

COMMUTATIVE DIFFERENTIAL GRADED ALGEBRA
WITH LATTICES

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Xiyuan Wang

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UNIVERSITY OF REGINA
FACULTY OF GRADUATE STUDIES AND RESEARCH
SUPERVISORY AND EXAMINING COMMITTEE

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External Examiner: Dr. Nader Mobed, Department of Physics

Supervisor: Dr. Donald Stanley, Department of Mathematics & Statistics

Committee Member: Dr. Bruce Gilligan, Department of Mathematics & Statistics

Committee Member: Dr. Soumen Sarkar, Department of Mathematics & Statistics

Chair of Defense: Dr. David Gerhard, Department of Computer Science

Abstract

We define an invariant called the “commutative differential graded algebra with lattice” of a topological space X . The commutative differential graded algebra with lattice encodes the information of rational homotopy type of X and the “lattice structure of X ”. We compute the quasi-isomorphism classes of several commutative differential graded algebra with lattices. Moreover, we define an integral Sullivan model. And we prove that, under some conditions, a commutative differential graded algebra with lattice has an integral Sullivan model.

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Notations and formulae

We will use the following notations in this work:

\mathbb{Z} : the ring of integers;

\mathbb{Q} : the rational number field;

\mathbb{R} : the real number field;

$\mathbb{Z}^{\geq 0}$: the non-negative integers;

X : the topological space;

\mathbb{R} : a commutative ring with identity;

σ_n : the standard topological n -simplex;

$S(X)$: the singular set of X ;

$S_{\bullet}(X; \mathbb{R})$: the singular chain complex of X ;

$S^{\bullet}(X; \mathbb{R})$: the singular cochain complex of X ;

$\Omega^*(X)$: the de Rham complex of a smooth manifold X ;

$H^n(M)$: the n th cohomology group of a cochain complex M ;

$H^*(X; \mathbb{R})$: the singular cohomology ring of X ;

Top: the category of topological space;

DGA: the category of differential graded algebras;

CDGA: the category of commutative differential graded algebras;

CGA + L: the category of commutative graded algebra with lattices;

CDGA + L: the category of commutative differential graded algebra with lattices;

Δ : the ordinary number caetegory;

sSets: the category of simplicial sets.

ΛV : the free commutative graded algebra generated by graded vector space V ;

$A_{PL}(X)$: the piecewise-linear de Rham forms on X ;

$K(G, n)$: the Eilenberg Maclane space;

$\mathbb{P}^n(B)$: the weighted projective space with weighted vectors B .

Chapter 1

Introduction

D. Sullivan shows that the diffeomorphism type of a simply connected closed manifold X of dimension greater than four is determined by three different invariants in his paper [13, Theorem 13.1]. The first one is the rational homotopy type of X . In rational homotopy theory, the rational homotopy type of a topological space X is closely related to the commutative differential graded algebra $A_{PL}(X)$. The $A_{PL}(X)$ encodes all information of the rational homotopy type of X . The second one is the “lattice structure of X ”. And the third one is the Pontrjagin classes of X .

The goal of this paper is to construct a new kind of invariant of topological space X . We call it “commutative differential graded algebra with lattice”. This invariant encodes the rational homotopy type of X as well as the lattice structure of X .

In our context, we provide some necessary preliminary knowledge of commutative differential graded algebras in Chapter 2. We first consider the basic definitions and properties

of commutative differential graded algebras. Then we discuss Sullivan algebras and Sullivan models with some examples. The minimal Sullivan model is a powerful tool for studying the topological space X . It captures the rational homotopy type of X . Lastly, we introduce model categories. The category of commutative differential graded algebras has a model category structure. Actually, there is an equivalence of subcategories of homotopy categories of topological spaces on the one hand and commutative differential graded algebras on the other hand.

In Chapter 3, we turn to the relation between the rational homotopy type of a topological space X and the commutative differential graded algebras. We first give a brief introduction to simplicial sets and show some examples of it. Then we define a functor A_{PL} from the category of rational homotopy types of simply connected, finite-type space to the category of commutative differential graded algebras over \mathbb{Q} . And we compute some minimal Sullivan model of $A_{PL}(X)$. At the end of the chapter 3, we discuss the relation between Postnikov towers and minimal Sullivan algebras.

In Chapter 4, we first discuss the lattice structure of a topological space X . Then we define the category of commutative differential graded algebra with lattices. And we define a functor from the category of simply connected topological spaces to the category of commutative differential graded algebra with lattices. We compute some examples of commutative differential graded algebra with lattices simultaneously. Moreover, we define the integral Sullivan models. An integral Sullivan model is a free commutative differential graded algebra over \mathbb{Z} . If we know the integral Sullivan model of a topological space X ,

we can understand the rational homotopy type of X and the lattice structure of X . We also prove that, under some conditions, a commutative differential graded algebra always has an integral Sullivan model. Then we compute the integral Sullivan models of some topological spaces.

Chapter 2

Commutative Differential Graded Algebra

A differential graded algebra is an algebra with grading and a differential map. It generalizes the idea of algebras and chain complexes. It is the main tool of this paper. For an introduction to differential graded algebra, the reader is referred to [5]. Throughout this chapter, we assume that R is a commutative ring with identity.

2.1 Differential graded algebras and quasi-isomorphisms

In this section, we shall introduce the concepts of differential graded algebras and define the quasi-isomorphism in the category of commutative differential graded algebras. We give two important examples of differential graded algebras, the de Rham complex of a smooth manifold and the singular cochain algebra of a topological space.

We begin with the definition of differential graded modules.

Definition 2.1.1. *A differential graded module over R is an R -module M together with a*

decomposition indexed by nonnegative integers:

$$M = \bigoplus_{i \in \mathbb{Z}} M^i \quad (2.1)$$

endowed with a differential $d_M = \{d^i\}_{i \in \mathbb{Z}}$, $d^i : M^i \rightarrow M^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for $i \in \mathbb{Z}$

A cochain complex is a differential graded module. An R -module is a differential graded module concentrated in degree zero with zero differential. If $a \in M^k$, we say the degree of a is k . We denote it by $|a|$.

Several examples of differential graded modules come from algebraic topology and differential topology.

Example 2.1.2. Let X be a topological space. For $n \geq 0$, let σ_n be the standard topological n -simplex; that is, the convex hull of the standard basis vectors in \mathbb{R}^{n+1} . We define

$$S(X) = \{S_n \mid n \in \mathbb{Z}^{\geq 0}, S_n(X) = \text{Hom}(\sigma_n, X)\}. \quad (2.2)$$

We will see that $S(-)$ is a functor in the next chapter. We call it the singular set of X . If we apply the free abelian group functor to $S(X)$ and take the alternating sum of the face map, we obtain the singular chain complex $S_\bullet(X; R)$. $S_\bullet(-; R)$ is the singular functor from the category of topological spaces to the category of R -modules. Then

$$S^\bullet(X; R) = \bigoplus_{i \in \mathbb{N}} \text{Hom}(S_i(X; R)) \quad (2.3)$$

is a differential graded module.

For the knowledge of singular homology and singular cohomology, the reader is referred to Allen Hatcher's book [8].

Definition 2.1.3. A differential graded algebra over R is a differential graded module M over R endowed with R -bilinear maps $M^i \times M^j \rightarrow M^{i+j}, (a, b) \mapsto ab$ such that

$$d^{i+j}(ab) = d^i(a)b + (-1)^i ad^j(b) \quad (2.4)$$

and such that $\bigoplus M^i$ becomes an associative and unital R -algebra.

Example 2.1.4. An R -algebra is a differential graded algebra concentrated in degree zero with zero differential. We can also turn any graded algebra into differential graded algebra by introducing a zero differential. For example, a given cohomology ring is a differential graded algebra.

Example 2.1.5. Let X be a smooth manifold. We have a de Rham complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \Omega^3(X) \longrightarrow \dots \quad (2.5)$$

where $\Omega^0(X)$ is the \mathbb{R} -module of smooth functions on X , $\Omega^1(X)$ is the module of 1-forms, and so on. Then the de Rham complex $\Omega^*(X)$ is a differential graded module over the real numbers. Moreover, under the wedge product, the de Rham complex becomes a differential graded algebra.

For the knowledge of differential forms and de Rham cohomology, the reader is referred to Bott's book [1].

Definition 2.1.6. (M, d_M) and (N, d_N) are differential graded modules. A homomorphism of differential graded modules $f : M \rightarrow N$ is a module map compatible with the gradings, such that $f \circ d_M = d_N \circ f$. A homomorphism of differential graded algebras is an algebra map as well as a differential graded module map.

Differential graded algebras form a category denoted by **DGA**.

For a differential graded module (resp. differential graded algebra) M , we define the n -th cohomology of M to be

$$H^n(M) = \text{Ker}(d^n) / \text{Im}(d^{n-1}) \quad (2.6)$$

as R -modules. Let $H(M) = \bigoplus_n H^n(M)$, then $H(M)$ is a graded module (resp. graded algebra). $H(-)$ is a functor from the category of differential graded modules to the category of graded modules. We say a homomorphism $f : M \rightarrow N$ of differential graded modules (resp. differential graded algebras) is a quasi-isomorphism if it induces an isomorphism $H(f) : H(M) \rightarrow H(N)$ of graded modules (resp. graded algebras)

We can generate the smallest equivalence relation by the quasi-isomorphisms in the category **DGA**. If two differential graded algebras are in the same equivalence class, we say the two differential graded algebras are quasi-equivalent. This means that there exist a sequence of zigzag quasi-isomorphisms between these two differential graded algebras.

Definition 2.1.7. *We say a differential graded algebra A is formal if A is quasi-equivalent to $H(A)$.*

Formal is an important property in rational homotopy theory. I will give some examples of formal differential graded algebras in the next chapter. We say a differential graded algebra $M = \bigoplus_i M^i$ is graded commutative or just commutative if, for any $a \in M^j$ and $b \in M^k$, $ab = (-1)^{jk}ba$. If M is a commutative differential graded algebra and the degree of $a \in M$ is an odd number, then $a^2 = -a^2$. This implies $2a^2 = 0$. If the characteristic of

R is not 2, then a^2 is 0. The commutative differential graded algebras form a category. We denote it by CDGA.

Theorem 2.1.8. *Let X be a smooth manifold. Then*

$$H(S^\bullet(X; \mathbb{R})) \cong H(\Omega^*(X)) \quad (2.7)$$

as commutative graded algebra.

For the proof of this theorem, the reader is referred to the chapter 2 and the chapter 3 of [1].

Remark 2.1.9. *$S^\bullet(X; \mathbb{R})$ is not commutative, but $H(S^\bullet(X; \mathbb{R}))$ is a commutative graded algebra with respect to the cup product.*

Definition 2.1.10. *Let (A, d_A) and (B, d_B) be differential graded algebras over R . The tensor product of A and B is an algebra $A \otimes_R B$ with multiplication defined by*

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd) \quad (2.8)$$

endowed with the differential d defined by $d(a \otimes b) = da \otimes b + (-1)^{|a|}a \otimes db$. $A \otimes_R B$ forms a differential graded algebra.

