ANALYTICAL MODELING OF TRANSIENT HEAT TRANSFER COUPLED WITH FLUID FLOW IN HEAVY OIL RESERVOIRS DURING THERMAL RECOVERY PROCESSES

A Thesis
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in Partial Fulfillment of the Requirements
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in
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By
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Kuizheng Yu, candidate for the degree of Master of Applied Science in Petroleum Systems Engineering, has presented a thesis titled, **Analytical Modeling of Transient Heat Transfer Coupled With Fluid Flow in Heavy Oil Reservoirs During Thermal Recovery Processes**, in an oral examination held on August 19, 2014. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Tremendous resources of heavy oil are located in western Canada. Among the many heavy oil recovery methods proposed, thermal recovery methods which employing hot fluid injection or in-situ combustion, have been conventionally utilized to enhance heavy oil recovery. Temperature profiles in heavy oil reservoirs are important factors for making operation and production plans in thermal recovery processes. A significant factor in these processes is to study the heat energy transfer in the oil formations.

In previous studies, heat transfer by conduction has been commonly considered as the main mechanism of heat transfer in porous media. However, this assumption is not reasonable for thermal recovery processes with fluid flow such as steam flooding, hot water flooding, steam-assisted gravity drainage (SAGD) and cyclic steam stimulation (CSS). Heat transfer and fluid flow occur simultaneously during thermal recovery processes, and heat transfer by convection cannot be neglected. Fluid flow motivates convective heat transfer, and increase the rate of energy transfer significantly. Heat conduction is dominated by temperature gradient, while heat convection is dominated by pressure gradient. By integrating conduction and convection, temperature and pressure domain can be coupled systematically.

In this study, novel heat transfer models, integrating both conduction and convection, have been developed to describe the one-dimensional transient heat transfer coupled with fluid flow. In the models, the properties of the reservoir and the fluid are integrated into two important parameters, i.e., the thermal diffusivity of a reservoir/fluid system and the thermal convection velocity of the fluid.
To derive the analytical solutions of these mathematical models, dimensionless variables are defined to reduce the models to the dimensionless form. After that, variable transformation and Laplace transformation are performed to derive the analytical solution in Laplace domain. By using the table of Laplace transformations, the solution in Laplace domain can be converted to dimensionless analytical solution in real time domain. Numerical simulations by COMSOL Multiphysics are conducted to validate the analytical solutions. Subsequently, case studies under both steady and unsteady flow conditions have been conducted. Satisfactory agreements of the results are achieved between analytical solutions and numerical simulation results.

To prove the mathematical models could have practical application in the oil and gas industry, results comparison between the analytical solution and CMG simulation is conducted. A numerical simulation model for transient heat transfer in heavy oil reservoirs during the SAGD process was used for comparison. It is found that the shapes of temperature distributions and propagations of the analytical solution and the CMG simulation have the similar trends. The studies showed good agreement between the test results and those from the CMG simulation.

The newly developed analytical solutions provide theoretical guidance for temperature transient analysis (TTA) and fluid injection strategies. These analytical solutions can be used to predict temperature profiles in heavy oil reservoirs during thermal recovery processes and improve the accuracy and efficiency of temperature transient analysis.
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DEDICATION

To my dearest parents,

Ms. Ruixiang Li and Mr. Yanming Yu,

and my dear brother,

Mr. Kuifeng Yu,

for their continuous support and unconditional love
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NOMENCLATURE

Notations

c Specific heat capacity, J/(kg·K)
ρc Volumetric heat capacity, J/(m^3·K)
C1, C2 Coefficients in general solution of homogeneous ODE
D Thermal diffusivity of reservoir/fluid system, m^2/s
D0 Initial thermal diffusivity of reservoir/fluid system, m^2/s
erfc Complementary error function
f(t), g(t) Time-dependent mathematical functions
F(x) Mathematical function with respect to distance x
G(t) Mathematical function with respect to time t
H(t-a) Heaviside unit step function
k Permeability, μm^2
K Thermal conductivity, W/(m·K)
K(xD,tD) New dependent variable defined in equation (3.32)
K(xD,s) Laplace transformation of K(xD,tD)
K_h General solution of homogeneous ODE
K_p Particular solution of non-homogeneous ODE
L^{-1} Inverse Laplace transformation
m Unsteady parameter, s^{-1}
P Pressure, Pa
qd Conductive heat flux, W/m^2
$q_v$  \hspace{1cm} \text{Convective heat flux, W/m}^2

$q_w$  \hspace{1cm} \text{Surface water rate of injector, m}^3/$\text{day}$

$q_l$  \hspace{1cm} \text{Surface liquid rate of producer, m}^3/$\text{day}$

$Q$  \hspace{1cm} \text{Rate of heat generation per unit volume, W/m}^3$

$r$  \hspace{1cm} \text{Characteristic roots of homogeneous ODE}

$s$  \hspace{1cm} \text{Laplace parameter}

$s_{oi}$  \hspace{1cm} \text{Initial oil saturation, fraction}

$T$  \hspace{1cm} \text{Temperature, K}

$T_D$  \hspace{1cm} \text{Dimensionless temperature}

$t$  \hspace{1cm} \text{Time, s}

$t_D$  \hspace{1cm} \text{Dimensionless time}

$u$  \hspace{1cm} \text{Thermal convection velocity of fluid, m/s}

$u_0$  \hspace{1cm} \text{Initial thermal convection velocity of fluid, m/s}

$V$  \hspace{1cm} \text{Darcy’s velocity, m/s}

$x$  \hspace{1cm} \text{Distance, m}

$x_D$  \hspace{1cm} \text{Dimensionless distance}

$X$  \hspace{1cm} \text{New space variable defined in equation (4.8)}

$\nabla T$  \hspace{1cm} \text{Temperature gradient, K/m}

$\nabla P$  \hspace{1cm} \text{Pressure gradient, Pa/m}

\textbf{Abbreviations}

AEUB  \hspace{1cm} \text{Alberta Energy and Utilities Board}

API  \hspace{1cm} \text{American Petroleum Institute}
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>BVP</td>
<td>Boundary Value Problem</td>
</tr>
<tr>
<td>CDE</td>
<td>Convection-Diffusion Equation</td>
</tr>
<tr>
<td>CMG</td>
<td>Computer Modelling Group</td>
</tr>
<tr>
<td>CO₂</td>
<td>Carbon Dioxide</td>
</tr>
<tr>
<td>CSS</td>
<td>Cyclic Steam Stimulation</td>
</tr>
<tr>
<td>CHOPS</td>
<td>Cold Heavy Oil Production with Sand</td>
</tr>
<tr>
<td>ERCB</td>
<td>Energy Resources Conservation Board</td>
</tr>
<tr>
<td>GOR</td>
<td>Gas-Oil Ratio</td>
</tr>
<tr>
<td>HOR</td>
<td>Heavy Oil Recovery</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
<tr>
<td>OOIP</td>
<td>Original Oil in Place</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>PTA</td>
<td>Pressure Transient Analysis</td>
</tr>
<tr>
<td>SAGD</td>
<td>Steam-Assisted Gravity Drainage</td>
</tr>
<tr>
<td>STARS</td>
<td>Steam, Thermal and Advanced processes Reservoir Simulator</td>
</tr>
<tr>
<td>TTA</td>
<td>Temperature Transient Analysis</td>
</tr>
<tr>
<td>VAPEX</td>
<td>Vapour Extraction</td>
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**Greek letters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>β</td>
<td>Exponential decay coefficient, m⁻¹</td>
</tr>
<tr>
<td>φ</td>
<td>Porosity, fraction</td>
</tr>
<tr>
<td>λ</td>
<td>Flow resistance coefficient, s⁻¹</td>
</tr>
<tr>
<td>μ</td>
<td>Viscosity, Pa·s</td>
</tr>
</tbody>
</table>
\( \rho \) Density, kg/m\(^3\)

