Robust Mini-Fracturing/DFIT Modeling and Analysis Using Semi-Analytical Strategy

A Thesis

Submitted to the Faculty of Graduate Studies and Research

In Partial Fulfillment of the Requirements

For the Degree of

Master of Applied Science

in

Petroleum Systems Engineering

University of Regina

By

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Regina, Saskatchewan

December, 2014

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Chang Su, candidate for the degree of Master of Applied Science in Petroleum Systems Engineering, has presented a thesis titled, *Robust Mini-Fracturing/DFIT Modeling and Analysis Using Semi-Analytical Strategy*, in an oral examination held on August 20, 2014. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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ABSTRACT

This thesis concentrates on mini-fracture propagation and closure, providing an innovative way to analyze this process. Source and sink function is implemented as a theoretical foundation to build the modeling strategy. A robust model is successfully developed to investigate fracture propagation by taking initial formation breakdown pressure, expansion of wellbore fluid after fracture initiation into consideration, which have always been ignored in literature by former researchers. Using injecting rate as the reference rate, dimensionless fracturing fluid leak-off rate normally described as flux term can be effectively coupled when source and sink function method is implemented. The dimensionless fracture volume growing rate during injection can be explicitly revealed through establishing the material balance equation among the injection fluid, wellbore fluid expansion and fluid leak-off. For shut-in period, the ultimate fracture half-length at shut-in can be found by diagnosing pressure data during injection and coupling with pressure decline during closure period, adding more confidence to the results analyzed.

Type curves are documented with sensitivity analysis including fracture propagating velocity, wellbore storage effect, formation breakdown pressure, and injecting rate. It is observed that higher fracture propagating velocity in length results in larger ultimate fracture volume. When fluid expansion in wellbore is considered, larger fracture volume is created and more fluid leakage into reservoir is happening, suggesting that fracturing fluid with larger compressibility or wellbore with larger volume can help hold up the bottom hole pressure from falling and contribute further to incremental fracture volume.
When fracturing fluid inside wellbore is compressed to reach a higher formation breakdown pressure level before fracture initiation, faster well bottom pressure decline and more fluid expansion from wellbore fluid occur, creating less fracture volume in mini-fracturing process. At wellhead, smaller injecting rate leads to faster bottom hole pressure decline, less fracture volume but more fluid volume leakage into reservoir.

Mini-fracture or DFIT analysis for the inference of fracturing fluid leak-off volume, fracture length and width is the main targeting application of this study. Moreover, it is capable of analyzing post-fracture performance with variable conductivity like fracture geometry change, and providing a base for hydraulic fracturing design.
ACKNOWLEDGEMENTS

I would like to express my sincere appreciation and gratitude to my supervisors, Dr. Gang Zhao, for his profound and clearly academic supervision during my graduate studies at the University of Regina. I have enjoyable and enlightened memories of discussions with Dr. Gang Zhao, whose inspiration and patience has always encouraged me to step forward in my research area.

I am also grateful to many friends I made in Canada. I won’t forget the happy times we had together, traveling to Calgary and Vancouver. You always encouraged me and built me up when I had problem in either my academic research or personal life in Regina. It was very confusing to be a new comer in Regina for the first few months. It is you guys who helped me to get accustomed to life here.

I would also like to give my thanks to everyone studying petroleum engineering and doing their research at the University of Regina. You created such a good environment for me to do research during these years.

I gratefully acknowledge the Faculty of Graduate Studies and Research at the University of Regina and the Petroleum Technology Research Centre for financial support in the form of scholarships.
DEDICATION

I would like to express special respect to my parents, Tie Su and Junying Xie. I thank them so much for their unlimited support and encouragement to me, as well as their understanding.
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NOMENCLATURE

$c_t =$ total compressibility, $L^2/m$, Mpa$^{-1}$

$c =$ traditional fluid loss coefficient, $L^2/\sqrt{t}$, m$^2/\sqrt{s}$

$C_D =$ dimensionless wellbore storage coefficient

$E =$ Young’s modulus, $m/L^2$, Mpa

$\dot{E} =$ strain plain modulus, $m/L^2$, Mpa

$f_e =$ fluid efficiency, $L^3/L^3$, m$^3$/m$^3$

$G =$ shear modulus

$H =$ reservoir depth, L, m

$h =$ reservoir thickness, L, m

ISIP =$ well instant shut-in pressure, Mpa

$k_r =$ reservoir permeability, $L^2$, m$^2$

$k_f =$ permeability in fracture, $L^2$, m$^2$

$L =$ fracture half length at current time, L, m

$p =$ pressure, $m/L^2$, Mpa

$p_c =$ pressure with constant rate, $m/L^2$, Mpa
$p_v =$ pressure with variable rate, m/L², Mpa

$p_w =$ well bottom pressure, Mpa

$p_b =$ formation breakdown pressure, m/L², Mpa

$p_{std} =$ standardized pressure with fixed half length $x_f$ and with constant rate

$q =$ fluid rate, L³/t, m³/s

$q_{inj} =$ injecting rate at wellhead, L³/t, m³/s

$q_f =$ flux along fracture at $(x, t), L^2/t, m^2/s$

$q_f =$ flux along fracture over a segment, L³/t, m³/s

$q_{std} =$ standardized flux along fracture with fixed half length $x_f, L^2/t, m^2/s$

$\Delta q_f =$ flux along fracture responding to $w(x, y, t), L^2/t, m^2/s$

$\bar{q}_N =$ flux at node in fracture at $(Y, t), L^2/t, m^2/s$

$\Delta \bar{q}_N =$ flux at node in fracture responding to $w(x, y, t), L^2/t, m^2/s$

$q_{wse} =$ rate caused by wellbore storage effect, L³/t, m³/s

$q_w =$ downhole fluid rate or fluid rate at well bottom, L³/t, m³/s

$q_l =$ fluid leak-off rate from fracture to reservoir, L³/t, m³/s

$Q_{sc} =$ cumulative leak-off rate from shut-in to fracture closure, L³/t, m³/s

$r_d =$ ratio of fracture diffusivity to that of reservoir’s
\( s = \) Laplace variable

\( t = \) time variable, t, s

\( t_{\text{shut}} = \) well shut-in time, t, s

\( V_{\text{frac}} = \) fracture volume, \( L^3 \), m^3

\( w_w = \) fracture width at well bottom L, m

\( w_{\text{ave}} = \) average fracture width L, m

\( x_f = \) fracture half length at time \( \tau \), L, m

\( \mu = \) viscosity, m/L, pa.s

\( v = \) Poisson’ ratio

\( \beta = \) formation breakdown pressure coefficient

\( \tau = \) previous time before current time, m/L, s

\( \tau = \) time, t, s

\( \phi = \) porosity, fraction

\( \eta = \) diffusivity, \( L^2/t \), m^2/s

**Subscripts**

\( D = \) dimensionless term

\( i = \) the ith time period
\( i = \) the \( i \)th fracture segment

\( r = \) reservoir

\( f = \) fracture
CHAPTER 1

INTRODUCTION

1.1 Background of mini-fracturing technology

A mini-fracturing test is an injection-falloff diagnostic test performed without proppant before a main fracture stimulation treatment. The intent is to break down the formation to create a short fracture during the injection period, and then to observe closure of the fracture system during the ensuing falloff period.

Mini-fracturing (DFIT) analysis has been a popular topic for many years as it provides a method of investigating fracture dimensions and fluid leak-off prior to hydraulic fracturing treatment. In addition, mini-fracturing is used as an important method to obtain the reservoir transmissibility \((kh/u)\) and initial reservoir pressure \((p_i)\), as it takes a long period of time and also is very expensive to do a production tests (drawdown or buildup test) in tight reservoirs. (Cramer et al., 2013)

DFIT consists of 3 stages as follows:

1. Injection period: A small volume of fluid is injected to initiate a fracture. The treatment lasts only tens of seconds and is normally used to find out the reservoir breaking pressure and fracture propagating nature.

2. Fracture closing period (stop pumping): During this time, most of information on factors such as fluid loss coefficient, fracture geometry, fracture efficiency, closing time, etc. is obtained.
3. After closing (Nolte et al., 1997): Two flow regimes having been identified within the pseudo-linear flow. These can be used to determine closing pressure over time and fracture length where the pseudo-radial flow uses the late time appearance of pseudo-radial pressure decline to provide information on reservoir transmissibility and initial pressure. This is a similar method to the traditional Honor plot during the buildup with the semi-log pressure derivative presenting a negative unit slope. The pseudo-radial flow originated with the idea-impulse test (Gu et al., 1993), since the shutting-in period is much longer than the pumping period.

1.2 Nature of the problem

Fracturing is a dynamic process rather than a steady performing process. It is very difficult to obtain a good understanding of the fluid leak-off rate from fracture body into reservoir while the fracture is growing. Therefore, simulating the wellbore pressure response is full of problems since there is a lack of fracture volume change rate and leak-off rate with time. To interpret the wellhead pressure response into both fracture geometry and reservoir transmissibility is not easy.

Many past studies did not address some key points such as the level of fluid being compressed before the fracture is initiated and the fluid expansion from wellbore while fracture is growing, etc. These factors could have a great influence on mini-fracturing.
1.3 Objectives of this study

1.3.1 Technical objective of current study

Generally, the ultimate objective of this research is to provide reliable information and efficient ways of interpreting DFIT for a field operator.

- This study will develop type curves of wellbore pressure response in dimensionless form during injection period, making it possible to match real pressure data and get information that is desired.
- Results are used to calculate dimensionless fracture volume growing rate and fluid leak-off rate versus time, based on the reference of injecting rate at wellhead in a semi-analytical way while the fracture is propagating.
- Analysis of the pressure decline response during shut-in closure, coupled with the results obtained during injection, is used to determine the fracture length and width at shut-in.

1.3.2 Long term objectives

By analyzing available field pressure and injecting rate historical data, a database summarizing mini-fracturing/DFIT information of all testing wells in Canada can be established.

1.4 Organization of this thesis

This thesis is comprised of five chapters. Chapter 1 briefly introduces the background and the proposed research. Both technical and long-term objectives are stated and possible field applications in mini-fracturing wells are outlined. Chapter 2 presents a literature
review of research on current technologies and strategies used in mini-fracturing including fluid leak-off rate, possible created fracture geometry, and transient pressure response of fractured well. Chapter 3 introduces a novel approach for modeling mini-fracturing/DFIT by using source and sink function. Synthetic case studies showcase the procedure and effectiveness of this approach. Chapter 4 presents model validation and results for fracture propagation, and sensitivity analysis is comprised of fracture propagating velocity, wellbore storage effect, formation breakdown pressure and injecting rate at wellhead. Finally, conclusions and recommendations are presented in Chapter 5.
During the hydraulic fracturing treatment of an oil or gas well, the liquid pressure in the borehole is increased to exceed the tensile stress in the surrounding rock. Once a tensile fracture is initiated, it is penetrated by liquid from the bore hole. Part of the liquid goes into the fracture, creating volume into reservoir and further fracture propagation under continuous hydraulic action.

Carter (Howard et al., 1957) and Nolte (1986) described very efficient ways to describe fracture propagation and closure including approximating fluid leak-off rate and fracture area change rate versus time. Geertsma et al. (1969) and Nordgren (1972) introduced simple geometric fracture-extension patterns as either radially or rectilinear propagation, known as penny or radial, KGD and PKN models. Through these breakthroughs, mini-fracture can now be described and interpreted in a proper way.

