\mathcal{A}) \otimes \text{span } \mathbb{F}_X$ in $\pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$, where $\mathbb{F}_X$ denotes the free group on X generators. Note that, such a choice is always possible by virtue of the fact that there exist a sequence $x_n \to x$ in the *max* norm such that $\{x_n\}_n \subset \pi(\mathcal{A}) \otimes \mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$ and choosing X to be such that $x_n \in \pi(\mathcal{A}) \otimes \text{span } \mathbb{F}_X$ for all n.

Since X is countable, we identify the free groups $\mathbb{F}_X$ and $\mathbb{F}_\infty$. Also, note that $\mathbb{F}_\infty = \mathbb{F}_X$ is a subgroup of $\mathbb{F}_{\mathbb{N}}$, therefore by the property of full group C*-algebra inclusions [3, Proposition 2.5.8], there is a natural inclusion $\Lambda_0 : \mathrm{C}^*(\mathbb{F}_\infty) \hookrightarrow \mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$ mapping the free unitaries in $\mathbb{F}_\infty$ onto the set $\mathbb{F}_X \subset \mathbb{F}_{\mathbb{N}} \subset \mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$.

Consider the natural homomorphism

$$\Lambda : \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \to \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_{\mathbb{N}}),$$

where $\Lambda = \mathrm{id}_{\pi(\mathcal{A})} \otimes \Lambda_0$ on $\pi(\mathcal{A}) \otimes \mathrm{C}^*(\mathbb{F}_\infty)$. Then $x \in \text{range } \Lambda$, since $\{x_n\}_n \subset \pi(\mathcal{A}) \otimes \text{span } \mathbb{F}_X \subset \text{range } \Lambda$. Let $y \in \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$ such that $x = \Lambda y$.

Now, we have

$$\pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \xrightarrow{\Lambda} \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_{\mathbb{N}}) \xrightarrow{\theta} \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_{\mathbb{N}}).$$

Let $\{y_m\}_m \subset \pi(\mathcal{A}) \otimes \mathrm{C}^*(\mathbb{F}_\infty)$ be a sequence such that $y_m \to y$ in $\pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$. By hypothesis, $\pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$, so $y_m \to y$ in $\pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$. Since the *min* tensor product respects inclusion, we have $\pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) \subset \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$ because $\mathrm{C}^*(\mathbb{F}_\infty)$ is a C*-subalgebra of $\mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$. Also, one has $y \in \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) \subset \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_{\mathbb{N}})$. So, in particular,

$y_m \to y$ in $\pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N})$. By definitions of θ and Λ, the map $\theta \circ \Lambda|_{\pi(\mathcal{A})\otimes \mathrm{C}^*(\mathbb{F}_\infty)} = \mathrm{id}_{\pi(\mathcal{A})\otimes \mathrm{C}^*(\mathbb{F}_\infty)}$. Thus, $\theta \circ \Lambda(y_m) \to \theta \circ \Lambda(y)$ in $\pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N})$. But, $\theta \circ \Lambda(y) = \theta(x) = 0$, so, $\theta \circ \Lambda(y_m) = \mathrm{id}_{\pi(\mathcal{A})\otimes \mathrm{C}^*(\mathbb{F}_\infty)}(y_m) = y_m \to 0$ in $\pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N})$. This shows that $y = 0$, which in turn shows that $x = 0$.

The arguments above show that the map θ is injective. Thus, we have

$$\pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\mathbb{N}) = \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N}).$$

From the equality above, we have

$$\pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\mathbb{N}) = \pi(\mathcal{A}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N}) \subset \mathcal{B}(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N}).$$

Let $\Phi_0 : \mathcal{B}(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\mathbb{N}) \to \mathcal{B}(\mathcal{H})$ be a unital completely positive Arveson extension of $\rho : \pi(\mathcal{A}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\mathbb{N}) \to \mathcal{B}(\mathcal{H})$. Define a unital completely positive map

$$\begin{aligned} \Phi : \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathcal{H}) \\ T &\longmapsto \Phi_0(T \otimes 1) \end{aligned}$$

From the definition above, one has, for $a \in \mathcal{A}$,

$$\Phi(\pi(a)) = \Phi_0(\pi(a) \otimes 1) = \rho(\pi(a) \otimes 1) = \pi(a).$$

Finally, to conclude our claim, we need to show that range $\Phi \subset \pi(\mathcal{A})''$, thereby obtaining the required weak expectation for the arbitrary but fixed π. To this end, note that the C*-subalgebra $1 \otimes \mathrm{C}^*(\mathbb{F}_\mathbb{N})$ is in the multiplicative domain of Φ_0 since

$\Phi_0|_{1\otimes \mathrm{C}^*(\mathbb{F}_{\mathcal{N}})} = \pi_{\mathcal{N}}$. Let $f \in \pi(\mathcal{A})' = \rho(1 \otimes \mathrm{C}^*(\mathbb{F}_{\mathcal{N}}))$. Then $f = \rho(1 \otimes f_0)$ for some $f_0 \in \mathrm{C}^*(\mathbb{F}_{\mathcal{N}})$. Next, for $T \in \mathcal{B}(\mathcal{H})$ one has

$$\begin{aligned}
\Phi(T)f = \Phi_0(T\otimes 1)\rho(1\otimes f_0) = \Phi_0(T\otimes 1)\Phi_0(1\otimes f_0) &= \Phi_0((T\otimes 1)(1\otimes f_0)) \\
&= \Phi_0((1\otimes f_0)(T\otimes 1)) \\
&= \Phi_0(1\otimes f_0)\Phi_0(T\otimes 1) \\
&= f\Phi(T).
\end{aligned}$$

This shows that $\Phi(T)$ commutes with all $f \in \pi(\mathcal{A})'$, thereby proving that

$$\text{range } \Phi \subset \pi(\mathcal{A})''.$$

(1) $\Rightarrow$ (2). The interested reader is directed to [14, Proposition 1.1] for a proof. $\square$

2.5.6 A matrix completion characterization

Recall that the definition of the weak expectation property of C*-algebras involves all faithful non-degenerate representations or equivalently the universal representation. In some recent works of much interest [7], [8] D. Farenick, A. Kavruk and V. Paulsen obtained a different characterization of the weak expectation property of *unital* C*-algebras, which is somewhat in stark contrast to the classical approach. Indeed, the contrast lies in the fact that their characterization requires the verification of a certain matrix completion property for any <u>one</u> chosen faithful non-degenerate representation of the C*-algebra.

Definition 2.5.9. *Let $\mathcal{A}$ be a unital C*-algebra. A strictly positive element a in $\mathcal{A}$, is a positive element such that, there exist a $\delta > 0$ satisfying $a \geq \delta 1_{\mathcal{A}}$.*

In the following result, choose and fix a faithful non-degenerate representation of the C*-algebra $\mathcal{A}$ on $\mathcal{H}$ and consider $\mathcal{A}$ as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Theorem 2.5.10. *(Matrix completion criterion, [8]) If $\mathcal{A}$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*

1. *$\mathcal{A}$ has the weak expectation property.*

2. *If, given $p \in \mathbb{N}$ and $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$, there exist strongly positive operators $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ such that $A + B + C = 1$ and*

$$Y = \begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2^* & C \end{bmatrix}$$

is strongly positive in $\mathcal{M}_{3p}(\mathcal{B}(\mathcal{H}))$, then there also exist $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}_p(\mathcal{A})$ with the same property.

2.6 Relative weak injectivity of C*-algebras

The concept of relative weak injectivity of C*-algebras was introduced by E. Kirchberg in [14] as an extension of Lance's weak expectation property with subtle differences as pointed out in Remark 2.5.7.

2.6.1 The *max* tensor product inclusion revisited

Recall that, if $\mathcal{A}$ is a C*-algebra with the weak expectation property then, for any C*-algebra $\mathcal{B} \supset \mathcal{A}$, the C*-algebra inclusion $\mathcal{A} \otimes_{\max} \mathcal{C} \subset \mathcal{B} \otimes_{\max} \mathcal{C}$ always holds true for any C*-algebra $\mathcal{C}$.

In this section, we discuss the case of *max* tensor product inclusion for a given pair of C*-algebras $\mathcal{A} \subset \mathcal{B}$, when $\mathcal{A}$ does not necessarily have the weak expectation property. An example illustrating this inclusion phenomenon is given below.

Example 2.6.1. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{J}$ be a ideal in $\mathcal{A}$. By an ideal we always mean a closed two sided ideal. It is a well known fact that an ideal is a C*-subalgebra. For the pair $\mathcal{J} \subset \mathcal{A}$, it is always true that for any C*-algebra $\mathcal{C}$,

$$\mathcal{J} \otimes_{\max} \mathcal{C} \subset \mathcal{A} \otimes_{\max} \mathcal{C}.$$

To see this, we directly appeal to the definition of the *max* C*-norm. Recall that, every non-degenerate representation $\pi : \mathcal{J} \otimes \mathcal{C} \to \mathcal{B}(\mathcal{H})$ is of the form $\pi = \pi_{\mathcal{J}} \cdot \pi_{\mathcal{C}}$ from Theorem 2.3.2, where $\pi_{\mathcal{J}}$ and $\pi_{\mathcal{C}}$ are non-degenerate representations with commuting ranges. For $x \in \mathcal{J} \otimes \mathcal{C}$,

$$\begin{aligned} \|x\|_{\mathcal{J} \otimes_{\max} \mathcal{C}} &= \sup_{\pi} \|\pi(x)\| \\ &= \sup_{\pi_{\mathcal{J}}, \pi_{\mathcal{C}}} \|\pi_{\mathcal{J}} \cdot \pi_{\mathcal{C}}(x)\| \end{aligned}$$

Now, every non-degenerate representation $\pi_{\mathcal{J}} : \mathcal{J} \to \mathcal{B}(\mathcal{H})$ has a unique extension to a representation $\widetilde{\pi_{\mathcal{J}}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by [5, Lemma I.9.14]. Further, for $a \in \mathcal{A}, c \in$

$\mathcal{C}$, $j \in \mathcal{J}$ and $h \in \mathcal{H}$

$$\begin{aligned}
\widetilde{\pi}_{\mathcal{J}}(a)\pi_{\mathcal{C}}(c)\pi_{\mathcal{J}}(j)h = \widetilde{\pi}_{\mathcal{J}}(a)\pi_{\mathcal{J}}(j)\pi_{\mathcal{C}}(c)h = \widetilde{\pi}_{\mathcal{J}}(aj)\pi_{\mathcal{C}}(c)h &= \pi_{\mathcal{J}}(aj)\pi_{\mathcal{C}}(c)h \\
&= \pi_{\mathcal{C}}(c)\pi_{\mathcal{J}}(aj)h \\
&= \pi_{\mathcal{C}}(c)\widetilde{\pi}_{\mathcal{J}}(aj)h \\
&= \pi_{\mathcal{C}}(c)\widetilde{\pi}_{\mathcal{J}}(a)\pi_{\mathcal{J}}(j)h.
\end{aligned}$$

Since $\pi_{\mathcal{J}}$ is a non-degenerate representation, the above equality show that

$$\widetilde{\pi}_{\mathcal{J}}(a)\pi_{\mathcal{C}}(c) = \pi_{\mathcal{C}}(c)\widetilde{\pi}_{\mathcal{J}}(a).$$

Therefore $\widetilde{\pi}_{\mathcal{J}}$ and $\pi_{\mathcal{C}}$ have commuting ranges.

Define a non-degenerate representation $\tilde{\pi} : \mathcal{A} \otimes \mathcal{C} \to \mathcal{B}(\mathcal{H})$ by $\tilde{\pi} = \widetilde{\pi}_{\mathcal{J}} \cdot \pi_{\mathcal{C}}$ corresponding to the non-degenerate representation $\pi = \pi_{\mathcal{J}} \cdot \pi_{\mathcal{C}} : \mathcal{J} \otimes \mathcal{C} \to \mathcal{B}(\mathcal{H})$. Thus, every representation π of $\mathcal{J} \otimes \mathcal{C}$ has an extension $\tilde{\pi}$ to $\mathcal{A} \otimes \mathcal{C}$. As a consequence, we have the norm inequality (considering $x \in \mathcal{J} \otimes \mathcal{C} \subset \mathcal{A} \otimes \mathcal{C}$) :

$$\|x\|_{\mathcal{J}\otimes_{\max}\mathcal{C}} = \sup_{\pi} \|\pi(x)\| = \sup_{\tilde{\pi}} \|\tilde{\pi}(x)\| \leq \sup_{\rho} \|\rho(x)\| = \|x\|_{\mathcal{A}\otimes_{\max}\mathcal{C}}$$

where the last supremum runs over all non-degenerate representation ρ of $\mathcal{A} \otimes \mathcal{C}$. The opposite inequality, that is $\|x\|_{\mathcal{A}\otimes_{\max}\mathcal{C}} \leq \|x\|_{\mathcal{J}\otimes_{\max}\mathcal{C}}$ follows easily from the universal property of the *max* tensor product mentioned in Proposition 2.3.4 and the fact that C*-homomorphisms are always contractive. Thus, for any $x \in \mathcal{J} \otimes \mathcal{C}$, $\|x\|_{\mathcal{J}\otimes_{\max}\mathcal{C}} = \|x\|_{\mathcal{A}\otimes_{\max}\mathcal{C}}$, which proves $\mathcal{J} \otimes_{\max} \mathcal{C} \subset \mathcal{A} \otimes_{\max} \mathcal{C}$.

2.6.2 Relative weak injectivity

Definition 2.6.2. *(Relative weak injectivity) Let $\mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{B}$. Call $\mathcal{A}$* ***relatively weakly injective*** *in $\mathcal{B}$ if every faithful non-degenerate representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ has a unital completely positive extension to $\mathcal{B}$, taking values in $\overline{\pi(\mathcal{A})}^{SOT}$. In other words, there exists unital completely positive map $\Phi_\pi : \mathcal{B} \to \overline{\pi(\mathcal{A})}^{SOT} \subset \mathcal{B}(\mathcal{H})$ such that $\Phi_\pi(a) = \pi(a)$ for all $a \in \mathcal{A}$.*

Remark 2.6.3. In the definition above, we assumed $\mathcal{B}$ to be unital. In general, it need not be so, in which case the map Φ_π needs be a contractive completely positive map. However, one may simply consider the unitization of $\mathcal{B}$ and consider the canonical unital extension of the contractive completely positive map thereby reducing the general situation to the unital one as defined above. Thus, without loss of generality, one may assume $\mathcal{B}$ to be unital.

Remark 2.6.4. Relative weak injectivity may as well be defined by just considering the universal representation of $\mathcal{A}$ similar to Remark 2.5.6. In that case, if $\pi_u : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u)$ denotes the universal representation of $\mathcal{A}$, then there exists a unital completely positive map $\Phi_{\pi_u} : \mathcal{B} \to \mathcal{A}^{**}$ extending π_u when $\mathcal{A}$ is relatively weakly injective in $\mathcal{B}$.