\( \gamma \) Specific gravity, fraction

\( \omega \) Coefficient defined in equation (3.73), m/s

\( \tau \) New time variable defined in equation (4.10)

**Subscript**

\( D \) Dimensionless

\( d \) Conduction

\( f \) Fluid

\( i \) Initial condition

\( l \) Liquid

\( o \) Oil

\( r \) Reservoir

\( s \) Solid/Rock

\( st \) Steam

\( v \) Convection

\( w \) Water
CHAPTER 1 INTRODUCTION

1.1 Heavy Oil Resources and Recovery Techniques

Heavy oil is characterized by high viscosities and low API gravities compared with that of conventional oil. Heavy oil is defined as any liquid petroleum with API gravity less than 20°. Heavy oil is closely related to bitumen from oil sands. The density of bitumen is less than 10 °API. It is present as solid and does not flow at ambient conditions (Attanasi and Meyer, 2010). The classification of heavy oil and bitumen are listed in TABLE 1-1.

Since conventional oil reserves are depleting in many countries and global energy consumption is still increasing, heavy oil resources have drawn much more attention to sustain the global energy demands. Heavy oil resources not only account for more than double the conventional oil resources in the world, but also offer potential to satisfy current and future oil demands (Speight, 2009).

Western Canada holds tremendous heavy oil and bitumen resources with an estimated original oil in place (OOIP) of 5.7 billion m³ in Alberta and 3.4 billion m³ in Saskatchewan (AEUB, 2007), respectively. Three important heavy oil and bitumen deposits in Alberta are Athabasca Wabiskwa-McMurray, Cold Lake Clearwater and Peace River Bluesky-Gething (ERCB, 2011). In Saskatchewan, heavy oil deposits are found in the sands of the Bakken formation (Mississippian) and the Mannville group (Wilson and Bennett, 1985). The geographical distributions of heavy oil and bitumen deposits in western Canada are shown in FIGURE 1-1.
TABLE 1-1 Classification of heavy oil and bitumen (Gibson, 1982; Farouq Ali, 2006)

<table>
<thead>
<tr>
<th>Classification</th>
<th>Viscosity$^{[1]}$, mPa⋅s</th>
<th>Density$^{[2]}$, g/cm$^3$</th>
<th>API gravity$^{[2]}$, °API$^{[3]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heavy oil</td>
<td>100 ~ 10,000</td>
<td>0.943 ~ 1.0</td>
<td>20 ~ 10</td>
</tr>
<tr>
<td>Bitumen</td>
<td>&gt; 10,000</td>
<td>&gt; 1.0</td>
<td>&lt; 10</td>
</tr>
</tbody>
</table>

Note:  
[1] Degassed oil viscosity at reservoir temperature, 0.1MPa

[2] Degassed oil density/API gravity at 15.6 °C, 0.1 MPa

[3] API gravity = \( \frac{141.5}{\gamma_o} - 131.5 \), and \( \gamma_o = \frac{\rho_o}{\rho_w} \)
FIGURE 1-1 Heavy oil and bitumen deposits in western Canada (ERCB, 2011)
Generally, two approaches have been used for recovery of heavy oil and bitumen: open pit mining and in-situ methods, depending upon the depth of the deposit (ERCB, 2011). The open pit method is used to recover the crude bitumen reserves that occur near the surface. In this way, overburden is removed, oil sand is mined, and bitumen is extracted from the sand in facilities using hot water. Although open pit mining has high recovery efficiency, mining susceptible reserves only account for approximately 5% of Canadian heavy oil and bitumen deposits. In contrast, the in-situ recovery method is applicable to the heavy oil reserves at a greater depth. In addition, it is more economical and environmentally friendly because the in-situ recovery method disturbs less of the surface environment and averts sand disposal (Das and Butler, 1994).

In general, in-situ recovery processes are classified into two categories: thermal recovery processes and non-thermal recovery processes. The key to heavy oil recovery (HOR) is to reduce oil viscosity and hence improve the oil mobility.

Thermal recovery processes, using heat to reduce oil viscosity in-situ, have been most successful, and currently account for the production of over one-half million barrels of oil per day in the U.S., Venezuela, and Canada (Farouq Ali, 2003). Thermal recovery techniques are effective as oil viscosity is very sensitive to temperature. Increasing the temperature of oil in the reservoir leads to the reduction of oil viscosity and the improvement of oil mobility. As can be seen in FIGURE 1-2, heating the heavy oil can reduce its viscosity by several orders of magnitude. At present, the most commonly used thermal recovery techniques are cyclic steam stimulation (CSS) (Vittoratos et al., 1990), steam flooding, steam-assisted gravity drainage (SAGD) (Butler et al., 1981), hot water flooding, and in-situ combustion (Moore et al., 1995).
FIGURE 1-2 A viscosity-temperature curve of heavy oil (Alboudwarej et al., 2006)
However, for those heavy oil reserves contained in thin formations with a thickness less than 10 meters, thermal recovery processes are ineffective or uneconomical due to excessive heat losses to overburden and underburden (Srivastava et al., 1999). Due to the burning of fuels for steam generation, a high environmental footprint also restricts the application of thermal recovery processes. Meanwhile, excessive water consumption is another restriction for thermal recovery processes in some areas (Zheng, 2012).

In comparison with thermal recovery processes, non-thermal recovery techniques have been applied in the past few decades because of their reduced energy consumption and good applicability to thin formations. The cold heavy oil production with sand (CHOPS) is one of the non-thermal recovery processes, in which heavy oil is produced with sand under solution-gas pressure by using progressive cavity pumps (Smith, 1988). CHOPS is an option for unconsolidated sand formations with good gas-oil ratio (GOR), but the effects of dilution caused by sand production may make it difficult to conduct follow-up approaches, such as steam injection (Farouq Ali, 2006). Recently, solvent-based non-thermal recovery processes have attracted increasing interest in enhancing heavy oil recovery. Such processes include vapour extraction (VAPEX) (Butler and Mokrys, 1991), cyclic solvent injection (Lim et al., 1995), CO₂ injection (Khatib et al., 1981; Dryer and Farouq Ali, 1989), and light hydrocarbon flooding (Garcia and Meneven, 1983). The primary oil recovery mechanism in these processes is to reduce oil viscosity through solvent dissolution rather than heat. The major challenges associated with thermal recovery processes, such as heat loss, water consumption, and produced water treatment, are avoided in solvent-based recovery processes (Das and Butler, 1994).
1.2 Purpose of This Thesis Study

The objective of this study is to comprehensively understand the heat transfer mechanisms in porous media and theoretically model one-dimensional transient heat transfer coupled with fluid flow. The primary objectives of this study include:

1) To develop novel mathematical models integrating conduction and convection to describe transient heat transfer process in heavy oil reservoirs;
2) To present the analytical solutions to mathematical models with a variety of heat-source configurations and boundary conditions;
3) To provide theoretical guidance on temperature transient analysis (TTA) and fluid injection strategies during thermal recovery processes.