2.1 Calculation of fracture penetration in length

2.1.1 Carter’s derivation

The rate of fluid leak-off controls the fracture dimensions and determines the fracturing fluid efficiency defined as the ratio of ultimate fracture volume to cumulative injected rate. The prediction of fluid leak-off behavior is crucial for the proper design of pumping schedules and the interpretation of pressure profiles during the fracture treatments.
Having pointed out the importance of fracture length on well fracturing results, the effects of fracturing-fluid characteristics and reservoir-fluid and rock characteristics on the areal extent of a fracture should be considered. The effect of these variables may be illustrated by considering how they influence the calculated areal extent of a fracture. Based on the derivation by Carter (Howard et al., 1969), the expression for estimating the extent of fracture with time elapsing is:

$$x_{f(t)} = \frac{q_{inj}w}{4\pi hc^2} \left( e^{\left(\frac{2c\sqrt{\pi t}}{w}\right)^2} \text{erfc} \left(\frac{2c\sqrt{\pi t}}{w}\right) + \frac{2}{\sqrt{\pi}} \frac{2c\sqrt{\pi t}}{w} - 1\right)$$ (2.1)

In the above, w is the constant fracture width, no growth in the width direction is assumed; c is the fluid leak-off coefficient. A low fracturing-fluid leak-off coefficient means low fluid flux from fracture to reservoir and thus a larger fracture area for a given volume and injection rate, whereas a high coefficient means smaller fractured area.

It may be shown from calculated results using the above equation that pumping rate and pumping time (cumulative injected volume) affect the fracture area, as the pumping rate or injected volume is increased.

### 2.1.2 Nolte’s derivation

An approximation is put forward related to current fracture area, time and those at some other time by Nolte (1986).

$$\left(\frac{t}{\ell}\right)^e = \frac{x_{f(t)}}{L}$$ (2.2)

In Eq. (2.2), t and L are the current time and fracture half length; $x_{f(t)}$ is fracture half length at another time $\ell$. The value of exponent e for upper and lower bounds respectively,
is between 1 and 1/2. Actually, because of width growth and fracture volume storage ability, the exponent is between 1 and 1/2.

\[
\frac{1}{2} < e < 1
\]  
(2.3)

This derivation enables fracture propagating velocity to be understood, based on a reference length which could be a fracture length at any time during injection.

2.2 Fluid leak-off coefficient

A commonly used expression for fracturing fluid leak-off was obtained by Carter. The leak-off model was inversely proportional to the square root of time with a constant called the fluid leak-off coefficient. A number of studies since Carter have evaluated experimentally and modeled theoretically leak-off behavior. Some authors explored two-dimensional flow in the reservoir containing both fracturing fluid and reservoir fluid. Yi and Peden and Mayerhofer, introduced models to describe flow through a three-region leak-off zone (Yi et al., 1993; Mayerhofer et al., 1993). These models have indicated other than linear between leak-off volume and square root of time which would have been expected from Carter’s theory.

The fracturing fluid leak-off coefficient, \( c \), contains three kinds of linear flow mechanisms encountered with fracturing fluids. These are (Howard et al., 1969): 1, viscosity and relative-permeability effect; 2, reservoir-fluid viscosity-compressibility effects; and 3, wall building effects. Although each mechanism is considered as acting alone, all may act simultaneously in a fracturing treatment so that the mechanisms may complement each other and increase the fluid's effectiveness.
2.2.1 High-viscosity fracturing fluid

For this kind of fluid, the leak-off rate follows a certain pattern shown below:

\[
v = \frac{Q}{A} = \left(\frac{k\Delta p}{2\mu t}\right)^{0.5} = \frac{c}{\sqrt{t}}
\]  

(2.4)

Hence, the coefficient here is:

\[
c = \sqrt{\frac{k\Delta p}{2\mu t}}
\]  

(2.5)

2.2.2 Viscosity and compressibility effects of fracturing fluid

When the compressibility of injected fluid plays an important role, the leak-off from fracture plains into reservoir does not follow Eq. (2.5) but has the following pattern:

\[
v = \Delta p \sqrt{\frac{k\phi c_f}{\mu \pi t}} = \frac{c}{\sqrt{t}}
\]  

(2.6)

Then, the fluid leak-off coefficient is:

\[
c = \Delta p \sqrt{\frac{k\phi c_f}{\mu \pi}}
\]  

(2.7)

2.2.3 Wall-building fracturing fluid

Work with oils containing fluid-loss additives have shown that the fluid leak-off characteristics of these materials, when pressured against porous media, can be determined by plotting the experimentally determined cumulative filtrate volume vs the square root of flow time. The relationship is:

\[V = m\sqrt{t}\]  

(2.8)

V is the total leak-off volume. The rate can be expressed as:
2.3 Fracture geometric extension

Geertsma et al. (1969) and Nordgren (1972) have introduced simple geometric fracture-extension patterns, either radially or rectilinear propagation, known as penny or radial, KGD and PKN models.

2.3.1 PKN model

A vertical fracture propagation is considered in a straight line away from the well. The vertical height of the fracture is assumed to be limited to a constant distance, \( h \), by layers of fracture resistant rock. The cross section of fracture is assumed to be elliptical and fracture becomes narrower toward the tip (Nordgern, 1972).

It considers a vertical fracture propagation to be in a straight line away from the well, Fig. 2.1 (Nordgern, 1972). The vertical height of the fracture is assumed to be limited to a constant distance, \( h \), by layers of fracture resistant rock.

Sneddon (1969) formulated a mathematical expression for a pressurized line fracture to evaluate the width of the fracture as:

\[
\begin{align*}
    w_w(t) &= \frac{4(p_w - p_c)}{E} x_f(t) \\
    w_w(t) &= 4 \left[ \frac{(1 - u)\mu q^2}{\pi^3 GC h} \right]^{0.125} t^{0.5} \\
    x_f(t) &= \frac{q t^{0.5}}{\pi Ch}
\end{align*}
\]  

Through a set of mathematical derivations (Nordgern, 1972), it becomes:

\[
\begin{align*}
    w_w(t) &= 4 \left( \frac{(1 - u)\mu q^2}{\pi^3 GC h} \right)^{0.125} t^{0.5} \\
    x_f(t) &= \frac{q t^{0.5}}{\pi Ch}
\end{align*}
\]  

And the half length becomes (Nordgern, 1972):

\[
\begin{align*}
    x_f(t) &= \frac{q t^{0.5}}{\pi Ch}
\end{align*}
\]
Fig. 2.1 Vertical fracture geometry of the PKN model (Nordgern, 1972)
2.3.2 KGD and radial model

Simple geometric fracture-extension pattern is assumed, rectilinear propagation originating from a line source. Such a rectilinear fracture must be located in the vertical plane. Its cross section is assumed to be rectangular and it narrows in a width direction toward tip (Geertsma et al., 1969).

The following assumptions are made by Geertsma et al. (1969): “

1. The formation is homogeneous and isotropic as regards those of its properties that influence the fracture-propagation process.

2. The deformations of the formation during fracture propagation can be derived from linear elastic stress-strain relations.

3. The fracturing fluid behaves like a purely viscous liquid; i.e., any peculiar flow behavior due to the addition of gelling agents or other additives is neglected. Moreover, the effect of the propping agent distribution on the distribution of fluid viscosity in the fracture is not taken into account.

4. Simple geometric fracture-extension patterns are assumed — either radially symmetrical propagation from a point source or rectilinear propagation originating from a line source, showing in Fig. 2.2 and Fig. 2.3 (Geerstma et al., 1969). In Fig 2.2 the periphery of the fracture is rectangular; in Fig. 2.3 it is circular.”

The width expression for rectilinear and radial models is (Geerstma et al., 1969):

\[ w_{w(t)} = \frac{4(p_w - p_c)}{E} x_f(t) \]  \hspace{1cm} (2.13)

\[ w_{w(t)} = 2.1 \sqrt{\frac{\mu q x^2_f(t)}{Gh}} \]  \hspace{1cm} (2.14)
Fig. 2.2 Rectangular fracture geometry of the KGD model (Geerstma et al., 1969)
Fig. 2.3 Circular fracture geometry (Geerstma et al., 1969)
2.4 G function and dynamic fractured area analysis

Nolte (1986) described the pressure decline response and fluid leak-off through mathematical derivations based on some of the ideas introduced above. By formulating the leak-off rate as \( q_t = \frac{c}{\sqrt{t}} \), the eventual expression is derived by Nolte (1986):

\[
q_l(t, A) = \frac{2cA}{\sqrt{t_{shut}}} f(t_D) \tag{2.15}
\]

\[
f(t_D) = 2 \left[ (1 + t_D)^{\frac{1}{2}} - t_D^{\frac{1}{2}} \right] \text{ or } \sin^{-1}(1 + t_D)^{-\frac{1}{2}} \tag{2.16}
\]

\[
t_D = \frac{t - t_{shut}}{t_{shut}} \tag{2.17}
\]

Eq. (2.16) depends on the upper and lower bounds for propagating velocity as described before.

The volume of leak-off is:

\[
V_l = \frac{2cA}{\sqrt{t_{shut}}} \left[ t_{shut} \int_0^{t_D} f(t_D) dt_D \right] = 2cA \sqrt{t_{shut}} \left[ g(t_D) - g(0) \right] \tag{2.18}
\]

This is the original form of the famous g function, a function with respect to time. By using g function, Nolte successfully related pressure response to leak-off coefficient c, and the coefficient is able to be obtained.

\[
c = \frac{(p(t_D) - p(\bar{t}_D)) c_f}{2 \sqrt{t_{shut}} \left[ g(t_D) - g(\bar{t}_D) \right]} \tag{2.19}
\]

\( c_f \) is the fracture’s compliance; \( t_D \) is current time and \( \bar{t}_D \) is a reference time before.
With the coefficient $c$, Nolte derived formulated expression of fracture length and width at shut-in:

$$x_f = \frac{Q_{inj}}{2ch\sqrt{t_{shut}g_c}}$$  \hspace{1cm} (2.20)

$Q_{inj}$ is cumulative injected volume during injection; $g_c$ is the $g$ function value at fracture closure.

$$w = 2c\sqrt{t_{shut}(g_c - g_0)}$$  \hspace{1cm} (2.21)

### 2.5 Pressure response from fractured well performance

Well bottom pressure is the physical response of the fracturing process. The simulation of mini-fracturing propagation in this work is similar to that of a static fractured well producing or injecting. Basically, it is categorized into two types: either a fractured well with infinite conductivity or finite conductivity.

#### 2.5.1 Vertically fractured well with infinite conductivity

Gringarten et al. (1974) modeled a plane (zero-thickness) vertical fracture totally penetrating a horizontal, homogeneous and isotropic reservoir initially at constant pressure. At time zero, a single-phase, slightly compressible fluid flows from the reservoir into the fracture at a constant total rate. The producing pressure is uniform over the fracture (infinite fracture conductivity). The pressure remains constant and equal to the initial pressure as distance from the well becomes infinitely large (infinite reservoir). An analytical expression for the pressure distribution created by the plane vertical
fracture may be obtained by the Green's function and product solution method, using the source function presented by Gringarten and Ramey.

2.5.2 Vertically fractured well with finite conductivity

Cinco et al. (1976) developed a mathematical model to study the transient behavior of a well with a finite-conductivity fracture in an infinite slab source. Reservoir is divided into two regions, fracture volume and reservoir. Pressure and flux continuity is coupled at the fracture and reservoir interface. Unlike infinite-conductivity fracture, pressure along fracture is not constant. But when conductivity is very larger, the pressure drop along fracture is barely detected.

2.6 Summary

A mini-fracture or DFIT involves pumping water at low rates into a reservoir to create a small fracture that cuts through near wellbore damage, thereby establishing communication between the wellbore and the formation. After the fracture is opened for a while, the pumps are shut down. As the bottom hole pressure declines, test fluid leaks off into the formation and the induced fracture closes.

Many efficient methods have been developed to describe the whole process of this treatment using both theoretical and experimental research. Three fracture geometries have been presented, namely KGD, PKN and radial models. Fluid leak-off rate is correlated to a coefficient and square root of time, which enables dynamic fracture propagation to be described. Fracture penetration tip, extending with time, is depicted either by Carter’s mathematical derivation or by Nolte’s approximation in formulating a relationship between current fracture length and that at another reference time.
Combing all the strategies above, theoretical relationships, based on the pressure decline response after fracturing, are derived by Nolte for the inference of fracture penetration, width, leak-off coefficient and fluid efficiency. These relationships are found for the three commonly applied fracture models, KGD, PKN and radial models. Those relationships are related to an upper and lower bound on the rate of fracture growth.