Remark 2.6.5. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a non-degenerate C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Combining Remarks 2.5.6 and 2.6.4 one sees that $\mathcal{A}$ has the weak expectation property if and only if $\mathcal{A}$ is relatively weakly injective in $\mathcal{B}(\mathcal{H})$.

The following theorem, due to E. Kirchberg, reminiscent of Theorem 2.5.5 and Theorem 2.5.8, is a characterization of relatively weakly injective pairs of C*-algebras and demonstrates its significance in *max* tensor product inclusion.

Theorem 2.6.6. *(Relative weak injectivity of a C*-algebra pair, [14]) Given C*-algebras $\mathcal{A} \subset \mathcal{B}$, where $\mathcal{B}$ is a unital C*-algebra, the following statements are equivalent :*

1. *$\mathcal{A}$ is relatively weakly injective in $\mathcal{B}$.*

2. *For any C*-algebra $\mathcal{C}$, $\mathcal{A} \otimes_{max} \mathcal{C} \subset \mathcal{B} \otimes_{max} \mathcal{C}$.*

3. *$\mathcal{A} \otimes_{max} C^*(\mathbb{F}_\infty) \subset \mathcal{B} \otimes_{max} C^*(\mathbb{F}_\infty)$.*

2.6.3 Existence of relatively weakly injective pairs

As discussed in the previous section, the concept of relative weak injectivity completely settles the question of *max* tensor product inclusion.

Definition 2.6.7. *(Nuclear C*-algebras) A C*-algebra $\mathcal{A}$ is said to be* ***nuclear*** *if for any other C*-algebra $\mathcal{C}$, the algebraic tensor product $\mathcal{A} \otimes \mathcal{C}$ has a unique C*-norm. In other words, $\mathcal{A} \otimes_{min} \mathcal{C} = \mathcal{A} \otimes_{max} \mathcal{C}$ for all $\mathcal{C}$.*

Examples 2.6.8. Some commonly known nuclear C*-algebras are :

1. Commutative C*-algebras.

2. Finite dimensional C*-algebras.

3. Group C*-algebras of discrete amenable groups like $\mathbb{Z}$.

The class of nuclear C*-algebras is rich with numerous more interesting examples and is an extensively studied, relatively well understood class.

By the definition of the nuclear C*-algebras it is easy to see using Theorem 2.5.8 that nuclear C*-algebras have the weak expectation property. Another example of C*-algebras with the weak expectation property are the *injective* C*-algebras.

Both the examples of C*-algebras listed above have the weak expectation property by virtue of more powerful intrinsic properties namely nuclearity and injectivity. Therefore, it is natural to ask if there exists any non-nuclear and non-injective C*-algebra with the weak expectation property. More generally, considering Remark 2.6.5, one may ask if there are non-trivial examples of relatively weakly injective C*-algebra pairs. Example 2.6.1 confirms the existence of non-trivial pairs. A very general and powerful existential result is known due to E. Kirchberg.

Theorem 2.6.9. *(Existence of relatively weakly injective pairs, [14]) Let $\mathcal{A}_0$ be a* <u>*separable*</u> *C*-subalgebra of $\mathcal{B}$. Then, there exist a* <u>*separable*</u> *C*-subalgebra $\mathcal{A}$ of $\mathcal{B}$, such that, $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ is relatively weakly injective in $\mathcal{B}$.*

Remark 2.6.10. The existence result above shows the abundance of relatively weakly injective pairs. In particular, choosing $\mathcal{B} = \mathcal{B}(\mathcal{H})$, and considering any *separable* non-degenerate C*-subalgebra $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H})$, we get a C*-algebra $\mathcal{A} \supset \mathcal{A}_0$ such that $\mathcal{A}$ is relatively weakly injective in $\mathcal{B}(\mathcal{H})$. By Remark 2.6.5, $\mathcal{A}$ ($\neq \mathcal{B}(\mathcal{H})$ since $\mathcal{A}$ is separable, while $\mathcal{B}(\mathcal{H})$ is not) is a C*-algebra with the weak expectation property, thereby showing the abundance of C*-algebras with the weak expectation property.

2.7 Tensor products of operator systems

Let $\mathcal{S}$ and $\mathcal{R}$ be operator systems. The algebraic tensor product $\mathcal{S} \otimes \mathcal{R}$ is a $*$-vector space. The goal is to put an operator system structure on $\mathcal{S} \otimes \mathcal{R}$. In other words, a tensor product structure on $\mathcal{S} \otimes \mathcal{R}$ comprises of a family of cones $\tau = \{\mathfrak{C}_n\}_{n \in \mathbb{N}}$, with $\mathfrak{C}_n \subset \mathcal{M}_n(\mathcal{S} \otimes \mathcal{R})$ such that :

1. $(\mathcal{S} \otimes \mathcal{R}, \{\mathfrak{C}_n\}_{n \in \mathbb{N}}, 1_{\mathcal{S}} \otimes 1_{\mathcal{R}})$ is an operator system denoted by $\mathcal{S} \otimes_\tau \mathcal{R}$,

2. $\mathcal{M}_n(\mathcal{S})^+ \otimes \mathcal{M}_m(\mathcal{R})^+ \subset \mathfrak{C}_{nm}$, for all $n, m \in \mathbb{N}$ and

3. If $\phi : \mathcal{S} \to \mathcal{M}_n$ and $\psi : \mathcal{R} \to \mathcal{M}_m$ are unital completely positive maps, then $\phi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{R} \to \mathcal{M}_{nm}$ is a unital completely positive map.

Given a tensor product operator system, say $\mathcal{S} \otimes_\tau \mathcal{R}$ as in above, it is often convenient to denote the positive cones $\mathfrak{C}_n$ by $\mathcal{M}_n(\mathcal{S} \otimes_\tau \mathcal{R})^+$.

A point to be noted here is that, unlike the C*-tensor product, the operator system tensor product does not require the resulting operator system to be closed.

The rich theory of tensor products in the operator systems category was introduced and extensively developed in [12], [13].

Just as in the C*-realm, the algebraic tensor product of two operator systems can be endowed with multiple tensor product structures. Similar to the C*-analogues, one has a spatial or the operator system ***min*** tensor product and an operator system ***max*** tensor product. But, in this category, there is another natural candidate, namely the ***commuting*** tensor product.

In [12] the authors explicitly describe the family of positive cones which define each of the three important tensor products above. It was also shown that, if any one (or both) of the operator system(s) is a C*-algebra, then the commuting tensor product coincides with the operator system *max* tensor product. Thereby, loosely speaking, the generalization of the max C*-tensor product has two *distinct* [12, Corollary 6.10] avatars in the operator system category.

For now we briefly describe below the operator system *min* and *max* tensor products and defer a detailed review of the *commuting* tensor product, which plays an integral role in this thesis, to Chapter 4.

2.7.1 The operator system *min* tensor product

The operator system *min* tensor product is the analog of the minimum tensor product in the C*-category. In fact, just like in the C*-scenario the operator system *min* tensor product is a spatial tensor product. The matricial positive cones are defined below.

Let $\mathcal{S}$ and $\mathcal{R}$ be operator systems. Denote by $\mathfrak{S}_k(\mathcal{S})$ the set of all unital completely positive maps from $\mathcal{S}$ into $\mathcal{M}_k$ (and similarly for $\mathcal{R}$). For $n \in \mathbb{N}$ define cones by :

$$\begin{aligned} \mathfrak{C}_n^{\min} &= \{(\eta_{ij}) \in \mathcal{M}_n(\mathcal{S} \otimes \mathcal{R}) : (\phi \otimes \psi(\eta_{ij})) \in \mathcal{M}_{nkm}^+, \\ &\qquad \text{for all } \phi \in \mathfrak{S}_k(\mathcal{S}), \psi \in \mathfrak{S}_m(\mathcal{R}) \text{ for all } k, m \in \mathbb{N}\} \end{aligned}$$

Remark 2.7.1. It was shown in [12] that these cones define a matrix ordering on $\mathcal{S} \otimes \mathcal{R}$ with $1_{\mathcal{S}} \otimes 1_{\mathcal{R}}$ as an archimedean matrix order unit such that, if $\tau = \{\mathfrak{C}_n\}_{n \in \mathbb{N}}$ is any other operator system tensor product structure on $\mathcal{S} \otimes \mathcal{R}$ then, for each $n \in \mathbb{N}$, $\mathfrak{C}_n \subset \mathfrak{C}_n^{\min}$.

Definition 2.7.2. *The operator system* $(\mathcal{S} \otimes \mathcal{R}, \{\mathfrak{C}_n^{min}\}_{n \in \mathbb{N}}, 1_{\mathcal{S}} \otimes 1_{\mathcal{R}})$ *is called the* ***min*** *tensor product of* $\mathcal{S}$ *and* $\mathcal{R}$ *and is denoted by* $\mathcal{S} \otimes_{min} \mathcal{R}$.

The spatial nature of the operator system *min* tensor product follows from the result below.

Theorem 2.7.3. *(Spatial nature of min tensor product, [12]) Let* $\mathcal{S}$ *and* $\mathcal{R}$ *be operator systems. Let* $i_{\mathcal{S}} : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ *and* $i_{\mathcal{R}} : \mathcal{R} \to \mathcal{B}(\mathcal{K})$ *be unital complete injections. The family* $\{\mathfrak{C}_n^{min}\}_{n \in \mathbb{N}}$ *is the operator system structure on* $\mathcal{S} \otimes \mathcal{R}$ *arising from the embedding* $i_{\mathcal{S}} \otimes i_{\mathcal{R}} : \mathcal{S} \otimes \mathcal{R} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

2.7.2 The operator system *max* tensor product

For operator systems $\mathcal{S}$ and $\mathcal{R}$, define a family of matricial cones $\{\mathfrak{C}_n^{\max}\}_{n\in\mathbb{N}}$ by :

$$\begin{aligned}\mathfrak{C}_n^{\max} &= \{\lambda(P\otimes Q)\lambda^* \in \mathcal{M}_n(\mathcal{S}\otimes\mathcal{R}) : P\in\mathcal{M}_k(\mathcal{S})^+,\\ &\qquad Q\in\mathcal{M}_m(\mathcal{R})^+, \lambda\in\mathcal{M}_{n,km}; k,m\in\mathbb{N}\}\end{aligned}$$

Remark 2.7.4. It was shown in [12] that these cones define a matrix ordering on $\mathcal{S}\otimes\mathcal{R}$ with $1_{\mathcal{S}}\otimes 1_{\mathcal{R}}$ as an archimedean matrix order unit such that, if $\tau=\{\mathfrak{C}_n\}_{n\in\mathbb{N}}$ is any other operator system tensor product structure on $\mathcal{S}\otimes\mathcal{R}$ then, for each $n\in\mathbb{N}$, $\mathfrak{C}_n\supset\mathfrak{C}_n^{\max}$.

Definition 2.7.5. *The operator system* $(\mathcal{S}\otimes\mathcal{R}, \{\mathfrak{C}_n^{max}\}_{n\in\mathbb{N}}, 1_{\mathcal{S}}\otimes 1_{\mathcal{R}})$ *is called the* ***max*** *tensor product of* $\mathcal{S}$ *and* $\mathcal{R}$ *and is denoted by* $\mathcal{S}\otimes_{max}\mathcal{R}$.

2.7.3 Inclusion of cones

Tensor product structures are defined on the algebraic tensor product of two operator systems by specifying the positive cones at every matricial level. Let γ and δ be two distinct tensor product structures (for example the *min* and the *max* tensor product). For operator systems $\mathcal{S}$ and $\mathcal{R}$, it may happen that for every $n\in\mathbb{N}$, $\mathcal{M}_n(\mathcal{S}\otimes_\gamma\mathcal{R})^+\subset\mathcal{M}_n(\mathcal{S}\otimes_\delta\mathcal{R})^+$. When such a situation arises, we say that the cones of the former are included in those of the latter. It is easy to see that this inclusion reflects on the norms on the operator system tensor product as $\|x\|_{\mathcal{S}\otimes_\delta\mathcal{R}}\le\|x\|_{\mathcal{S}\otimes_\gamma\mathcal{R}}$ for any $x\in\mathcal{S}\otimes\mathcal{R}$, where the norms are defined as in Subsection 2.1.2. In particular, the above statements always hold true when $\gamma = max$ and $\delta = min$. Indeed, a consequence of Remarks 2.7.1 and 2.7.4 is that, for every $x \in \mathcal{S} \otimes \mathcal{R}$

$$\|x\|_{\mathcal{S}\otimes_{\min}\mathcal{R}} \leq \|x\|_{\mathcal{S}\otimes_{\tau}\mathcal{R}} \leq \|x\|_{\mathcal{S}\otimes_{\max}\mathcal{R}}$$

for any operator system tensor product τ.

Chapter 3

Weak expectation property and crossed product C*-algebras

In this chapter we discuss a permanence property of **unital** C*-algebras with the weak expectation property and their crossed products by **discrete** groups. We briefly review *amenable groups* and *amenable actions* and proceed to study the cases of *amenable actions* due to *amenable groups* followed by the general case of *amenable actions*. The contents of this chapter appears in [2].

3.1 A permanence question

A C*-algebra $\mathcal{A}$ has the quotient weak expectation property (QWEP) if $\mathcal{A}$ is a quotient of a C*-algebra with the weak expectation property. The class of C*-algebras with QWEP enjoys a number of permanence properties, many of which are enumerated in [17, Proposition 4.1] and originate with E. Kirchberg [14]. For example, if $\mathcal{A}$ is a unital C*-algebra with QWEP and if α is an amenable action of a discrete group G on $\mathcal{A}$, then the crossed product C*-algebra $\mathcal{A} \rtimes_\alpha G$ has QWEP [17, Proposition 4.1(vi)].

In contrast to QWEP, the weak expectation property appears to have few permanence properties. For example, $\mathcal{A} \otimes_{\min} \mathcal{B}$ may fail to have the weak expectation property if $\mathcal{A}$ and $\mathcal{B}$ have the same; one such example is furnished by $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathcal{H})$ [18]. In comparison, if $\mathcal{A}$ and $\mathcal{B}$ are nuclear, then so is $\mathcal{A} \otimes_{\min} \mathcal{B}$, and if $\mathcal{A}$ and $\mathcal{B}$ are exact, then so is $\mathcal{A} \otimes_{\min} \mathcal{B}$ [3, Sections 10.1,10.2].

The purpose of this chapter is to establish the following permanence result for the weak expectation property (Theorem 3.6.3):

If α is an amenable action of a discrete group G on a unital C-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property if and only if $\mathcal{A}$ does.*

In this regard, the weak expectation property is consistent with the analogous permanence results for nuclear and exact C*-algebras [3, Theorem 4.3.4].

3.2 Amenable groups and amenable actions

Definition 3.2.1. *(Amenable groups) A group G is amenable if there exist a state μ on $l^\infty(G)$ which is invariant under the left translation action, that is, for all $g \in G$ and $f \in l^\infty(G)$*

$$\mu(g.f) = \mu(f)$$

where $g.f$ is the left translation given by $g.f(t) = f(g^{-1}t)$, $t \in G$.