1.3 Outline of the Thesis

This thesis is composed of five chapters. Chapter 1 introduces the research topic together with its major research objectives. Chapter 2 provides an updated literature review on mechanisms of heat transfer coupled with fluid flow in porous media. It also includes previous mathematical studies of heat transfer in porous media. Chapter 3 presents mathematical models that describe the one-dimensional transient heat transfer under steady flow condition. Ten cases with different initial and boundary conditions are studied. Chapter 4 presents a one-dimensional heat transfer model under unsteady flow condition. Four cases with different time-dependent thermal diffusivity and thermal convection velocity are studied. Finally, Chapter 5 summarizes the conclusions of current research and provides some recommendations for future work.
2.1 Mechanisms of Heat Transfer in Porous Media

Conductive and convective heat transfers occur simultaneously in porous media with fluid flow, and usually the rate of heat transfer by conduction is slower than that by convection (Kaviany, 1995). The stationary fluids and reservoir matrix are heated by conduction, while the displacement of oil and movement of injection fluid in reservoir are the sources of convective heat transfer. Heat conduction is dominated by temperature gradient, while heat convection is dominated by pressure gradient. Conduction transfers heat to the surrounding area in all directions, while convection mainly transports heat along the direction of fluid flow, which is illustrated in FIGURE 2-1.

2.1.1 Conduction

Heat transfer by conduction occurs at the molecular scale, by means of collisions and interactions between molecules at different energy states. Conductive heat transfer is caused by temperature difference between adjacent particles (Kaviany, 1995). When hot fluid is injected into the reservoir, its heat transferred to the molecules it contacts, which, in turn, conduct the heat to neighboring molecules. The Fourier’s equation below shows the conductive heat flux (Irani and Ghannadi, 2013):

\[ q_d = -K \nabla T \]  

(2.1)

where \( q_d \) is conductive heat flux, \( K \) is thermal conductivity, \( \nabla T \) is temperature gradient.
FIGURE 2-1 Heat transfer mechanisms in molecules of porous media
An equation to describe transient heat conduction can be constructed by performing an energy balance on a small control volume, using Fourier’s law. The result is a second order partial differential equation, often called the “heat diffusion equation” or simply the “heat equation” (Carslaw and Jaeger, 1959):

\[ \nabla \cdot (K \nabla T) + Q = \rho c \frac{\partial T}{\partial t} \]

(2.2)

where \( Q \) is the rate of heat generation per unit volume, \( \rho \) is the density, and \( c \) is the specific heat capacity.

If the heat is conducted through an isotropic medium (in which the thermal conductivity is equal in all directions), then the heat equation can be expanded as follows in three-dimensional Cartesian coordinates (Baston, 2008):

\[
\frac{\partial}{\partial x} \left( K \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( K \frac{\partial T}{\partial z} \right) + Q = \rho c \frac{\partial T}{\partial t}
\]

(2.3)

If the medium is also homogeneous (having a spatially constant value of thermal conductivity), then the equation may be further simplified as:

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q}{K} = \frac{1}{D} \frac{\partial T}{\partial t}
\]

(2.4)

where \( D \) represents the thermal diffusivity, which is the ratio of thermal conductivity to volumetric heat capacity:

\[
D = \frac{K}{\rho c}
\]

(2.5)
2.1.2 Convection

Heat may also be transferred through the movement of heated fluid. Convective heat transfer occurs when hot fluid flows in the heavy oil reservoir, and heat is transferred by the movements of particles within the fluid. The velocity of fluid flow can be calculated from Darcy’s law. The following equations show the convective heat flux and Darcy’s velocity (Baston, 2008; Irani and Ghannadi, 2013):

$$\overrightarrow{q}_c = \rho_f c_f \overrightarrow{V}(T - T_r)$$  \hspace{1cm} (2.6)

$$\overrightarrow{V} = -\frac{k}{\mu} \nabla P$$  \hspace{1cm} (2.7)

where $\overrightarrow{q}_c$ is convective heat flux, $\rho_f$ is fluid density, $c_f$ is the specific heat capacity of injection fluid, $\overrightarrow{V}$ is Darcy’s velocity, $k$ is reservoir permeability, $\mu$ is fluid viscosity, $P$ is pressure, $T$ is the temperature of injection fluid and $T_r$ is reservoir temperature.

When a fluid moves with a constant velocity, the effect of convective heat transfer can be modeled by the addition of this convective heat flux into the heat equation (2.2):

$$\nabla \cdot (K \nabla T) - \nabla \cdot (\rho_f c_f \overrightarrow{V}(T - T_r)) + Q = \rho c \frac{\partial T}{\partial t}$$  \hspace{1cm} (2.8)

Pure conduction is a static heat transfer process, which transfers the energy of a static heat source to a surrounding environment. Convection-Conduction is a dynamic heat transfer process, which transfers the energy of a moving heat source over a much further distance. A comparison of heat transfer efficiency between pure conduction (Carslaw and Jaeger, 1959) and convection-conduction is performed, which indicates that convection plays an important role in accelerating heat transfer (see FIGURE 2-2).
FIGURE 2-2 (a) Temperature distributions with pure conduction; (b) Temperature distributions with convection-conduction
2.2 Mathematical Studies of Heat Transfer in Porous Media

The heat transfer through a permeable domain coupled with migration of fluids is a common problem in reservoir engineering.

A widely known mathematical study dealing with the injection of a hot liquid into a cold reservoir was presented by Lauwerier (1955). Lauwerier assumed that the injection rate and fluid temperature would remain constant, the flow system was linear, thermal conductivity in the direction of flow was zero, and that the thermal conductivity in the flooded layer perpendicular to the direction of flow was infinite so that the temperature in the flooded layer was always uniform at a given location in the flooded zone. Steady flow of the constant density fluid is uncoupled from the unsteady heat flow caused by the hot fluid injection. Thus the fluid flow field is given and unchanging, while the temperature field varies with time. Lauwerier developed a compact analytical expression for temperature propagation in the linear flow geometry. He neglected reservoir and surrounding formation thermal conductivities in the direction of fluid flow, and assumed uniform temperatures in the reservoir at any distance along it. This is equivalent to assuming infinite thermal conductivity in the reservoir in the direction perpendicular to fluid flow, and has been called the “Lauwerier assumption” by Prats (1969).

Marx and Langenheim (1959) presented a mathematical model for reservoir heating. It is assumed in this model that the reservoir has uniform thickness and properties and that steam is distributed uniformly in a vertical direction so that temperature is uniform through the vertical section. They described a method for estimating thermal invasion rates, cumulative heated area and theoretical economic limits for sustained heat injection at a constant rate into a radial flow system. Rather than solving for temperature
distributions in the reservoir, they considered the total heated area of the reservoir to be at constant temperature, and proceeded to develop an expression for the heated area as a function of time. Such an expression would be useful for determining the rate of growth of the heated area.

Ramey (1962) presented a theoretical model and provided an approximate analytical solution for wellbore heat transmission. In his model, Ramey assumed that fluid is non-compressible and flow is single phase with constant thermal and physical properties along the wellbore. He considered that heat flows radially away the wellbore and the overall heat transfer coefficient is independent of depth. He did not take into account frictional pressure loss and kinetic energy effects in his calculation. He presented a general expression for the overall heat transfer coefficient for a wellbore based on various resistances against heat transfer to the surrounding formation. He also suggested that well radius was to be considered insignificant in acting as a line source.