Most mini-fracture analysis comes from Nolt’s equation derivations addressing fluid loss with either a constant or a pressure-dependant parameter called fluid loss coefficient, having a relationship with square root time \( q_t = \frac{c}{\sqrt{t}} \). Sometimes this can result in unrealistic estimate especially when the fracture boundary is moving, and it is difficult to find the correct coefficient c.
CHAPTER 3

MODELING AND METHODOLOGY

In our modeling approach, transient pressure response in fracture and reservoir is described by using source and sink function, which is simply a rate multiplying a source or sink function in Laplace domain. In the following, we will illustrate the modeling philosophy by applying it to fracture- reservoir system.

3.1 Fracture propagation geometry and assumptions

In this study, rectilinear propagation fracture and rectangular cross-section were used, the geometry as described in the KGD model.

During pumping, fracture boundaries in both length and width directions are moving. The process of fracturing is very complex in reality and it can’t be monitored either. To simplify the propagating model and flowing model, the following assumptions are made:

1. The fracture propagation is symmetrical and the tip lies on the same horizontal line at any time (Fig. 3.1).
2. An isotropic, homogeneous, infinite reservoir is bounded by an upper and lower impermeable cap rock. The reservoir has uniform thickness and is fully penetrated by a vertical fracture.
3. Fluid injected is partly stored in the fracture and the rest flows into reservoir from fracture; flow from the tip into reservoir is neglected because the width compared to length is very small (Cinco et al., 1976). It makes a prescribed boundary
moving in length direction, a flowing boundary moving in width direction (Fig. 3.2), and an average width along fracture is used.

4. The dynamic propagating fracture is regarded as static in each time period with proper initial and boundary conditions, since each event is controlled to be a very short time period.

5. Fluid injected at wellhead has the same property as that in reservoir.
Fig. 3.1 Fracture geometry growth with time lapse
Fig. 3.2 Fluid flow graph in a propagating fracture
3.2 Solution to a static fracture performance

The dynamic fracturing propagation has some similar behaviors to the performance of a static hydraulically fractured well, which is the basis of the construction of the mini-fracturing model. This research was based on the study of the transient behavior of a finite-conductivity vertically fractured well that is introduced with the strategy from Zhao et al. (2002).

Based on this physical philosophy, the whole system consists of two regions: a fracture body and the rest of two semi-infinite reservoirs (the fracture is very narrow). They are separated over the extent of fracture plains by a hydraulic boundary with flowing abilities. For the reservoir region (Fig. 3.3), it behaves as if there were a single injecting planar fractured well (injecting at wellhead) at the area of communication between the two semi-infinite reservoirs. In addition, the flux along fracture is considered to be non-uniform, and then flux along fracture should be discretized into a series of segments with different flux for each. As long as each segment is small enough, the flux over its length can be treated as uniform. Considering fracture body region (Fig. 3.4), the ideology is able to be applied. It is discretized into as many segments and each one is bounded by sealing faults with slab sources at the interface between each connected discretized fracture segment and the reservoir, and since. Because all the fluid leaving fracture is entering reservoir, the rate along fracture in reservoir region and in fracture region must be the same.

The following section shows the derivation of the mathematical model. More details can be found elsewhere (Zhao et al., 2002).
In the fracture region, the pressure difference inside one segment of fracture is based on the source and sink function is expressed as:

\[ \Delta p_{f,i(x,y,t)} = \frac{1}{(\rho \mu c_f)_f} \int_0^{1/2w_{ave}} \int_{0}^{t} \bar{q}_{N,i-1} (Y, t) \Delta p_{sx,seal}(x, x_{i-1}, t, \tau) \Delta p_{sy,seal}(y, Y, t, \tau) dY d\tau \]

\[ - \int_{x_{i-1}}^{x_i} \int_{0}^{1/2w_{ave}} \bar{q}_{f,i-1} (x, t) \Delta p_{sx,seal}(x, x_i, t, \tau) \Delta p_{sy,seal}(y, Y, t, \tau) dY d\tau \]

\[ - \int_{x_{i-1}}^{x_i} \int_{0}^{1/2w_{ave}} \bar{q}_{f,i-1} (x, t) \Delta p_{sx,seal}(x, x_i, t, \tau) \Delta p_{sy,seal}(y, Y, t, \tau) dY d\tau \]

\[ - \int_{0}^{1/2w_{ave}} \int_{0}^{1/2w_{ave}} \bar{q}_{N,i} (Y, t) \Delta p_{sx,seal}(x, x_i, t, \tau) \Delta p_{sy,seal}(y, Y, t, \tau) dY d\tau \]

The expression of point source function in x direction is:

\[ \Delta p_{sx}(x, \chi, t, \tau) = \frac{1}{2(x_i - x_{i-1})} \sqrt{\eta/(t-\tau)} \sum_{n=1}^{\infty} \exp \left[ -\frac{(x-x_i)_{n}}{4\eta/(t-\tau)} \right] \]

Due to the assumption that the flux in each segment can be regarded as uniform, flux can be approximated as:

\[ \bar{q}_{N,i-1} = \frac{q_{N,i-1}}{w_{ave}} \quad \text{for} \quad -\frac{1}{2} wd < Y < \frac{1}{2} wd \]

And

\[ \bar{q}_{f,i-1} = \frac{q_{f,i-1}}{x_i - x_{i-1}} \quad \text{for} \quad x_{i-1} < \chi < x_i \]

The double integration in Eq. (3.1) can be evaluated as:
\[ \int_{0}^{t} \int_{x_{i-1}}^{x_{i}} \bar{q}_{f,i-1}(\chi, \tau) \Delta p_{sx}(x, \chi, t, \tau) \Delta p_{sy} \left( y, \frac{1}{2} wd, t, \tau \right) d\chi d\tau \]

\[ \approx \int_{0}^{t} \frac{q_{f,i-1}}{x_{i} - x_{i-1}} \int_{x_{i-1}}^{x_{i}} \Delta p_{sx}(x, \chi, t, \tau) \Delta p_{sy} \left( y, \frac{1}{2} wd, t, \tau \right) d\chi d\tau \quad (3.5) \]

And

\[ \int_{x_{i-1}}^{x_{i}} \Delta p_{sx}(x, \chi, t, \tau) d\chi \]

\[ = \frac{1}{2(x_{i} - x_{i-1})} \sum_{n=0}^{\infty} \left[ \text{erf} \left( \frac{x_{i} - \chi + n}{2\sqrt{\eta_{f}(t - \tau)}} \right) - \text{erf} \left( \frac{x_{i-1} - \chi + n}{2\sqrt{\eta_{f}(t - \tau)}} \right) \right] \]

\[ + \frac{1}{2(x_{i} - x_{i-1})} \sum_{n=0}^{\infty} \left[ \text{erf} \left( \frac{x_{i} + \chi + n}{2\sqrt{\eta_{f}(t - \tau)}} \right) - \text{erf} \left( \frac{x_{i-1} + \chi + n}{2\sqrt{\eta_{f}(t - \tau)}} \right) \right] \quad (3.6) \]

**In the reservoir region**, the pressure difference in reservoir is expressed as:

\[ \Delta p_{r(x,y,t)} = \frac{1}{\varphi \mu c_{t}} \int_{0}^{t} \int_{-x_{f}}^{x_{f}} \bar{q}_{f}(\chi, \tau) \Delta p_{sx,inf}(x, \chi, t, \tau) \Delta p_{sy,inf}(y, 0, t, \tau) d\chi d\tau \quad (3.7) \]

\( \Delta p_{sx,inf} \) and \( \Delta p_{sy,inf} \) are point source functions in infinite domain for \( x \)- and \( y \)- direction.

\[ \Delta p_{sx,inf} = \frac{1}{2\sqrt{\pi \eta_{x}(t - \tau)}} \exp \left( -\frac{(x - x')^{2}}{4\eta_{x}(t - \tau)} \right) \quad (3.8) \]

Using the same tactic as in reservoir system:

\[ \Delta p_{r(x,y,t)} \approx 2 \int_{0}^{t} \sum_{1}^{n} \frac{q_{f,i}}{x_{i} - x_{i-1}} \int_{x_{i-1}}^{x_{i}} \Delta p_{sx,inf}(x, \chi, t, \tau) \Delta p_{sy,inf}(y, 0, t, \tau) d\chi d\tau \quad (3.9) \]

\[ \int_{x_{i-1}}^{x_{i}} \Delta p_{sx,inf}(x, \chi, t, \tau) d\chi = \frac{1}{2} \left[ \text{erf} \left( \frac{x_{i} - \chi}{2\sqrt{\eta_{r}(t - \tau)}} \right) - \text{erf} \left( \frac{x_{i-1} - \chi}{2\sqrt{\eta_{r}(t - \tau)}} \right) \right] \quad (3.10) \]
Laplace transforms with time are taken for solutions in both fracture and reservoir systems. In Laplace domain, the fracture system and reservoir system are coupled together at the fracture plains, based on pressure and flux continuity conditions, as well as inside fracture itself at the interface of each segment. Let $L[]$ denote the Laplace transform operator and $s$ denote the Laplace variable. Numerical Laplace transform of products of source functions were sourced from Zhao et al. (2002).

$$L(\Delta p) = \frac{1}{\rho c_i \mu} L(q)L(\Delta p_s)$$

Results can then be inverted numerically with the Stehfest Inversion algorithm (Stehfest, H 1970).
Fig. 3.3 Reservoir region of a fractured well reservoir
Fig. 3.4 Fracture region of a fractured well reservoir
Fig. 3.5 Comparison of the results with KAPPA Saphir for a finite conductivity fracture
Fig. 3.5 presents the type curves of dimensionless pressure and its derivative for a finite conductivity fracture with conductivity $\frac{K_{f, a} W_{f, a}}{K_{r, s} X_f} = 400$. KAPPA Saphir V.4.02 is used to compare with the results obtained based on strategy explained above with 10 segments on each half fracture length.

In addition, the source and sink function method explained above can also be used to simulate transient pressure response in naturally fractured reservoir, either natural fracture being conductive or unconductive/sealing fault. By using this method, fracture and reservoir are explicitly divided into different regions, each doing their own work. Results about pressure response in reservoir with a natural fracture are shown in Fig. 3.7, where the conductive fracture and sealing fault respond to totally different reactions. Fig. 3.6 shows the system structure that simply removes the well from Fig. 3.4 into the reservoir region while no-flow outer boundaries are applied.

Fracture is seen as porous medium as above, where $c_r$ is defined as the ratio of fracture diffusivity to that of reservoir. Fig. 3.7 compares the results containing the diffusivity ratio 10000, 1 and 0.00001, representing conductive natural fracture, original homogenous reservoir and sealing fault respectively. With conductive natural fracture, the production well pressure derivative drops when natural fracture, with much bigger flowing ability, starts to affect production. After pressure transient reaches boundary, pressure derivative presents a unit slope rise. On the contrary, sealing fault makes pressure derivative approaches 1 but always less than that before pressure transient reaches boundary, for the reason that sealing fault is not all across the whole reservoir, shown in Fig. 3.6.
Fig. 3.6 Naturally fractured reservoir system
Fig. 3.7 Comparison of results for different natural fracture and reservoir diffusivity ratio.
3.3 Solution pursuit to the dynamic fracture propagation

3.3.1 Fluid leak-off rate

Before the fracture is created, fluid keeps being injected in the wellbore and compressed. It is reasonable to assume that none or very little, if there is any, goes into the formation in tight reservoir. Due to the compressibility of injected fluid, the compressed fluid will expand when a highly conductive path (fracture) is initiated. Since the formation breakdown pressure is far above the initial reservoir pressure, the flowing rate at well bottom is very likely to be more than that at well head, due to expansion as long as pressure keeps declining. In addition, the fracture volume grows and part of injected fluid will be used to fill up the fracture, which does not flow into the reservoir. It is similar to wellbore storage effect when doing an injection test that part of the fluid injected is stored in the wellbore. Hence, the fluid leak-off rate should equal the real bottom hole rate minus the fracture volume change rate.