The state μ in the definition above is called an *invariant mean.*

Examples 3.2.2. (Amenable groups)

1. Finite groups.

2. Abelian groups.

3. Solvable groups.

An example of a non-amenable group is given below.

Example 3.2.3. (Non-amenable group) The free group on two generators $\mathbb{F}_2$ is non-amenable.

Amenable groups have numerous characterizations and nice properties. We mention (without proof) a few of the equivalent formulations below which are relevant to our study. First we define a well known important concept for groups.

Definition 3.2.4. *(Følner condition) A group G satisfies the Følner condition if for any finite subset $E \subset G$ and $\epsilon > 0$, there exist a finite subset $F \subset G$ such that*

$$\max_{g \in E} \frac{|gF \Delta F|}{|F|} < \epsilon$$

where $gF = \{gt : t \in F\}$ and Δ denotes the symmetric difference of two sets. A net of finite subsets $\{F_\omega\}_{\omega \in \Omega}$ of G such that

$$\frac{|gF_\omega \Delta F_\omega|}{|F_\omega|} \to 0$$

for every $g \in G$ is called a Følner net.

Theorem 3.2.5. *(Characterizations of amenable groups, [3, Theorem 2.6.8]) Let G be a discrete group. The following conditions are equivalent :*

1. *G is amenable.*
2. *G satisfies the Følner condition.*
3. *$C^*(G) = C^*_\lambda(G)$.*
4. *$C^*_\lambda(G)$ is nuclear.*

Remark 3.2.6. A discrete group G is amenable if and only if it admits a Følner net. However, for the purpose of this chapter, we only need the fact that amenable groups admit a Følner net.

Remark 3.2.7. A well known, simple but useful reformulation (as used in Theorem 3.4.3) of the characteristic property of Følner nets

$$\frac{|gF_\omega \Delta F_\omega|}{|F_\omega|} \to 0$$

is that

$$\frac{|gF_\omega \cap F_\omega|}{|F_\omega|} \to 1$$

for every $g \in G$. To see this, simply note that

$$gF_\omega \Delta F_\omega = [gF_\omega \setminus gF_\omega \cap F_\omega] \cup [F_\omega \setminus gF_\omega \cap F_\omega],$$

therefore

$$|gF_\omega \Delta F_\omega| = |gF_\omega \setminus gF_\omega \cap F_\omega| + |F_\omega \setminus gF_\omega \cap F_\omega| = 2|F_\omega| - 2|gF_\omega \cap F_\omega|$$

from which the required limit follows.

3.3 Crossed products of C*-algebras by amenable groups

Throughout the next two sections let G be a discrete amenable group. Recall that a G-C*-algebra $\mathcal{A}$ is a C*-algebra such that there is an action

$$\alpha : G \to \mathrm{Aut}(\mathcal{A})$$

of G on $\mathcal{A}$ by automorphisms. In this section we shall record (without proof) some well known permanence properties of crossed product C*-algebras by discrete amenable groups.

The following lemma is the key to one of our main results presented in the following section and the permanence properties mentioned above.

Lemma 3.3.1. *(Factorization of the identity map on crossed products by amenable groups, [3, Lemma 4.2.3]) If $\mathcal{A}$ is a G-C^*-algebra and $F \subset G$ is a finite set, then there exist contractive completely positive maps $\varphi : A \rtimes_{\alpha,r} G \to \mathcal{A} \otimes \mathcal{M}_{|F|}(\mathbb{C})$ and $\psi : \mathcal{A} \otimes \mathcal{M}_{|F|}(\mathbb{C}) \to C_c(G, \mathcal{A}) \subset A \rtimes_{\alpha,r} G$ such that for all $a \in \mathcal{A}$ and $g \in G$ we have*

$$\psi \circ \varphi(a\lambda_g) = \frac{|F \cap gF|}{|F|} a\lambda_g.$$

We omit the proof of the lemma above as a similar factorization will be used in Theorem 3.4.3 whereby the factoring maps shall be described in detail.

The following permanence result motivates our main results in this chapter.

Theorem 3.3.2. *(Permanence properties of nuclear and exact C^*-algebras, [3, Theorem 4.2.4]) The following statements are equivalent :*

1. $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$
2. *A is nuclear if and only if $A \rtimes_\alpha G$ is nuclear.*
3. *A is exact if and only if $A \rtimes_\alpha G$ is exact.*

3.4 Crossed product C*-algebras and the weak expectation property : the amenable group case

In this section we present one of the main results of this chapter. The following lemmas are useful for the proof.

Lemma 3.4.1. *([3, Exercise 4.1.3]) Let $(\mathcal{A}, \alpha, G)$ be a dynamical system. Let $id : G \to Aut(\mathcal{B})$ denote the trivial action of G on another C*-algebra $\mathcal{B}$. The action $\alpha \otimes id$ defined by $(\alpha \otimes id)_g = \alpha_g \otimes id_{\mathcal{B}}$ on $\mathcal{A} \otimes_{min} \mathcal{B}$ is such that*

$$(\mathcal{A} \otimes_{min} \mathcal{B}) \rtimes_{\alpha \otimes id, r} G \cong (A \rtimes_{\alpha, r} G) \otimes_{min} \mathcal{B}.$$

Proof. If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{K})$ are faithful representations of $\mathcal{A}$ and $\mathcal{B}$ then, $\mathcal{A} \otimes_{\min} \mathcal{B} \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is a faithful representation. For notational convenience we refer to the regular representations of $C_c(G, \mathcal{A})$ simply as $C_c(G, \mathcal{A})$ (and similarly for other C*-algebras when necessary). Recalling the construction of the reduced crossed product, we consider the representation

$$\begin{aligned}
\pi : \mathcal{A} \otimes_{\min} \mathcal{B} &\to \mathcal{B}((\mathcal{H} \otimes \mathcal{K}) \otimes l^2(G)) \\
\sum_i a_i \otimes b_i &\longmapsto \bigoplus_{g \in G} (\alpha \otimes \mathrm{id})_g^{-1} (\sum_i a_i \otimes b_i) \\
&= \bigoplus_{g \in G} \sum_i \alpha_g^{-1}(a_i) \otimes b_i \\
&= \sum_i \left(\bigoplus_{g \in G} \alpha_g^{-1}(a_i) \right) \otimes b_i
\end{aligned}$$

A typical element of $C_c(G, \mathcal{A} \otimes_{\min} \mathcal{B}) \subset (\mathcal{A} \otimes_{\min} \mathcal{B}) \rtimes_{\alpha\otimes \mathrm{id},\mathrm{r}} G \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K} \otimes l^2(G))$ is a finite linear span of elements like

$$\begin{aligned}
\left(\pi(\sum_i a_i \otimes b_i)\right)(1_{\mathcal{A}\otimes_{\min}\mathcal{B}} \otimes \lambda_{g_0}) &= \left(\sum_i \left(\bigoplus_{g\in G} \alpha_g^{-1}(a_i)\right) \otimes b_i\right)(1_{\mathcal{A}\otimes_{\min}\mathcal{B}} \otimes \lambda_{g_0}) \\
&= \left(\sum_i \left(\sum_{g\in G} \alpha_g^{-1}(a_i) \otimes E_{g,g}\right) \otimes b_i\right)(1_{\mathcal{A}\otimes_{\min}\mathcal{B}} \otimes \lambda_{g_0}) \\
&= \sum_i \left(\sum_{g\in G} \alpha_g^{-1}(a_i) \otimes E_{g,g}\lambda_{g_0}\right) \otimes b_i \\
&= \sum_i \left(\left(\bigoplus_{g\in G} \alpha_g^{-1}(a_i)\right)(1_{\mathcal{A}} \otimes \lambda_{g_0})\right) \otimes b_i
\end{aligned}$$

where $E_{g,g}$'s are the diagonal matrix units in $\mathcal{B}(l^2(G))$. Noting that,

$$\left(\left(\bigoplus_{g\in G} \alpha_g^{-1}(a_i)\right)(1_{\mathcal{A}} \otimes \lambda_{g_0})\right) \in C_c(G, \mathcal{A}) \subset A \rtimes_{\alpha,r} G \subset \mathcal{B}(\mathcal{H} \otimes l^2(G))$$

and thus

$$\begin{aligned}
\sum_i \left(\left(\bigoplus_{g\in G} \alpha_g^{-1}(a_i)\right)(1_{\mathcal{A}} \otimes \lambda_{g_0})\right) \otimes b_i \in C_c(G, \mathcal{A}) \otimes \mathcal{B} &\subset (A \rtimes_{\alpha,r} G) \otimes \mathcal{B} \\
&\subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K} \otimes l^2(G))
\end{aligned}$$

This shows that the subalgebras $C_c(G, \mathcal{A} \otimes_{\min} \mathcal{B})$ and $C_c(G, \mathcal{A}) \otimes \mathcal{B}$ of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K} \otimes l^2(G))$ are the same, which in turn proves the required result as $(\mathcal{A} \otimes_{\min} \mathcal{B}) \rtimes_{\alpha\otimes \mathrm{id},\mathrm{r}} G$ and $(A \rtimes_{\alpha,r} G) \otimes_{\min} \mathcal{B}$ are the norm closures of $C_c(G, \mathcal{A} \otimes_{\min} \mathcal{B})$ and $C_c(G, \mathcal{A}) \otimes \mathcal{B}$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K} \otimes l^2(G))$ respectively. $\square$

Lemma 3.4.2. *(Positive elements in $A \rtimes_{\alpha,r} G$, [3, Lemma 4.2.2]) If $\mathcal{A}$ is a G-C*-algebra, and $F \subset G$ is a finite set, then for each set $\{a_g\}_{g\in F} \subset \mathcal{A}$, the element*

$$\sum_{g,g'\in F} \alpha_g(a_g^* a_{g'})\lambda_{gg'^{-1}} \in C_c(G, \mathcal{A})$$

is a positive element in $A \rtimes_{\alpha,r} G$.

Proof. The element above is equal to $(\sum_{g\in F} a_g\lambda_{g^{-1}})^*(\sum_{g\in F} a_g\lambda_{g^{-1}})$. □

We now present our first main result of this chapter.

Theorem 3.4.3. *If α is an action of an amenable discrete group G on a unital C*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property if and only if $\mathcal{A}$ does.*

Proof. Assume first that $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property. To show that $\mathcal{A}$ has the weak expectation property, it is sufficient to show that if $\mathcal{A}$ is represented faithfully as a unital C*-subalgebra of $\mathcal{B}(\mathcal{K})$, for some Hilbert space $\mathcal{K}$, and if $\pi_u^{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}})$ is the universal representation of $\mathcal{A}$, then there a unital completely positive map $\omega : \mathcal{B}(\mathcal{K}) \to \mathcal{A}^{**}$ such that $\omega(a) = \pi_u^{\mathcal{A}}(a)$ for every $a \in \mathcal{A}$.

To this end, let $\mathcal{A} \rtimes_\alpha G \subset \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G})$ be the universal representation of $\mathcal{A} \rtimes_\alpha G$. Because $\mathcal{A}$ is unital, $\mathcal{A}$ is a unital C*-subalgebra of $\mathcal{A} \rtimes_\alpha G$. Hence,

$$\mathcal{A} \subset \mathcal{A} \rtimes_\alpha G \subset (\mathcal{A} \rtimes_\alpha G)^{**} \subset \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G})$$

and we therefore, on the one hand, consider $\mathcal{A}$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{K})$, where $\mathcal{K} = \mathcal{H}_u^{\mathcal{A} \rtimes_\alpha G}$. On the other hand,

$$\begin{aligned} \mathcal{A} \subset \mathcal{A} \rtimes_\alpha G = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} G \ &\subset \ \mathcal{B}(\mathcal{H}_u^{\mathcal{A} \rtimes_\alpha G}) \otimes_{\min} \mathrm{C}^*_{\mathrm{r}}(G) \\ &\subset \ \mathcal{B}(\mathcal{H}_u^{\mathcal{A} \rtimes_\alpha G}) \,\overline{\otimes}\, \mathcal{B}\left(l^2(G)\right) \\ &\subset \ \mathcal{B}\left(\mathcal{K} \otimes l^2(G)\right), \end{aligned}$$

where $\overline{\otimes}$ denotes the von Neumann algebra tensor product, yields another faithful representation of $\mathcal{A} \rtimes_\alpha G$, in this case, as a unital C*-subalgebra of $\mathcal{B}\left(\mathcal{K} \otimes l^2(G)\right)$. Let $(\mathcal{A} \rtimes_\alpha G)''$ denote the double commutant of $\mathcal{A} \rtimes_\alpha G$ in $\mathcal{B}\left(\mathcal{K} \otimes l^2(G)\right)$.

Using the vector state τ on $\mathcal{B}\left(l^2(G)\right)$ defined by $\tau(x) = \langle x\delta_e, \delta_e \rangle$ together with the identity map $\mathrm{id}_{\mathcal{B}(\mathcal{K})} : \mathcal{B}(\mathcal{H}_u^{\mathcal{A} \rtimes_\alpha G}) \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A} \rtimes_\alpha G})$, we obtain a normal unital completely positive map

$$\psi = \mathrm{id}_{\mathcal{B}(\mathcal{K})} \overline{\otimes} \tau : \mathcal{B}(\mathcal{K}) \,\overline{\otimes}\, \mathcal{B}\left(l^2(G)\right) \to \mathcal{B}(\mathcal{K}).$$

If $\mathcal{E} : \mathcal{A} \rtimes_{\alpha,\mathrm{r}} G \to \mathcal{A}$ denotes the conditional expectation of $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} G$ onto $\mathcal{A}$ whereby $\mathcal{E}\left(\sum_g a_g \lambda_g\right) = a_e$, then, using the identification $\mathcal{A} \rtimes_\alpha G = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} G$, the restriction of ψ to $(\mathcal{A} \rtimes_\alpha G)''$ is a normal extension of $\rho \circ \mathcal{E}$, where $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is the faithful representation of $\mathcal{A} \subset \mathcal{B}\left(\mathcal{K} \otimes l^2(G)\right)$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{K})$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} \rtimes_\alpha G & \xrightarrow{\mathcal{E}} & \mathcal{A} \\ \big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ (\mathcal{A} \rtimes_\alpha G)'' & \xrightarrow[\psi]{} & \mathcal{B}(\mathcal{K}). \end{array}$$

Because ψ is normal, the range of $\psi_{|(\mathcal{A}\rtimes_\alpha G)''}$ is determined by

$$\psi\left((\mathcal{A}\rtimes_\alpha G)''\right) = \overline{(\psi(\mathcal{A}\rtimes_\alpha G))}^{\mathrm{SOT}} = \overline{(\rho(\mathcal{A}))}^{\mathrm{SOT}}.$$

In other words, the range of $\psi_{|(\mathcal{A}\rtimes_\alpha G)''}$ is the strong-closure of the C*-subalgebra $\mathcal{A}$ of $\mathcal{A}\rtimes_\alpha G$ in the enveloping von Neumann algebra $(\mathcal{A}\rtimes_\alpha G)^{**}$ of $\mathcal{A}\rtimes_\alpha G$. Therefore, by [20, Corollary 3.7.9], there is an isomorphism $\theta : \overline{(\rho(\mathcal{A}))}^{\mathrm{SOT}} \to \mathcal{A}^{**}$ such that $\pi_u^{\mathcal{A}} = \theta_{|\rho(\mathcal{A})}$.