Vinsome and Westerveld (1980) developed a semi-analytical approach for the problem of non-isothermal fluid injection into a permeable layer that is sandwiched between impermeable base and cap rocks. Their method greatly simplifies the heat conduction problem, while providing satisfactory accuracy. They considered that heat conduction perpendicular to the conductive boundary will be more important than parallel to the boundary. Noting that heat conduction will tend to wipe out sharp temperature differences, they suggested that the temperature profile in the conductive domain may be approximated by means of a simple trial function that contains a few adjustable parameters. More specifically, they proposed that the temperature profile in the cap or base rock may be represented by a low-order polynomial with an exponential tail.
Wu and Pruess (1990) proposed a new analytical solution for wellbore heat transmission in layered formations without Ramey’s assumptions. In their mathematical model, they introduced an overall heat transfer coefficient to consider the wellbore heat resistance, and treated the surrounding earth as consisting of non-homogeneous layered formations, with different thermal and physical properties, and arbitrary initial temperature distributions. They obtained analytical solutions, in both real domain and Laplace domain, for accurately representing transient heat transfer effects in layered formations and predicting wellbore heat transmission for engineering designs or reservoir simulation studies in petroleum and geothermal reservoir development.

Butler (1991) developed a mathematical model to describe the conduction ahead of an advancing front which occurs when oil drains downward across a steam front which is advancing horizontally through a reservoir. Although there is a temperature profile along the heated zone, in this model it is assumed that the temperature in the steam zone is constant, so, there are two temperatures - steam temperature and reservoir temperature. Heat transfer by convection is neglected, and the velocity of advancing front is assumed as constant. The result gives the steady state solution for the one-dimensional flow of heat ahead of a boundary at a constant temperature moving at a constant velocity.

Hasan and Kabir (1994) developed an analytical model to determine the flowing fluid temperature inside the well. They started with a steady-state energy balance equation and combined it with definition of fluid enthalpy in terms of heat capacity and Joule-Thompson coefficient. Then, by adapting some simplifications, they converted the original partial differential equation to an ordinary differential equation and solved it with appropriate boundary conditions.
Livescu et al. (2008) developed a comprehensive novel numerical non-isothermal multiphase wellbore model. After their initial attempts to solve the fully coupled conservation equations, they decoupled the wellbore energy balance equation from the mass balance equation in most of their investigations. They reported that decoupling can be justified when the density difference in each phase, with respect to temperature, is much less than that with respect to pressure. Additionally, they found that this decoupling approach can decrease computation time of simulations without violating stability. They further showed that if several simplifying assumptions were imposed, their model can be reduced to Ramey’s model.

Following Lauwerier’s concept, Barend (2010) improved the solution by considering both heat conduction and convection in porous media. He proposed complete solution for transient heat transport in porous media during steady-state flow. Barend’s solution is more complete than those offered in previous studies. It could be readily used for engineering applications as long as users can analytically or numerically evaluate the integrals involved in this solution.

Bahonar (2011) presents a numerical transient wellbore model for computing the wellbore fluid temperature, pressure, density, and velocity profiles in steam injection wells. This model couples mass, momentum, and energy balance equations and provides all the necessary data in the well with respect to depth and time for a predetermined surface condition. While the model is designed for steam injection wells, with some minor modifications it can be extended to modeling the injection of other fluids.
CHAPTER 3  MODELING OF 1-D TRANSIENT HEAT TRANSFER COUPLED WITH STEADY FLUID FLOW

3.1 Convection-Diffusion Equation

Heat transfer in porous media is described by a partial differential equation known as convection-diffusion equation (CDE). This partial differential equation is of parabolic type (Guenther and Lee, 1988). Analytical solutions to convection-diffusion equations are of continuous interest because they present benchmark solutions to problems in the assessment of heat transfer and distribution in porous media.

The energy equation for fluid flow through a homogeneous and isotropic porous medium can be derived by using the first law of thermodynamics. For simplicity, the radiative effects, viscous dissipation, and the work done by pressure changes are negligible (Bejan and Kraus, 2003).

For most cases, it is acceptable to assume there is local thermal equilibrium where \( T_s = T_f \), and \( T_s \) and \( T_f \) are the temperatures of the solid and fluid phases, respectively. A further assumption is that there is a parallel conduction heat transfer taking place in the solid and fluid phases, so there is no net heat transfer from one phase to the other. Taking the average over an elemental volume of the porous medium, we have the following energy equations for solid and fluid phase (Nield and Bejan, 1999):

\[
(1 - \phi) \rho_s c_s \frac{\partial T_s}{\partial t} = (1 - \phi) \nabla \cdot (K_s \nabla T_s) + (1 - \phi) Q_s \quad (3.1)
\]

\[
\phi \rho_f c_f \frac{\partial T_f}{\partial t} + \rho_f c_f \vec{V} \cdot \nabla T_f = \phi \nabla \cdot (K_f \nabla T_f) + \phi Q_f \quad (3.2)
\]
Here the subscripts \(s\) and \(f\) refer to the solid and fluid phases, respectively; \(\rho\) is density, \(\text{kg/m}^3\); \(c\) is the specific heat capacity, \(\text{J/(kg} \cdot \text{K)}\); \(\phi\) is the porosity of reservoir, fraction; \(Q\) is rate of heat generation per unit volume, \(\text{W/m}^3\); \(\vec{V}\) is Darcy’s velocity, \(\text{m/s}\); \(K\) is thermal conductivity, \(\text{W/(m} \cdot \text{K)}\); and \(T\) is temperature, \(\text{K}\).

The conductive heat flux through the solid is \(-K_s \nabla T_s\), and thus \(\nabla \cdot (K_s \nabla T_s)\) is the net rate of heat conduction into a unit volume of the solid. In equation (3.1), this appears multiplied by the factor \((1-\phi)\), which is the ratio of the cross-sectional area occupied by solid to the total cross-sectional area of the medium. The other two terms in equation (3.1) also contain the factor \((1-\phi)\), because this is also the ratio of volume occupied by solid to the total volume of the element. In equation (3.2) there appears a convective term, due to the seepage velocity. It is recognized that \(\vec{V} \cdot \nabla T_f\) is the rate of change of temperature in the elemental volume due to the convection of fluid into it, so that, multiplied by \(\rho_f c_f\), must be the rate of change of thermal energy, per unit volume of fluid, due to convection (Nield and Bejan, 1999).