From the material balance equation:

\[
q_w(t) = q_l(t) + \frac{dV_{frac}(t)}{dt} \tag{3.11}
\]

Showing that well bottom rate is used to provide the fracture volume growing rate and leak-off rate \(q_l\).

The mass-balance components between wellhead and well bottom are considered as the constant rate of fluid in minus rate of fluid out equals the rate of fluid accumulation in wellbore or called rate caused by wellbore storage effect. It is:
\[ q_w(t) - q_{inj} B = q_{wse}(t) = \frac{dV_{wb}}{dt} \quad (3.12) \]

According to the chain rule:

\[ \frac{dV_{wb}}{dt} = \frac{dV_{wb}}{dp_w} \frac{dp_w}{dt} \quad (3.13) \]

The definition of fluid’s compressibility gives:

\[ \epsilon_{wb} = -\frac{dV_{wb}}{V_{wb}dp_w} \quad (3.14) \]

Then, the fluid accumulation rate is:

\[ q_{wse}(t) = \frac{dV_{wb}}{dt} = -V_{wb}\epsilon_{wb} \frac{dp_w}{dt} \quad (3.15) \]

Placing the injecting rate to the right side of Eq. (3.12):

\[ q_w(t) = q_{inj} B + q_{wse}(t) \quad (3.16) \]

Eq. (3.11) equals Eq. (3.12).

Then, fluid leak-off rate is:

\[ q_l(t) = q_{inj} B + q_{wse}(t) - \frac{dV_{frac}(t)}{dt} \quad (3.17) \]

Equaling:

\[ q_l(t) = q_{inj} B - C \frac{dp_w}{dt} - \frac{dV_{frac}(t)}{dt} \quad (3.18) \]

In the above, C is the traditional wellbore storage coefficient \( C = V_{wb}\epsilon_{wb} \).
The final dimensionless leak-off rate expression is:

\[
q_{\text{ID}}(t_D) = 1 - \left( C_D + \frac{x_{fD}^2(t_D)}{E\phi c_l} \right) \frac{dp_{\text{WD}}}{dt_D} - \frac{2}{E\phi c_l} x_{fD}(t_D) \frac{dx_{fD}(t_D)}{dt_D} (p_{\text{WD}} - \alpha \beta p_{\text{BD}}) \tag{3.19}
\]

And it also indicates the fluid expansion rate at well bottom and the fracture growing rate with time.

**Appendix A** presents the mathematical derivation in detail and its dimensionless form.

### 3.3.2 Fracture dimension growth

When dealing with fracture propagation, it is essential to know the velocity or fracture’s rate of areal growth that can be bounded by an upper, assuming negligible fluid leak-off, or lower, fluid leak off domination, case. Based on this assumption, an approximation is put forward related to current fracture area, time versus those at some other time by Nolte (1986).

\[
\frac{t}{t'}^e = \frac{x_f(t_D)}{L} \tag{3.20}
\]

In Eq. (3.20), t and L are the current time and fracture half length; \( x_f(t) \) is fracture half length at another time \( t' \). The value of exponent \( e \) for upper and lower bound respectively are 1 and \( 1/2 \). Actually, because of width growth and fracture volume storage ability the exponent is between 1 and \( 1/2 \).

\[
\frac{1}{2} < e < 1 \tag{3.21}
\]

As is shown in Eq. (3.20), fracture length at any time point is a function to that at another time, hence in dimensionless form fracture changing length is able to obtain based on a
reference length that could be the length at any time. Illustrating by Fig. 3.2, the fracture boundaries move in both width (Eq. (A.2)) and length with time. From $t_i$ to $t_{i+1}$, fracture tip is at $x_{fi+1}$ with width $w_{ave,i+1}$; at $t_{i+1}$, the half length grows to $x_{fi+2}$ with its corresponding width; at $t_{i+2}$, fracture half length reaches $x_{fi+3}$. With semi-analytical solution, it was assumed that the fracture geometry is stable or static in each time period. Tackling these “moving boundary” problems is difficult because reservoir condition keeps changing from each time period as the result of fracture growth or its boundary movement.

It is occurred that transient pressure response, caused by dynamic propagating fracture, is equivalent, in value, to the superposition of dozens of fracture’s performance with different geometry in its history. In another words, it is the superposition of the reservoirs with fractured wells in history containing certain relationships among each other. In traditional variable rate well test, only wellbore/rates superposition is required.

Detailed mathematical derivation and physical explanation of this dynamic process are presented in Appendix B.

3.4 Shut-in closure analysis

Traditionally, pressure decline after treatment can provide information for analysis of fluid leak-off coefficient $c$, fracture length and width (Nolte 1986). In this research, fluid leak-off rate is not dealt with traditional fluid leak-off coefficient, which describes flux flowing from fracture to reservoir as $q_t = \frac{c}{\sqrt{t}}$. Instead, use of the source and sink function enabled this study to relate flux and pressure in an analytical way. The strategy to simulate fracture growth has been discussed above and results are provided in the
following section **modeling results analysis.** In this study, fracture half length $x_f$ is evaluated in a simple way during the shut-in closure period by drawing on results simulated during injection, on the assumption that fracture ceased to grow after shut-in at wellhead or no additional extension of fracture tip.

Fracture volume on scale of cumulative injected fluid volume, referred to as the fluid efficiency, at shut-in and is expressed in Eq. (A.15) as:

$$f_e = \frac{V_{frac}}{q_{inj} t_{shut}} = \int_0^{t_{shut}} \frac{2}{E \phi c_t} x_f(t) \frac{dx_f(t)}{dt} (p_{WD} - p_{CD}) + \frac{x_f^2(t) dp_{WD}}{E \phi c_t} dt$$

The value of the integration portion of this formula is able to obtain by using numerical integration algorithm based on the simulation results during injection when curve matching with real pressure data is achieved.

Fracture volume is:

$$V_{frac} = f_e q_{inj} t_{shut} \quad (3.22)$$

The continuity equation with downhole shut-in (no afterflow effect $C_D = 0$) is:

$$q_t = -\frac{\partial V_{frac}}{\partial t} \quad (3.23)$$

It implies that after shut-in and before fracture closes, the fluid leak-off rate is equal to the rate of fracture volume change.

From Eq. (A.2), the average fracture width is:
\[ w_{ave} = \frac{\pi x_f (p_w - p_c)}{\dot{E}} \]

Therefore:

\[ q_t = -\frac{\pi h x_f^2 \partial p_w}{\dot{E}} \frac{\partial}{\partial t} \]

(3.24)

Integrating the two sides:

\[ \int_{t_{shut}}^{t_{closure}} q_t \, dt = \int_{t_{shut}}^{t_{closure}} \left( -\frac{\pi h x_f^2}{\dot{E}} \frac{\partial p_w}{\partial t} \right) \, dt \]

(3.25)

In Eq. (3.25), \( t_{closure} \) is the time when fracture is completely closed.

Result of the integration in Eq. (3.25) is:

\[ Q_{t,sc} = \frac{\pi h x_f^2}{\dot{E}} (ISIP - p_{closure}) \]

(3.26)

Note that \( Q_{t,sc} \) is the total leak-off volume during the fracture closing period; \( p_{closure} \) is closure pressure and ISIP is the well instant shut-in pressure.

It is clear that the total leak-off volume during fracture closure should be equal to the fracture volume at shut-in, assuming fracture is completely closed in width with no afterflow effect. Then, Eq. (3.22) equals to Eq. (3.26) in value.

\[ f_e q_{inj} t_{shut} = \frac{\pi h x_f^2}{\dot{E}} (ISIP - p_{closure}) \]

(3.27)

\[ x_f = \frac{f_e q_{inj} t_{shut} \dot{E}}{\sqrt{\pi h (ISIP - p_{closure})}} \]

(3.28)
Clearly, width can be attained by Eq. (A.2).

There are many ways to estimate closure pressure. Generally, it can be found from the pump-in/flow back procedure (Nolte 1986). Also, the decline data plotted versus the square root of time after shut-in shows a break in the curve at a particular time point and pressure, at which pressure can be interpreted as closure pressure. In addition, some simple approximation still seems work well like regarding closure pressure as being equal to the minimum principal stress, or being proportional to the formation breakdown pressure.

With this simplified method, real fracture half length can be estimated by Eq. (3.28). The key to this result is to find out the fluid efficiency $f_e$ from fracture propagation period.

3.5 Summary

This chapter presents a general methodology for computing pressure response and flux profile along fractured well. This method consists of dividing complex reservoir system into 2 simple regions, original reservoir and fracture, which interact with each other through pressure and fluid transfer continuity over their contact interface. Source and sink function is used as the fundamental method to compute the pressure response in Laplace domain, and results are inverted with Stehfest algorithm. A fast and accurate method of taking numerical Laplace transformation of the source/sink solutions is presented that makes the computations reasonably efficient.

Fluid leak-off rate is derived analytically from the view of material balance, connecting injecting rate, wellbore storage effect and fracture volume growing rate as a function of bottom hole pressure. This derivation put the solution into a variable rate well test.
problem combined with a moving boundary of fracture in reservoir. The compressed level of fracturing fluid before fracture is created is considered as well, which is determined by initial formation breakdown pressure.

A method was developed to account for fracture boundary growth and variable downhole rate. The essence of this method is the superposition of both the reservoir and wellbore in the solution.
4.1 Model validation

To the author’s knowledge, current commercial simulation software has very limited ability to simulate the performance of a fractured well with its changing fracture geometry. Therefore, no direct validation is available. Nevertheless, given some simple situations this model is able to approach results we already know. Fig. 4.1 shows some simple cases when fracture half length extends an additional part, ΔL, after a period of time’s producing with original half length L from dimensionless time 0 to 1.

When ΔL=L, dimensionless pressure declines for a short period of time resulting from a newly exposed fracture which starts to conduct fluid toward wellbore, and it gradually increases. In late stage, pressure eventually approaches that of fractured well with half length 2L all the time due to long production time with new fracture length. When ΔL =0.1L, the result almost overlaps that of fractured well with half length L all the time because the growth is small and it barely affects the well performance. When ΔL=0.25L, the curve starts to become nearly flat for a period and then rises.

To validate that this model is robust, gradually consistent fracture length increments over a period of time is applied. Two scenarios are compared in Fig. 4.2 below. The length increment is ΔL=L for both of them but the scenario mentioned above in Fig. 4.1, has an
instantaneous increment at \( t_D = 1 \) while the other scenario goes through a number of increments from \( t_D = 1 \) to 2 which add up to \( L \), and where \( \Delta L = \sum_{i}^{n} L_i = L \).
Fig. 4.1 Model validation by comparison of different values of instantaneous fracture length increment.
Fig. 4.2 Model validation for model robustness by going through a series of fracture length increments during a period of time.
4.2 Modeling results analysis

This model dealing with fracture propagation is programmed by using C++ language and type curves are documented. Due to the complex nature of this problem, some basic parameters, listed in table 1, are come up with for computing rock/fluid property, and formation breakdown pressure gradient in dimensionless form.