Now let $\pi_0 : (\mathcal{A}\rtimes_\alpha G)^{**} \to (\mathcal{A}\rtimes_\alpha G)''$ be the normal epimorphism that extends the identity map of $\mathcal{A}\rtimes_\alpha G$. Because $\mathcal{A}\rtimes_\alpha G$ has the weak expectation property, there is a unital completely positive map $\phi_0 : \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G}) \to (\mathcal{A}\rtimes_\alpha G)^{**}$ that fixes every element of $\mathcal{A}\rtimes_\alpha G$. Hence, if $\Upsilon = \theta\circ\psi_{|(\mathcal{A}\rtimes_\alpha G)''}\circ\pi_0\circ\phi_0$, then Υ is a unital completely positive map of $\mathcal{B}(\mathcal{K}) \to \mathcal{A}^{**}$ for which $\Upsilon(a) = \pi_u^{\mathcal{A}}(a)$ for every $a\in\mathcal{A}$. That is, $\mathcal{A}$ has the weak expectation property.

Conversely, assume that $\mathcal{A}$ has the weak expectation property and that $\mathcal{A}$ is (represented faithfully as) a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Thus, we consider $\mathcal{A}$ and $\mathcal{A}\rtimes_\alpha G$ faithfully represented via

$$\mathcal{A}\subset\mathcal{A}\rtimes_\alpha G = \mathcal{A}\rtimes_{\alpha,\mathrm{r}} G \subset \mathcal{B}\left(\mathcal{H}\otimes l^2(G)\right).$$

Note that $\mathfrak{u} : G\to\mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G})$ whereby $\mathfrak{u}(g) = \pi_u^{\mathcal{A}\rtimes_\alpha G}(1\otimes\lambda_g)$ is a unitary representation of G such that $(1\otimes\lambda)\times\pi$ is the regular (covariant) representation associated with the dynamical system $(\mathcal{A},\alpha,G)$.

Let $\pi_u^{\mathcal{A}\rtimes_\alpha G} : \mathcal{A} \rtimes_\alpha G \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G})$ be the universal representation of $\mathcal{A} \rtimes_\alpha G$ and define $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G})$ by $\pi = \pi_u^{\mathcal{A}\rtimes_\alpha G}{}_{|\mathcal{A}}$. For simplicity, put $\mathcal{H} = \mathcal{H}_u^{\mathcal{A}\rtimes_\alpha G}$. Because π is a faithful representation of $\mathcal{A}$ and $\mathcal{A}$ has the weak expectation property, there is a unital completely positive map

$$\phi_0 : \mathcal{B}(\mathcal{H}) \to \pi(\mathcal{A})'' \subset \pi_u^{\mathcal{A}\rtimes_\alpha G}(\mathcal{A} \rtimes_\alpha G)''$$

such that $\phi_0\,(\pi(a)) = \pi(a)$ for every $a \in \mathcal{A}$.

As in [3, Proposition 4.5.1], if $F \subset G$ is a finite nonempty subset and if $p_F \in \mathcal{B}(l^2(G))$ is the projection with range $\mathrm{Span}\{\delta_f : f \in F\}$, then $p_F \mathcal{B}(l^2(G)) p_F$ is isomorphic to the matrix algebra $\mathcal{M}_n$ for $n = |F|$, and so we obtain a unital completely positive map

$$\phi_F : \mathcal{B}(\mathcal{H} \otimes l^2(G)) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_n$$

defined by

$$\phi_F(x) = (1 \otimes p_F)x(1 \otimes p_F).$$

Next, let $\{e_{f,h}\}_{f,h\in F}$ denote the matrix units of $\mathcal{M}_n$ and define an action β of G on $\pi(\mathcal{A})''$ by

$$\beta_g(y) = \mathfrak{u}(g)y\mathfrak{u}(g)^*,$$

for $y \in \pi(\mathcal{A})''$. Observe that $\pi(\mathcal{A})'' \rtimes_\beta G \subset \pi_u^{\mathcal{A}\rtimes_\alpha G}(\mathcal{A} \rtimes_\alpha G)''$.

The linear map

$$\psi_F : \pi(\mathcal{A})'' \otimes \mathcal{M}_n \to \pi(\mathcal{A})'' \rtimes_\beta G$$

for which

$$\psi_F(y \otimes e_{f,h}) = |F|^{-1}\beta_f(y)\mathfrak{u}(fh^{-1})$$

for $y \in \pi(\mathcal{A})''$, is a unital completely positive map. To see this, it is enough to verify that it is positive because, by virtue of Lemma 3.4.1 one has a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_p \otimes (\pi(\mathcal{A})'' \otimes \mathcal{M}_n) & \xrightarrow{\cong} & (\mathcal{M}_p \otimes \pi(\mathcal{A})'') \otimes \mathcal{M}_n \\
{\scriptstyle \psi_F^{(p)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{\psi_F}} \\
\mathcal{M}_p \otimes (\pi(\mathcal{A})'' \rtimes_\beta G) & \xrightarrow{\cong} & (\mathcal{M}_p \otimes \pi(\mathcal{A})'') \rtimes_{\mathrm{id}\otimes\beta} G\,.
\end{array}$$

where the map $\widetilde{\psi_F}$ is defined exactly like ψ_F but replacing $\pi(\mathcal{A})'' \otimes \mathcal{M}_p$ for $\pi(\mathcal{A})''$ and the action $\mathrm{id} \otimes \beta$ for β. Thus, p-positivity of ψ_F is equivalent to positivity of $\widetilde{\psi_F}$. Now, every positive element in $(\mathcal{M}_p \otimes \pi(\mathcal{A})'') \otimes \mathcal{M}_n$ is of the form

$$\sum_{g,g' \in F} w_g^* w_{g'} \otimes e_{g,g'}$$

where $\{w_g\}_{g\in F} \subset \mathcal{M}_p \otimes \pi(\mathcal{A})''$. The image of this element under $\widetilde{\psi_F}$ is of the form

$$|F|^{-1} \sum_{g,g' \in F} (\mathrm{id} \otimes \beta)_g(w_g^* w_{g'}) \left(1_{\mathcal{M}_p} \otimes u(g{g'}^{-1})\right)$$

which is positive by Lemma 3.4.2.

Hence,

$$\theta_F := \psi_F \circ (\phi_0 \otimes \mathrm{id}_{\mathcal{M}_n}) \circ \phi_F$$

is a unital completely positive map $\mathcal{B}\left(\mathcal{H} \otimes l^2(G)\right) \to \pi_u^{\mathcal{A}\rtimes_\alpha G}(\mathcal{A} \rtimes_\alpha G)''$.

Hence, if $\{F_\omega\}_\omega$ is a Følner net in G and if $\theta_\omega : \mathcal{B}\left(\mathcal{H} \otimes l^2(G)\right) \to \pi_u^{\mathcal{A}\rtimes_\alpha G}(\mathcal{A} \rtimes_\alpha G)''$ is the unital completely positive map constructed above, for each ω, then the net

$\{\theta_\omega\}_\omega$ admits a cluster point θ relative to the point-ultraweak topology. Now, for every $\omega \in \Omega$, $a\lambda_g \in \mathcal{A} \rtimes_{\alpha,\mathrm{r}} G$, and $\xi, \eta \in \mathcal{H}^{\mathcal{A}\rtimes_\alpha G}$,

$$\begin{aligned}\left|\left\langle\left(\theta(a\lambda_g) - \pi_u^{\mathcal{A}\rtimes_\alpha G}(a\lambda_g)\right)\xi, \eta\right\rangle\right| &\leq \left|\left\langle\left(\theta(a\lambda_g) - \theta_{F_\omega}(a\lambda_g)\right)\xi, \eta\right\rangle\right| \\ &\quad + \left|\left\langle\left(\theta_{F_\omega}(a\lambda_g) - \pi_u^{\mathcal{A}\rtimes_\alpha G}(a\lambda_g)\right)\xi, \eta\right\rangle\right| \\ &= \left|\left(1 - \frac{|F_\omega \cap gF_\omega|}{|F_\omega|}\right)\langle \pi_u^{\mathcal{A}\rtimes_\alpha G}(a\lambda_g)\xi, \eta\rangle\right|.\end{aligned}$$

Because θ is a cluster point of $\{\theta_\omega\}_\omega$, we deduce that $\theta(a\lambda_g) = \pi_u^{\mathcal{A}\rtimes_\alpha G}(a\lambda_g)$. Hence, by continuity, $\theta : \mathcal{B}\left(\mathcal{H} \otimes l^2(G)\right) \to \pi_u^{\mathcal{A}\rtimes_\alpha G}(\mathcal{A} \rtimes_\alpha G)''$ is a unital completely positive map for that extends the identity map on $\pi_u^{\mathcal{A}\rtimes_\alpha G}(\mathcal{A} \rtimes_\alpha G)$, which proves that $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property. □

3.5 Crossed products of C*-algebras by amenable actions

There are groups which are *non-amenable* but have actions nice enough to exhibit properties close to those of amenable groups. Such groups are said to have *amenable actions.* Before embarking on the precise definition and properties, the following norm needs be introduced.

Definition 3.5.1. *Let $\mathcal{A}$ be a G-C^*-algebra. Let f_1, f_2 be finitely supported functions on G, that is, $f_1, f_2 \in C_c(G, \mathcal{A})$. Set*

$$\langle f_1, f_2\rangle = \sum a_i^* b_i \in \mathcal{A},$$

where $f_1 = \sum_i a_i \delta_{g_i}$ *and* $f_2 = \sum_j b_j \delta_{g'_j}$. *Finally, define*

$$\|f_1\|_2 = \|\langle f_1, f_1\rangle\|^{\frac{1}{2}}.$$

Definition 3.5.2. *(Amenable actions, [3, Definition 4.3.1]) An action* $\alpha : G \to Aut(\mathcal{A})$ *on a unital G-C*-algebra* $\mathcal{A}$ *is* ***amenable*** *if there exist finitely supported functions* $f_l : G \to \mathcal{A}$, $f_l = \sum_i a_{li}\delta_{g_i}$ *with the following properties :*

1. $0 \le a_{li} \in \mathcal{Z}(\mathcal{A})$ *for all* $l \in \mathbb{N}$ *and* $g_i \in G$.
2. $\langle f_l, f_l\rangle = \sum_i a_{li}^2 = 1_{\mathcal{A}}$.
3. $\|\delta_g f_l - f_l\|_2 \to 0$ *for all* $g \in G$ *(the product* $\delta_g f_l \in C_c(G, \mathcal{A})$ *is as defined in Subsection 2.4.3).*

In the definition above $\mathcal{Z}(\mathcal{A})$ stands for the center of $\mathcal{A}$, that is

$$\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba \text{ for all } b \in \mathcal{A}\}.$$

Remark 3.5.3. The set of functions $\{f_l\}_l$ exhibit Følner like properties. As a consequence a factorization result like Lemma 3.3.1 is true for crossed products of C*-algebras by groups with amenable actions. However, the details may be omitted due to it's similarity with Lemma 3.3.1 (or the factorization described in Theorem 3.4.3) or simply because of the lack of necessity of appealing directly to this for our next main result described in the following section, which involves the theorem below (which in turn is a consequence of the factorization in this case).

Theorem 3.5.4. *(Permanence properties of nuclear and exact C*-algebras for amenable actions, [3, Theorem 4.3.4]) For any amenable action α of G on $\mathcal{A}$, the following statements are equivalent :*

1. $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$
2. *A is nuclear if and only if $A \rtimes_\alpha G$ is nuclear.*
3. *A is exact if and only if $A \rtimes_\alpha G$ is exact.*

3.6 Crossed product C*-algebras and the weak expectation property : the amenable action case

In this section we generalize Theorem 3.4.3 in the case of groups with amenable actions. The proof relies on (Theorem 3.5.4(1)), Kirchberg's characterization of the weak expectation property (Theorem 2.5.8) and the matrix completion criterion for detecting weak expectation property (Theorem 2.5.10). The following lemmas are also used in the proof.

Lemma 3.6.1. *Let α be an amenable action of G on $\mathcal{A}$. If $\iota : G \to Aut(\mathcal{B})$ denotes the trivial action of G on a unital C*-algebra $\mathcal{B}$, then the action $\alpha \otimes \iota$, given by $\alpha \otimes \iota(g)[a \otimes b] = \alpha_g(a) \otimes b$ for all $g \in G$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, of G on $\mathcal{A} \otimes_{max} \mathcal{B}$ is amenable. (See [24, Remark 2.74]).*

Proof. Let $\{f_l\}_l$ denote a net of finitely supported positive-valued functions $f_l : G \to \mathcal{Z}(\mathcal{A})$, $f_l = \sum_i a_{li}\delta_{g_i}$ such that $\langle f_l, f_l\rangle = \sum_i a_{li}^2 = 1_{\mathcal{A}}$ and

$$\|\delta_g f_l - f_l\|_2 \to 0$$

for all $g \in G$ as in Definition 3.5.2.

Define finitely supported positive-valued functions $\tilde{f}_l : G \to \mathcal{Z}\left(\mathcal{A} \otimes_{\max} \mathcal{B}\right)$ by

$$\tilde{f}_l(g_i) = a_{li} \otimes_{\max} 1_{\mathcal{B}}.$$

Then $\langle \tilde{f}_l, \tilde{f}_l\rangle = 1_{\mathcal{A}\otimes_{\max}\mathcal{B}}$ and the limiting equation above holds with f_l replaced with $\tilde{f}_l$, δ_g replaced by $\tilde{\delta}_g$ ($\mathcal{A} \otimes_{\max} \mathcal{B}$ valued dirac function at $g \in G$) and α replaced with $\alpha \otimes \iota$. Hence, the action $\alpha \otimes \iota$ of G on $\mathcal{A} \otimes_{\max} \mathcal{B}$ is amenable. $\square$

Lemma 3.6.2. *([24, Lemma 2.75]) For a dynamical system $(\mathcal{A}, \alpha, G)$ and a C*-algebra $\mathcal{B}$, we have*

$$(\mathcal{A} \otimes_{max} \mathcal{B}) \rtimes_{\alpha\otimes\iota} G \cong (\mathcal{A} \rtimes_{\alpha} G) \otimes_{max} \mathcal{B}.$$

We now present the main result of this section.