More generally, let $f : C \rightarrow A$ and $g : C \rightarrow B$ be two commutative differential graded algebra maps. Let $A \otimes_C B$ be the quotient of $A \otimes B$ by the subgraded vector space spanned by elements of the form $af(c) \otimes b - a \otimes g(c)b$, $a \in A, b \in B$ and $c \in C$. Then $A \otimes_C B$ is a commutative differential graded algebra such that the quotient map $A \otimes B \rightarrow A \otimes_C B$ is a commutative differential graded algebra map.

2.2 Sullivan algebras and minimal models

A minimal model is a particularly computable commutative differential graded algebra that can be associated to any nice commutative differential graded algebra or any nice topological space. The most important feature of a minimal model is that the minimal model encode all rational homotopy information about a space. Some references on minimal Sullivan algebras are [5] and [7].

In this section, we always let R be a ring of characteristic 0. We introduce the basic definitions and the main properties of Sullivan algebras. In particular, we discuss the minimal condition of a Sullivan algebra and compute several examples.

We begin with the free tensor algebras. Given a non-negatively graded vector space $V = \bigoplus_{i \geq 0} V^i$. We define the tensor algebra $TV = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ with the concatenation multiplication, that is, $V^{\otimes i} = V \otimes V \otimes \dots \otimes V$ of i copies of V and $(a_1 \otimes \dots \otimes a_s)(b_1 \otimes \dots \otimes b_l) = a_1 \otimes \dots \otimes a_s \otimes b_1 \otimes \dots \otimes b_l$. Then let ΛV denote the free commutative graded algebra generated by V , that is,

$$\Lambda V = TV / (a \otimes b - (-1)^{|a||b|} b \otimes a)_{a,b \in V} = S[V^{even}] \otimes E[V^{odd}] \quad (2.9)$$

Note that if $|a|$ is odd, then $a^2 = 0$ (here we require that $\text{char } R$ is 0). So if $|a|$ is odd, $\Lambda(a)$ has basis $1, a$, which is the exterior algebra $E(a)$ generated by a . If $|a|$ is even, then $\Lambda(a)$ has basis a^n for $n \geq 0$, which is the symmetric algebra $S(a)$ generated by a . More generally, ΛV is the tensor product of the symmetric algebra on the even degree vectors and of exterior algebra on the odd degree vectors $S[V^{even}] \otimes E[V^{odd}]$. For instance, when G is a compact connected Lie group, its rational cohomology is a free commutative graded

algebra.

We usually write $\Lambda V = \Lambda(x_i)$, where x_i is a homogeneous basis of V . We denote by $\Lambda^l V$ the vector space generated by the product $x_1 \dots x_l$ with the x_i in V . We also write $\Lambda^+ V = \bigoplus_{n \geq 1} \Lambda^n V$ and $\Lambda^{\geq q} V = \bigoplus_{n \geq q} \Lambda^n V$. The elements of $\Lambda^{\geq 2} V$ are referred to as decomposable elements.

For a commutative graded algebra W , a graded vector space map $V \rightarrow W$ has a unique extension to a commutative graded algebra map $\Lambda V \rightarrow W$.

Let $(\Lambda V, d)$ be a free commutative differential graded algebra generated by V . The differential d is determined by its restriction to V .

Definition 2.2.1. *A Sullivan algebra is of the form $(\Lambda V; d)$ such that V has a basis $\{v_i | i \in I\}$, where I is a well-ordered set, such that $dv_i \in \Lambda V_{<i}$ for all $i \in I$, where $V_{<i}$ is the graded vector space generated by $\{v_j | j < i\}$.*

In the next section, I will give another explanation of the Sullivan algebras.

Example 2.2.2. $(\Lambda(a, b), db = a^2)$, where $|b| = |2a - 1|$, is a Sullivan algebra. But $(\Lambda(a, b, c), da = bc, db = ca, dc = ab)$, where $|a| = |b| = |c| = 1$, is an example of a commutative differential graded algebra with free underlying graded algebra that is not a Sullivan algebra.

From now on, we assume $V^0 = V^1 = 0$. Then $(\Lambda V, d)$ is always a Sullivan algebra.

Definition 2.2.3. *We say a Sullivan algebra is minimal if $dV \subseteq \Lambda^{\geq 2} V$, where $\Lambda^{\geq 2} V$ is the subspace of words of length at least two.*

Example 2.2.4. $(\Lambda(a, b), da = b^2)$ is a minimal Sullivan algebra. But $(\Lambda(a, b), da = b)$ is not a minimal Sullivan algebra.

Proposition 2.2.5. Each Sullivan algebra is isomorphic to the tensor product of a minimal Sullivan algebra and a tensor product of acyclic Sullivan algebra's of the form $(\Lambda(x_i, y_i), d)$ with $dx_i = 0$ and $dy_i = x_i$.

The proof of this proposition can be found in [5, Theorem 14.9].

Definition 2.2.6. A (minimal) Sullivan model of a differential graded algebra A is a (minimal) Sullivan algebra which is quasi-isomorphic to A .

Proposition 2.2.7. Minimal Sullivan models of commutative differential graded algebra are unique up to isomorphism.

The proof of this proposition can be found in [5, Theorem 14.12].

A differential graded algebra A is said to be n -reduced if $A_i = 0$ for $0 < i < n$ and $A_0 = R$. A differential graded algebra A is said to be n -connected if $H^0(A) = R$ and $H^i(A) = 0$ for $0 < i \leq n$.

Theorem 2.2.8. If the differential graded algebra A is 1-connected and has finite-dimensional cohomology, then it has a minimal Sullivan model.

The proof of this theorem can be found in [1, Proposition 19.4].

2.3 Model categories

The notation of a model category was introduced by Quillen in the 60's and has been developed since then. In this section we introduce the concept of model categories and give some examples. Generally, we are interested in homotopy theory in different categories. Model categories is an efficient machinery, which encodes this structure. It encodes the deeper homotopical structures, making a large class of arguments formal. In particular, we are interested in the model categories structure on the category of commutative differential graded algebras. Some general references for model categories are [4] and [9].

An object X of a category C is said to be a retract of an object Y if there exist morphisms $i : X \rightarrow Y$ and $r : Y \rightarrow X$ such that $ri = id_X$. Let C^D be the functor category, where D is the category $\{a \rightarrow b\}$. If f and g are morphisms of C , we will say that f is a retract of g if the object of the functor category C^D represented by f is a retract of the object of C^D represented by g . In other words, there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ f \downarrow & & g \downarrow & & f \downarrow \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

in which the composites ri and $r'i'$ are the appropriate identity maps.

Definition 2.3.1. *Let C be a category, and $i : A \rightarrow B$ and $p : X \rightarrow Y$ be two maps in C . We say that p has the right lifting property with respect to i (and that i has the left lifting*

property with respect to p) if for all commutative diagrams

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

in C , there is a map $h : B \rightarrow X$ making the following diagram commutative

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow h & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

Definition 2.3.2. A model category is a category C together with three classes of maps fibrations (\twoheadrightarrow), cofibrations (\rightarrowtail), and weak equivalences ($\xrightarrow{\sim}$) each of which is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called an acyclic fibration (resp. cofibration). We require the following axioms.

- (1) Finite limit and colimit exists in C .
- (2) If f and g are maps in C such that gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.
- (3) If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f .
- (4) The cofibrations have the left lifting property with respect to the acyclic fibrations and the fibrations have the right lifting property with respect to the acyclic cofibrations.
- (5) Any map f can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and (ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

Actually, the axioms for a model category are overdetermined in some sense. If C is a model category, the fibrations (resp. cofibration) are completely determined by the cofibrations (resp. fibrations) and the weak equivalences.

Remark 2.3.3. *The factorizations of a map in a model category provided by axiom (5) are not required to be functorial.*

We now give some examples of model categories. The first natural example is \mathbf{Top} , the category of topological space.

Example 2.3.4. *The category \mathbf{Top} of topological spaces can be given the structure of a model category by defining $f : X \rightarrow Y$ to be*

- (1) *a weak equivalence if f is a weak homotopy equivalence,*
- (2) *a cofibration if f is a retract of a map $X \rightarrow Y'$ in which Y' is obtained from X by attaching cells, and*
- (3) *a fibration if f is a Serre fibration.*

This is not the unique model category structure on \mathbf{Top} .

Example 2.3.5. *The category \mathbf{Top} of topological spaces can be given the structure of a model category by defining $f : X \rightarrow Y$ to be*

- (1) *a weak equivalence if f is a homotopy equivalence,*
- (2) *a cofibration if f is a closed Hurewicz cofibration, and*
- (3) *a fibration if f is a Hurewicz fibration.*

The reader can find the details about these examples in [4]. In the next chapter, we will describe the model category of simplicial sets.

Since model categories C have any finite limits and colimits, they have both an initial object \emptyset and a terminal object $*$. An object $A \in C$ is said to be cofibrant if $\emptyset \rightarrow A$ is a cofibration and fibrant if $A \rightarrow *$ is a fibration. For any object A , we can factor the map $f : \emptyset \rightarrow A$ as

$$\begin{array}{ccc} & PA & \\ \nearrow & & \searrow p \\ \emptyset & \xrightarrow{f} & A \end{array}$$

where PA is a cofibrant object and p is an acyclic fibration. We say PA is a cofibrant replacement of A . For any object B , we can factor the map $g : B \rightarrow *$ as

$$\begin{array}{ccc} B & \xrightarrow{g} & * \\ & \searrow i & \nearrow \\ & QB & \end{array}$$

where QB is a fibrant object and i is an acyclic cofibration. We say QB is a fibrant replacement of B .

We now go back to the category of commutative differential graded algebras.

Theorem 2.3.6. *The category of commutative differential graded algebras admits a model category structure in which fibrations are degreewise surjections and weak equivalences are quasi-isomorphisms.*

For the detail of this theorem, the reader is referred to [2] and [9].

In this model category, the initial object is R , all commutative differential graded algebras are fibrant, and the Sullivan algebras are the cofibrant objects. Let A be any one connected commutative differential graded algebra and $f : R \rightarrow A$ be the unique map, f

can be factored as

$$\begin{array}{ccc} & PA & \\ \nearrow & & \searrow p \\ R & \xrightarrow{f} & A \end{array}$$

where PA is a cofibrant object and p is a quasi-isomorphism. So a Sullivan model of a commutative differential graded algebra A is just a cofibrant replacement of A . We know that a cofibrant replacement is not unique. So taking a minimal Sullivan model of a commutative differential graded algebra is not a functor. It depends on the quasi-isomorphism.