Table 4.1-Basic reservoir parameters for model computation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir Thickness h, m</td>
<td>5</td>
</tr>
<tr>
<td>Reservoir Deepness H, m</td>
<td>2000</td>
</tr>
<tr>
<td>Initial Reservoir Pressure Gradient, $\frac{p_i}{\rho_w g H}$</td>
<td>1</td>
</tr>
<tr>
<td>Reservoir Permeability $k_r$, D</td>
<td>0.001</td>
</tr>
<tr>
<td>Oil Viscosity $\mu_o$, cp</td>
<td>1</td>
</tr>
<tr>
<td>Rock Plain Strain Modulus, $\hat{E}$, Mpa</td>
<td>$4 \times 10^4$</td>
</tr>
</tbody>
</table>

This model investigated different cases with some sensitivity analysis that lead to different results. The following analysis focuses on illustrating how reservoir condition and fracturing treatment affect the created fracture volume, fluid leak-off and pressure response.
4.2.1 Effect of fracture propagating velocity

Fig. 4.3 gives the wellbore downhole pressure and its derivative response under different propagating velocity exponents $e=0.65, 0.6$ and $0.55$ with injecting rate $q_{inj}=1$ bbl/min, dimensionless wellbore storage coefficient $C_D=10$ and formation breakdown pressure coefficient $\beta=2$. For all 3 cases, fracture is initiated at dimensionless pressure $p_D=0.237$, at which point formation breaks down resulted from fluid being injected into wellbore leading to pressure explosion. It is easy to see that a faster propagating fracture has more pressure drop due to a larger high-conductive path being created, which leads to a greater pressure release.

Fig. 4.4 and Fig. 4.5 present their respective fracture volume growing rate and fluid expansion rate in wellbore after fracture is created. In Fig. 4.4, it shows that faster propagating velocity fracture grows larger in volume than those with a slower propagating velocity. However, at later time observations, there are intersects indicating slower propagating fracture has larger volume growth rates. This is thought to be because pressure at well bottom implodes for faster propagating fracture leading to that fracture width almost ceases to grow, Fig. 4.6. Nevertheless, Fig. 4.7 clarifies the total fracture volume created before stop pumping, is comparable to the cumulative amount of injected fluid, defined as fluid efficiency, and is proportional to the propagating velocity.

Fig. 4.5 illustrates that fluid in wellbore expands dramatically for a certain period of time after fracture begins to grow, and then expansion rate slows down. For fracture volume growing velocity slows down, fluid expands less or pressure drops slower when a high-conductive path is exposed on a slower rate.
Fig. 4.3 pressure response for the effects of different fracture propagating velocity
Fig. 4.4 fracture volume growing rate for the comparison of different propagating velocity
Fig. 4.5 Fluid expansion rate after fracture initiation from wellbore under different propagating velocity
Fig. 4.6 comparison of width growing rate under different propagating rate
Fig. 4.7 comparison of final fracture volume under different propagating velocity

Fluid efficiency, $fe = \frac{V_{fra}}{q_{inj\text{shut}}}$
4.2.2 Effect of wellbore storage

To investigate the effects of wellbore contribution, different wellbore storage coefficients $C_D$ are set at 10, 5 and 2.5 with injecting rate $q_{inj}=1$ bbl/min, propagating rate $e=0.6$ and formation breakdown pressure coefficient $\beta=2$.

Fig. 4.8 compares the bottom hole pressure and its derivative response for the 3 cases with different wellbore storage coefficients. The results show that pressure falls more slowly with larger $C_D$ which suggests that fracturing fluid with larger compressibility or wellbore with larger volume is able to better maintain the pressure from falling.

Fig. 4.9 and Fig. 4.10 present fracture volume growing rate and fluid expansion rate from wellbore. Wells with larger $C_D$ have higher rates for both parameters. Analyzing from the two figs, it can be concluded that fluid expansion helps contribute to the growth in fracture volume. Fig. 4.11 shows the fracture fluid efficiency.

Fig. 4.12 gives the fluid leak-off rate over time for the three scenarios. Not only does larger fluid expansion from wellbore contributes to fracture volume, it also contributes to increased fluid leaking into reservoir due to a greater contribution from the wellbore. In case hot fluid is preferred to spread over reservoir while fracturing, well with large wellbore storage coefficient holds advantage.
Fig. 4.8 pressure response with different wellbore storage coefficient
Fig. 4.9 fracture volume growing rate with different wellbore contribution ability under different wellbore storage coefficient.

\[ \text{dimensionless fracture volume growing rate, } \frac{dV_{\text{frac}}}{dt}/q_{\text{inj}} \]
Fig. 4.10 fluid expansion from wellbore after fracture initiation with different wellbore storage coefficient
Fig. 4.11 comparison of final fracture volume under different wellbore storage coefficient
Fig. 4.12 fluid leak-off rate from fracture into reservoir with time under different wellbore storage coefficient
4.2.3 Effect of formation breakdown pressure

The formation breakdown pressure coefficient, defined as the ratio of initial formation breakdown pressure to initial reservoir pressure, normally ranges from 1.7 to 2.3. To clarify the differences within this range, this study compares the results with different formation breakdown pressure coefficient $\beta=1.7$, 2 and 2.3 determining different level of fluid compressed in wellbore before fracture is initiated under injecting rate $q_{inj}=1\text{bbl/min}$, wellbore storage coefficient $C_D=10$ and propagating velocity $e=0.6$.

Formation breaks down at dimensionless pressure $p_D=0.1659$, 0.237 and 0.3081 for each case respectively. Fig. 4.13 provides the bottom hole pressure and its derivative response, indicating that pressure releases more steeply when fluid is compressed to a higher point of pressure before formation breaks down.

Fig. 4.14 and Fig. 4.15 provide curves for fracture volume growing rate and wellbore fluid expansion rate. It is noteworthy that the case with higher breakdown pressure has a larger contribution from wellbore fluid expansion but has slower fracture volume growing rate. This is likely due to the fluid being compressed to a higher level with higher breakdown pressure and the downhole pressure release is very fast, resulting in a narrower fracture width and a slower growth rate in width. Fig. 4.16 explains this and Fig. 4.17 shows their fluid efficiency.

Fig. 4.18 compares the results of fluid leak-off rate into reservoir for these three scenarios. It clearly shows that more fluid is forced into reservoir with higher breakdown pressure and fracture volume is smaller due to large pressure drop.
Fig. 4.13 pressure response for cases with different initial formation breakdown pressure

Formation breakdown pressure for $\beta=2.3$
Formation breakdown pressure for $\beta=2$
Formation breakdown pressure for $\beta=1.7$
Fig. 4.14 Fracture volume growing rate with different initial formation breakdown pressure
Fig. 4.15 fluid expansion rate from wellbore after fracture initiation with different initial formation breakdown pressure
Fig. 4.16 fracture width with different initial formation breakdown pressure
Fig. 4.17 final fracture volume with different formation breakdown pressure

\[
\text{fluid efficiency, } \text{fe} = \frac{V_{\text{frac}}}{q_{\text{inj}}t_{\text{shut}}}
\]
Fig. 4.18 fluid leak-off rate from fracture into reservoir with different initial formation breakdown pressure.
4.2.4 Effect of injecting rate

The comparison of results caused by different injecting rate at wellhead is given as follow. In this paper, injecting rate $q_{inj} = 2$, 1 and 0.5 bbl/min is picked under the situation where fracture propagating velocity $c = 0.6$, wellbore storage coefficient $C_B = 10$ and formation breakdown pressure coefficient $\beta = 2$.

Fig. 4.19 offers the bottom hole pressure and its derivative response in dimensionless form all based on $q_{inj} = 1$ bbl/min. It shows that pressure releases very quickly with a small injecting rate, while it is able to maintain at a comparatively stable level with large injecting rate. As is shown, pressure from the well with 2 bbl/min rises a little at very beginning for the reason that a large amount of fluid is forced into reservoir with a relatively small fracture volume, which has to overcome much resistance. This is followed by a growing fracture volume as it becomes easier for injecting fluid to go into reservoir, therefore pressure drops. Similar idea goes to small injecting rate scenario.

Fig. 4.20 and Fig. 4.21 provide fracture growing rate and fluid expansion rate in wellbore. For the case, an injection rate 0.5 bbl/min is used. Fracture growing rate is high at the initial stage, resulting from a high fluid expansion rate which then decreases dramatically. This may be due to much of the fluid being forced into reservoir with little going into fracture or due to large pressure drop, leading to the sacrifice of width growing rate, see Fig. 4.23. For the case of an injection rate of 2 bbl/min, fracture volume growing rate is steady and the eventual fluid efficiency is the highest among the three, see Fig. 4.22.

Fig. 4.24 compares the leak-off rate for the 3 scenarios on their own each injecting rate basis. On the scale of their own injecting rate, the case with the smallest injecting rate has
the most of fluid to leak into reservoir and the least fluid to go into fracture; the one with largest injecting rate has the least fluid to leak into reservoir and most to go into fracture. It seems that a larger pressure drop leads to the consequence of more fluid leaking into reservoir as well as less occupying the fracture.
Fig. 4.19 pressure response with different injecting rate

Formation breakdown pressure

Dimensionless downhole pressure and derivative, $P_D, \frac{dP_D}{d\ln(t_D)}$

$q_{inj}=2\, \text{bbl/min}$
$q_{inj}=1\, \text{bbl/min}$
$q_{inj}=0.5\, \text{bbl/min}$
Fig. 4.20 fracture volume growing rate with different injecting rate
Fig. 4.21 fluid expansion rate from wellbore after fracture initiation with different injecting rate.
Fig. 4.22 final fracture volume with different injecting rate

Fluid efficiency, $fe = \frac{V_{frac}}{q_{inj} t_{shut}}$
Fig. 4.23 fracture width with different injecting rate
Fig. 4.24 fluid leak-off rate from fracture into reservoir with different injecting rate
4.3 Summary

This chapter shows results during injection under several scenarios, as well as model validation. Due to lack of direct validation from current existing commercial software, given some simple situations this model is able to get close to results we already know.

Fig. 4.1 shows the effect of fracture increment length. When the increment is close to the original fracture length itself, the result gets close to the standard pressure response from that with 2 times the original fracture length. When the increment is very short at approximately 10% of original fracture length, the resulting pressure barely changes because effect from the fracture increment was very small. Fig. 4.2 shows the model is very robust while fracture goes through a couple of increments over a period of time.

Results during injection are shown in section 4.2. It is observed that higher fracture propagating velocity in length results in larger final fracture volume. When fluid expansion in wellbore is considered, a larger fracture volume is created and more fluid leaks into reservoir; it suggests that fracturing fluid with larger compressibility or wells with larger wellbore volume can help maintain the bottom hole pressure and have greater contribution to the creation of fracture volume. By examining different formation breakdown pressure, the study showed when fluid pressure is increased in the wellbore before fracture is initiated, this resulted in faster well bottom pressure release, increased fluid expansion in wellbore but less fracture volume was created. With different injecting rate at wellhead, a smaller injecting rate leads to a faster bottom hole pressure decline, less fracture volume but more fluid leaking into reservoir on the scale of its cumulative injecting rate.
CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

Source and sink function method is used to build up modeling foundation for fractured well, and it can be extended to apply in naturally fractured reservoir.

Fast and accurate method of taking numerical Laplace transformation of the source/sink solutions is presented that makes the computations reasonably efficient.

A physical based strategy to deal with fracture growth (moving boundary) problem is come up with, the essence of which is the superposition of both wellbore and reservoir in the solution.

Using injecting rate as the reference rate, dimensionless pressure, fracture volume growing rate, fluid leak-off rate, and expansion rate from wellbore storage effect are obtained from mini-fracturing test through semi-analytical strategy.

Wellbore storage effect helps create larger fracture volume, suggesting that fluid with larger compressibility or larger wellbore volume contributes to incremental fracture volume.

When fracturing fluid inside wellbore is compressed to reach a higher formation breakdown pressure level prior to fracture initiation, more fluid expansion from wellbore and more fluid leakage occur, but less ultimate fracture volume is created.
At wellhead, smaller injecting rate leads to faster bottom hole pressure decline, less fracture volume but more fluid volume leakage into reservoir.