Using the assumption of local thermal equilibrium, i.e., \(T_s = T_f\), and adding equations (3.1) and (3.2) yields:

\[
\rho_c \frac{\partial T}{\partial t} + \rho_f c_f \vec{V} \cdot \nabla T = \nabla \cdot (K \nabla T) + Q
\]

(3.3)

where \(\rho_c\), \(K\), \(\vec{V}\), and \(Q\) are the overall volumetric heat capacity of reservoir/fluid system, \(\text{J/(m}^3 \cdot \text{K)}\); overall thermal conductivity of reservoir/fluid system, \(\text{W/(m} \cdot \text{K)}\); Darcy’s velocity of injection fluid, \(\text{m/s}\); and overall rate of heat generation per unit volume of the porous medium, \(\text{W/m}^3\). All these parameters are defined as follows:
\[ \rho c = (1-\phi)\rho_s c_s + \phi \rho_f c_f \quad (3.4) \]

\[ K = (1-\phi)K_s + \phi K_f \quad (3.5) \]

\[ \bar{V} = -\frac{k}{\mu} \nabla P \quad (3.6) \]

\[ Q = (1-\phi)Q_s + \phi Q_f \quad (3.7) \]

Assuming the medium is homogeneous (having a spatially constant value of thermal conductivity), and then the equation (3.3) may be further simplified as:

\[ \frac{\partial T}{\partial t} = \frac{K}{\rho c} \nabla^2 T - \frac{\rho_f c_f}{\rho c} \bar{V} \cdot \nabla T + \frac{Q}{\rho c} \quad (3.8) \]

Here, two important parameters are introduced, i.e., thermal diffusivity of the reservoir/fluid system and thermal convection velocity of the fluid:

\[ D = \frac{K}{\rho c} \quad (3.9) \]

\[ \bar{u} = \frac{\rho_f c_f}{\rho c} \bar{V} \quad (3.10) \]

Then, the energy balance equation becomes:

\[ \frac{\partial T}{\partial t} = D \nabla^2 T - \bar{u} \cdot \nabla T + \frac{Q}{\rho c} \quad (3.11) \]

And neglecting internal heat generation, i.e., \( Q = 0 \), the energy balance equation is a typical convection-diffusion equation (CDE):

\[ \frac{\partial T}{\partial t} = D \nabla^2 T - \bar{u} \cdot \nabla T \quad (3.12) \]
3.2 Mathematical Model

3.2.1 Governing heat transfer equation

Under steady flow conditions, the pressure gradient between the injected fluid zone and the crude oil zone that determines thermal convection velocity of fluid is constant, which means constant thermal convection velocity in steady flow. The thermal diffusivity of the heavy oil reservoir is assumed to be constant.

The partial differential equation (PDE) describing one-dimensional heat transfer under steady fluid flow conditions is taken as:

\[
\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x}
\]

(3.13)

where thermal diffusivity \(D\) and thermal convection velocity \(u\) are constant coefficients, and \(0 \leq x < \infty\), \(0 \leq t < \infty\).

3.2.2 Initial and boundary conditions

3.2.2.1 Initial condition

The general initial condition is:

\[
T(x, 0) = F(x), \quad x \geq 0
\]

(3.14)

where \(F(x)\) can take several forms, such as a constant value, an exponentially increasing or decreasing function with distance.

3.2.2.2 Upper boundary condition

For the upper boundary condition at \(x=0\), two different general boundary conditions are applied in our models (van Genuchten and Alves, 1982):
1) Temperature-type boundary condition:

\[ T(0,t) = G(t), \quad t \geq 0 \]  \hspace{1cm} (3.15)

2) Heat flux-type boundary condition:

\[ -D \frac{\partial T}{\partial x} + uT = uG(t), \quad t \geq 0 \]  \hspace{1cm} (3.16)

where \( G(t) \) also can take on several distributions, such as a constant value, a pulse-type distribution, or an exponentially increasing or decreasing function with time.

### 3.2.2.3 Lower boundary condition

For the lower boundary condition at \( x=\infty \), zero heat flow boundary (zero temperature gradient) is applied:

\[ \frac{\partial T}{\partial x} \bigg|_{x=\infty} = 0, \quad t \geq 0 \]  \hspace{1cm} (3.17)

### 3.3 Analytical Solution

The objective is to characterize the temperature as a function of distance and time. For convenience, mathematical model are first reduced to the dimensionless form by introducing dimensionless variables. Then variable transformation is performed to simplify the convection-diffusion equation to the diffusion equation which can be solved more easily. After that, Laplace transformation is applied to derive the analytical solution in Laplace domain. And using the table of Laplace transformations in the Appendix A, the solution in Laplace domain can be converted to dimensionless solution in real time domain. Finally, substituting the variable transformation and dimensionless variables back to dimensionless solution, the final analytical solution is obtained.

The detailed analytical solution procedure is shown in **FIGURE 3-1**.
FIGURE 3-1 Flowchart of analytical solution procedure
3.4 Model Validation

COMSOL Multiphysics is used to validate the analytical solutions. COMSOL is a general-purpose software platform based on advanced numerical methods. It models and simulates coupled or multiphysics phenomena based on the finite element method, which is commonly used in solving engineering problems. It also has the physics and equation-based modeling interfaces, and the automatic and semi-automatic meshing tools.

In COMSOL Multiphysics simulation, a 2-D simulation model is developed to simulate the 1-D heat transfer process (see FIGURE 3-2). This model contains 308 triangular elements, and the length of the model in $x$-direction is much larger than that in $y$-direction. The initial condition either remains at constant or exponentially decreases with distance. The upper boundary in $x$-direction is kept at certain temperature or heat flux. The two boundaries in the $y$-direction and the lower boundary in the $x$-direction are set to be zero heat flow boundaries. At the end of the simulation, the temperature surface profiles are obtained and the simulated temperature data can be exported. The basic parameters used in the simulation models under steady flow condition are listed in TABLE 3-2. Taking case A-1 in TABLE 3-1 as an example, its temperature distributions by COMSOL simulation are shown in FIGURE 3-3.

The simulated temperature data by COMSOL was exported to plot the curves of temperature distributions and propagations. And MATLAB programming was used to obtain the temperature distributions and propagations by the analytical solutions. The comparisons between the COMSOL numerical simulation results and analytical solutions are conducted in an attempt to obtain the agreements between them.
FIGURE 3-2 Simulation model and mesh system used in COMSOL Multiphysics
FIGURE 3-3 Temperature surface profiles simulated by COMSOL in case A-1
3.5 Case Studies

Under steady flow condition, ten cases with different initial and boundary conditions are studied. The initial and boundary conditions of the ten cases are listed in TABLE 3-1. The basic parameters used in mathematical and simulation models under steady flow condition are listed in TABLE 3-2.

For the first three cases A-1 to A-3, the initial reservoir temperatures are remained at constant. Case A-1 describes a continuous fluid injection at constant temperature. Case A-2 describes a continuous fluid injection with exponentially decreasing temperature. Case A-3 describes a periodic fluid injection, which initially injects high-temperature fluid for a period of time and then changes to inject low-temperature fluid.

For cases B-1 to B-3, the initial reservoir temperatures are also remained at constant. However, the upper boundary conditions are all heat flux-type. Case B-1 describes a continuous fluid injection at constant heat flux. Case B-2 describes a continuous fluid injection with exponentially decreasing heat flux. Case B-3 describes a periodic fluid injection, which initially has an injection fluid with a high heat flux for a period of time and then the injection fluid was changed to a lower heat flux.