Chapter 3

Background Knowledge of Rational Homotopy

Theory

Rational homotopy theory is an attempt to do homotopy theory “modulo torsion”. The basic idea is that we can identify the rational homotopy theory of simply connected spaces with the homotopy theory of commutative differential graded algebras. We can construct a Sullivan functor A_{PL} from the category of topological spaces to the category of commutative differential graded algebras. If X is a topological space, then $A_{PL}(X)$ is a commutative differential graded algebra which is quasi-equivalent to $S^\bullet(X)$. Then we can compute the minimal Sullivan model of $A_{PL}(X)$. This minimal Sullivan model captures the whole rational homotopy type of the topological space X .

In the chapter, without special mention, we always assume that the ground ring is a field of characteristic 0 and the topological space is connected and simply connected.

3.1 Simplicial categories

Here we give an introduction to the simplicial categories which play an essential role in the definition of the functor A_{PL} . We do not have time to develop the theory of simplicial sets in any detail, instead we refer the reader to [5] and [6].

We begin with the definition of ordinary number category. Let $[k]$ be the set $\{0, 1, \dots, k\}$. A weakly order preserving map $\phi : [n] \rightarrow [m]$ is a set map satisfied that $\phi(i) \leq \phi(j)$ for $1 \leq i < j \leq n$.

Definition 3.1.1. *The ordinal number category Δ has an object the ordered sets $[n] = \{0, 1, \dots, n\}$ and a morphism weakly order preserving maps $\phi : [n] \rightarrow [m]$. In particular we have the maps*

$$d^i : [n - 1] \rightarrow [n], 0 \leq i \leq n \quad (3.10)$$

$$\{0, 1, \dots, i - 1, i, i + 1, \dots, n - 1\} \mapsto \{0, 1, \dots, i - 1, i + 1, \dots, n\} \quad (3.11)$$

which skip i and the maps

$$s^j : [n + 1] \rightarrow [n], 0 \leq j \leq n \quad (3.12)$$

$$\{0, 1, \dots, n\} \mapsto \{0, 1, \dots, j, j, \dots, n\} \quad (3.13)$$

which double up j .

We can show that all the morphisms in Δ are compositions of these two types of morphisms.

Now we can define the simplicial objects in a category.

Definition 3.1.2. A simplicial object in a category \mathcal{C} is a contravariant functor from Δ to \mathcal{C} :

$$F : \Delta^{op} \rightarrow \mathcal{C}. \quad (3.14)$$

A morphism of simplicial objects is a natural transformation. The category of simplicial objects in \mathcal{C} will be denoted $s\mathcal{C}$. As a matter of notation, we will write F_n for $F([n])$,

$$d_i = F(d^i) : F_n \rightarrow F_{n-1} \text{ and } s_i = F(s^i) : F_{n-1} \rightarrow F_n \quad (3.15)$$

These are respectively, the face and degeneracy maps. More generally, if ϕ is morphism in Δ , we will write ϕ^* for $F(\phi)$.

It is not easy to handle the idea of simplicial objects. Here we give three basic examples of simplicial objects.

Example 3.1.3. Let X be a topological space. The singular set $S(X)$ is a simplicial set. $S(-)$ is a functor from the category of topological spaces to the category of simplicial sets.

Example 3.1.4. By the Yoneda Lemma, the functor from simplicial sets to sets sending X to X_n is representable. Indeed, define the standard n -simplex Δ^n of $s\mathbf{Sets}$ by

$$\Delta^n = \Delta(-, [n]) : \Delta^{op} \rightarrow \mathbf{Sets}. \quad (3.16)$$

Then the Yoneda Lemma supplies the isomorphism $s\mathbf{Sets}(\Delta^n, X) \cong X_n$. The morphisms in Δ yield morphisms $\Delta^m \rightarrow \Delta^n$.

We also have the horns of the standard simplex

$$\Delta_k^n = \bigcup_{i \neq k} d^i \Delta^{n-1} \subseteq \Delta^n. \quad (3.17)$$

Example 3.1.5. Let \mathcal{C} be a small category; that is, a category with a set of objects. Then we define a simplicial set with n -simplices as all strings of composable arrows in \mathcal{C} :

$$BC_n = \{x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n\}. \quad (3.18)$$

If $\phi : [m] \rightarrow [n]$ is a morphism in Δ , then the induced function $\phi^* : BC_n \rightarrow BC_m$ is given by

$$\phi^*(\{x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n\}) = \{x_{\phi(0)} \rightarrow x_{\phi(1)} \rightarrow \dots \rightarrow x_{\phi(m)}\} \quad (3.19)$$

To be concrete, if \mathcal{C} is the category

$$0 \rightarrow 1 \rightarrow \dots \rightarrow n, \quad (3.20)$$

then $BC \cong \Delta^n$.

Unlike the product of simplicial complexes, the product of simplicial sets is easy to describe. If X and Y are simplicial sets, then $(X \times Y)_n = X_n \times Y_n$ and ϕ is a morphism in Δ , then $(X \times Y)(\phi) = X(\phi) \times Y(\phi)$.

Definition 3.1.6. For a simplicial sets X , we define the geometric realization of X a topological space $|X|$ given by

$$|X| = \coprod_{n=0}^{\infty} X_n \times \sigma_n / \sim, \quad (3.21)$$

where \sim is the equivalence relation generated by the relations $(x, D_i(p)) \sim (d_i(x), p)$ for $x \in X_{n+1}, p \in \sigma_n$ and the relations $(x, S_i(p)) \sim (s_i(x), p)$ for $x \in X_{n-1}, p \in \sigma_n$. Here D_i and S_i are the face inclusions and collapse maps.

Let CGH be the category of compactly generated Hausdorff spaces. The next proposition collects some facts about the geometric realization functor.

Proposition 3.1.7. *The geometric realization functor has the following properties:*

(1) *The functor $| - | : s\mathbf{Sets} \rightarrow \mathbf{CGH}$ is the left adjoint to the singular set functor $S(-)$.*

(2) *If X and Y are simplicial sets, then the natural morphism $|X \times Y| \rightarrow |X| \times |Y|$ in \mathbf{CGH} is a homeomorphism.*

(3) $|\Delta^n| \cong \sigma_n$.

The proof of this proposition can be found in [6].

We now come to the model category structure on simplicial sets. It is still one of the deeper and important result in the theory of simplicial sets. But we do not use this result in this paper. We just mention this theorem.

Theorem 3.1.8. *The category $s\mathbf{Sets}$ has a model category structure with a morphism $f : X \rightarrow Y$ such that*

(1) *f is a weak equivalence if $|f| : |X| \rightarrow |Y|$ is a weak equivalence of topological spaces;*

(2) *f is a cofibration if $f_n : X_n \rightarrow Y_n$ is a monomorphism for all n ; and*

(3) *f is a cofibration if f has the left lifting property with respect to the inclusions $\Delta_k^n \rightarrow \Delta^n, n \geq 1, 0 \leq k \leq n$.*

The proof of this theorem can be found in the chapter 17.5 of [12].

3.2 The A_{PL} functor

We now explain the passage from topology to algebra. The first step in constructing the functor A_{PL} is the construction of the simplicial commutative differential graded algebra.

Recall first that the standard n -simplex Δ^n is the convex hull of the standard basis e_0, e_1, \dots, e_n in \mathbb{R}^{n+1} :

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_j \geq 0, j = 0, \dots, n\}. \quad (3.22)$$

Definition 3.2.1. *The algebra of polynomial differential forms, denoted A_{PL} , is the simplicial commutative differential graded algebra given by*

$$A_{PL}[n] = (\Lambda(t_0, \dots, t_n, y_0, \dots, y_n) / \{1 - \sum_{i=0}^n t_i, \sum_{j=0}^n y_j\}, d) \quad (3.23)$$

where $\deg t_i = 0$ and $dt_i = y_i$ for all i . The faces and degeneracies are specified by

$$\partial_i : A_{PL}[n] \rightarrow A_{PL}[n-1] : t_k \mapsto \begin{cases} t_k & : k < i \\ 0 & : k = i \\ t_{k-1} & : k > i \end{cases}$$

and

$$s_i : A_{PL}[n] \rightarrow A_{PL}[n+1] : t_k \mapsto \begin{cases} t_k & : k < i \\ t_k + t_{k+1} & : k = i \\ t_{k+1} & : k > i \end{cases}$$

$A_{PL}[n]$ is a cyclic commutative differential graded algebra. It can be viewed as an algebra of polynomial \mathbb{Q} -forms on Δ^n . We call the elements of $A_{PL}[n]$ polynomial differential forms with degree n .

Definition 3.2.2. For a topological space X , we define $A_{PL}(X) = \mathbf{sSet}(S_{\bullet}(X), A_{PL})$, which we call the commutative differential graded algebra of piecewise-linear de Rham forms on X .

Actually, it is not easy to compute the commutative differential graded algebra $A_{PL}(X)$ of a topological space X by using the definition of A_{PL} . For example, we cannot even compute the piecewise-linear de Rham forms of a disc directly, which is a contractible space. But this formal definition does have a lot of nice properties.

Theorem 3.2.3. For a topological space X , there is a quasi-equivalence between the singular cochain complex $S^{\bullet}(X)$ and the PL de Rham form $A_{PL}(X)$. This induces natural isomorphisms

$$H^*(X) \cong H(A_{PL}(X)). \quad (3.24)$$

The proof of this theorem can be found in [5, Corollary 10.9].

Definition 3.2.4. A minimal Sullivan model of a topological space is the minimal Sullivan model of its algebra of piecewise-linear de Rham forms

$$\varphi : (\Lambda V, d) \rightarrow A_{PL}(X). \quad (3.25)$$

Let X be a simply connected space and let $(\Lambda V, d)$ be its minimal Sullivan model. Then, by construction, we have an isomorphism $H(\Lambda V, d) \cong H^*(X; \mathbb{Q})$. And we have the next important relation.

Theorem 3.2.5. Let X be a simply connected manifold and A be its minimal model. Then

the dimension of the vector space $\pi_p(X) \otimes \mathbb{Q}$ is the number of generators of the minimal model A in degree p .

The proof of this theorem can be found in [5, Theorem 15.11].

Example 3.2.6. Let G be a compact connected Lie group, then the minimal Sullivan model of G is $\Lambda(x_{2p_1+1}, \dots, x_{2p_r+1})$, where $|x_{2p_i+1}| = 2p_i + 1$ for $i = 1, \dots, r$. The generators of the minimal model are only in odd degrees. Therefore, $\dim \pi_{\text{even}} \otimes \mathbb{Q} = 0$.

Let recall that a commutative differential graded algebra A is formal if there is a zigzag sequence of quasi-isomorphism between A and $H(A)$.

Definition 3.2.7. A space X is formal if $A_{PL}(X)$ is formal.

For such spaces, the rational homotopy type can be determined easily. If a space is a formal space, the cohomology ring will be a nice model of this space. Examples are spheres, complex projective spaces and H -spaces. Formality is preserved under taking wedges, products and connected sum of spaces. The details can be found in [5]. Furthermore, Deligne, Griffiths, Morgan and Sullivan showed in [2] that compact Kähler manifolds are also formal.