For the shut-in period, the ultimate fracture half-length at shut-in can be found by diagnosing pressure data during injection and coupling it with pressure decline during closure period, adding more confidence to the results analyzed.

5.2 Recommendations

It is important to analyze field mini-fracturing data with this model, for both injection and shut-in period.

Once reservoir properties are evaluated using this model, effective comparison can be made to those obtained from drawdown or buildup well tests.

This source and sink function method strategy to deal with “moving boundary” problem used in this work, may be used in further research areas such as post-fracture performance analysis with variable conductivity, SAGD, VEPEX, etc.
LIST OF REFERENCE


APPENDIX A

According to geo-mechanics, the maximum fracture width at wellbore is a function of net pressure and length (Nolte, 1986):

\[ w_w(t) = \frac{4p_n x_f(t)}{\hat{E}} \quad (A.1) \]

And fracture width is replaced by average width along fracture (Nolte, 1968).

\[ w_{ave}(t) = \frac{\pi}{4} w_w(t) \quad (A.2) \]

\( \hat{E} \) is Plain Strain Modulus, equaling to \( \frac{E}{1 - \nu^2} \). E is Young’s Modulus; \( \nu \) is Poisson’s ratio.

\( p_n \) is the net pressure, equaling pressure at wellbore minus closure pressure; \( p_c \) is closure pressure.

\[ p_n = p_w - p_c \quad (A.3) \]

Furthermore, the formation breakdown pressure gradient is always within a certain range compared to that of initial reservoir. There are many ways to find out closure pressure discussed in the shut-in closure section. Below demonstrates a very simple form, correlating to formation breakdown pressure by a multiplier.

\[ p_c = \alpha p_b \quad p_b = \beta p_l \quad (A.4) \]

\( p_b \) is formation breakdown pressure. Normally \( \alpha \) is between 0.8-0.9 and \( \beta \) is 1.7-2.4 for industry norms.
The fracture volume growing rate is:

\[
\frac{dV_{frac}(t)}{dt} = 2h \frac{d(x_f(t)w_{ave}(t))}{dt} = \frac{2\pi h}{E} \left( x_f^2(t) \frac{dp_w}{dt} + 2p_n x_f(t) \frac{dx_f(t)}{dt} \right) \quad (A.5)
\]

Since fluid leak-off rate is expressed as Eq. (3.17) and Eq. (3.18):

\[
q_l(t) = q_{inj} B + q_{wse}(t) - \frac{dV_{frac}(t)}{dt}
\]

\[
q_l(t) = q_{inj} B - C \frac{dp_w}{dt} - \frac{2\pi h}{E} \left( x_f^2(t) \frac{dp_w}{dt} + 2p_n x_f(t) \frac{dx_f(t)}{dt} \right) \quad (A.6)
\]

The mathematical model for the leak-off rate and pressure can be expressed in dimensionless forms defined as following groups:

\[
p_D = \frac{2\pi k_r h}{q_{inj} u B} (p - p_i) \quad p_{bD} = \frac{2\pi k_r h}{q_{inj} u B} (p_b - p_i) \quad (A.7)
\]

\[
t_D = \frac{k_r t}{\phi c_t u x_f^2} \quad (A.8)
\]

\[
x_{fD} = \frac{x_f(t)}{x_f} \quad (A.9)
\]

\[
q_D = \frac{q(t)}{q_{inj} B} \quad (A.10)
\]

\[
w_D = \frac{w_{ave}(t)}{x_f} \quad (A.11)
\]

\[
c_D = \frac{C}{2\pi \phi c_t h x_f^2} \quad (A.12)
\]

Then, the dimensionless fluid leak-off rate is:
\[ q_{LD}(t_D) = 1 - \left( c_D + \frac{x_f^2(t_D)}{E \varnothing c_t} \right) \frac{dp_{WD}}{dt_D} - \frac{2}{E \varnothing c_t} x_f(t_D) \frac{dx_f(t_D)}{dt_D} (p_{WD} - p_{CD}) \] (A.13)

Note that fracture ultimate half length \( x_f \) at shut-in \( t_{shut} \) is chosen as reference length in dimensionless terms. Eq. (3.20) describes fracture propagating velocity, which can form a correlation between eventual half length and that at a previous time point.

\[
\left( \frac{t}{t_{shut}} \right)^e = \frac{x_f(t)}{x_f}
\]

\[ x_f(t_D) = \left( \frac{t_D}{t_{shut_D}} \right)^e \]

Therefore fracture half length at any time \( x_f(t_D) \) is a known on the basis of current time and shut-in time; as well propagating velocity is a known too.

\[
\frac{dx_f(t_D)}{dt_D} = e \left( \frac{t_D}{t_{shut_D}} \right)^{e-1} t_{shut_D}
\]

The dimensionless fracture volume growing rate, on injecting rate scale, is

\[
\frac{dV_{frac}(t_D)}{dt_D} \left/ q_{inj} \right. = \frac{2}{E \varnothing c_t} x_f(t_D) \frac{dx_f(t_D)}{dt_D} (p_D - p_{CD}) + \frac{x_f^2(t_D)}{E \varnothing c_t} \frac{dp_{WD}}{dt_D} \] (A.14)

The fluid efficiency \( f_e \) defined as the ratio of fracture volume to total cumulative injected fluid volume at shut-in is:

\[
f_e = \int_0^{t_{shut_D}} \left[ \frac{2}{E \varnothing c_t} x_f(t_D) \frac{dx_f(t_D)}{dt_D} (p_{WD} - p_{CD}) + \frac{x_f^2(t_D)}{E \varnothing c_t} \frac{dp_{WD}}{dt_D} \right] dt_D \] (A.15)
APPENDIX B

B.1 Expression of the first exposed fracture segment:

Starting from time \( t=0 \), fracture is initiated with a half length \( x_{f1} \) till \( t_1; \Delta t_1 = t_1 - 0 \).

As described in Chapter 3 section 2, pressure is expressed by using source function method. Reservoir system is separated into two sub-regions, fracture and original reservoir. Following the assumption in this thesis, the injection well source is located inside the fracture and the direct fluid injection from wellbore to fracture tip is minimal and is neglected. The pressure in fracture region is:

\[
\Delta p_{f(x,y,t)} = \frac{1}{(\rho \mu c_t)_{f}} \left[ \int_{0}^{t_1} q_{\text{inj}}(\tau) \Delta p_{s,x,\text{seal}}(x,0,\tau) \Delta p_{s,y,\text{seal}}(y,0,\tau) dy d\tau \\
- \int_{0}^{t_1} \int_{x_{f1}}^{x_{f1}} \frac{q_{f}(\chi,\tau,x_{f1}) \Delta p_{s,x,\text{seal}}(x,\chi,\tau) \Delta p_{s,y,\text{seal}}(0,\frac{1}{2}w_{ave1},\tau)}{d\chi d\tau} \\
- \int_{0}^{t_1} \int_{x_{f1}}^{x_{f1}} \frac{q_{f}(\chi,\tau,x_{f1}) \Delta p_{s,x,\text{seal}}(x,\chi,\tau) \Delta p_{s,y,\text{seal}}(0,-\frac{1}{2}w_{ave1},\tau)}{d\chi d\tau} \right] \quad (B.1)
\]
\( \Delta p_{sx,seal} \) and \( \Delta p_{sy,seal} \) are point source functions with bounded sealing faults for x and y direction respectively.

\[
\Delta p_{sx,seal}(x, \chi, t, \tau) = \frac{1}{2(x_i - x_{i-1}) \sqrt{n_f(t - \tau)}} \sum_{n=0}^{\infty} \left( \exp \left[ -\frac{(x - \chi + n)^2}{4n_f(t - \tau)} \right] + \exp \left[ \frac{(x - \chi - n)^2}{4n_f(t - \tau)} \right] \right)
\]

(B.2)

As is seen, pressure in fracture region is expressed with no-flow boundary. It is based on the image principle, and is successfully achieved by Zhao et al (2002).

In Eq. (B.1), \( \bar{q}_f(x, t, x_{f1}) \) is the flux along fracture-reservoir contact with half length \( x_{f1} \).

It is assumed that flux can be regarded as uniform on a small segment. If the fluid rate from the segment located at \( x_i < x < x_{i+1} \) is denoted by \( q_{f,i} \), then the fluid flux on this segment can be approximated as

\[
\bar{q}_f(x, t, x_{f1}) \approx \frac{q_{f,i}}{x_{i+1} - x_i}, \quad \text{for} \quad x_i < \chi < x_{i+1}
\]

(B.3)

Eq. (B.1) can be written with the flux along fracture-reservoir contact \( \bar{q}_f(x, t, x_{f1}) \) replaced by Eq. (B.3).

The fluid rate flowing out fracture must equal to that flowing into reservoir. By using source function method, pressure in reservoir region can be expressed as:

\[
\Delta p_{r(x,y,t)} = \frac{1}{\phi \mu c_t} \int_0^{\chi_{f1}} \int_{-\chi_{f1}}^{\chi_{f1}} 2\bar{q}_f(\chi, \tau, x_{f1}) \Delta p_{sx,inf}(x, \chi, t, \tau) \Delta p_{sy,inf}(y, 0, t, \tau) d\chi d\tau
\]

(B.4)

\( \Delta p_{sx,inf} \) and \( \Delta p_{sy,inf} \) are point source functions in infinite domain for x- and y- direction.
\[ \Delta p_{sx, inf} = \frac{1}{2\sqrt{\pi \eta_x(t - \tau)}} \exp \left[ \frac{(x - x')^2}{4\eta_x(t - \tau)} \right] \]

Eq. (B.4) can also be written with the flux in reservoir \( \tilde{q}_f(x, t, x_{f1}) \) replaced by Eq. (B.3).

The computation detail is provided in chapter 3. Fracture and reservoir is coupled at their contact location based on pressure and flux continuity. Well bottom pressure and flux along fracture-reservoir contact of a finite conductivity fractured well is expressed below in Fig. B-2 and Fig. B-3 with a presumed constant wellbore flow rate and with fracture conductivity 400.
Fig. B-2 Pressure and its derivative behavior for a finite conductivity fracture
Fig. B-3 Flux along fracture and reservoir contact
B.2 Expression of fluid fracture leak-off:

Fluid leak-off rate causes pressure response inside reservoir. The derivation of leak-off rate equation is expressed in Appendix A. Applying Duhamel principle, the pressure drop caused by a variable rate process takes form in Laplace domain as below:

\[
\overline{\Delta p_v(s)} = L(\Delta p_v(t)) = L\left(\frac{q_v(t)}{q_c}\right) L\left(\frac{\partial \Delta p_c(t)}{\partial t}\right) = s \frac{q_v(s)}{q_c} \overline{\Delta p_c(s)} \quad (B.6)
\]

Where \(q_c\) represents a constant rate process and \(\Delta p_c(t)\) is its corresponding pressure response; \(q_v(t)\) represents a variable rate process and \(\Delta p_v(t)\) is its corresponding pressure response.

Recall that leak-off rate in dimensionless form is shown in Eq. (A.13):

\[
q_{ID}(t_D) = 1 - \left(C_D + \frac{x_f^2(t_D)}{E\phi c_t}\right) \frac{dp_{WD}}{dt_D} - \frac{2}{E\phi c_t} x_D(t_D) \frac{dx_f(t_D)}{dt_D} (p_{WD} - p_{cD}) \quad (B.7)
\]

In order to make Eq. (B.4) seem more concise, defining:

\[
\delta = \frac{1}{E\phi c_t} \quad (B.8)
\]

\[
\gamma = x_D \frac{\partial x_D}{\partial t_D} p_{cD} \quad (B.9)
\]

With the propagating velocity description proposed by Nolte (1986), Eq. (9) could be evaluated. Then one rewrites the expression of \(q_{ID}(t_D)\) into Laplace domain as:

\[
\bar{p}_{ID}(s) = \frac{(1 + 2\delta \gamma)}{s} - (C_D + \delta x_{f1D}) \bar{p}_{WD}(s) - 2\delta \gamma \bar{p}_{WD}(s) + (C_D + \delta x_{f1D}) p_{WD}^i \quad (B.10)
\]

where, \(p_{WD}^i\) is the dimensionless well bottomhole pressure at the end of i-th time period in real time domain. Actually, \(p_{WD}^i\) represents well bottomhole pressure at the end of last
time period or the beginning of current time period. Here $p_{wD}^0$ is the formation breakdown pressure at $t=0$, when formation breaks down $p_{wD}^0 = p_{bD}$.