For the last four cases, the initial reservoir temperatures are all exponentially decreasing with distance. Case C-1 describes a continuous fluid injection at constant temperature. Case C-2 describes a continuous fluid injection with exponentially decreasing temperature. Case D-1 describes a continuous fluid injection at constant heat flux. Case D-2 describes a continuous fluid injection with exponentially decreasing heat flux.
<table>
<thead>
<tr>
<th>Case</th>
<th>Mathematical Model</th>
<th>Analytical Solution</th>
<th>Result Figures</th>
</tr>
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<td>Equation (3.18)</td>
<td>Equation (3.52) to Equation (3.54)</td>
<td>Figure 3-4 Figure 3-5</td>
</tr>
<tr>
<td>A-2</td>
<td>Equation (3.55)</td>
<td>Equation (3.77) to Equation (3.81)</td>
<td>Figure 3-6 Figure 3-7</td>
</tr>
<tr>
<td>A-3</td>
<td>Equation (3.82)</td>
<td>Equation (3.111) to Equation (3.113)</td>
<td>Figure 3-8 Figure 3-9</td>
</tr>
<tr>
<td>B-1</td>
<td>Equation (3.114)</td>
<td>Equation (3.134) to Equation (3.136)</td>
<td>Figure 3-10 Figure 3-11</td>
</tr>
<tr>
<td>B-2</td>
<td>Equation (3.137)</td>
<td>Equation (3.158) to Equation (3.162)</td>
<td>Figure 3-12 Figure 3-13</td>
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<tr>
<td>B-3</td>
<td>Equation (3.163)</td>
<td>Equation (3.191) to Equation (3.192)</td>
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<tr>
<td>C-1</td>
<td>Equation (3.193)</td>
<td>Equation (3.220) to Equation (3.223)</td>
<td>Figure 3-16 Figure 3-17</td>
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<tr>
<td>C-2</td>
<td>Equation (3.224)</td>
<td>Equation (3.247) to Equation (3.252)</td>
<td>Figure 3-18 Figure 3-19</td>
</tr>
<tr>
<td>D-1</td>
<td>Equation (3.253)</td>
<td>Equation (3.275) to Equation (3.278)</td>
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<tr>
<td>D-2</td>
<td>Equation (3.279)</td>
<td>Equation (3.302) to Equation (3.307)</td>
<td>Figure 3-22 Figure 3-23</td>
</tr>
</tbody>
</table>
**TABLE 3-2** Basic parameters in mathematical and simulation models under steady flow condition (Irani and Ghannadi, 2013)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal diffusivity, $D$</td>
<td>0.075</td>
<td>m$^2$/day</td>
</tr>
<tr>
<td>Thermal convection velocity, $u$</td>
<td>5.0</td>
<td>cm/day</td>
</tr>
<tr>
<td>Initial reservoir temperature, $T_i$</td>
<td>12</td>
<td>°C</td>
</tr>
<tr>
<td>High fluid temperature, $T_0$</td>
<td>250</td>
<td>°C</td>
</tr>
<tr>
<td>Low fluid temperature, $T_z$</td>
<td>150</td>
<td>°C</td>
</tr>
<tr>
<td>Temperature increment, $T_1$</td>
<td>50</td>
<td>°C</td>
</tr>
<tr>
<td>Temperature increment, $T_2$</td>
<td>30</td>
<td>°C</td>
</tr>
<tr>
<td>Temperature changing time, $t_0$</td>
<td>2</td>
<td>year</td>
</tr>
<tr>
<td>Flow resistance coefficient, $\lambda$</td>
<td>$2.5\times10^{-8}$</td>
<td>s$^{-1}$</td>
</tr>
<tr>
<td>Exponential decay coefficient, $\beta$</td>
<td>$3.0\times10^{-2}$</td>
<td>m$^{-1}$</td>
</tr>
</tbody>
</table>
3.5.1 Case A-1

The mathematical formulations for case A-1 are taken as follows:

\[
\begin{align*}
\frac{\partial T}{\partial t} &= D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
T(x,0) &= T_r, \quad x \geq 0 \\
T(0,t) &= T_o, \quad t \geq 0 \\
\left.\frac{\partial T}{\partial x}\right|_{x\to\infty} &= 0, \quad t \geq 0
\end{align*}
\]

(3.18)

3.5.1.1 Dimensionless form

For convenience in solving the problem, the partial differential equation system and solution are often expressed in dimensionless form. Hence, the following dimensionless variables are defined as follows:

\[
T_D = \frac{T - T_i}{T_0 - T_i}
\]

(3.19)

\[
x_D = \frac{ux}{D}
\]

(3.20)

\[
t_D = \frac{u^2 t}{D}
\]

(3.21)

Rewriting the above three equations in the following forms:

\[
T = T_i + (T_0 - T_i)
\]

(3.22)

\[
x = \frac{D}{u} x_D
\]

(3.23)

\[
t = \frac{D}{u^2} t_D
\]

(3.24)
Performing the partial differentials, the space and time derivatives are obtained:

\[
\frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \left( \frac{D}{u} x_D \right) = \frac{D}{u} \frac{\partial}{\partial x_D} \frac{D}{D} \frac{\partial T}{\partial x_D} = \frac{D}{u} \frac{\partial}{\partial x_D} \frac{D}{D} \frac{\partial T}{\partial x_D} 
\]  (3.25)

\[
\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{D}{u} \frac{\partial T}{\partial x_D} \right) = \frac{D}{u} \frac{\partial}{\partial x_D} \frac{D}{D} \frac{\partial^2 T}{\partial x_D^2} 
\]  (3.26)

\[
\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \left( \frac{D}{u^2} t_D \right) = \frac{D}{u^2} \frac{\partial}{\partial t_D} \frac{D}{D} \frac{\partial T}{\partial t_D} = \frac{D}{u^2} \frac{\partial}{\partial t_D} \frac{D}{D} \frac{\partial T}{\partial t_D} 
\]  (3.27)

Substituting equations (3.19) – (3.27) into mathematical formulations (3.18), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D} 
\]  (3.28)

\[
T_D(x_D, 0) = 0 
\]  (3.29)

\[
T_D(0, t_D) = 1 
\]  (3.30)

\[
\frac{\partial T_D}{\partial x_D} \bigg|_{x_D \to \infty} = 0 
\]  (3.31)

### 3.5.1.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable transformation (Ogata and Banks, 1961; Farlow, 1982):

\[
T_D(x_D, t_D) = e^{-\frac{2x_D-x_0}{4}} K(x_D, t_D) 
\]  (3.32)
Performing the partial differentials, the space and time derivatives are obtained:

\[
\frac{\partial T_D}{\partial t_D} = e^{\frac{2x_D - t_D}{4}} \left( \frac{\partial K}{\partial t_D} - \frac{1}{4} K \right) \tag{3.33}
\]

\[
\frac{\partial T_D}{\partial x_D} = e^{\frac{2x_D - t_D}{4}} \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \tag{3.34}
\]

\[
\frac{\partial^2 T_D}{\partial x_D^2} = e^{\frac{2x_D - t_D}{4}} \left( \frac{\partial^2 K}{\partial x_D^2} + \frac{\partial K}{\partial x_D} + \frac{1}{4} K \right) \tag{3.35}
\]

Substituting equations (3.33) – (3.35) into equation (3.28) gives:

\[
\frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D} \tag{3.36}
\]

Substituting equations (3.32) and (3.34) into equations (3.29) – (3.31), the initial and boundary conditions are transformed to:

\[
K(x_D, 0) = 0 \tag{3.37}
\]

\[
K(0, t_D) = e^{\frac{t_D}{4}} \tag{3.38}
\]

\[
\left. \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \right|_{t_0 \to \infty} = 0 \tag{3.39}
\]

### 3.5.1.3 Laplace transformation

Applying Laplace transformation on equations (3.36) – (3.39), the following ordinary boundary value problem (BVP) is obtained:

\[
\frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = 0 \tag{3.40}
\]

\[
\overline{K}(0, s) = \frac{1}{s - \frac{1}{4}} \tag{3.41}
\]
\[
\left( \frac{d\overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right)_{x_D \to \infty} = 0
\] (3.42)

where \(\overline{K}(x_D, s) = \int_0^\infty K(x_D, t_D) e^{-st_D} dt_D\), \(s\) is the Laplace parameter.