Example 3.2.8. $A = (\Lambda(x, y, z), dz = xy)$, where $|x| = 3, |y| = 5$ and $|z| = 7$. We can use knowledge of model categories to prove A is not a formal commutative differential graded algebra.

We can now show some basic examples of the minimal Sullivan model of some formal spaces. Since the topological spaces are formal, we can use the cohomology ring of these spaces to compute the minimal Sullivan model.

Example 3.2.9. (The sphere S^n)

The rational cohomology ring of S^n is $\mathbb{Q}[x]/\langle x^2 \rangle$, $|x| = n$.

If n is an odd number. The minimal Sullivan model is

$$f : \Lambda(a) \rightarrow \mathbb{Q}[x]/\langle x^2 \rangle, a \mapsto x \quad (3.26)$$

where $|a| = n$.

If n is an even number. The minimal Sullivan model is

$$f : (\Lambda(a, b), db = a^2) \rightarrow \mathbb{Q}[x]/\langle x^2 \rangle, a \mapsto x, b \mapsto 0 \quad (3.27)$$

where $|a| = n$, $|b| = 2n - 1$.

Example 3.2.10. (The complex projective space $\mathbb{C}P^n$)

The rational cohomology ring of $\mathbb{C}P^n$ is $\mathbb{Q}[x]/\langle x^{n+1} \rangle$, $|x| = 2$. The minimal Sullivan model is

$$f : (\Lambda(a, b), db = a^{n+1}) \rightarrow \mathbb{Q}[x]/\langle x^{n+1} \rangle, a \mapsto x, b \mapsto 0 \quad (3.28)$$

where $|a| = 2$, $|b| = 2n - 1$.

Example 3.2.11. (The Eilenberg-MacLane space $K(\mathbb{Z}, n)$)

Let G be an abelian group, and let $n \geq 2$. The Eilenberg-MacLane space $K(G, n)$ is a CW complex such that $\pi_n(K(G, n)) = G$, and such that the other homotopy groups are zero.

If $G = \mathbb{Z}$, then the rational cohomology ring of $K(\mathbb{Z}, n)$ is $\mathbb{Q}[x]$, $|x| = n$, if n is a odd number, or $\mathbb{Q}[x]/\langle x^2 \rangle$, $|x| = n$, if n is a even number. The minimal Sullivan model is

$$f : \Lambda(a) \rightarrow \mathbb{Q}[x] \text{ (or } \mathbb{Q}[x]/\langle x^2 \rangle), a \mapsto x \quad (3.29)$$

where $|a| = n$. More generally, the minimal Sullivan model of $K(G, n)$ is $(\Lambda V, d)$, where $V^* \cong \text{Hom}(\pi_*(K(G, n)), \mathbb{Q})$, $d = 0$, $V = V^n$ and $\dim V = \dim G$.

Using the Theorem 2.2.7 and the Theorem, we have the following theorem.

Theorem 3.2.12. *Let X be simply connected topological space such that each $H_i(X; \mathbb{Q})$ is finite dimensional. Then X has a minimal Sullivan model*

$$m : (\Lambda V, d) \rightarrow A_{PL}(X) \tag{3.30}$$

such that $V = \{V^i\}_{i \geq 2}$ and each V^i is finite dimensional.

The proof of this theorem can be found in [5, Proposition 12.2].

The minimal Sullivan model of a simply connected topological space is unique up to isomorphism. It encodes all the information of rational homotopy type of this space.

3.3 Postnikov towers and minimal models

Let X be a simply connected topological space with minimal Sullivan model $(\Lambda V, d)$. In this section, we explain the relations between the minimal Sullivan model of X and its Postnikov tower. We will need some knowledge of homotopy theory, including the definitions of fibrations, Eilenberg MacLane space and Postnikov towers. Readers are referred to [11] and [12].

We begin with the definition of Hirsch extension.

Definition 3.3.1. *Let A be a commutative differential graded algebra. A Hirsch extension*

of A is an extension

$$A \rightarrow A \otimes \Lambda(V^k) \quad (3.31)$$

where V^k is a finite dimensional vector space of homogeneous degree k , and $d : V^k \rightarrow A^{k+1}$ is a map whose image is contained in $\ker d_A$. We extend d to the graded algebra $A \otimes \Lambda(V^k)$ by the Leibniz rule. We denote this differential graded algebra by $A \otimes_d \Lambda(V^k)$.

Two Hirsch extension $A \rightarrow A \otimes_d \Lambda(V^k)$ and $A \rightarrow A \otimes_{d'} \Lambda((V')^k)$ are equivalent if there is a commutative diagram

$$\begin{array}{ccc} & A \otimes_d \Lambda(V^k) & \\ & \nearrow & \downarrow \varphi \\ A & & \\ & \searrow & \\ & A \otimes_{d'} \Lambda((V')^k) & \end{array}$$

with φ an isomorphism which is the identity on A .

Proposition 3.3.2. *Two Hirsch extensions $A \rightarrow A \otimes_d \Lambda(V^k)$ and $A \rightarrow A \otimes_{d'} \Lambda((V')^k)$ are equivalent if and only if there is an isomorphism $\psi : V \rightarrow V'$ so that the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{d} & H^{k+1}(A) \\ \psi \downarrow & & \downarrow id \\ V' & \xrightarrow{d'} & H^{k+1}(A) \end{array}$$

Proof. If $\varphi : A \otimes_d \Lambda(V^k) \rightarrow A \otimes_{d'} \Lambda((V')^k)$ is an isomorphism extending the identity on A . For $v \in V$, we write $\varphi(v) = a_v + \psi(v)$. Since φ is an isomorphism, ψ must be an isomorphism. Since $\varphi(dv) = d'(\varphi(v)) = d'(a_v) + d'\psi(v)$, we have $[dv] = [d'\psi(v)] \in H^{k+1}(A)$.

Conversely, if we have $\psi : V \rightarrow V'$ such that $[dv] = [d'\psi(v)] \in H^{k+1}(A)$, then $dv - d'\psi(v) = da_v$ for some $a_v \in A$. We choose a_v linearly in v . Define $\varphi(v) = a_v + \psi(v)$. This defines a map $\varphi : A \otimes_d \Lambda(V^k) \rightarrow A \otimes_{d'} \Lambda((V')^k)$, which is easily seen to be an isomorphism and to commute with the differentials. \square

By this proposition, we know that the equivalence classes of Hirsch extensions of A with a fixed vector space of new generators, V , are in natural one-to-one correspondence with maps

$$d : V \rightarrow H^{k+1}(A). \quad (3.32)$$

Note that if A is free as a commutative graded algebra, then so is any Hirsch extension of A . Now suppose we build a commutative differential graded algebra by Hirsch extensions, starting with some $A = A(0)$ equal to the ground field, and where successive extensions are in degree 2,3,4 and so on. Set $A(n+1) = A(n) \otimes_d \Lambda(V^{n+1})$. Since every dv has degree $n+2$, and since $A(n)$ is inductively generated by elements of degrees smaller than $n+1$, it follows that d is decomposable, and therefore $A = \bigcup A(n)$ is a minimal Sullivan algebra.

On the other hand, every 1-reduced minimal Sullivan algebra arises this way:

Proposition 3.3.3. *Let A be a 1-reduced minimal Sullivan algebra, and let $A(n)$ be the subalgebra generated by elements of degree smaller than $n+1$. Then $A(0) = A(1) \subset A(2) \subset \dots$ with $\bigcup A(n) = A$, and with each $A(n) \hookrightarrow A(n+1)$ a Hirsch extension.*

Proof. Since each $A(i)$ is free as a commutative graded algebra, $A(n+1) = A(n) \otimes \Lambda(V^{n+1})$ for some $\Lambda(V)$ in degree $n+1$, as commutative graded algebras. Since $A(1)$ is

the ground field and d is decomposable, every dv is a sum of products of elements of degree smaller than $n + 1$, that is, $dv \in A(n)$. Hence each extension is a Hirsch extension. \square

Let recall the definition of Postnikov tower.

Definition 3.3.4. *Let X be a topological space. A Postnikov tower for X is a sequence of fibrations $p_i : X_i \rightarrow X_{i-1}$ with fibers $K(\pi_i(X), i)$ and compatible maps $f_i : X \rightarrow X_i$ such that*

(1) *for each i , the homotopy groups of X_i vanish in dimension bigger than i ; and*

(2) *each map $f_i : X \rightarrow X_i$ induces an isomorphism in homotopy groups in dimensions smaller than $i + 1$;*

(3) *$p_i : X_i \rightarrow X_{i-1}$ is a principal fibration obtained as a pullback of the path fibration $K(\pi_i(X), i) = \Omega(K_i(X), i + 1) \rightarrow PK(\pi_i(X), i + 1) \rightarrow K(\pi_i(X), i + 1)$ along a map $k_i : X_{i-1} \rightarrow K(\pi_i(X), i + 1)$.*

The maps k_i are called the associated k -invariants.

Theorem 3.3.5. *A connected CW complex X has a Postnikov tower of fibrations if $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n > 1$.*

The proof of this theorem can be found in [8]. Notice that if X is simply connected space, $\pi_1(X) = 0$, the π_1 action is trivial. Then X has a Postnikov tower.

Let X be a simply connected space. Each fibration $K(\pi_{i+1}(X), i + 1) \rightarrow X_{i+1} \rightarrow X_i$ can be extended to a fibration $X_{i+1} \rightarrow X_i \rightarrow K(\pi_{i+1}(X), i + 2)$ which is therefore classified by the map $k_i : X_i \rightarrow K(\pi_{i+1}(X), i + 2)$, or equivalently, by $[k_i] \in H^{i+2}(X_i; \pi_{i+1}(X))$.

Such an X can be recovered (up to homotopy) from the groups $\pi_i(X)$ and the k -invariants k_i .

Let $(\Lambda V, d)$ be the minimal model of a simply connected space X . We then have the following properties:

(1) The minimal Sullivan model of X_i is given by the $(\Lambda^{\leq i} V, d)$.

(2) The minimal Sullivan model of the map f_i and p_i are, respectively, given by the injections

$$(\Lambda^{\leq i} V, d) \hookrightarrow (\Lambda V, d), \quad (\Lambda^{\leq i-1} V, d) \hookrightarrow (\Lambda^{\leq i} V, d). \quad (3.33)$$

(3) The minimal Sullivan model of the i th k -invariant $k_i : X_{i-1} \rightarrow K(\pi_i(X), i+1)$ is given by the map $d_i : (\Lambda(s^{-1}V^i), 0) \rightarrow (\Lambda^{\leq i-1} V, d)$, where $(s^{-1}V^i)^{i+1} = V^i$, $(s^{-1}V^i)^q = 0$, $q \neq i+1$, $d_i(s^{-1}v) = dv$.

For the details of these properties, the reader is referred to [7].