With the knowledge of leak-off rate expressed in Eq.(10), Duhamel principle can be applied to generate the bottom hole pressure as

$$
\tilde{p}_{wD}(s) = s\left[\frac{1 + 2\delta\gamma}{s} - (C_D + \delta x_{f1D}^2)s\tilde{p}_{wD}(s) - 2\delta\gamma\tilde{p}_{wD}(s)
\right. \\
\left. + (C_D + \delta x_{f1D}^2)p_{wD}^0]\tilde{p}_{eD}(s) \right]
$$

(B.11)

Rearranging Eq.(B.11) presents the bottom hole pressure corresponding to a variable rate process in Laplace domain as:

$$
\tilde{p}_{wD}(s) = \frac{\tilde{p}_D(s)[1 + 2\delta\gamma + (C_D + \delta x_{f1D}^2)s p_{wD}^0]}{1 + (C_D + \delta x_{f1D}^2)s^2\tilde{p}_D(s) + 2s\delta\gamma\tilde{p}_D(s)} 
$$

(B.12)

**B.3 Modeling dynamic fracture propagating during $t_1$ to $t_2$, $\Delta t_2 = t_2 - t_1$:**

**B.3.1 Conceptual description of instantaneous fracture increment at $t_1$:**

As dynamic fracture propagation is considered, it is assumed that at the time point of $t_1$, the fracture tip instantaneously propagates to $x_{f2}$; the fracture half length is assumed to remain at $x_{f2}$ till the end of this period at $t_2$, the duration of this period is expressed as $\Delta t_2 = t_2 - t_1$. An increment of fracture half length, $(x_{f2} - x_{f1})$, is then attained before fracture goes into future increment.
Fig. B-4 Fractured reservoir with fracture half length $x_{f2}$ with the instantaneous incremental fracture portions indicated by red dashed lines

### B.3.2 Partial Differential Equation in fracture for the period from $t_1$ to $t_2$:

From the view of material balance

$$-\left(\frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y}\right) + q_{inj} = \frac{\partial (\rho \phi)}{\partial t} \quad (B.13)$$

Based on Darcy’s law in describing two-dimensional flow, velocity is expressed as

$$v_x = -\frac{k_f}{\mu} h \frac{\partial p}{\partial x} \quad (B.14)$$

Eq. (B.13) becomes

$$\frac{\partial}{\partial x} \left( \frac{k_f}{\mu} \rho h \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{k_f}{\mu} \rho h \frac{\partial p}{\partial y} \right) + q_{inj} = \rho \frac{\partial \phi}{\partial t} + \phi \frac{\partial p}{\partial t} \quad (B.15)$$

Because the injection well is located inside fracture, as assumed before, the flow equation for a slightly compressible fluid can be written as

$$\frac{\partial^2 \Delta p_f}{\partial x^2} + \frac{\partial^2 \Delta p_f}{\partial y^2} + \frac{q_{inj} \mu}{hk_f} = \frac{1}{\eta_f} \frac{\partial \Delta p_f}{\partial t} \quad (B.16)$$

with initial condition
\[ \Delta p_f = f_f(x, y), \text{ at } t = t_1 \quad -x_{f2} < x < x_{f2}, 0 < y < w_{ave2} \]  \hspace{1cm} (B.17) \\

and boundary condition

\[ \frac{k_f}{\mu} \frac{\partial \Delta p_f}{\partial x} \bigg|_{x = x_{f2}} = 0 \quad -w_{ave2}/2 < y < w_{ave2}/2, \quad t > t_1 \]  \hspace{1cm} (B.18) \\

\[ \frac{k_f}{\mu} \frac{\partial \Delta p_f}{\partial y} \bigg|_{y = w_{ave2}/2} = \bar{q}_f(x, t, x_{f2}), \quad -x_{f2} < x < x_{f2}, \quad t > t_1 \]  \hspace{1cm} (B.19) \\

The term \( q_{inj} \) in flow equation is the contribution of injection; \( \bar{q}_f(x, t, x_{f2}) \) is the actual flux along fracture-reservoir contact after increment of half length at \( t_1 \); the initial condition \( f_f(x, y) \) is the pressure distribution in fracture at \( t_1 \) caused by previous fractured well during \( \Delta t_1 \). The fracture rate notation at the right hand sides of Eq. (B.19) is simplified due to the symmetry of the problem under study.

\section*{B.3.3 Partial Differential Equation in reservoir for the period from \( t_1 \) to \( t_2 \)}

Flow equation

\[ \frac{\partial^2 \Delta p_r}{\partial x^2} + \frac{\partial^2 \Delta p_r}{\partial y^2} = \frac{1}{\eta_r} \frac{\partial \Delta p_r}{\partial t} \]  \hspace{1cm} (B.20) \\

With initial condition

\[ \Delta p_r(x, y, t) = f_r(x, y), \text{ at } t = t_1, \quad x \rightarrow \infty, y \rightarrow \infty \]  \hspace{1cm} (B.21) \\

Boundary condition

\[ \Delta p_r(x \rightarrow \infty, y \rightarrow \infty, t) = 0 \]  \hspace{1cm} (B.22) \\

\[ \frac{hk_r}{\mu} \frac{\partial \Delta p_r}{\partial y} \bigg|_{y = 0} = \bar{q}_f(x, t, x_{f2}), \quad -x_{f2} \leq x \leq x_{f2}, t > t_1 \]  \hspace{1cm} (B.23) \\

The initial condition term of \( f_r(x, y) \) is the pressure distribution inside reservoir caused by previous fractured well performance. Note that in those PDEs above, time resets back to zero, or \( t_1 \) is regarded as “mathematical” zero time point by putting on initial
conditions for both fracture and reservoir systems, which are the corresponding pressure distributions inside fracture and reservoir at $t_1$.

**B.4 Decomposition of the problem for the period from $t_1$ to $t_2$**

Pressure in fracture and reservoir as well as the flux has to satisfy their flow equations and the initial and boundary conditions discussed above. Applying superposition principle, a decomposition strategy is proposed as the following for both the fracture and reservoir systems

\[
\Delta p_f(x, y, t) = u_f(x, y, t) + v_f(x, y, t) + w_f(x, y, t) \quad (B.24)
\]

\[
\Delta p_r(x, y, t) = u_r(x, y, t) + v_r(x, y, t) + w_r(x, y, t) \quad (B.25)
\]

Where,

- $u(x, y, t)$ takes the contribution of injecting source in new fracture geometry with half length $x_{f_2}$.

- $v(x, y, t)$ describes the continuous transient pressure diffusion with pressure distribution over the reservoir-fracture system at $t_1$ caused by previous fractured well performance.

- $w(x, y, t)$ helps manage the diffusion of the pressure domain difference caused by the implementation of the instantaneous dynamic fracture propagation scheme proposed in this research. The pressure domain initial condition occurs at $t = t_1$. Physically, the sudden change of the fracture half length from $x_{f_1}$ to $x_{f_2}$ and the requirement of pressure and flux continuity contribute to the inclusion of this new term.

**B.4.1 Simplification in fracture region for the period from $t_1$ to $t_2$**
The flow equation, Eq. (13), in fracture can be written as

\[
\frac{\partial^2 (u_f + v_f + w_f)}{\partial x^2} + \frac{\partial^2 (u_f + v_f + w_f)}{\partial y^2} + \frac{q_{inj} \mu}{h k_f} = \frac{1}{\eta_f} \frac{\partial (u_f + v_f + w_f)}{\partial t} \quad (B.26)
\]

Let each satisfies

\[
\frac{\partial^2 u_f}{\partial x^2} + \frac{\partial^2 u_f}{\partial y^2} + \frac{q_{inj} \mu}{h k_f} = \frac{1}{\eta_f} \frac{\partial u_f}{\partial t} \quad (B.27)
\]

And

\[
\frac{\partial^2 (v_f)}{\partial x^2} + \frac{\partial^2 (v_f)}{\partial y^2} = \frac{1}{\eta_f} \frac{\partial (v_f)}{\partial t} \quad (B.28)
\]

\[
\frac{\partial^2 (w_f)}{\partial x^2} + \frac{\partial^2 (w_f)}{\partial y^2} = \frac{1}{\eta_f} \frac{\partial (w_f)}{\partial t} \quad (B.29)
\]

With initial condition

\[
u_f(x, y) + v_f(x, y) + w_f(x, y) = f_f(x, y) \text{ at } t_1 \quad (B.30)
\]

Let each satisfy

\[
u_f(x, y) = 0 \text{ at } t_1 \quad (B.31)
\]

\[
v_f(x, y) = f_f(x, y) \text{ at } t_1, \quad -x_{f2} < x < x_{f2}, 0 < y < w_{ave2} \quad (B.32)
\]

\[
w_f(x, y) = 0 \text{ at } t_1 \quad (B.33)
\]

And boundary condition

\[
\frac{k_f}{\mu} h \frac{\partial (u_f + v_f + w_f)}{\partial y} \bigg|_{y=0} = \bar{q}_f(x, t) \quad -x_{f2} < x < x_{f2} \quad (B.34)
\]
Let each satisfy

$$\frac{k_f}{\mu} \frac{\partial (u_f)}{\partial y} \big|_{y=0} = \tilde{q}_{f_{std}}(x, t, x_{f2}), \quad t_1 < t < t_2, \quad -x_{f2} < x < x_{f2} \quad (B.35)$$

$$\frac{k_f}{\mu} \frac{\partial (v_f)}{\partial y} \big|_{y=0} = \tilde{q}_{f_{std}}(x, t, x_{f1}) - \tilde{q}_{f_{std}}(x, t, x_{f2}), \quad t_1 < t < t_2, \quad -x_{f2} < x < x_{f2} \quad (B.36)$$

$$\frac{k_f}{\mu} \frac{\partial (w_f)}{\partial y} \big|_{y=0} = \Delta \tilde{q}_f(x, t, x_{f2}), \quad t_1 < t < t_2, \quad -x_{f2} < x < x_{f2} \quad (B.37)$$

Therefore

$$\tilde{q}_f(x, t) = \tilde{q}_{f_{std}}(x, t, x_{f2}) + [\tilde{q}_{f_{std}}(x, t, x_{f1}) - \tilde{q}_{f_{std}}(x, t, x_{f1})]$$

$$+ \Delta \tilde{q}_f(x, t, x_{f2}), \quad t_1 < t \leq t_2 \quad (B.38)$$

Note that $\tilde{q}_{f_{std}}(x, t, x_{f1})$ and $\tilde{q}_{f_{std}}(x, t, x_{f2})$ represent standardized fracture flux along fracture-reservoir contact responding to fractured well with fixed half length $x_{f1}$ and $x_{f2}$ respectively, which can be expressed in Fig. (B-3). Note that the $\Delta \tilde{q}_f(x, t - t_1, x_{f2})$ item takes care of the rate variation caused by the assumption of an instantaneous change of the fracture dynamic propagation at $t_1$ through sudden change of fracture from $x_{f1}$ to $x_{f2}$ at $t_1$. Therefore, the mathematical involvement of Eq.(B.37) is because of the semi-analytical modeling strategy, which is based on the assumption just mentioned above.