Theorem 3.6.3. *If α is an amenable action of a discrete group G on a unital C*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property if and only if $\mathcal{A}$ does.*

Proof. Assume that $\mathcal{A}$ has the weak expectation property. By Kirchberg's criterion (Theorem 2.5.8), $\mathcal{A} \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$. Let $\iota : G \to \mathrm{Aut}\left(\mathrm{C}^*(\mathbb{F}_\infty)\right)$ denote the trivial action of G on $\mathrm{C}^*(\mathbb{F}_\infty)$. Then by Lemma 3.6.1, $\alpha \otimes \iota$ is an amenable action. Hence,

$$\begin{aligned}
(\mathcal{A} \rtimes_\alpha G) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) &= (\mathcal{A} \rtimes_{\alpha,\mathrm{r}} G) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) \\
&= (\mathcal{A} \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)) \rtimes_{\alpha\otimes\iota,\mathrm{r}} G \\
&= (\mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)) \rtimes_{\alpha\otimes\iota,\mathrm{r}} G \\
&= (\mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)) \rtimes_{\alpha\otimes\iota} G \\
&= (\mathcal{A} \rtimes_\alpha G) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)\,,
\end{aligned}$$

where the first equality is due to Theorem 3.5.4(1) and the final equality holds by Lemma 3.6.2. Another application of Kirchberg's Criterion implies that $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property.

Conversely, assume that $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property and that $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} G$ is represented faithfully on a Hilbert space $\mathcal{H}$. Thus,

$$\mathcal{A} \subset \mathcal{A} \rtimes_{\alpha,\mathrm{r}} G = \mathcal{A} \rtimes_\alpha G \subset \mathcal{B}(\mathcal{H})$$

also represents $\mathcal{A}$ faithfully on $\mathcal{H}$. Let $\mathcal{E} : \mathcal{A} \rtimes_{\alpha,\mathrm{r}} G \to \mathcal{A}$ denote the canonical conditional expectation of $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} G$ onto $\mathcal{A}$ [3, Proposition 4.1.9]. We now use the criterion of Theorem 2.5.10 for the weak expectation property.

Suppose that $p \in \mathbb{N}$, $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$, and $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ are such that $A + B + C = 1$ and the matrix

$$Y = \begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2^* & C \end{bmatrix} \in \mathcal{M}_{3p}(\mathcal{B}(\mathcal{H}))$$

is strongly positive. Because $\mathcal{A} \subset \mathcal{A} \rtimes_\alpha G$ and because $\mathcal{A} \rtimes_\alpha G$ has the weak expectation property, there are, by Theorem 2.5.10, $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}_p(\mathcal{A} \rtimes_\alpha G)$ such that

$$\tilde{Y} = \begin{bmatrix} \tilde{A} & X_1 & 0 \\ X_1^* & \tilde{B} & X_2 \\ 0 & X_2^* & \tilde{C} \end{bmatrix} \in \mathcal{M}_{3p}(\mathcal{A} \rtimes_\alpha G)$$

is strongly positive and $\tilde{A} + \tilde{B} + \tilde{C} = 1$. Because unital completely positive maps preserve strong positivity, the matrix

$$(\mathcal{E} \otimes \mathrm{id}_{\mathcal{M}_3})[\tilde{Y}] = \begin{bmatrix} \mathcal{E}(\tilde{A}) & X_1 & 0 \\ X_1^* & \mathcal{E}(\tilde{B}) & X_2 \\ 0 & X_2^* & \mathcal{E}(\tilde{C}) \end{bmatrix} \in \mathcal{M}_{3p}(\mathcal{A})$$

is strongly positive and the diagonal elements sum to $1 \in \mathcal{M}_{3p}(\mathcal{A})$. Thus, $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ satisfies the criterion of Theorem 2.5.10 for the weak expectation property. $\square$

Remark 3.6.4. The arguments to establish the main results of this chapter depend crucially on the fact that $\mathcal{A}$ is a unital C*-algebra, and it would be of interest to know to what extent such results remain true for non-unital C*-algebras.

Chapter 4

Relative weak injectivity in the operator system category

This chapter attempts to resolve a natural question about operator system tensor product inclusions in the case of commuting tensor product (a generalization of the *max* C*-tensor product to the operator system category). This is achieved by the generalization of the concept of relative weak injectivity for operator systems. The contents of this chapter appears in [1].

4.1 Motivation

As recent as in 2011, the theory of operator system tensor products was introduced in [12, 13] and various aspects systematically studied. Generalizing the *min* and *max* C*-tensor products to the operator system category was a cornerstone of this establishment and fundamental to the rich and extensive development of the theory therein. However, interestingly enough the *max* C*-tensor product has two distinct counterparts in the operator system category, namely the *max* and *commuting* operator system tensor products, which collapses to the same when either or both the operator systems under consideration is order isomorphic to a C*-algebra. This curious departure from the C*-algebra theory somewhat make the study of the maximal tensor products of the operator system stand out from their C*-algebra analogs. The category $\mathcal{O}_1$ comprises of operator systems as objects and unital completely positive maps as morphisms.

Recall that in Section 2.5, the case of *max* tensor product inclusion for C*-algebras were discussed. It was recorded that, in general it is not true that for a given C*-algebra pair $\mathcal{A}_1 \subset \mathcal{A}_2$, $\mathcal{A}_1 \otimes_{\max} \mathcal{B}$ is a C*-subalgebra of $\mathcal{A}_2 \otimes_{\max} \mathcal{B}$ for every other C*-algebra $\mathcal{B}$. Such a property holds true when $\mathcal{A}_1$ is *relatively weakly injective* in $\mathcal{A}_2$ (Section 2.6).

With the advent of the operator system tensor products, it is natural to ask the same question as above in the category $\mathcal{O}_1$. More precisely, one may ask : for an operator system pair $\mathcal{S} \subset \mathcal{T}$, is it true that $\mathcal{S} \otimes_\beta \mathcal{R}$ is an operator subsystem of $\mathcal{T} \otimes_\beta \mathcal{R}$, for any choice of an operator system $\mathcal{R}$, where β is the *min*, *commuting* or *max* operator system tensor product.

Owing to Theorem 2.7.3 and arguing exactly like as in Subsection 2.5.2, one sees that the answer to the question above is always affirmative for the case $\beta = min$. An attempt to answer the question for the cases of *commuting* and *max* tensor products leads to a generalization of the concept of relative weak injectivity to the category $\mathcal{O}_1$. However, it turns out that the *commuting* case is more tractable than the case of *max* in $\mathcal{O}_1$.

This chapter aims at settling the question of tensor product inclusions for the *commuting* case under a mild condition, while pointing out a major obstruction to the methods applied here for the case of the *max* tensor product in $\mathcal{O}_1$.

4.2 The *commuting* tensor product of operator systems

If $\mathcal{S}$ and $\mathcal{T}$ are operator systems, then the notation $\mathcal{S} \subset \mathcal{T}$ means that $\mathcal{S}$ is a unital operator subsystem of $\mathcal{T}$. That is, if $1_\mathcal{S}$ and $1_\mathcal{T}$ denote the distinguished Archimedean order units for $\mathcal{S}$ and $\mathcal{T}$ respectively, then $1_\mathcal{S} = 1_\mathcal{T}$. Unless the context is not clear, the order unit for an operator system will be denoted simply by 1.

If $\mathcal{S}_1 \subset \mathcal{T}_1$ and $\mathcal{S}_2 \subset \mathcal{T}_2$ are inclusions of operator systems, and if $\iota_j : \mathcal{S}_j \to \mathcal{T}_j$ are the inclusion maps, then for any operator system structures τ and σ on $\mathcal{S}_1 \otimes \mathcal{S}_2$ and

$\mathcal{T}_1 \otimes \mathcal{T}_2$, respectively, the notation (as used in [9] also)

$$\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \ \subset_+ \ \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$$

expresses the fact that the linear vector-space embedding $\iota_1 \otimes \iota_2 : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2$ is a ucp map $\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \to \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$. That is, $\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \ \subset_+ \ \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$ if and only if $M_n(\mathcal{S}_1 \otimes_\tau \mathcal{S}_2)_+ \subset M_n(\mathcal{T}_1 \otimes_\sigma \mathcal{T}_2)_+$ for every $n \in \mathbb{N}$. If, in addition, $\iota_1 \otimes \iota_2$ is a complete order isomorphism onto its range, then this is denoted by

$$\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_{\mathrm{coi}} \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2 \,.$$

Thus, $\mathcal{S} \otimes_\tau \mathcal{T} \ = \ \mathcal{S} \otimes_\sigma \mathcal{T}$ means $\mathcal{S} \otimes_\tau \mathcal{T} \subset_{\mathrm{coi}} \mathcal{S} \otimes_\sigma \mathcal{T}$ and $\mathcal{S} \otimes_\sigma \mathcal{T} \subset_{\mathrm{coi}} \mathcal{S} \otimes_\tau \mathcal{T}$.

The commuting operator system tensor product $\otimes_{\mathrm{c}}$ was introduced and studied in [12] and will be defined below. A slight simplification in the definition is afforded by the following lemma, which allows one to restrict to unital completely positive maps rather than use all completely positive maps.

Lemma 4.2.1. *(Unital vs non-unital completely positive maps, [4, Lemma 2.2], [6, Lemma 5.1.6]) Let $\mathcal{S} \subset \mathcal{B}(\mathcal{K})$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists a unital completely positive map $\tilde{\phi} : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ such that*

$$\phi(\cdot) = \phi(1)^{\frac{1}{2}} \tilde{\phi}(\cdot) \phi(1)^{\frac{1}{2}}.$$

Remark 4.2.2. The proof of the lemma above describes the map $\tilde{\phi}$ as a strong limit of $\tilde{\phi}_n$ in $\mathcal{B}(\mathcal{H})$, that is

$$\tilde{\phi} = \mathrm{SOT} - \lim_n \tilde{\phi}_n,$$

where

$$\tilde{\phi}_n : \mathcal{S} \to \mathcal{B}(\mathcal{H})$$

$$\tilde{\phi}_n(s) = \left(\phi(1) + \frac{1}{n}\right)^{\frac{1}{2}} \phi(s) \left(\phi(1) + \frac{1}{n}\right)^{-\frac{1}{2}} + \langle s\eta, \eta\rangle(1 - \mathrm{P}_{\phi(1)}),$$

for $\eta \in \mathcal{K}$, and $\mathrm{P}_{\phi(1)}$ is the projection onto the closure of the range of $\phi(1)$. Thus, for operator systems $\mathcal{S} \subset \mathcal{B}(\mathcal{K}_{\mathcal{S}})$ and $\mathcal{T} \subset \mathcal{B}(\mathcal{K}_{\mathcal{T}})$, if $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{T} \to \mathcal{B}(\mathcal{H})$ are completely positive maps with commuting ranges, then the projections $\mathrm{P}_{\phi(1)}$ and $\mathrm{P}_{\psi(1)}$ commute since $\phi(1)$ commutes with $\psi(1)$. Further, $\mathrm{P}_{\phi(1)}$ commutes with $\psi(1)$ and $\mathrm{P}_{\psi(1)}$ commutes with $\phi(1)$. This leads to the fact that the maps $\tilde{\phi}_n$ and $\tilde{\psi}_n$ (defined similarly) have commuting ranges for all $n \in \mathbb{N}$. As a consequence, the corresponding unital completely positive maps $\tilde{\phi}$ and $\tilde{\psi}$ which are strong limits of $\tilde{\phi}_n$'s and $\tilde{\psi}_n$'s respectively, also have commuting ranges.

Denote by $\mathrm{ucp}(\mathcal{S}, \mathcal{T})$ the set of all pairs (ϕ, ψ) of unital completely positive maps from $\mathcal{S}$ and $\mathcal{T}$, respectively, into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, such that $\phi(\mathcal{S})$ commutes with $\psi(\mathcal{T})$.

Recall that, for each $(\phi, \psi) \in \mathrm{ucp}(\mathcal{S}, \mathcal{T})$ let $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(\mathcal{H})$ be the unique linear map whose value on elementary tensors is given by

$$\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y).$$

Define cones by

$$\mathcal{C}_n^{\text{comm}} = \{\eta \in \mathcal{M}_n(\mathcal{S} \otimes \mathcal{T}) : (\phi \cdot \psi)^{(n)}(\eta) \geq 0, \text{ for all } (\phi, \psi) \in \text{ucp}(\mathcal{S}, \mathcal{T})\}.$$

It was shown in [12] that the collection of cones above is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with Archimedean matrix order unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$.

Definition 4.2.3. *(Commuting operator system tensor product) The operator system $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^{comm}\}_{n\in\mathbb{N}}, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ is called the commuting operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$ and is denoted by $\mathcal{S} \otimes_c \mathcal{T}$.*

Remark 4.2.4. The definition above considers all pairs of commuting *unital* completely positive maps. However, in [12], the definition of the *commuting* tensor product, all pairs of commuting completely positive maps were used. Nonetheless, by virtue of Remark 4.2.2, the two definitions are equivalent. More precisely, for $\eta \in \mathcal{M}_n(\mathcal{S} \otimes \mathcal{T})$, and for given completely positive maps ϕ_0 and ψ_0 with commuting ranges, $(\phi_0 \cdot \psi_0)^{(n)}(\eta) \geq 0$ if and only if their unital counterpart $(\tilde{\phi}_0 \cdot \tilde{\psi}_0)^{(n)}(\eta) \geq 0$. Thus the defining cones in our case and the ones in [12] are the same.

The following notation, introduced in [13], will be used.

Notation 4.2.5. If $\mathcal{X}$ and $\mathcal{Y}$ are operator systems, then $\mathcal{X}\hat{\otimes}_c\mathcal{Y}$ shall denote the norm-completion of $\mathcal{X} \otimes_c \mathcal{Y}$. For any subspaces $\mathcal{X}_0 \subset \mathcal{X}$ and $\mathcal{Y}_0 \subset \mathcal{Y}$, $\mathcal{X}_0\overline{\otimes}\mathcal{Y}_0$ denotes the closure of $\mathcal{X}_0\otimes\mathcal{Y}_0$ in $\mathcal{X}\hat{\otimes}_c\mathcal{Y}$.

As mentioned earlier, an important fact is : if two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are considered as operator systems, then $\mathcal{A}\hat{\otimes}_{\mathrm{c}}\mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ [12, Theorem 6.6].

In principle an abstract operator system $\mathcal{S}$ generates many different C*-algebras. The largest such C*-algebra is called the universal C*-algebra generated by $\mathcal{S}$. That is, a unital C*-algebra $\mathcal{A}$ is *universal* for $\mathcal{S}$ if :

1. there is a unital complete order injection $\iota_{\mathrm{u}} : \mathcal{S} \to \mathcal{A}$,
2. $\mathcal{A}$ is generated by $\iota_{\mathrm{u}}(\mathcal{S})$, and
3. if $\phi : \mathcal{S} \to \mathcal{B}$ is a ucp map into another C*-algebra $\mathcal{B}$, then there is a homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that $\phi = \pi \circ \iota_{\mathrm{u}}$.