Chapter 4

Commutative Differential Graded Algebra with Lattice

This chapter is the core of this thesis. We introduce a new invariant called the commutative differential graded algebra with lattice of a simply connected topological space X and give some examples. This invariant captures the rational homotopy type of X as well as the lattice structure of X . Moreover, we define an integral Sullivan model for a commutative differential graded algebra with lattice. And we prove that, under some conditions, a commutative differential graded algebra has an integral Sullivan model.

4.1 Lattices of a space

Let X be topological space. We have a natural map $\varphi : Hom(S_*(X; \mathbb{Z}), \mathbb{Z}) \rightarrow Hom(S_*(X; \mathbb{Z}), \mathbb{Q})$. If $f \in Hom(S_*(X; \mathbb{Z}), \mathbb{Z})$, then $\varphi(f) = i \circ f$ where i is the canonical map from \mathbb{Z} to \mathbb{Q} . Actually, this map is just the tensor \mathbb{Q} map. It induces a natural map

$l : H^*(X; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Q})$. We say the image of l is the lattice of X denoted by L_X . We should mention that the lattice is a \mathbb{Z} -subalgebra. Different multiplication structures of the graded \mathbb{Z} -submodule of $H^*(X; \mathbb{Q})$ will induce different lattice structures. So the multiplication structure of this \mathbb{Z} -algebra is important.

Recall first that to every map from S^n to S^n one can associate an integer called the degree. The degree is a homotopy invariant. We have a group isomorphism $deg : \pi_n(S^n) \rightarrow \mathbb{Z}$. For the detail of degree maps of sphere, the reader is referred to the chapter 2.2 of [8]. The lattice is an invariant of topological spaces. Two spaces with the same rational homotopy type may have different lattice structure.

Example 4.1.1. Let $\eta : S^3 \rightarrow S^2$ be the Hopf map and $f_n : S^3 \rightarrow S^3$ be the degree n map, where n is a positive integer. $i : S^3 \rightarrow D^4$ is the boundary map. We take the homotopy cofiber of the map $\eta \circ f_n$, then we get a pushout diagram.

$$\begin{array}{ccc} S^3 & \xrightarrow{\eta \circ f_n} & S^2 \\ \downarrow i & & \downarrow \\ D^4 & \dashrightarrow & X[n] \end{array}$$

It is easy to compute the cohomology of $X[n]$ by using cellular cohomology theory. $H^i(X[n]; \mathbb{Z}) = \mathbb{Z}$, for $i = 0, 2, 4$, and $H^i(X[n]; \mathbb{Z}) = 0$, otherwise. Let $\alpha[n]$ be the generator of $H^2(X[n]; \mathbb{Z})$ and $\beta[n]$ be the generator of $H^4(X[n]; \mathbb{Z})$.

$$\alpha[n]^2 = H(\eta \circ f_n)\beta[n] = deg(f_n)H(\eta)\beta[n] = n\beta[n]. \quad (4.34)$$

If n and m are positive integers and $n \neq m$, $X[n]$ and $X[m]$ have different lattice structures. But it is easy to construct an isomorphism between the rational cohomology ring

of $X[n]$ and the rational cohomology ring of $X[m]$. So $X[n]$ and $X[m]$ have the same rational homotopy type. In particular, $X[1]$ is $\mathbb{C}P^2$.

For the detail of the definition and the properties of Hopf invariants, the reader is referred to the additional topics 4.B of [8].

We give other examples.

Example 4.1.2. $S^2 \times S^2$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ are formal spaces. Then, if we want to prove these two spaces have the same rational homotopy type, we just need to prove that their rational cohomology rings are quasi-equivalent to each other.

The cohomology ring of $S^2 \times S^2$ is

$$R[x, y]/\langle x^2, y^2 \rangle, |x| = |y| = 2. \quad (4.35)$$

The cohomology ring of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ is

$$R[a, b]/\langle a^2 + b^2, ab \rangle, |a| = |b| = 2. \quad (4.36)$$

If $R = \mathbb{Q}$,

$$f : R[x, y]/\langle x^2, y^2 \rangle \rightarrow R[a, b]/\langle a^2 + b^2, ab \rangle, x \mapsto a + b, y \mapsto a - b \quad (4.37)$$

is a quasi-isomorphism. So they have the same rational homotopy type. But, if $R = \mathbb{Z}$, it is impossible to construct an isomorphism between the cohomology ring of $S^2 \times S^2$ and the cohomology ring of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. So these two cohomology rings are not isomorphic. They have different lattice structures.

4.2 The category of commutative differential graded algebra with lattices

We begin with the definition of a commutative graded algebra with lattice.

Definition 4.2.1. A commutative graded algebra with lattice (A, L) consists of a commutative graded algebra and a commutative graded \mathbb{Z} -subalgebra L of A , such that

(1) L is free as a graded \mathbb{Z} -module, and

(2) There exists an isomorphism $\varphi : L \otimes \mathbb{Q} \rightarrow A$, such that $\varphi \circ i = I$, where $i : L \rightarrow$

$L \otimes \mathbb{Q}$ and $I : L \rightarrow A$ are the inclusion maps.

$$\begin{array}{ccc}
 L & \xrightarrow{i} & L \otimes \mathbb{Q} \\
 & \searrow I & \swarrow \varphi \\
 & & A
 \end{array}$$

Let's recall that if A is a commutative differential graded algebra, $H(A)$ is a commutative graded algebra.

Definition 4.2.2. Let A be a commutative differential graded algebra and L be a free graded \mathbb{Z} -module. We say (A, L) is a commutative differential graded algebra with lattice if $(H(A), L)$ is a commutative graded algebra with lattice.

Remark 4.2.3. The lattice is choosing generators for a graded \mathbb{Z} -submodule of a commutative graded algebra $H(A)$. For example, $(\mathbb{Q}\langle a \rangle, \mathbb{Z}\langle ta \rangle)$, where $|a| = 0$ and t is a rational number, is a commutative differential graded algebra with lattice. Also, we have to mention that, because $\text{Aut}(\mathbb{Z}) = \{\pm 1\}$, $(\mathbb{Q}\langle a \rangle, \mathbb{Z}\langle ta \rangle)$ and $(\mathbb{Q}\langle a \rangle, \mathbb{Z}\langle -ta \rangle)$ are the same commutative differential graded algebra with lattices.

Definition 4.2.4. A homomorphism of commutative graded algebra with lattices $f : (A, L_1) \rightarrow (B, L_2)$ is a homomorphism of commutative graded algebras $f : A \rightarrow B$, such that $f(L_1) \subseteq L_2$.

Definition 4.2.5. We say $f : (A, L_1) \rightarrow (B, L_2)$ is a homomorphism of commutative differential graded algebra with lattices if $f : A \rightarrow B$ is a homomorphism of commutative differential graded algebras and $H(f) : (H(A), L_1) \rightarrow (H(B), L_2)$ is a homomorphism of commutative graded algebra with lattices.

Let (A, L_1) and (B, L_2) be commutative differential graded algebra with lattices. The tensor product $(A, L_1) \otimes (B, L_2)$ is defined to be $(A \otimes_{\mathbb{Q}} B, L_1 \otimes_{\mathbb{Z}} L_2)$.

We have defined two categories, the category of commutative graded algebra with lattices denoted by $\mathbf{CGA} + \mathbf{L}$ and the category of commutative differential graded algebra with lattices denoted by $\mathbf{CDGA} + \mathbf{L}$. In the $\mathbf{CDGA} + \mathbf{L}$ we define a homomorphism $f : (A, L_1) \rightarrow (B, L_2)$ to be a quasi-isomorphism if and only if $f : A \rightarrow B$ is a quasi-isomorphism in the category of commutative differential graded algebras and $H(f)(L_1) = L_2$.

The Theorem 3.2.3 states that there is a natural isomorphism F from $H^*(X; \mathbb{Q})$ to $H(A_{PL}(X))$. And we have a natural map $l : H^*(X; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Q})$. Then $F \circ l : H^*(X; \mathbb{Z}) \rightarrow H(A_{PL}(X))$ is a natural homomorphism of graded \mathbb{Z} -algebra. $F(L_X) = F \circ l(H^*(X; \mathbb{Z}))$ is a commutative graded \mathbb{Z} -subalgebra of $H(A_{PL}(X))$ and $F(L_X) \otimes \mathbb{Q}$ is isomorphic to $H(A_{PL}(X))$. So for any topological space X , $(A_{PL}(X), F(L_X))$ is a commutative differential graded algebra with lattice. Let f be a continuous map from

topological space X to Y . Since $F \circ l$ is a natural map, f induces a map from $F(L_X)$ to $F(L_Y)$. So we can define a functor

$$AL : \mathbf{Top} \rightarrow \mathbf{CDGA} + \mathbf{L} \quad (4.38)$$

where $AL(X) = (A_{PL}(X), F(L_X))$.

4.3 The quasi-isomorphism classes of commutative graded algebra with lattices

Now We focus on the lattice structures of some basic commutative differential graded algebras. We compute the quasi-isomorphism classes of the commutative differential graded algebra with lattices.

Example 4.3.1. *(The rational homotopy type of sphere S^n)*

If n is odd, the minimal model is $\Lambda(a), |a| = n$. The lattice structure is $\mathbb{Z}\langle ta \rangle$, where t is a positive rational number. We can define a commutative differential graded algebra with lattice $(\Lambda(a_0), \mathbb{Z}\langle a_0 \rangle), |a_0| = |a|$ and a map

$$f : (\Lambda(a_0), \mathbb{Z}\langle a_0 \rangle) \rightarrow (\Lambda(a), \mathbb{Z}\langle ta \rangle), a_0 \mapsto ta \quad (4.39)$$

where f is a quasi-isomorphism. There is one lattice structure up to quasi-isomorphism on the minimal model of odd sphere of commutative differential graded algebra with lattices.

If n is even, the minimal model is $(\Lambda(a, b), db = a^2), |a| = n$. The lattice structure is $\mathbb{Z}\langle ta \rangle$, where t is a positive rational number. We can define a commutative differential

graded algebra with lattice $(\Lambda(a_0, b_0), \mathbb{Z}\langle a_0 \rangle)$, $db_0 = a_0^2$, $|a_0| = |a|$ and a map

$$f : (\Lambda(a_0, b_0), \mathbb{Z}\langle a_0 \rangle) \rightarrow (\Lambda(a, b), \mathbb{Z}\langle ta \rangle), a_0 \mapsto ta, b_0 \mapsto t^2b \quad (4.40)$$

where f is a isomorphism. There is one lattice structure up to quasi-isomorphism on the minimal model of even sphere of commutative differential graded algebra with lattices.

Let a and b are nonzero rational numbers. We say a is a divisor of b denoted by $a|b$ if there is a integer n such that $an = b$. And we define $lcm(a_1, a_2, \dots, a_k)$, where a_1, a_2, \dots, a_k are nonzero rational numbers, to be the least positive integer n such that $a_1|n, a_2|n, \dots, a_k|n$.