The characteristic equation (Zwillinger, 1997) of ordinary differential equation (ODE) (3.40) is:

\[
r^2 - s = 0
\] (3.43)

The characteristic roots are obtained:

\[
r_1 = \sqrt{s}, \quad r_2 = -\sqrt{s}
\] (3.44)

Then, the general solution (Zwillinger, 1997) to the homogeneous ODE (3.40) is:

\[
\overline{K}(x_D, s) = C_1 e^{\sqrt{s}r} + C_2 e^{-\sqrt{s}r}
\] (3.45)

Applying the general solution (3.45) to equations (3.41) and (3.42) gives:

\[
C_1 = 0
\] (3.46)

\[
C_2 = \frac{1}{s - \frac{1}{4}}
\] (3.47)

Then, the solution of the ordinary boundary value problem in Laplace domain is:

\[
\overline{K}(x_D, s) = \frac{e^{-\sqrt{s}r}}{s - \frac{1}{4}}
\] (3.48)

### 3.5.1.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformation is given:

\[
L^{-1} \left[ \frac{e^{-\sqrt{s}r}}{s - \frac{1}{4}} \right] = \frac{1}{2} \left[ e^{\frac{t_D - 2x_D}{4}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{\frac{t_D + 2x_D}{4}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right]
\] (3.49)
Applying the inverse Laplace transformation to equation (3.48) gives:

\[
K(x_D, t_D) = L^{-1}\left\{ \frac{e^{-x_D\sqrt{s}}}{s - \frac{1}{4}} \right\} = \frac{1}{2} \left[ e^{\frac{t_D}{4} - x_D^2} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + e^{\frac{t_D}{4} + x_D^2} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \right] \tag{3.50}
\]

Substituting equation (3.50) into equation (3.32) gives:

\[
T_b(x_D, t_D) = \frac{1}{2} \left[ \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + e^{\frac{t_D}{4}} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \right] \tag{3.51}
\]

Recalling the dimensionless variables defined in equations (3.19) – (3.21), the analytical solution is derived:

\[
T(x, t) = T_i + \frac{1}{2}(T_0 - T_i) \left[ \text{erfc}\left(\frac{x - ut}{2\sqrt{Dt}}\right) + e^{\frac{ux}{4}} \text{erfc}\left(\frac{x + ut}{2\sqrt{Dt}}\right) \right] \tag{3.52}
\]

The final analytical solution can be simplified as follows:

\[
T(x, t) = T_i + (T_0 - T_i) A(x, t) \tag{3.53}
\]

\[
A(x, t) = \frac{1}{2} \left[ \text{erfc}\left(\frac{x - ut}{2\sqrt{Dt}}\right) + e^{\frac{ux}{4}} \text{erfc}\left(\frac{x + ut}{2\sqrt{Dt}}\right) \right] \tag{3.54}
\]

Case A-1 in TABLE 3-1 describes a continuous fluid injection at constant temperature. The initial reservoir temperature is remained at constant, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case A-1 is given in equation (3.52) to equation (3.54). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in FIGURE 3-4 and FIGURE 3-5, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-4 Temperature distribution at different times in case A-1
FIGURE 3-5 Temperature propagation at different locations in case A-1
3.5.2 Case A-2

The mathematical formulations for case A-2 are taken as follows:

\[
\begin{cases}
    \frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
    T(x,0) = T_i, \quad x \geq 0 \\
    T(0,t) = T_0 + T_i e^{-\lambda t}, \quad t \geq 0 \\
    \frac{\partial T}{\partial x} \bigg|_{x \to \infty} = 0, \quad t \geq 0
\end{cases}
\]

(3.55)

3.5.2.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D}
\]

(3.56)

\[
T_D(x_D,0) = 0
\]

(3.57)

\[
T_D(0,t_D) = 1 + \frac{T_i}{T_0 - T_i} e^{\frac{\lambda D}{u^2} t_D}
\]

(3.58)

\[
\frac{\partial T_D}{\partial x_D} \bigg|_{x_D \to \infty} = 0
\]

(3.59)

3.5.2.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable transformation (Ogata and Banks, 1961; Farlow, 1982):
Substituting equation (3.60) into equations (3.56) – (3.59) gives:

\[
\frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D} \tag{3.61}
\]

\[K(x_D, 0) = 0 \tag{3.62}\]

\[K(0, t_D) = e^{\frac{i \lambda_t}{4}} + \frac{T_i}{T_0 - T_i} e^{\left(\frac{1}{4} \lambda_D \right) t_D} \tag{3.63}\]

\[
\left. \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \right|_{x_D \to \infty} = 0 \tag{3.64}
\]

### 3.5.2.3 Laplace transformation

Applying Laplace transformation on equations (3.61) – (3.64), the following ordinary boundary value problem (BVP) is obtained:

\[
\frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = 0 \tag{3.65}
\]

\[
\overline{K}(0, s) = \frac{1}{s - \frac{1}{4}} + \frac{T_i}{T_0 - T_i} \frac{1}{s - \frac{1}{4} \lambda D + \frac{1}{u^2}} \tag{3.66}
\]

\[
\left. \left( \frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right) \right|_{x_D \to \infty} = 0 \tag{3.67}
\]

Then, the general solution (Zwillinger, 1997) to the homogeneous ODE (3.65) is:

\[
\overline{K}(x_D, s) = C_1 e^{\frac{i \lambda x_D}{4}} + C_2 e^{-\frac{i \lambda x_D}{4}} \tag{3.68}
\]

Applying general solution (3.68) to equations (3.66) and (3.67) gives:
\[ C_1 = 0 \] (3.69)

\[ C_2 = \frac{1}{s - \frac{1}{4}} + \frac{T_i}{T_0 - T_i} \frac{1}{s - \frac{1}{4}} + \frac{\lambda D}{u^2} \] (3.70)

Then, the exact solution in Laplace domain becomes:

\[ \overline{K}(x_D, s) = \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} + \frac{T_i}{T_0 - T_i} \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} + \frac{\lambda D}{u^2} \] (3.71)

### 3.5.2.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformations are given:

\[ L^{-1} \left[ \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} \right] = \frac{1}{2} \left[ e^{-\frac{T_i - 2x_D}{4u}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{-\frac{T_i + 2x_D}{4u}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \] (3.72)

\[ L^{-1} \left[ \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} + \frac{\lambda D}{u^2} \right] = \frac{1}{2} \left[ e^{-\frac{\omega^2 T_D}{2u^2}} \text{erfc} \left( \frac{u x_D - \omega t_D}{2u\sqrt{t_D}} \right) + e^{-\frac{\omega^2 T_D}{2u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u\sqrt{t_D}} \right) \right] \] (3.73)

where \( \omega = \sqrt{u^2 - 4\lambda D} \).