It was shown in [15, Proposition 8] that every operator system has a universal C*-algebra, unique up to isomorphism, and an explicit construction was given. Therefore, $\mathrm{C}^*_u(\mathcal{S})$ shall unambiguously denote the universal C*-algebra generated by $\mathcal{S}$.

The following key fact facilitates the analysis carried out in this chapter.

Theorem 4.2.6. *(Structural inheritance of the commuting tensor product, [13, Lemma 2.5]) For all operator systems $\mathcal{S}$ and $\mathcal{T}$,*

$$\mathcal{S} \otimes_c \mathcal{T} \subset_{coi} \mathcal{S} \otimes_c C^*_u(\mathcal{T}) \subset_{coi} C^*_u(\mathcal{S}) \otimes_{max} C^*_u(\mathcal{T}) \,.$$

Corollary 4.2.7. *For every unital C*-algebra $\mathcal{A}$, operator system $\mathcal{S}$, and $n \in \mathbb{N}$, the operator systems $\mathcal{M}_n(\mathcal{S} \otimes_c \mathcal{A})$ and $\mathcal{S} \otimes_c \mathcal{M}_n(\mathcal{A})$ are completely order isomorphic.*

4.3 Relative weak injectivity in the operator system category

The definition of relative weak injectivity in the category of operator systems is given below :

Definition 4.3.1. *A pair* $(\mathcal{S}, \mathcal{T})$ *of operator systems is a relatively weakly injective pair if, for every operator system* $\mathcal{R}$*,* $\mathcal{S} \otimes_c \mathcal{R}$ *is a unital operator subsystem of* $\mathcal{T} \otimes_c \mathcal{R}$*, that is*

$$\mathcal{S} \otimes_c \mathcal{R} \subset_{coi} \mathcal{T} \otimes_c \mathcal{R}\,.$$

It is also convenient to say that $\mathcal{S}$ *is relatively weakly injective in* $\mathcal{T}$ if $(\mathcal{S}, \mathcal{T})$ is a relatively weakly injective pair.

Remark 4.3.2. Comparing the definition above with Definition 2.6.2, one observes that in the case of C*-algebras, an equivalent reformulation of the definition of relative weak injectivity in terms of *max* tensor product inclusion of C*-algebras is equally good due to Theorem 2.6.6.

4.4 Some preliminary results

In this section we will use the fact that the matricial order on an operator system $\mathcal{S}$ gives rise to a norm $\|\cdot\|_{\mathcal{M}_n(\mathcal{S})}$ on each matrix space $\mathcal{M}_n(\mathcal{S})$ (See page 8).

The following lemma is an obvious perturbation of [12, Corollary 6.5]. We include a proof for the convenience of the reader.

Lemma 4.4.1. *Let $\mathcal{S}$ be an operator system and $\mathcal{A}$ be a unital C^*-algebra. A linear map $\phi : \mathcal{S} \otimes_c \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a ucp map if and only if there is a Hilbert space $\mathcal{K}$, homomorphisms $\pi : C_u^*(\mathcal{S}) \to \mathcal{B}(\mathcal{K})$ and $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ with commuting ranges, and an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $\phi(s \otimes a) = V^*\pi(s)\rho(a)V$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}$.*

Proof. Because $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{A} \subset_{\mathrm{coi}} \mathrm{C}_u^*(\mathcal{S}) \otimes_{\max} \mathcal{A}$ by Proposition 4.2.6, ϕ admits a ucp extension $\Phi : \mathrm{C}_u^*(\mathcal{S}) \otimes_{\max} \mathcal{A} \to \mathcal{B}(\mathcal{H})$. A minimal Stinespring dilation implies the existence of a Hilbert space $\mathcal{K}$ and an isometry $V : \mathcal{H} \to \mathcal{K}$ and a homomorphism $\pi_0 : \mathrm{C}_u^*(\mathcal{S}) \otimes_{\max} \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that $\Phi(u) = V^*\pi_0(x)V$ for all $x \in \mathrm{C}_u^*(\mathcal{S}) \otimes_{\max} \mathcal{A}$. By Theorem 2.3.2, there exist representations $\pi : \mathrm{C}_u^*(\mathcal{S}) \to \mathcal{B}(\mathcal{K})$ and $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ with commuting ranges such that $\pi_0 = \pi \cdot \rho$. Evaluating on elementary tensors gives the required result.

Conversely, if π, ρ and V are as in the statement above, $\pi \cdot \rho$ is completely positive on $\mathrm{C}_u^*(\mathcal{S}) \otimes_{\max} \mathcal{A}$ and thus on $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{A}$. Conjugation by V gives the result. □

Lemma 4.4.2. *Let $\mathcal{S}$ be a operator system. Let $\{\mathcal{S}_i\}_{i \in \mathcal{I}}$ be the set of all separable non-trivial operator subsystems of $\mathcal{S}$ (that is, $\mathcal{S}_i \subset \mathcal{S}$). Then, there is a co-final ultrafilter $\mathcal{U}$ on $\mathcal{I}$ such that the map $\Psi : \mathcal{S} \to \prod^{\mathcal{U}} C_u^*(\mathcal{S}_i)$ given by*

$$x \longmapsto (\psi_i(x))_{\mathcal{U}},$$

where $\psi_i(x) = x$ if $x \in \mathcal{S}_i$ or 0 otherwise, is a unital completely positive linear map, where $\prod^{\mathcal{U}}$ denotes the C^*-ultraproduct.

Proof. Note that the set $\mathcal{I}$ is partially ordered by inclusion of the corresponding operator subsystems $\mathcal{S}_i$ and that $\mathcal{S} = \bigcup \mathcal{S}_i$. Consider a co-final ultrafilter $\mathcal{U}$ on the directed set $\mathcal{I}$. The map Ψ defined in the statement of the lemma is linear because of the structure of C*-ultraproducts (see [10] for relevant definitions of ultraproducts). To show that Ψ is ucp it is sufficient to show that Ψ is a complete isometry (following the discussion after [21, Remark 2.8.4]).

If $x \in \mathcal{S}$, note that the set $\{i \mid x \in \mathcal{S}_i\} \in \mathcal{U}$. To see this, simply observe that $\{i \mid x \in \mathcal{S}_i\} = \{i \mid i \geq i_x\}$, where $\mathcal{S}_{i_x} = \text{span}\{1, x, x^*\}$. Now, for $n = 1$,

$$\|\Psi(x)\| = \|(\psi_i(x))_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|\psi_i(x)\| = \|x\|$$

by the preceding comment.

For $n > 1$, we use a similar argument as follows. Let $X = (x_{kl}) \in M_n(\mathcal{S})$. Now, an ultrafilter is closed under finite intersections. So,

$$I_X = \{i \mid x_{kl} \in \mathcal{S}_i \ \forall \ k, l\} = \bigcap_{k,l} \{i \mid x_{kl} \in \mathcal{S}_i\}$$

is in $\mathcal{U}$. Finally, using the identification $M_n(\prod^{\mathcal{U}} \mathrm{C}^*_u(\mathcal{S}_i)) = \prod^{\mathcal{U}} M_n(\mathrm{C}^*_u(\mathcal{S}_i))$ (see Remark on page 60 of [21]) we obtain

$$\begin{aligned} \|\Psi^{(n)}(X)\| &= \|(\Psi(x_{kl}))_{k,l}\| = \|((\psi_i(x_{kl}))_{\mathcal{U}})_{k,l}\| = \|((\psi_i(x_{kl}))_{k,l})_{\mathcal{U}}\| \\ &= \lim_{\mathcal{U}} \|(\psi_i(x_{kl}))_{k,l}\| = \|(x_{kl})_{k,l}\|_{M_n(S_i), i \in I_X} = \|X\|, \end{aligned}$$

thereby showing that Ψ is a complete isometry. $\square$

The following result is of central importance in what follows.

Lemma 4.4.3. *Assume that $\mathcal{A}$ is a C^*-algebra and $\mathcal{T}$ is an operator system, and fix $x \in \mathcal{T} \otimes \mathcal{A}$. If $\{\mathcal{T}_i\}_{i \in \mathcal{I}(x)}$ is the directed set of all separable unital operator subsystems of $\mathcal{T}$ for which $x \in \mathcal{T}_i \otimes \mathcal{A}$, then*

$$\|x\|_{\mathcal{T} \otimes_c \mathcal{A}} = \lim_{\mathcal{I}(x)} \|x\|_{\mathcal{T}_i \otimes_c \mathcal{A}}.$$

Proof. Let us denote by $\|x\|_{(\cdot)}$ the norm $\|x\|_{(\cdot)\otimes_c \mathcal{A}}$. If $x \in \mathcal{T}_1 \subset \mathcal{T}_2$, then

$$\mathcal{T}_1 \otimes_c \mathcal{A} \subset_+ \mathcal{T}_2 \otimes_c \mathcal{A} \quad \text{implies that} \quad \|x\|_{\mathcal{T}_2} \leq \|x\|_{\mathcal{T}_1}.$$

Thus, $\lim_{\mathcal{I}} \|x\|_{\mathcal{T}_i}$ exists, since it is a decreasing net, and

$$\|x\|_{\mathcal{T}} \leq \lim_{\mathcal{I}} \|x\|_{\mathcal{T}_i}.$$

To establish the opposite inequality, following the techniques in the proof of [17, Proposition 3.4],we proceed as follows.

Assume that $\|x\|_{\mathcal{T}_i} \geq 1$ for all $i \in \mathcal{I}$. Thus, $\|x\|_{\mathcal{T}_i} = \|x\|_{C^*_u(\mathcal{T}_i)\otimes_{\max}\mathcal{A}} \geq 1$. Therefore, there exists representations π_i, ρ_i of $C^*_u(\mathcal{T}_i)$ and $\mathcal{A}$ respectively, on $\mathcal{B}(\mathcal{H}_i)$ with commuting ranges such that

$$\|\pi_i \cdot \rho_i(x)\| \geq 1.$$

Using the map Ψ from Lemma 4.4.2 above and the injective $*$-homomorphism $\iota : \mathcal{A} \hookrightarrow \prod^{\mathcal{U}} \mathcal{A}$, where $\mathcal{U}$ is the same ultrafilter over the same index set $\mathcal{I}$ as in Lemma 4.4.2 or above, we have unital completely positive maps $\phi : \mathcal{T} \to \mathcal{B}(\mathcal{H}_\mathcal{T})$ and $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\mathcal{T})$ with commuting ranges and such that

$$\|\phi \cdot \rho(x)\| \geq 1,$$

where $\mathcal{H}_\mathcal{T} = \prod^\mathcal{U} \mathcal{H}_i$, $\phi = (\prod^\mathcal{U} \pi_i) \circ \Psi$ and $\rho = (\prod^\mathcal{U} \rho_i) \circ \iota$.

Now, $\phi \cdot \rho$ is a unital completely positive map of $\mathcal{T} \otimes_{\mathrm{c}} \mathcal{A}$. By Lemma 4.4.1, there exist representations π_0 and ρ_0 of $\mathrm{C}^*_u(\mathcal{T})$ and $\mathcal{A}$ with commuting ranges and an isometry V such that

$$\phi \cdot \rho(x) = V^* \pi_0 \cdot \rho_0(x) V.$$

Since $\|\phi \cdot \rho(x)\| \geq 1$, we have $\|\pi_0 \cdot \rho_0(x)\| \geq 1$ because V is an isometry. But then,

$$\|x\|_\mathcal{T} = \|x\|_{\mathrm{C}^*_u(\mathcal{T}) \otimes_{\max} \mathcal{A}} \geq 1,$$

thereby showing that $\|x\|_\mathcal{T} = \lim_\mathcal{J} \|x\|_{\mathcal{T}_i}$. □

Remark 4.4.4. Lemma 4.4.3 is also true if $\mathcal{A}$ is only an operator system, as in that case, one may simply carry out the argument above with $\mathrm{C}^*_u(\mathcal{A})$ and arrive at the conclusion by virtue of Theorem 4.2.6.

Remark 4.4.5. Observe that at the beginning of the proof of Lemma 4.4.3, the inequality $\|x\|_{\mathcal{T}_2} \leq \|x\|_{\mathcal{T}_1}$ may not hold true if $1_{\mathcal{T}_1} \neq 1_{\mathcal{T}_2}$. This is where the unital subsystem criterion comes into play.

4.5 A characterization of relative weak injectivity for operator systems

Our first main result is an operator system version of Kirchberg's theorem [14, Proposition 3.1]. For a subset $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$, the double commutant of $\mathcal{X}$ in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{X}''$.

Theorem 4.5.1. *The following statements are equivalent for operator systems $\mathcal{S}$ and $\mathcal{T}$ for which $\mathcal{S} \subset \mathcal{T}$:*

1. *$(\mathcal{S}, \mathcal{T})$ is a relatively weakly injective pair of operator systems;*
2. *$\mathcal{S} \otimes_c C^*(\mathbb{F}_\infty) \subset_{coi} \mathcal{T} \otimes_c C^*(\mathbb{F}_\infty)$;*
3. *For any unital completely positive map $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$, there exist a unital completely positive map $\Phi : \mathcal{T} \to \phi(\mathcal{S})''$ such that $\Phi|_{\mathcal{S}} = \phi$;*
4. *$(C_u^*(\mathcal{S}), C_u^*(\mathcal{T}))$ is a relatively weakly injective pair of C^*-algebras.*

Proof. The order of implications to be proved is (4) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (2). Assume that $X \in \mathcal{M}_n(\mathcal{S} \otimes \mathrm{C}^*(\mathbb{F}_\infty))$ is positive in $\mathcal{M}_n(\mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty))$. We need to show that $X \in \mathcal{M}_n(\mathcal{S} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty))_+$. Because

$$X \in \mathcal{M}_n(\mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty))_+ \subset \mathcal{M}_n(\mathrm{C}_u^*(\mathcal{T}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty))_+ ,$$

hypothesis (4) implies $X \in \mathcal{M}_n(\mathrm{C}_u^*(\mathcal{S}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty))_+$, and so X is positive in $\mathcal{M}_n(\mathcal{S} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty))$ because $\mathcal{S} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty) \subset_{\mathrm{coi}} \mathrm{C}_u^*(\mathcal{S}) \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$.

(2) $\Rightarrow$ (1). Let $\mathcal{R}$ be an arbitrary operator system. By Theorem 4.2.6, $\mathcal{W} \otimes_{\rm c} \mathcal{R} \subset_{\rm coi} \mathcal{W} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R})$ for every operator system $\mathcal{W}$; thus, if we can show that $\mathcal{S} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R}) \subset_{\rm coi} \mathcal{T} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R})$, then we deduce immediately that $\mathcal{S} \otimes_{\rm c} \mathcal{R} \subset_{\rm coi} \mathcal{T} \otimes_{\rm c} \mathcal{R}$.