Example 4.3.2. (The minimal Sullivan model of complex projective space $\mathbb{C}P^n$)

The minimal model of $\mathbb{C}P^n$ is $(\Lambda(a, b), db = a^{n+1})$, $|a| = 2$. The lattice structure is $\mathbb{Z}\langle t_1a, t_2a^2, \dots, t_na^n \rangle$, where t_i are positive rational numbers and $t_{i+j}|t_it_j$, $2 \leq i + j \leq n$, $1 \leq i \leq j \leq n$. There are $n\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor^2$ divisor relations. We can define a commutative differential graded algebra with lattice

$$(\Lambda(a_0, b_0), \mathbb{Z}\langle s_1^{-1}a_0, s_2^{-1}a_0^2, \dots, s_n^{-1}a_0^n \rangle), |a_0| = |a|, |b_0| = |b|, \quad (4.41)$$

such that

$$s_1 = 1, s_{m+1} = \frac{t_1^{m+1}}{t_{m+1}} = \frac{t_1^{m+1}}{t_{m+1}lcm_{i+j=m+1}\{s_i s_j\}} lcm_{i+j=m+1}\{s_i s_j\} \quad (4.42)$$

for $1 \leq m < n$. We let

$$k_{m+1} = \frac{t_1^{m+1}}{t_{m+1}lcm_{i+j=m+1}\{s_i s_j\}}. \quad (4.43)$$

Then we can easily prove that k_{m+1} is an integer by induction.

$$f : (\Lambda(a_0, b_0), \mathbb{Z}\langle s_1^{-1}a_0, s_2^{-1}a_0^2, \dots, s_n^{-1}a_0^n \rangle) \rightarrow (\Lambda(a, b), \mathbb{Z}\langle t_1a, t_2a^2, \dots, t_na^n \rangle) \quad (4.44)$$

where $f(a_0) = t_1 a$, $f(b_0) = t_1^{n+1} b$. f forms an isomorphism. It is easy to see that the lattice structure up to quasi-isomorphism on $(\Lambda(a, b), db = a^{n+1})$, $|a| = 2$ is determined by $n - 1$ integers k_{m+1} , $1 \leq m < n$. By the way, if $|a|$ is even, we can get a similar result.

To be concrete, if $n = 4$. $s_1 = 1$. $\text{lcm}_{i+j=2}\{s_i s_j\} = 1$, then $s_2 = k_2$. $\text{lcm}_{i+j=3}\{s_i s_j\} = \text{lcm}\{k_2\} = k_2$, then $s_3 = k_3 k_2$. $\text{lcm}_{i+j=4}\{s_i s_j\} = \text{lcm}\{k_3 k_2, k_2^2\}$, then $s_4 = k_4 \text{lcm}\{k_3 k_2, k_2^2\}$. The quasi-isomorphism classes are represented by

$$\{(\Lambda(a_0, b_0), \mathbb{Z}\langle a_0, k_2^{-1} a_0^2, (k_2 k_3)^{-1} a_0^3, (k_4 \text{lcm}\{k_3 k_2, k_2^2\})^{-1} a_0^4 \rangle)\} \quad (4.45)$$

where k_2, k_3 and k_4 are positive integers.

Example 4.3.3.

Let $(\Lambda(a, b, c), dc = ab)$, $|a| = 3$, $|b| = 5$ be a commutative differential graded algebra with lattice. Then $H(\Lambda(a, b, c), dc = ab) = \Lambda(x, y, z, w) / \langle xw - yz, xy, xz, zw, yw \rangle$, $|x| = 3$, $|y| = 5$, $|z| = 10$, $|w| = 12$, is a commutative graded algebra with lattice.

The lattice structure on $(\Lambda(a, b, c), dc = ab)$ is $\mathbb{Z}\langle t_1 a, t_2 b, t_3 ac, t_4 bc, t_5 abc \rangle$, where t_1, t_2, t_3, t_4, t_5 are positive rational numbers and $t_5 | t_1 t_4$ and $t_5 | t_2 t_3$. We can define a commutative differential graded algebra with lattice

$$((\Lambda(a_0, b_0, c_0), dc_0 = a_0 b_0), \mathbb{Z}\langle a_0, b_0, \frac{t_5}{t_1^2 t_2^2} \frac{t_2 t_3}{t_5} a_0 c_0, \frac{t_5}{t_1^2 t_2^2} \frac{t_1 t_4}{t_5} b_0 c_0, \frac{t_5}{t_1^2 t_2^2} a_0 b_0 c_0 \rangle) \quad (4.46)$$

and a commutative differential graded algebra map such that $f(a_0) = t_1 a$, $f(b_0) = t_2 b$, $f(c_0) = t_1^2 t_2^2 c$. f forms an isomorphism. Let

$$\frac{t_5}{t_1^2 t_2^2} = r, \quad \frac{t_2 t_3}{t_5} = n, \quad \frac{t_1 t_4}{t_5} = m. \quad (4.47)$$

Then r is a positive rational number. n and m are positive integers. It is easy to see that the lattice structures up to quasi-isomorphism on $(\Lambda(a, b, c), dc = ab)$ are determined by n, m and r .

The lattice structure on $\Lambda(x, y, z, w)/\langle xw - yz, xy, xz, zw, yw \rangle$ is $\mathbb{Z}\langle t_1x, t_2y, t_3z, t_4w, t_5xw \rangle$, where $t_5|t_1t_4$ and $t_5|t_2t_3$. We can define

$$(\Lambda(x_0, y_0, z_0, w_0)/\langle x_0w_0 - y_0z_0, x_0y_0, x_0z_0, z_0w_0, y_0w_0 \rangle, \mathbb{Z}\langle x_0, y_0, \frac{t_2t_3}{t_5}z_0, \frac{t_1t_4}{t_5}w_0, x_0w_0 \rangle) \quad (4.48)$$

a commutative graded algebra with lattice and a commutative graded algebra map such that $f(x_0) = t_1x, f(y_0) = t_2y, f(z_0) = \frac{t_5}{t_2}z, f(w_0) = \frac{t_5}{t_1}w$. f forms a isomorphism. Let

$$\frac{t_2t_3}{t_5} = n, \frac{t_1t_4}{t_5} = m. \quad (4.49)$$

Then n and m are positive integers. It is easy to see that the lattice structures up to quasi-isomorphism on $\Lambda(x, y, z, w)/\langle xw - yz, xy, xz, zw, yw \rangle$ are determined by n and m .

Let $(A, L_1) = ((\Lambda(a_0, b_0, c_0), dc_0 = a_0b_0), \mathbb{Z}\langle a_0, b_0, na_0c_0, mb_0c_0, a_0b_0c_0 \rangle)$ and $(A, L_2) = ((\Lambda(a_0, b_0, c_0), dc_0 = a_0b_0), \mathbb{Z}\langle a_0, b_0, 2na_0c_0, 2mb_0c_0, 2a_0b_0c_0 \rangle)$. They are not quasi-isomorphic to each other. But $(H(A), L_1) \cong (\Lambda(x_0, y_0, z_0, w_0)/\langle x_0w_0 - y_0z_0, x_0y_0, x_0z_0, z_0w_0, y_0w_0 \rangle, \mathbb{Z}\langle x_0, y_0, nz_0, mw_0, x_0w_0 \rangle) \cong (H(A), L_2)$. So A is quasi-isomorphic to A and L_1 is isomorphic to L_2 , but (A, L_1) is not quasi-isomorphic to (A, L_2) ,

This example shows that commutative differential graded algebra with lattices include more information than commutative differential graded algebras and lattices separately.

4.4 Integral Sullivan models

Now we want to define an integral Sullivan model for the commutative differential graded algebra with lattice. It is similar to the Sullivan model of a commutative differential graded algebra.

Let $A_{\mathbb{Z}}$ be a commutative differential graded algebra over integers. By the universal coefficient theorem, $(A_{\mathbb{Z}} \otimes \mathbb{Q}, H(A_{\mathbb{Z}})/torsion)$ is a commutative differential graded algebra with lattice.

Definition 4.4.1. *An integral model of a commutative differential graded algebra with lattice (A, L) is a commutative differential graded algebra $A_{\mathbb{Z}}$ over the integers such that there exists a commutative differential graded \mathbb{Z} -algebra map $f : A_{\mathbb{Z}} \rightarrow A$ and*

$$f \otimes \mathbb{Q} : (A_{\mathbb{Z}} \otimes \mathbb{Q}, H(A_{\mathbb{Z}})/torsion) \rightarrow (A, L) \quad (4.50)$$

is a quasi-isomorphism.

Definition 4.4.2. *We say an integral model $A_{\mathbb{Z}}$ of a commutative differential graded algebra with lattice (A, L) is a Sullivan model if $A_{\mathbb{Z}}$ is a Sullivan algebra. We say $A_{\mathbb{Z}}$ is the integral Sullivan model of a topological space X if $A_{\mathbb{Z}}$ is the integral Sullivan model of $(A_{PL}(X), F(L_X))$.*

Integral Sullivan models of a topological space X carry two pieces of information about X , the rational homotopy type of X and the lattice structure of X . If $A_{\mathbb{Z}}$ is the integral Sullivan model of X , then $A_{\mathbb{Z}} \otimes \mathbb{Q}$ is just the Sullivan model of X . But $A_{\mathbb{Z}} \otimes \mathbb{Q}$ may not be minimal.

Theorem 4.4.3. *Let (A, L) be a commutative differential graded algebra with lattice. If A is 1-connected and the dimension of $H(A)$ as a graded vector space is finite, then (A, L) has an integral Sullivan model.*

Proof. Let $T_1 = \{b_i\}$ be a finite subset of A such that $\{[b_i]\}$ is a basis of the graded \mathbb{Z} -module L . Then we find a set $T_2 = \{b_j\}$ of A such that $[b_j] = 0$ and $T_1 \cup T_2$ is a basis of cocycles of A as a graded \mathbb{Q} -vector space. Then we find a subset $T_3 = \{b_k\}$ of A such that $T = T_1 \cup T_2 \cup T_3$ is a basis of the graded \mathbb{Q} -vector space A .

We define an integral commutative differential graded algebra $A_0 = \Lambda_{\mathbb{Z}}(\{e_i\}, |e_i| = |b_i|)$, and an integral commutative differential graded algebra homomorphism

$$f : (\Lambda_{\mathbb{Z}}(\{e_i\}), 0) \rightarrow A, e_i \mapsto b_i \quad (4.51)$$

It induces a map $f \otimes \mathbb{Q} : (A_0 \otimes \mathbb{Q}, H(A_0)/torsion) \rightarrow (A, L)$. At this stage $H(f \otimes \mathbb{Q}) : H(A_0 \otimes \mathbb{Q}) \rightarrow H(A)$ is an isomorphism in degrees less than 4. $H(f) : (H(A_0)/torsion) \rightarrow L$ is an isomorphism in degrees less than 4. The map

$$H(f) : (H(A_0)/torsion) \rightarrow L \quad (4.52)$$

is a surjective map.