**B.4.2 Simplification in reservoir region for the period from $t_1$ to $t_2$**

The flow equation, Eq. (B.20), in reservoir can be written as
\[
\frac{\partial^2(u_r + v_r + w_r)}{\partial x^2} + \frac{\partial^2(u_r + v_r + w_r)}{\partial y^2} = \frac{1}{\eta_r} \frac{\partial(u_r + v_r + w_r)}{\partial t} \quad (B.39)
\]

With initial condition

\[u_r(x, y, t) + v_r(x, y, t) + w_r(x, y, t) = f_r(x, y), \quad at \ t = t_1, \ x \to \infty, y \to \infty \quad (B.40)\]

Let each satisfy

\[u_r(x, y, t) = 0, \ at \ t = t_1, \ x \to \infty, y \to \infty \quad (B.41)\]

\[v_r(x, y, t) = f_r(x, y, t), \ at \ t = t_1, \ x \to \infty, y \to \infty \quad (B.42)\]

\[w_r(x, y, t) = 0, \ at \ t = t_1, \ x \to \infty, y \to \infty \quad (B.43)\]

And boundary condition

\[\frac{hk_r}{\mu} \left. \frac{\partial(u_r + v_r + w_r)}{\partial y} \right|_{y=0} = \bar{q}_f(x, t, x_{f2}), \ x \to \infty, y \to \infty, \ t > t_1 \quad (B.44)\]

Let each satisfy

\[\frac{k_r}{\mu h} \left. \frac{\partial(u_r)}{\partial y} \right|_{y=0} = \bar{q}_{f_{std}}(x, t - t_1, x_{f2}) \quad x \to \infty, y \to \infty, \ t > t_1 \quad (B.45)\]

\[\frac{k_r}{\mu h} \left. \frac{\partial(v_r)}{\partial y} \right|_{y=0} = \bar{q}_{f_{std}}(x, t, x_{f1}) - \bar{q}_{f_{std}}(x, t - t_1, x_{f1}) \quad x \to \infty, y \to \infty, \ t > t_1 \quad (B.46)\]

\[\frac{k_r}{\mu h} \left. \frac{\partial(w_r)}{\partial y} \right|_{y=0} = \Delta \bar{q}_f(x, t - t_1, x_{f2}) \quad x \to \infty, y \to \infty, \ t > t_1 \quad (B.47)\]

**B. 4. 3.1 Further explanation of the mathematical and physical meaning of** \( u(x, y, t) \)
From the flow equation, initial and boundary condition of \( u_f(x, y, t) \) (that in reservoir is not put down here)

\[
\frac{\partial^2 u_f}{\partial x^2} + \frac{\partial^2 u_f}{\partial y^2} + \frac{q_{\text{inj}} u}{h k_f} = \frac{1}{\eta_f} \frac{\partial u_f}{\partial t} \quad (B.48)
\]

\( u_f(x, y) = 0 \) at \( t_1 \) \( (B.49) \)

\[
\frac{k_f}{\mu} h \frac{\partial (u_f)}{\partial y} \bigg|_{y=0} = \tilde{q}_{f,\text{stand},x_{f2}}(x, t - t_1) \quad -x_{f2} < x < x_{f2} \quad (B.50)
\]

\[
\frac{k_f}{\mu} h \frac{\partial u_f}{\partial x} \bigg|_{x=x_{f2}} = 0 \quad 0 < y < w_{\text{ave2}} \quad (B.51)
\]

It is easy to see when fracture and reservoir are coupled with each other as discussed in chapter 3 or above in section B.1, \( u(x, y, t) \) represents the pressure response of a fractured well with half length \( x_{f2} \), starting injection at \( t_1 \) and staying active during the period of \( (t_2 - t_1) \) or \( \Delta t_2 \). It is

\[
u(x, y, t) = \Delta p_{\text{c,stand}}(x, y, t - t_1, x_{f2}), \quad t_1 < t \leq t_2 \quad (B.52)
\]

\( \Delta p_{\text{c,stand}}(x, y, t, x_{f2}) \) is the standard pressure behavior responding to constant fracture half length \( x_{f2} \) with constant rate injection. See Fig. (B-3).
B. 4. 3.2 Further explanation of the mathematical value and physical meaning of
\( v(x, y, t) \)

From the flow equation, initial and boundary condition of \( v_f(x, y, t) \) (that in reservoir is not put down here):

\[
\frac{\partial^2(v_f)}{\partial x^2} + \frac{\partial^2(v_f)}{\partial y^2} = \frac{1}{\eta_f} \frac{\partial(v_f)}{\partial t} \quad (B.53)
\]

\[
v_f(x, y) = f_f(x, y) \text{ at } t_1, \quad -x_{f2} < x < x_{f2}, 0 < y < w_{ave2} \quad (B.54)
\]

\[
\frac{k_f h}{\mu} \frac{\partial(v_f)}{\partial y} \bigg|_{y=0} = \bar{q}_{f_{std}}(x, t, x_{f1}) - \bar{q}_{f_{std}}(x, t - t_1, x_{f1}) - x_{f2} < x < x_{f2} \quad (B.55)
\]

As is seen, \( v_f(x, y, t) \) does not take the contribution of wellhead injection \( q_{inj} \), but instead it takes the corresponding response of continuous diffusion of pressure distribution \( f_f(x, y) \) at \( t_1 \) without injection. Note that the boundary condition here is set as \( \bar{q}_{f_{std}}(x, t, x_{f1}) - \bar{q}_{f_{std}}(x, t - t_1, x_{f1}) \), which in value is equivalent to that of a “shut-in” test of the last time period’s fractured well with half length \( x_{f1} \).

\[
v_f(x, y, t) = \Delta p_{c_{std}}(x, y, t, x_{f1}) - \Delta p_{c_{std}}(x, y, t - t_1, x_{f1}), \quad t_1 < t \leq t_2 \quad (B.56)
\]

\( \Delta p_{c_{std}}(x, y, t, x_{f1}) \) is the standardized pressure behavior responding to fixed fracture half length \( x_{f1} \) with constant rate injection.
B. 4. 3.3 Further explanation of the mathematical and physical meaning of \( w(x, y, t) \)

From the flow equation, initial and boundary condition of \( w_f(x, y, t) \) (that in reservoir is not put down here)

\[
\frac{\partial^2 w_f}{\partial x^2} + \frac{\partial^2 w_f}{\partial y^2} = \frac{1}{\eta_f} \frac{\partial w_f}{\partial t} \quad (B.57)
\]

\( w_f(x, y) = 0 \) at \( t_1 \) \( (B.58) \)

\[
\frac{k_f}{\mu} h \frac{\partial w_f}{\partial y} \bigg|_{y=0} = \Delta \tilde{q}_f(x, t - t_1, x_{f_2}) \quad - x_{f_2} < x < x_{f_2} \quad (B.59)
\]

It can be seen that \( w_f(x, y, t) \) did not take any of wellhead injection contribution or previous pressure distribution. It helps manage the diffusion of the pressure domain difference caused by the implementation of the instantaneous dynamic fracture propagation scheme proposed in this research. It adds flux on the boundary of fracture to complete the response dynamic fracture propagation besides the contribution of \( u(x, y, t) \) and \( v(x, y, t) \).
B.5 Coupling of fracture and reservoir systems during the period from \( t_1 \) to \( t_2 \)

B.5.1 Discretization of fracture

As described in the previous section, \( \Delta \tilde{q}_f(x, t - t_1, x_{f_1}) \) is the unknown that needs to be computated. First, fracture is discretized at \( x = |x_{f_1}| \) where increment started from the tip of the previous. It is explained before that pressure response in fracture is expressed with no-flow boundary by using image principle, which leads to the executing strategy that the pressure in the previous fracture part and the newly grown part can be formulated independently.

B.5.2 Coupling in fracture during \( t_1 \sim t_2 \) at \( x = |x_{f_1}| \)

By superposition of the three items namely \( u(x, y, t) \), \( v(x, y, t) \) and \( w(x, y, t) \), after \( t_1 \) the pressure in former fracture region \( x < |x_{f_1}| \) can be formulated as
\[
\Delta p_f(x, t) = \Delta p_{c, \text{std}}(x, y, t - t_1, x_{f2}) + [\Delta p_{c, \text{std}}(x, y, t, x_{f1}) - \Delta p_{c, \text{std}}(x, y, t - t_1, x_{f1})] \\
- \frac{1}{(\rho \mu c_f)_f} \left[ \int_{t-t_1}^{t-t_1} \int_{-x_{f1}}^{x_{f1}} \Delta \tilde{q}_f(x, \tau, x_{f2}) \Delta p_{s, \text{sea}}(x, x, t - t_1, \tau) \Delta p_{s, \text{sea}}(y, y, \frac{1}{2} w_{\text{ave}}, t) \\
- t_1, \tau \right] d\chi \, d\tau + \int_{0}^{t-t_1} \int_{-x_{f1}}^{x_{f1}} \Delta \tilde{q}_f(x, \tau, x_{f2}) \Delta p_{s, \text{sea}}(x, x, \tau) \\
- t_1, \tau \Delta p_{s, \text{sea}}(y, y, \frac{1}{2} w_{\text{ave}}, t - t_1, \tau) d\chi \, d\tau \\
+ \int_{0}^{t-t_1} \int_{-x_{f1}}^{x_{f1}} \Delta \tilde{q}_N(y, \tau, x_{f2}) \Delta p_{s, \text{sea}}(x, x_{f1}, t - t_1, \tau) \Delta p_{s, \text{sea}}(y, y, t - t_1, \tau) d\chi \, d\tau \\
+ \int_{0}^{t-t_1} \int_{-x_{f1}}^{x_{f1}} \Delta \tilde{q}_{N,x_{f2}}(y, \tau, x_{f2}) \Delta p_{s, \text{sea}}(x, x_{f1}, t - t_1, \tau) \Delta p_{s, \text{sea}}(y, y, t \left( B.60 \right)
\]

The term \( \Delta \tilde{q}_N(y, \tau, x_{f2}) \) is the flux contributed by \( w(x, y, t) \) term inside fracture at node \( x = x_{f1} \). Note that Eq.(B.60) is going to be evaluated at node \( x = x_{f1} \) to help build hydraulic connection.

After \( t_1 \) the pressure inside the incremental fracture part, \( |x_{f1}| < x < |x_{f2}| \), can be formulated as
Pressure continuity needs to be achieved at \( x = x_{f1} \). Therefore, Eq. (B.60) and Eq. (B.61) are equal at \( x = x_{f1} \). This builds the effective hydraulic communication inside the dynamically propagating fracture.

**B.3.3 Coupling at fracture-reservoir contact**

By superposition of the three items namely \( u(x, y, t) \), \( v(x, y, t) \) and \( w(x, y, t) \), after \( t \) the pressure response over reservoir region can be formulated as

\[
\Delta p_r(x, y, t) = \Delta p_{c_{std}}(x, y, t - t_1, x_{f2}) + \left[ \Delta p_{c_{std}}(x, y, t, x_{f1}) - \Delta p_{c_{std}}(x, y, t - t_1, x_{f1}) \right] \\
+ \frac{1}{\phi \mu c_t} \int_0^{t-t_1} \int_{-x_{f2}}^{x_{f2}} 2\tilde{q}_f(x, \tau, x_{f2}) \Delta p_{sx,inf1}(x, \chi, t, \tau) \Delta p_{sy,inf1}(y, 0, t, \tau) d\chi d\tau
\]

(B.62)

Then the pressure in reservoir region, Eq. (B.62), is equal to that in fracture region, described by Eqs. (B.60) and (B.61), to achieve pressure continuity at fracture-reservoir contact. Then the fracture flux contributed by \( w(x, y, t) \), *i.e.*, \( \Delta \tilde{q}_f(x, t - t_1, x_{f2}) \) and
$\Delta q_N(y, t - t_1, x_{f_2})$, can be successfully computed, the flux connection problem is therefore resolved. Proceeding in a similar way, more fracture increment can be modeled by the strategy described above.