To begin, assume that $\mathcal{R}$ is separable. Hence, there is an ideal $\mathcal{K}$ of $\mathrm{C}^*(\mathbb{F}_\infty)$ such that $\mathrm{C}^*_u(\mathcal{R}) = \mathrm{C}^*(\mathbb{F}_\infty)/\mathcal{K}$. By [13, Corollary 5.17], and using Notation 4.2.5,

$$\mathcal{S} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R}) \subset_{\rm coi} \mathcal{S} \hat{\otimes}_{\rm c} \mathrm{C}^*_u(\mathcal{R}) \; = \; \frac{\mathcal{S} \hat{\otimes}_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty)}{\mathcal{S} \overline{\otimes} \mathcal{K}} \, .$$

The hypothesis $\mathcal{S} \otimes_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty) \subset_{\rm coi} \mathcal{T} \otimes_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty)$ implies that $\mathcal{S} \otimes_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty) \subset_{\rm coi} \mathrm{C}^*_u(\mathcal{T}) \otimes_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty)$, again by Theorem 4.2.6. Therefore, [13, Proposition 5.14] yields

$$\frac{\mathcal{S} \hat{\otimes}_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty)}{\mathcal{S} \overline{\otimes} \mathcal{K}} \; \subset_{\rm coi} \; \frac{\mathrm{C}^*_u(\mathcal{T}) \hat{\otimes}_{\rm c} \mathrm{C}^*(\mathbb{F}_\infty)}{\mathrm{C}^*_u(\mathcal{T}) \overline{\otimes} \mathcal{K}} \; = \; \mathrm{C}^*_u(\mathcal{T}) \otimes_{\max} \mathrm{C}^*_u(\mathcal{R}) \, .$$

Thus, $\mathcal{S} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R}) \subset_{\rm coi} \mathrm{C}^*_u(\mathcal{T}) \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R})$, which implies $\mathcal{S} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R}) \subset_{\rm coi} \mathcal{T} \otimes_{\rm c} \mathrm{C}^*_u(\mathcal{R})$ and, hence, $\mathcal{S} \otimes_{\rm c} \mathcal{R} \subset_{\rm coi} \mathcal{T} \otimes_{\rm c} \mathcal{R}$.

Now assume that $\mathcal{R}$ is an arbitrary nonseparable operator system. We have proved above that $\mathcal{S} \otimes_{\rm c} \mathcal{R}_0 \subset_{\rm coi} \mathcal{T} \otimes_{\rm c} \mathcal{R}_0$ for every separable operator system $\mathcal{R}_0$. Fix $x \in \mathcal{S} \otimes \mathcal{R}$ and choose a separable operator subsystem $\mathcal{R}_1 \subset \mathcal{R}$ such that $x \in \mathcal{S} \otimes \mathcal{R}_1$. Thus, $\mathcal{S} \otimes_{\rm c} \mathcal{R}_1 \subset \mathcal{T} \otimes_{\rm c} \mathcal{R}_1$. By the beginning of the proof of Lemma 4.4.3 we have the inequality

$$\|x\|_{\mathcal{S} \otimes_{\rm c} \mathcal{R}} \leq \|x\|_{\mathcal{S} \otimes_{\rm c} \mathcal{R}_1} = \|x\|_{\mathcal{T} \otimes_{\rm c} \mathcal{R}_1} .$$

This inequality above holds for any separable operator subsystem $\mathcal{R}_1 \subset \mathcal{R}$ for which $x \in \mathcal{S} \otimes \mathcal{R}_1$. Lemma 4.4.3 (or Remark 4.4.4) thus implies $\|x\|_{\mathcal{S} \otimes_{\rm c} \mathcal{R}} \leq \|x\|_{\mathcal{T} \otimes_{\rm c} \mathcal{R}}$, which in turn implies

$$\|x\|_{\mathcal{S}\otimes_{\mathrm{c}}\mathcal{R}} = \|x\|_{\mathcal{T}\otimes_{\mathrm{c}}\mathcal{R}}.$$

Next, for $n > 1$, fix $X \in \mathcal{M}_n(\mathcal{S} \otimes \mathcal{R}) \subset \mathcal{M}_n(\mathcal{S} \otimes \mathrm{C}^*_u(\mathcal{R})) \cong \mathcal{S} \otimes \mathcal{M}_n(\mathrm{C}^*_u(\mathcal{R}))$. One also has $\mathcal{M}_n(\mathcal{S} \otimes_{\mathrm{c}} \mathrm{C}^*_u(\mathcal{R})) \cong \mathcal{S} \otimes_{\mathrm{c}} \mathcal{M}_n(\mathrm{C}^*_u(\mathcal{R}))$ (see [13, Theorem 7.1]). Now, just as in the case $n = 1$, there exists a separable operator system $\mathcal{R}_n^0 \subset \mathcal{M}_n(\mathrm{C}^*_u(\mathcal{R}))$ such that $X \in \mathcal{S} \otimes \mathcal{R}_n^0$ and therefore, for any separable operator system $\mathcal{R}_n \subset \mathcal{M}_n(\mathrm{C}^*_u(\mathcal{R}))$ for which $X \in \mathcal{S} \otimes \mathcal{R}_n$, we have the inequality

$$\|X\|_{\mathcal{M}_n(\mathcal{S}\otimes_{\mathrm{c}}\mathrm{C}^*_u(\mathcal{R}))} = \|X\|_{\mathcal{S}\otimes_{\mathrm{c}}\mathcal{M}_n(\mathrm{C}^*_u(\mathcal{R}))} \leq \|X\|_{\mathcal{S}\otimes_{\mathrm{c}}\mathcal{R}_n} = \|X\|_{\mathcal{T}\otimes_{\mathrm{c}}\mathcal{R}_n}.$$

This implies (as in case of $n = 1$) that

$$\|X\|_{\mathcal{M}_n(\mathcal{S}\otimes_{\mathrm{c}}\mathrm{C}^*_u(\mathcal{R}))} \leq \|X\|_{\mathcal{T}\otimes_{\mathrm{c}}\mathcal{M}_n(\mathrm{C}^*_u(\mathcal{R}))} = \|X\|_{\mathcal{M}_n(\mathcal{T}\otimes_{\mathrm{c}}\mathrm{C}^*_u(\mathcal{R}))},$$

which in turn implies that $\|X\|_{\mathcal{M}_n(\mathcal{S}\otimes_{\mathrm{c}}\mathrm{C}^*_u(\mathcal{R}))} = \|X\|_{\mathcal{M}_n(\mathcal{T}\otimes_{\mathrm{c}}\mathrm{C}^*_u(\mathcal{R}))}$. That is, the inclusion map $\mathcal{S} \otimes \mathcal{R} \to \mathcal{T} \otimes \mathcal{R}$ is a unital complete isometry $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{R} \to \mathcal{T} \otimes_{\mathrm{c}} \mathcal{R}$ and, hence, is a complete order injection.

(1) $\Rightarrow$ (3). Let $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a unital completely positive map. Since $(\mathcal{S}, \mathcal{T})$ is a relatively weakly injective pair, and because the commutant $\phi(\mathcal{S})' \subset \mathcal{B}(\mathcal{H})$ of $\phi(\mathcal{S})$ is a C*-algebra,

$$\mathcal{S} \otimes_{\mathrm{c}} \phi(\mathcal{S})' \subset_{\mathrm{coi}} \mathcal{T} \otimes_{\mathrm{c}} \phi(\mathcal{S})' \subset_{\mathrm{coi}} \mathrm{C}^*_u(\mathcal{T}) \otimes_{\max} \phi(\mathcal{S})'.$$

By the definition of commuting tensor product, $\phi \cdot \mathrm{id}_{\phi(\mathcal{S})'}$ is a unital completely positive map on $\mathcal{S} \otimes_{\mathrm{c}} \phi(\mathcal{S})'$ with values in $\mathcal{B}(\mathcal{H})$. Take an Arveson extension Ψ of $\phi \cdot \mathrm{id}_{\phi(\mathcal{S})'}$ to

$\mathrm{C}_u^*(\mathcal{T}) \otimes_{\max} \phi(\mathcal{S})'$ and define a unital completely positive map Φ on $\mathcal{T}$ by

$$\Phi(t) = \Psi(t \otimes 1),$$

for all $t \in \mathcal{T}$. Obviously, $\Phi|_{\mathcal{S}} = \phi$. Finally, to see that Φ takes values in $\phi(\mathcal{S})''$, one invokes the usual multiplicative domain argument for completely positive maps as given below. For $f \in \phi(\mathcal{S})'$, one has $f = \Psi(1 \otimes f)$ and compute :

$$\begin{aligned} \Phi(t)f = \Psi(t \otimes 1)\Psi(1 \otimes f) &= \Psi((t \otimes 1)(1 \otimes f)) \\ &= \Psi((1 \otimes f)(t \otimes 1)) \\ &= \Psi(1 \otimes f)\Psi(t \otimes 1) \\ &= f\Psi(t). \end{aligned}$$

This concludes our claim that (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4). Since $\mathcal{S} \subset \mathcal{T}$, $\mathrm{C}_u^*(\mathcal{S})$ is a unital C*-subalgebra of $\mathrm{C}_u^*(\mathcal{T})$ [15, Proposition 9]. Let $\pi_U : \mathrm{C}_u^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}_U)$ be the universal representation of $\mathrm{C}_u^*(\mathcal{S})$. Then $\pi_U|_{\mathcal{S}} : \mathcal{S} \to \mathcal{B}(\mathcal{H}_U)$ is a unital completely positive map. By hypothesis, $\pi_U|_{\mathcal{S}}$ extends to $\phi : \mathcal{T} \to (\pi_U|_{\mathcal{S}}(\mathcal{S}))'' \subset (\pi_U(\mathrm{C}_u^*(\mathcal{S})))''$. Now, since $\mathrm{C}_u^*(\mathcal{T})$ is generated as an algebra by $\mathcal{T}$, the unique homomorphism from $\mathrm{C}_u^*(\mathcal{T})$ extending ϕ takes values in $(\pi_U(\mathrm{C}_u^*(\mathcal{S})))''$. Further, since this homomorphism extends $\pi_U|_{\mathcal{S}}$, it fixes π_U, which completes the proof. $\square$

4.6 Existence of relatively weakly injective pairs of operator systems

Our second main result shows the abundant existence of pairs of relatively weakly injective operator systems and is a generalisation of [14, Lemma 3.4].

Theorem 4.6.1. *If $\mathcal{S}$ is a separable unital operator subsystem of an operator system $\mathcal{T}$, then there exists a separable operator system $\mathcal{R}$ such that $\mathcal{S} \subset \mathcal{R} \subset \mathcal{T}$ and $\mathcal{R}$ is relatively weakly injective in $\mathcal{T}$.*

Proof. Let $\{s_k\}_{k\in\mathbb{N}}$ be a dense sequence in $\mathcal{S} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$. Using Lemma 4.4.3, we choose separable operator subsystems $\mathcal{S}_n$ of $\mathcal{T}$ such that, $\mathcal{S} \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots$ and $\|s_k\|_{\mathcal{S}_n} \leq \|s_k\|_{\mathcal{T}} + \frac{1}{n}$ for $1 \leq k \leq n$. Let $\mathcal{S}^{(1)} = \overline{\bigcup \mathcal{S}_i}$. Then $\mathcal{S}^{(1)}$ is a separable operator system containing $\mathcal{S}$, such that, for all $x \in \mathcal{S} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$, one has $\|x\|_{\mathcal{S}^{(1)}} = \|x\|_{\mathcal{T}}$. By iterating the argument above with $\mathcal{S}^{(1)}$ instead of $\mathcal{S}$ we obtain a sequence of separable operator systems $\mathcal{S} \subset \mathcal{S}^{(1)} \subset \mathcal{S}^{(2)} \subset \ldots$ such that $\|\cdot\|_{\mathcal{S}^{(n)}} = \|\cdot\|_{\mathcal{T}}$ on $\mathcal{S}^{(n-1)} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$. Define $\mathcal{X}_1 = \overline{\bigcup \mathcal{S}^{(k)}}$. Thus, $\mathcal{X}_1$ is a separable operator system containing $\mathcal{S}$ such that $\|\cdot\|_{\mathcal{X}_1} = \|\cdot\|_{\mathcal{T}}$.

Replacing $\mathcal{X}_1$ for $\mathcal{S}$ and $\mathcal{M}_2(\mathrm{C}^*(\mathbb{F}_\infty))$ for $\mathrm{C}^*(\mathbb{F}_\infty)$, repeat the procedure described above to obtain a separable operator system $\mathcal{X}_2$ such that, for all $x \in \mathcal{X}_2 \otimes_{\mathrm{c}} \mathcal{M}_2(\mathrm{C}^*(\mathbb{F}_\infty))$, we have

$$\|x\|_{\mathcal{X}_2 \otimes_{\mathrm{c}} \mathcal{M}_2(\mathrm{C}^*(\mathbb{F}_\infty))} = \|x\|_{\mathcal{T} \otimes_{\mathrm{c}} \mathcal{M}_2(\mathrm{C}^*(\mathbb{F}_\infty))}.$$

In other words, using the identification $\mathcal{W} \otimes_{\mathrm{c}} \mathcal{M}_2(\mathrm{C}^*(\mathbb{F}_\infty)) = \mathcal{M}_2(\mathcal{W} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty))$ for operator systems $\mathcal{W}$, we have that the inclusion map $\mathcal{X}_2 \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty) \to \mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$ is a 2-isometry.

Further iterations of the procedure above gives us $\mathcal{S} \subset \mathcal{X}_1 \subset \mathcal{X}_2 \subset \mathcal{X}_3 \subset \ldots \mathcal{T}$ such that the inclusion map $\mathcal{X}_k \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty) \to \mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$ is a k-isometry.

Finally, set $\mathcal{R} = \bigcup \mathcal{X}_k$. To show that $\mathcal{R}$ is relatively weakly injective in $\mathcal{T}$, it is enough, by Theorem 4.5.1, to show that the inclusion map $\mathcal{R} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty) \to \mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$ is a complete isometry.

For $Y \in \mathcal{R} \otimes \mathcal{M}_n(\mathrm{C}^*(\mathbb{F}_\infty))$ there exists an integer $k_Y > n$ such that $Y \in \mathcal{X}_k \otimes \mathcal{M}_n(\mathrm{C}^*(\mathbb{F}_\infty))$ for all $k > k_Y$. Now recall the fact that the inclusion maps $\mathcal{X}_k \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty) \to \mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$ are k-isometries. As a consequence, for $n < k_Y < k$ the inclusions $\mathcal{X}_k \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty) \to \mathcal{T} \otimes_{\mathrm{c}} \mathrm{C}^*(\mathbb{F}_\infty)$ are also n-isometries. Therefore, by Lemma 4.4.3 we have

$$\begin{aligned}
\|Y\|_{\mathcal{R} \otimes_{\mathrm{c}} \mathcal{M}_n(\mathrm{C}^*(\mathbb{F}_\infty))} &= \lim_k \|Y\|_{\mathcal{X}_k \otimes_{\mathrm{c}} \mathcal{M}_n(\mathrm{C}^*(\mathbb{F}_\infty))} \\
&= \lim_{k > k_Y} \|Y\|_{\mathcal{X}_k \otimes_{\mathrm{c}} \mathcal{M}_n(\mathrm{C}^*(\mathbb{F}_\infty))} \\
&= \|Y\|_{\mathcal{T} \otimes_{\mathrm{c}} \mathcal{M}_n(\mathrm{C}^*(\mathbb{F}_\infty))}.
\end{aligned}$$

This shows that $\mathcal{R}$ is relatively weakly injective in $\mathcal{T}$, contains $\mathcal{S}$, and is separable, thereby concluding the proof. $\square$

Remark 4.6.2. Several aspects of the proofs of the preliminary and main results depend in the fact that the *commuting* tensor product cones are inherited from the positive cones of a C*-*max* tensor product. This is not true for the *operator system max* tensor product, wherein lies the problem of applicability of the techniques employed here to study relative weak injectivity with respect to the *operator system max* tensor product.