We will prove inductively that for any n there are an integral free commutative differential graded algebra A_n together with an integral commutative differential graded algebra map $f : A_n \rightarrow A$ and two subset $S[n]$ and $B[n]$ of A_n such that

- (1) The algebra A_n has no elements in dimension 1 and no generators in degrees greater than $n + 1$ except e_i ;

(2) The $S[n]$ is a finite set. $S[n] = \{\alpha_1[n], \alpha_2[n], \dots, \alpha_{i_n}[n]\}$ ($\{e_i\} \subseteq S[n]$) is a basis of the graded algebra A_n . $B[n] = \{[\beta_1[n]], [\beta_2[n]], \dots, [\beta_{j_n}[n]], \dots\}$ is a basis of graded module A_n . The elements in $B[n]$ are finite products of elements in $S[n]$. $B[n]$ may not be a finite set. But, for any positive integer M , the subset $\{\beta | \beta \in B[n], |\beta| < M\}$ is a finite set.

(3) The map $f \otimes \mathbb{Q} : A_n \otimes \mathbb{Q} \rightarrow A$ is a commutative differential graded algebra map. And $H(f \otimes \mathbb{Q}) : H(A_n \otimes \mathbb{Q}) \rightarrow H(A)$ is an isomorphism in degrees less than $n + 2$. For any $\beta \in B[n]$ and $\beta \notin \{e_i\}$, $f(\beta) = \sum_i K_{i\beta} b_i + \sum_j l_{j\beta} b_j + \sum_k m_{k\beta} b_k$, $K_{i\beta}$ are integers. $H^t(f) : (H^t(A_n)/torsion) \rightarrow L^t$ is an isomorphism for $t \leq n + 1$.

So suppose this is true for $n = q - 1$. Let $H^{q+1}(A_{q-1})/torsion = \mathbb{Z}\langle [e_{1q+1}], \dots, [e_{i_{q+1}q+1}], [\delta_{1q+1}], \dots, [\delta_{j_{q+1}q+1}] \rangle$ where $\{e_{1q+1}, \dots, e_{i_{q+1}q+1}\} \subseteq \{e_i\}$. By hypothesis we know that, for $u \in \{1, 2, \dots, j_{q+1}\}$,

$$\delta_{uq+1} = \sum_{\beta \in B[q-1], |\beta|=q+1} V_\beta \beta. \quad (4.53)$$

where V_β are integers. Then

$$f(\delta_{uq+1}) = \sum_{\beta \in B[q-1], |\beta|=q+1} V_\beta \left(\sum_i K_{i\beta} b_i + \sum_j l_{j\beta} b_j + \sum_k m_{k\beta} b_k \right) \quad (4.54)$$

δ_{uq+1} is a cocycle, then $f(\delta_{uq+1})$ is a cocycle. Then $\sum V_\beta \sum m_{k\beta} b_k = 0$. So $f(\delta_{uq+1})$ is of the form $\sum_i K_{uq+1i} b_i + \sum_j l_{uq+1j} b_j$ where K_{uq+1i} are integers.

Now we know that $H(f)([e_i]) = [b_i]$ and $H(f)([\delta_{uq+1}]) = \sum_i K_{uq+1i} [b_i] + \sum_j l_{uq+1j} [b_j] = \sum_i K_{uq+1i} [b_i]$. Then $H^{q+1}(f)(H^{q+1}(A_n)/torsion) \subseteq L^{q+1}$ and

$$\ker H^{q+1}(f) = \mathbb{Z}\langle [\delta_{1q+1} - \sum_i K_{1q+1i} e_i], [\delta_{2q+1} - \sum_i K_{2q+1i} e_i], \dots, [\delta_{j_{q+1}q+1} - \sum_i K_{j_{q+1}q+1i} e_i] \rangle \quad (4.55)$$

We let $\alpha_1 = \delta_{1q+1} - \sum_i K_{1q+1i}e_i$, K_{1q+1i} are integers. We choose a element $r_{\alpha_1} \in A$ with degree q such that $f(\alpha_1) = dr_{\alpha_1}$ (If $f(\alpha_1) = 0$, we let $r_{\alpha_1} = 0$).

$$\begin{array}{ccc} & r_{\alpha_1} & \\ & \downarrow d & \\ \alpha_1 & \xrightarrow{f} & f(\alpha_1) \end{array}$$

For any $\gamma \in B[q-1]$ and l a non-negative integer, we let $(r_{\alpha_1})^l f(\gamma) = \sum_i C_{li\gamma} b_i + \sum_j D_{lj\gamma} b_j + \sum_k E_{lk\gamma} b_k$, where $C_{li\gamma}$, $D_{lj\gamma}$ and $E_{lk\gamma}$ are rational numbers. We fix l and γ . Let recall that $\{b_i\}$ is a finite set. If there exist an i_0 such that $C_{li_0\gamma} \neq 0$, we can choose a positive integer $M_{l\gamma}$ such that $M_{l\gamma} C_{li\gamma}$ are integers, for all i . If $C_{li\gamma} = 0$ for all i , then we choose $M_{l\gamma}$ to be 0.

It is easy to see that there exist two positive integers L and K such that, if $l > L$ or $|\gamma| > K$, $|(r_{\alpha_1})^l f(\gamma)| = l|r_{\alpha_1}| + |f(\gamma)| > Lq + K > \max_i \{|b_i|\}$ (For example, let $L = K = \max_i \{|b_i|\} + 1$). Then $C_{li\gamma} = 0$ if $l > L$, $|\gamma| > K$. This imply that $M_{l\gamma} = 0$ for $l > L$ or $|\gamma| > K$. By hypothesis we know that $\{M_{l\gamma} | l \leq L, |\gamma| \leq K, M_{l\gamma} \neq 0\}$ is a finite set. We can define $M_{\alpha_1} = lcm\{M_{l\gamma} | l \leq L, |\gamma| \leq K, M_{l\gamma} \neq 0\}$.

Define

$$A'_q = A_{q-1} \otimes \Lambda_{\mathbb{Z}}(\theta_1), |\theta_1| = q,$$

Then A'_q is again a free algebra, with differential

$$d(\theta_1) = M_{\alpha_1}(\delta_{1q+1} - \sum_i K_{1q+1i}e_i).$$

And we define

$$S'[q] = S[q-1] \cup \{\theta_1\}, B'[q] = B[q-1] \cup \{\theta_1^l \gamma | l \in \mathbb{Z}^{>0}, \gamma \in B[q-1]\}.$$

We extend $f : A_{q-1} \rightarrow A$ to $f : A'_q \rightarrow A$ by

$$f(\theta_1) = M_{\alpha_1} r_{\alpha_1}.$$

Since $d \circ f(\theta_1) = M_{\alpha_1} dr_{\alpha_1} = M_{\alpha_1} f(\alpha_1) = f(M_{\alpha_1} \alpha_1) = f \circ d(\theta_1)$, this new f is again a chain map. We have $f(\theta_1^l \gamma) = M_{\alpha_1}^l (r_{\alpha_1})^l f(\gamma) = \sum_i M_{\alpha_1}^l C_{li\gamma} t_i b_i + \sum_j M_{\alpha_1}^l D_{lj\gamma} b_j + \sum_k M_{\alpha_1}^l E_{lk\gamma} b_k$. By our definition of M_{α_1} , $M_{\alpha_1}^l C_{li\gamma} = M_{\alpha_1}^{l-1} (M_{\alpha_1} C_{li\gamma})$ are integers. The rank of $\ker H^{q+1}(f)$ is strictly smaller. We treat $A'_q, S'[q]$ and $B'[q]$ as $A_{q-1}, S[q-1]$ and $B[q-1]$. Then we repeat this procedure $j_{q+1} - 1$ times. We can get a commutative graded algebra A_q and a new map $f : A_q \rightarrow A$. The $\ker H^{q+1}(f)$ is 0. This means that $H(f) : H(A_q)/torsion \rightarrow L$ is an isomorphism in degree $q+1$ and the basis of the free \mathbb{Z} -module $H^{q+1}(A_q)/torsion$ is $\{[e_{1q+1}], \dots, [e_{i_{q+1}q+1}]\}$. Actually, we add j_{q+1} elements $\{\theta_1, \dots, \theta_{j_{q+1}}\}$ in degree q to kill the cocycles $\delta_{1q+1} - \sum_i K_{1q+1i} e_i, \dots, \delta_{j_{q+1}q+1} - \sum_i K_{j_{q+1}q+1i} e_i$. Also, we will get a set $S[q] = \{\theta_1, \dots, \theta_{j_{q+1}}\} \cup S[q-1]$. The elements in $B[q]$ are finite products of elements in $S[q]$. $S[q]$ and $B[q]$ satisfy the hypothesis (2). So $A_q, S[q], B[q]$ and f satisfy the hypotheses in degree q . \square

Corollary 4.4.4. *Let X be a simply connected space with $\sum_{i=0}^{\infty} rank H^i(X; \mathbb{Q}) < \infty$.*

Then X has an integral Sullivan model.

Proof. Let $(\Lambda V, d)$ be the minimal Sullivan model of X . Then $H^*(X; \mathbb{Q}) \cong H((\Lambda V, d))$.

Since $\sum_{i=0}^{\infty} rank H^i(X; \mathbb{Q}) < \infty$, the dimension of $H(\Lambda V)$ as a graded vector space is finite. By the theorem 4.4.3. X has an integral Sullivan model. \square

Unlike the case of rational numbers, we cannot define the minimal property of an integral Sullivan model.

According to the proof of the theorem 4.4.3, we have a method to construct the integral Sullivan algebra of a commutative differential graded algebra with lattice.

Example 4.4.5.

Let $(\Lambda(a_0), \mathbb{Z}\langle a_0 \rangle)$, $|a_0|$ be an odd number, be a commutative differential graded algebra with lattice. An integral Sullivan model is

$$f : \Lambda_{\mathbb{Z}}(a) \rightarrow \Lambda(a_0), a \mapsto a_0 \quad (4.56)$$

where $|a| = n$. Then $f \otimes \mathbb{Q}$ is a quasi-isomorphism in $\mathbf{CDGA} + \mathbf{L}$.

Let $((\Lambda(a_0, b_0), db_0 = a_0^2), \mathbb{Z}\langle a_0 \rangle)$, $|a_0|$ be an even number, be commutative differential graded algebra with lattice. The integral Sullivan model is

$$f : (\Lambda_{\mathbb{Z}}(a, b), db = a^2) \rightarrow (\Lambda(a_0, b_0), db_0 = a_0^2), a \mapsto a_0, b \mapsto b_0 \quad (4.57)$$

where $|a| = n, |b| = 2n - 1$. Then $f \otimes \mathbb{Q}$ is a quasi-isomorphism in $\mathbf{CDGA} + \mathbf{L}$.

Example 4.4.6.

For $((\Lambda(a_0, b_0), db_0 = a_0^{n+1}), \mathbb{Z}\langle s_1^{-1}a_0, s_2^{-1}a_0^2, \dots, s_n^{-1}a_0^n \rangle)$, $|a_0| = 2, |b_0| = 2n + 1$, where $s_1 = 1, s_{m+1} = (t_1^{m+1}/t_{m+1} \text{lcm}_{i+j=m+1}\{s_i s_j\}) \text{lcm}_{i+j=m+1}\{s_i s_j\}$ are integers, we define $A_0 = \Lambda_{\mathbb{Z}}(x_1, x_2, \dots, x_n)$, $|x_1| = 2, |x_2| = 4, \dots, |x_n| = 2n$ and a map

$$f : \Lambda_{\mathbb{Z}}(x_1, x_2, \dots, x_n) \rightarrow (\Lambda(a_0, b_0), db_0 = a_0^{n+1}), x_i \mapsto s_i^{-1}a_0^i \quad (4.58)$$

Then $H(f) : H(A_0) \rightarrow L$ is a surjective map. We know

$$H(f)([x_1^i - s_i x_i]) = [a_0^i - s_i (s_i^{-1} a_0^i)] = [a_0^i - a_0^i] = 0 \quad (4.59)$$

So we need to kill the cocycle $x_1^i - s_i x_i$. So the integral Sullivan model is

$$f : (\Lambda_{\mathbb{Z}}(x_1, \dots, x_n, y, y_2, \dots, y_n), dy = x_1^{n+1}, dy_i = x_1^i - s_i x_i) \rightarrow (\Lambda(a_0, b_0), db_0 = a_0^{n+1}) \quad (4.60)$$

where $f(x_i) = s_i^{-1} a_0^i, 1 \leq i \leq n, f(y_j) = 0, 1 < j \leq n, f(y) = b$. It is easy to check $f \otimes \mathbb{Q}$ is a quasi-isomorphism in $\mathbf{CDGA} + \mathbf{L}$.

Example 4.4.7.

For $((\Lambda(a, b, c), dc = ab), \mathbb{Z}\langle a, b, rnac, rmbc, rabc \rangle), |a| = 3, |b| = 5, |c| = 7$, where r is a positive rational number and n, m are positive integers, let $A_{\mathbb{Z}} = (\Lambda_{\mathbb{Z}}(e_1, \dots, e_5, b_1, \dots, b_4), db_1 = rnme_1 e_2, db_2 = me_3 - e_1 b_1, db_3 = ne_4 - e_2 b_1, db_4 = nme_5 - e_1 e_2 b_1), |e_1| = 3, |e_2| = 5, |e_3| = 10, |e_4| = 12, |e_5| = 15, |b_1| = 7, |b_2| = 9, |b_3| = 11, |b_4| = 14$. The integral Sullivan model is

$$f : A_{\mathbb{Z}} \rightarrow (\Lambda(a_0, b_0, c_0), dc_0 = a_0 b_0) \quad (4.61)$$

where $f(e_1) = a, f(e_2) = b, f(e_3) = rnac, f(e_4) = rmbc, f(e_5) = rnmac, f(b_1) = rnmc, f(b_2) = f(b_3) = f(b_4) = 0$. We can check $f \otimes \mathbb{Q}$ is a quasi-isomorphism in $\mathbf{CDGA} + \mathbf{L}$.

In theorem 4.4.3, we show that if A is 1-connected and $\sum_{i=0}^{\infty} \text{rank} H^i(X; \mathbb{Q}) < \infty$, then (A, L) has an integral Sullivan model. We next show a counterexample to this theorem satisfying $\sum_{i=0}^{\infty} \text{rank} H^i(X; \mathbb{Q}) = \infty$.

Let $A = (\Lambda(x, y, z), dz = xy), |x| = 3, |y| = 4, |z| = 6$. Then $H^i(A) = \mathbb{Q}\langle y^n \rangle$, if $i = 4n, n \geq 0$. $H^i(A) = \mathbb{Q}\langle xz^n \rangle$, if $i = 3 + 6n, n \geq 0$. $H^i(A) = 0$, otherwise.

Proposition 4.4.8. *There exist a lattice structure $L = \mathbb{Z}\langle u_1y^1, \dots, u_ny^n, \dots, v_0xz^0, \dots, v_nxz^n, \dots \rangle$ on $A = (\Lambda(x, y, z), dz = xy, |x| = 3, |y| = 4, |z| = 6)$ such that (A, L) doesn't have an integral Sullivan model.*

Proof. We assume (A, L) has an integral Sullivan model $A_{\mathbb{Z}}$. Then $A_{\mathbb{Z}} \otimes \mathbb{Q}$ is a Sullivan model of the minimal Sullivan algebra A . And $H(f) : H(A_{\mathbb{Z}})/torsion \rightarrow L$ is a isomorphism. So we can choose $a \in A_{\mathbb{Z}}$ and $b \in A_{\mathbb{Z}}$ such that $f(a) = v_0x$ and $f(b) = u_1y$. Then $f(ab) = f(a)f(b) = v_0u_1xy$ and $H(f)([ab]) = [f(ab)] = v_0u_1[xy] = 0$. Then there exist a elements $c \in A_{\mathbb{Z}}$ such that $dc = rab$ with r a nonzero rational numbers. f forms a chain map, then $f(c) = rv_0u_1z$. We have

$$f(ac^n) = f(a)f(c^n) = v_0^{n+1}u_1^n r^n xz^n = \frac{v_0^{n+1}u_1^n r^n}{u_n} u_n xz^n. \quad (4.62)$$

Since $ac^n \in A_{\mathbb{Z}}$, $H(f)([ac^n]) \in L$, so $u_n | v_0^{n+1}u_1^n r^n$ for all n . This is impossible. For example, we let u_n be the n th prime number and let $v_i = 1$ for $i \in \mathbb{Z}^{\geq 0}$, it is easy to see that there exist a u_{n_0} such that $u_{n_0} \nmid v_0^{n+1}u_1^n r^n$. It is a contradiction. So this (A, L) doesn't have an integral Sullivan models. □

4.5 Constructing a topological space for a commutative differential graded algebra with lattice

A basic problem in rational homotopy theory is to construct a topological space with a given rational homotopy type. So we want to ask the similar question for the commutative differential graded algebra with lattices. More precisely, for a commutative differential graded algebra with lattice (A, L) , can we construct a topological space X such that $(A_{PL}(X), F(L_X))$ is quasi-equivalent to the (A, L) ?

We give some observations on this question in a special case. $A = (\Lambda(a, b), db = a^{n+1})$. Recall that in the example 4.3.2. we have already found all possible lattice structures (up to quasi-isomorphism) on A .

If $n = 1$, there is one possible lattice structure on A . So X is $\mathbb{C}P^1$ (or S^2). If $n = 2$, the lattice structures on A is $L(m) = \mathbb{Z}\langle a, m^{-1}a^2 \rangle$ completely determined by positive integer m . Actually, we have done this case in the example 4.1.1. $AL(X[m]) = (A, L(m))$.

If $n > 2$, the problem gets complicated. We first give a brief introduction of weighted projective space. Let $B = (b_0, \dots, b_n)$ be a vector of positive integers. The associated weighted projective space is defined to be the quotient

$$\mathbb{P}^n(B) = S^{2n+1}/S^1(b_0, \dots, b_n), \quad (4.63)$$

where the numbers b_0, \dots, b_n indicate the weighted with S^1 acts on the unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$,

$$g \cdot (x_0, \dots, x_n) = (g^{b_0}x_0, \dots, g^{b_n}x_n). \quad (4.64)$$

If $B = (1, \dots, 1)$, then $\mathbb{P}^n(B) = \mathbb{C}P^n$. It is easy to see that $H^*(\mathbb{P}^n(B); \mathbb{Q}) \cong H^*(\mathbb{C}P^n; \mathbb{Q})$.

The rational cohomology ring of weighted projective spaces does not depend on the weighted vectors. But different weighted vectors may give different lattice structures. In Testuro Kawasaki's paper [10], he computes the cohomology of weighted projective space with integer coefficients. If a_1 is a generator of the group $H^2(\mathbb{P}^n(B))$, then $H^*(\mathbb{P}^n(B))$ is generated as a graded ring by the elements

$$a_m = \frac{\text{lcm}\{\prod_{i \in I} b_i : |I| = m\}}{\text{lcm}(b_0, \dots, b_n)^m} a_1^m \in H^{2m}(\mathbb{P}^n(B)) \quad (4.65)$$

with $1 \leq m \leq n$, with the obvious multiplication. So the lattice structure of the weighted projective space is

$$L = \mathbb{Z}\langle r_1^{-1}a, r_2^{-1}a^2, \dots, r_n^{-1}a^n \rangle, r_m = \frac{\text{lcm}(b_0, \dots, b_n)^m}{\text{lcm}\{\prod_{i \in I} b_i : |I| = m\}} \quad (4.66)$$

with $1 \leq m \leq n$. Unfortunately, it does not give all the possible lattice structures on the $(\Lambda(a, b), db = a^{n+1})$ when n is bigger than 2. For example, $n = 3$

$$r_1 = 1, r_2 = \frac{\text{lcm}(b_0, b_1, b_2, b_3)^2}{\text{lcm}\{b_0b_1, b_0b_2, b_0b_3, b_1b_2, b_1b_3, b_2b_3\}}, r_3 = \frac{\text{lcm}(b_0, b_1, b_2, b_3)^3}{\text{lcm}\{b_1b_2b_3, b_0b_2b_3, b_0b_1b_3, b_0b_1b_2\}}. \quad (4.67)$$

We can show that $(r_2)^2 | r_3$. If p is a prime number, we just need to show that $2\text{ord}_p(r_2) \leq \text{ord}_p(r_3)$. Let $e_i = \text{ord}_p(b_i)$ for $i = 1, 2, 3, 4$. We can assume $e_1 \leq e_2 \leq e_3 \leq e_4$. Then $2\text{ord}_p(r_2) = 4e_4 - 2e_3 - 2e_4 = 2e_4 - 2e_3$ and $\text{ord}_p(r_3) = 3e_4 - e_2 - e_3 - e_4 = (e_4 - e_3) + (e_4 - e_2)$. So $2\text{ord}_p(r_2) \leq \text{ord}_p(r_3)$.

Let $n = r_2$ and $m = r_3/r_2$. All the possible lattice structures given by weighted projective space are $\mathbb{Z}\langle a, n^{-1}a^2, (nm)^{-1}a^3 \rangle$ with n, m are positive integers and $n|m$. On

the other hand, we know that all the possible lattice structures are $\mathbb{Z}\langle a, n^{-1}a^2, (nm)^{-1}a^3 \rangle$ with n, m are positive integers. So the weighted projective spaces does not provide all lattice structures on $(\Lambda(a, b), db = a^4)$.

Naturally, we will ask whether we can construct a topological space X such that $AL(X)$ is quasi-isomorphic to $((\Lambda(a, b), db = a^4), \mathbb{Z}\langle a, n^{-1}a^2, (nm)^{-1}a^3 \rangle)$ with n, m are positive integers and $n \nmid m$. It is still an interesting question.

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