4.7 Examples

4.7.1 Operator systems generated by free unitaries

Denote the generators of the free group $\mathbb{F}_\infty$ by $\{u_j\}_{j\in\mathbb{N}}$. In $\mathrm{C}^*(\mathbb{F}_\infty)$, each u_j is a unitary and so, for each $n \in \mathbb{N}$, define

$$\mathbb{S}_n = \mathrm{span}\{u_{-n}, \ldots, u_{-1}, 1, u_1, \ldots, u_n\},$$

which is an operator subsystem of $\mathrm{C}^*(\mathbb{F}_n)) \subset \mathrm{C}^*(\mathbb{F}_\infty)$.

Example 4.7.1. *If $n < m$, then $(\mathbb{S}_n, \mathbb{S}_m)$ is a relatively weakly injective pair of operator systems.*

The proof of this assertion is adapted from the proof of [8, Lemma 4.1] and makes use of our main result, Theorem 4.5.1. Let $\phi : \mathbb{S}_n \to \mathcal{B}(\mathcal{H})$ be a unital completely positive map. By Theorem 4.5.1, it is enough to show that ϕ extends to $\mathbb{S}_m$, taking values in $\phi(\mathbb{S}_n)''$. For each contraction $\phi(u_i)$, $i \in \mathbb{N}$, consider its Halmos unitary dilation W_i on $\mathcal{H} \oplus \mathcal{H}$ given by

$$W_i = \begin{bmatrix} \phi(u_i) & (1 - \phi(u_i)\phi(u_{-i}))^{\frac{1}{2}} \\ (1 - \phi(u_{-i})\phi(u_i))^{\frac{1}{2}} & -\phi(u_{-i}) \end{bmatrix}$$

Let $T \in \phi(\mathbb{S}_n)'$ and consider the operator $\tilde{T} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Now, by functional calculus, $\tilde{T}$ commutes with W_i for all $1 \leq i \leq n$. Since $u_1, \dots, u_m$ are universal unitaries in $\mathrm{C}^*(\mathbb{F}_m) \subset \mathrm{C}^*(\mathbb{F}_\infty)$, there is a unique homomorphism $\pi : \mathrm{C}^*(\mathbb{F}_m)) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, such that $\pi(u_i) = W_i$, for $1 \leq i < n$, and $\pi(u_j) = W_n$ for $n \leq j \leq m$. Let $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Define unital completely positive map $\tilde{\phi} : \mathrm{C}^*(\mathbb{F}_m) \to \mathcal{B}(\mathcal{H})$ by $\tilde{\phi}(\cdot) = P\pi(\cdot)|_{\mathcal{H}}$. Note that, $\tilde{\phi}$ extends ϕ and $\tilde{T}$ commutes with P. Since, $\tilde{T}$ commutes with every W_i, it commutes with $\pi(\mathrm{C}^*(\mathbb{F}_m)))$. Thus, for $x \in \mathbb{S}_m$ we have

$$\tilde{\phi}(x)T = P\pi(x)P\tilde{T}P = P\pi(x)\tilde{T}P = P\tilde{T}\pi(x)P = P\tilde{T}P\pi(x)P = T\tilde{\phi}(x).$$

So, $\tilde{\phi}(x) \in \phi(\mathbb{S}_n)''$ as T was chosen arbitrarily in $\phi(\mathbb{S}_n)'$. This concludes our claim.

4.7.2 Inclusion in the double dual

The dual $\mathcal{S}^*$ of an operator system is a matricially normed space, but the double dual $\mathcal{S}^{**}$ is an operator system containing $\mathcal{S}$ as an operator subsystem [13]. The following example is established in [13, Corollary 6.6].

Example 4.7.2. $(\mathcal{S}, \mathcal{S}^{**})$ *is a relatively weakly injective pair of operator systems, for every operator system* $\mathcal{S}$*.*

4.7.3 Operator systems with DCEP

An operator system $\mathcal{S}$ is said to have the *double commutant expectation property* (DCEP) if, for every complete order embedding $\mathcal{S} \to \mathcal{B}(\mathcal{H})$, there exists a completely positive linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{S}'' \subset \mathcal{B}(\mathcal{H})$, fixing $\mathcal{S}$.

Example 4.7.3. *If* $\mathcal{S}$ *has the double commutant expectation property, then* $(\mathcal{S}, \mathcal{T})$ *is a relatively weakly injective pair of operator systems, for every operator system* $\mathcal{T}$ *that contains* $\mathcal{S}$ *as an operator subsystem.*

The assertion above is a consequence of [13, Theorem 7.3, Theorem 7.1], which states that if $\mathcal{S} \subset \mathcal{T}$ and $\mathcal{S}$ has the double commutant expectation property, then $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{R} \subset_{\mathrm{coi}} \mathcal{T} \otimes_{\mathrm{c}} \mathcal{R}$ for every operator system $\mathcal{R}$.

Chapter 5

Conclusion and further questions

5.1 Summary

This dissertation delves on multiple facets of weak expectations and related concepts. In the first half of this thesis we investigated a permanence result of unital C*-algebras with the weak expectation property. For a unital C*-algebra $\mathcal{A}$, we showed that for either an amenable group or a group with amenable action, say G, the crossed product C*-algebra $\mathcal{A} \rtimes G$ has the weak expectation property if and only if $\mathcal{A}$ has the weak expectation property. Two distinct methods were applied for proving the two cases. Similar results were known for *nuclear* and *exact* C*-algebras. So, our result generalizes those known results to the case of C*-algebras with the weak expectation property, which are a substantially larger class of C*-algebras than the *nuclear* C*-algebras.

The second half of this dissertation formulates and studies the concept of relative weak injectivity in the category of operator systems. Relative weak injectivity for C*-algebras characterizes C*-algebraic inclusions, say $\mathcal{A} \subset \mathcal{B}$, which respects the *max* C*-tensor product inclusion, that is, for any C*-algebra $\mathcal{C}$,

$$\mathcal{A} \otimes_{\max} \mathcal{C} \subset \mathcal{B} \otimes_{\max} \mathcal{C}$$

if and only if $\mathcal{A}$ is relatively weakly injective in $\mathcal{B}$ or $(\mathcal{A}, \mathcal{B})$ is a relatively weakly injective pair. In the operator system scenario, the maximal tensor product has two distinct counterparts, the *commuting tensor product* and the *operator system max tensor product*, which reduces to the same if one of the operator system in the tensor product is a C*-algebra. We formulate this property for the *commuting* tensor product of operator systems and prove that the C*-algebra of the free group on countably many generators characterizes this formulation, just like in the case of C*-algebras. In other words, we show that for a unital inclusion of operator systems $\mathcal{S} \subset \mathcal{T}$, and for every other operator system $\mathcal{R}$,

$$\mathcal{S} \otimes_{\mathrm{c}} \mathcal{R} \subset_{\mathrm{coi}} \mathcal{T} \otimes_{\mathrm{c}} \mathcal{R}$$

is true if and only if

$$\mathcal{S} \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty) \subset_{\mathrm{coi}} \mathcal{T} \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty).$$

Further, we extend results on the existence of relatively weakly injective pairs of C*-algebras to the operator system pair setting and show that, for any operator system inclusion $\mathcal{S}_0 \subset \mathcal{T}$, where $\mathcal{S}_0$ is separable, there exist a *separable* operator system

$$\mathcal{S}_0 \subset \mathcal{S} \subset \mathcal{T},$$

such that $(\mathcal{S}, \mathcal{T})$ is a relatively weakly injective pair, where each of the above inclusions are unital. Finally, we mention a few naturally occurring examples to consolidate the validity of this study.

5.2 Further questions and methodology

Several research questions culminating in possible future work are outlined below.

Constructing new C*-algebras with the weak expectation property from old C*-algebras with the weak expectation property is a topic explored in this thesis in the form of crossed product C*-algebras by groups with amenable actions. Various constructions have already been investigated in detail for the cases of nuclear and exact C*-algebras. One such is the construction of continuous bundles of nuclear or exact C*-algebras due to E. Kirchberg and S. Wassermann. A natural question which presents itself is the construction of continuous bundles of C*-algebras with the weak expectation property. More precisely, for a given family of C*-algebras (the fiber C*-algebras) with the weak expectation property, one may aim to determine if the bundle C*-algebra thus constructed (possibly under some equivalent conditions) has the weak expectation property or not. In a recent paper [7], D. Farenick and V. Paulsen studied an operator matrix completion problem, as an application of which a new characterization of the C*-algebras with the weak expectation property was obtained. One may attempt to study conditions under which such a matrix completion problem may have an affirmative solution over a continuous bundle of C*-algebras, where each of the fiber C*-algebras has the aforementioned matrix completion property. An application of this, if answered in the positive, is likely to answer the question posed above.

It is known that the multiplier algebra of a C*- algebra with a strictly positive element has the weak expectation property if and only if the original C*-algebra has the weak expectation property. A C*-algebra sits as an *essential* ideal [3, Definition 8.4.1] in its multiplier algebra. As shown in this thesis, an ideal (in particular, the *essential* ones) always has the weak expectation property if the parent C*-algebra does. Conversely, the extent to which the study of *essential* ideals affects a C*-algebra with the weak expectation property is currently unknown. More precisely, is it true that, if all *essential* ideals of a C*-algebra have the weak expectation property, then the C*-algebra has weak expectation property? The definition of weak expectation property involves the existence of "weak expectations" for every non-degenerate representation of a C*-algebra. A partial result that, if an *essential* ideal of a C*-algebra has the weak expectation property, then every representation of the C*-algebra has a sub-representation admitting a weak expectation, is recorded below :

Proposition 5.2.1. *Let J be a essential ideal in a C*-algebra A. If J has the weak* *expectation property, then every faithful non-degenerate representation of A has a faithful non-degenerate sub-representation which admits a weak expectation.*

Proof. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a faithful non-degenerate representation of A. Now, consider $\mathcal{H}_0 = \overline{\pi(J)\mathcal{H}}$. Since J is an ideal of A, the subspace $\mathcal{H}_0$ is invariant for $\pi(A)$. Thus, π decomposes as $\pi = \pi_0 \oplus \pi_0^{\perp}$ on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$.

Since $\pi_0^{\perp}(J) = 0$, we have that $\pi_o|_J = \pi|_J$. Thus, $\pi_0|_J$ is a faithful representation of J in $\mathcal{B}(\mathcal{H}_0)$ (and incidentally in $\mathcal{B}(\mathcal{H})$ also).

We have the obvious inclusion $\pi_0(J) \trianglelefteq \pi_0(A) \subseteq \mathcal{B}(\mathcal{H}_0)$. Since J has WEP, there exists a ucp map $\Phi_0 : \mathcal{B}(\mathcal{H}_0) \to \pi_0(J)'' \subseteq \pi_0(A)''$ extending the identity on $\pi_0(J)$. Thus we may consider Φ_0 as taking values in $\pi_0(A)''$. Next, we show that $\Phi_0|_{\pi_0(A)} = \mathrm{id}_{\pi_0(A)}$ as below :

$$\Phi_0(\pi_0(a)\pi_0(j)) = \Phi_0(\pi_0(aj)) = \pi_0(aj) = \pi_0(a)\pi_0(j).$$

Now, $\pi_0(j)$ is in the multiplicative domain of Φ_0. So, one has :

$$\Phi_0(\pi_0(a)\pi_0(j)) = \Phi_0(\pi_0(a))\Phi_0(\pi_0(j)) = \Phi_0(\pi_0(a))\pi_0(j).$$

Combining the above two equations we get

$$(\Phi_0(\pi_0(a)) - \pi_0(a))\pi_0(j) = 0$$

for all $j \in J$, which shows that $\Phi_0|_{\pi_0(A)} = \mathrm{id}_{\pi_0(A)}$.

Finally, to complete the proof, we show that π_0 is a faithful representation of A on $\mathcal{H}_0$. To this end, we proceed as follows.

For $a \in A, a > 0$, consider :

$$\pi(a) = \pi_0(a) \oplus \pi_0^{\perp}(a)$$

For $\lambda \in \mathbb{C}$, $\lambda \in \sigma(a) = \sigma(\pi(a))$. The inverse of an element in $\pi(A)$, if exists, must be in $\pi(A)$, since a C*-algebra is inverse closed. Thus, it must be of the form $\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$. Thus $\pi(a) - \lambda I_{\mathcal{H}}$ is not invertible implies, either or neither $\pi_0(a) - \lambda I_{\mathcal{H}_0}$ and $\pi_0^{\perp}(a) - \lambda I_{\mathcal{H}_0^{\perp}}$ are not invertible. This means that $\lambda \in \sigma(\pi_0(a)) \cup \sigma(\pi_0^{\perp}(a))$. Thus, $\sigma(\pi(a)) = \sigma(\pi_0(a)) \cup \sigma(\pi_0^{\perp}(a))$.

Next, suppose $\exists\ \lambda_0 \in \sigma(\pi_0^{\perp}(a)) \setminus \sigma(\pi_0(a))$. By Urysohn's lemma, there exists a continuous function f_0 on $\sigma(\pi(a))$ such that $f_0(\lambda_0) = 1$ and $f_0(\sigma(\pi_0(a))) = 0$. Define $a_0 = f_0(a)$. Note that $a_0 \neq 0$ and $\pi(a_0) = \begin{pmatrix} 0 & 0 \\ 0 & \pi_0^{\perp}(a_0) \end{pmatrix}$.

The ideal K_0 generated by $\pi(a_0)$ given by $\pi(a_0 A a_0)$ is such that $K_0 \cap \pi(J) = \{0\}$, which is a contradiction, for $\pi(J)$ is an essential ideal in $\pi(A)$. Thus, we have $\sigma(\pi_0^{\perp}(a)) \subseteq \sigma(\pi_0(a))$, which implies $\sigma(a) = \sigma(\pi(a)) = \sigma(\pi_0(a))$. Therefore,

$$\|a\| = \|\pi_0(a)\|$$

for all $a > 0$, and hence for all $a \in A$ in general. □

An affirmative progress towards this direction is likely to extend the permanence property from the multiplier algebra of a C*-algebra scenario to the case of the "local multiplier algebra" of a C*-algebra with the weak expectation property.

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