Applying the inverse Laplace transformation to equation (3.71) gives:

\[ K(x_D, t_D) = \frac{1}{2} \left[ e^{-\frac{T_i - 2x_D}{4u}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{-\frac{T_i + 2x_D}{4u}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \]

\[ + \frac{1}{2} \frac{T_i}{T_0 - T_i} \left[ e^{-\frac{\omega^2 T_D}{2u^2}} \text{erfc} \left( \frac{u x_D - \omega t_D}{2u\sqrt{t_D}} \right) + e^{-\frac{\omega^2 T_D}{2u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u\sqrt{t_D}} \right) \right] \] (3.74)

Substituting equation (3.74) into equation (3.60) gives:
Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[
T_D(x_D,t_D) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{\omega} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \\
+ \frac{1}{2 T_0 - T_i} \left[ e^{-\frac{(u^2 - \omega^2)\lambda}{4D}} \text{erfc} \left( \frac{u x_D - \omega t_D}{2u\sqrt{t_D}} \right) + e^{-\frac{(u^2 - \omega^2)\lambda}{2D}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u\sqrt{t_D}} \right) \right]
\]  

(3.75)

Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[
T(x,t) = T_i + \frac{1}{2} (T_0 - T_i) \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^\lambda \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \\
+ \frac{1}{2} T_i \left[ e^{-\frac{(u^2 - \omega^2)\lambda}{4D}} \text{erfc} \left( \frac{x - \omega t}{2\sqrt{Dt}} \right) + e^{-\frac{(u^2 - \omega^2)\lambda}{2D}} \text{erfc} \left( \frac{x + \omega t}{2\sqrt{Dt}} \right) \right]
\]  

(3.76)

Since \( \omega = \sqrt{u^2 - 4\lambda D} \), then \( \lambda = \frac{u^2 - \omega^2}{4D} \). Equation (3.76) is then reduced to:

\[
T(x,t) = T_i + \frac{1}{2} (T_0 - T_i) \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^\lambda \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \\
+ \frac{1}{2} T_i e^{-\lambda t} \left[ e^{-\frac{(u^2 - \omega^2)\lambda}{2D}} \text{erfc} \left( \frac{x - \omega t}{2\sqrt{Dt}} \right) + e^{-\frac{(u^2 - \omega^2)\lambda}{2D}} \text{erfc} \left( \frac{x + \omega t}{2\sqrt{Dt}} \right) \right]
\]  

(3.77)

The final analytical solution can be simplified as follows:

\[
T(x,t) = T_i + (T_0 - T_i) A(x,t) + T_i B(x,t)
\]  

(3.78)

\[
A(x,t) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^\lambda \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right]
\]  

(3.79)

\[
B(x,t) = \frac{1}{2} e^{-\lambda t} \left[ e^{-\frac{(u^2 - \omega^2)\lambda}{2D}} \text{erfc} \left( \frac{x - \omega t}{2\sqrt{Dt}} \right) + e^{-\frac{(u^2 - \omega^2)\lambda}{2D}} \text{erfc} \left( \frac{x + \omega t}{2\sqrt{Dt}} \right) \right]
\]  

(3.80)

\[
\omega = \sqrt{u^2 - 4\lambda D}
\]  

(3.81)
Case A-2 in TABLE 3-1 describes a continuous fluid injection with exponentially decreasing temperature. The initial reservoir temperature is remained at constant, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case A-2 is given in equation (3.77) to equation (3.81). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in FIGURE 3-6 and FIGURE 3-7, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-6 Temperature distribution at different times in case A-2
FIGURE 3-7 Temperature propagation at different locations in case A-2
3.5.3 Case A-3

The mathematical formulations for case A-3 are taken as follows:

\[
\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x}
\]

\(T(x,0) = T_i, \quad x \geq 0\)

\(T(0,t) = \begin{cases} T_0, & 0 < t \leq t_0 \\ T_z, & t \geq t_0 \end{cases}\)

\(\left. \frac{\partial T}{\partial x} \right|_{x \to \infty} = 0, \quad t \geq 0\)  \hspace{1cm} (3.82)

### 3.5.3.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D}
\]

\(T_D(x_D, 0) = 0\)  \hspace{1cm} (3.83)

\(T_D(0, t_D) = \begin{cases} 1, & 0 < t_D \leq \frac{u^2 t_0}{D} \\ T_z - T_i, & t_D \geq \frac{u^2 t_0}{D} \end{cases}\)

\(\left. \frac{\partial T_D}{\partial x_D} \right|_{x_D \to \infty} = 0\)  \hspace{1cm} (3.85)

### 3.5.3.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be
solved more easily, a new dependent variable is introduced by the following variable transformation (Ogata and Banks, 1961; Farlow, 1982):

\[ T_D(x_D, t_D) = e^{\frac{2\tau_0 - t_D}{4}} K(x_D, t_D) \quad (3.87) \]

Substituting equation (3.87) into equations (3.83) – (3.86), we get:

\[ \frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D} \quad (3.88) \]

\[ K(x_D, 0) = 0 \quad (3.89) \]

\[ K(0, t_D) = \begin{cases} \frac{\tau_0}{4} e^{\frac{-t_D}{4}}, & 0 < t_D \leq \frac{u^2 t_0}{D} \\ \frac{T_i - T_0}{T_i - T_0} e^{\frac{-t_D}{4}}, & t_D \geq \frac{u^2 t_0}{D} \end{cases} \quad (3.90) \]

\[ \left. \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \right|_{x_D \to \infty} = 0 \quad (3.91) \]

3.5.3.3 Laplace transformation

Applying Laplace transformation on equations (3.88) – (3.91), the following ordinary boundary value problem (BVP) is obtained:

\[ \frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = 0 \quad (3.92) \]

\[ \overline{K}(0, s) = \frac{1}{s - \frac{1}{4} + \frac{T_i - T_0}{T_i - T_0}} \left( 1 + e^{\left( -\frac{1}{4} \right) \frac{u^2 t_0}{D}} \right) \quad (3.93) \]

\[ \left. \left( \frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right) \right|_{x_D \to \infty} = 0 \quad (3.94) \]
Then, the general solution (Zwillinger, 1997) to equation (3.92) is:

\[ \bar{K}(x_D, s) = C_1 e^{x_D \sqrt{s}} + C_2 e^{-x_D \sqrt{s}} \]  

(3.95)

Applying general solution (3.95) to equations (3.93) and (3.94) gives:

\[ C_1 = 0 \]  

(3.96)

\[ C_2 = \frac{1}{s - \frac{1}{4}} + \frac{T_z - T_0}{T_0 - T_i} \frac{1}{s - \frac{1}{4}} e^{-\left(\frac{1}{4}\right)\frac{u_{t_0}^2}{D}} \]  

(3.97)

Then, the exact solution of the ordinary boundary value problem in Laplace domain becomes:

\[ \bar{K}(x_D, s) = e^{-x_D \sqrt{s}} + \frac{u_{t_0}^2}{s - \frac{1}{4}} + e^{\frac{u_{t_0}^2}{4D}} \frac{T_z - T_0}{T_0 - T_i} \frac{1}{s - \frac{1}{4}} e^{-\left(\frac{1}{4}\right)\frac{u_{t_0}^2}{D}} \]  

(3.98)

**3.5.3.4 Inverse Laplace transformation**

In Appendix A, the following inverse Laplace transformation is given:

\[ L^{-1} \left[ \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} \right] = \frac{1}{2} \left[ e^{-\frac{x_D - t_D}{2\sqrt{D}}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{D}} \right) + e^{-\frac{t_D - x_D}{4}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{D}} \right) \right] \]  

(3.99)

The **Second Shifting Property** of Laplace transformation (Debnath and Bhatta, 2007) is described as:

\[ L^{-1} \left[ e^{-as} \bar{F}(x_D, s) \right] = F(x_D, t_D - a)H(t_D - a) \]  

(3.100)

where \( a > 0 \), \( L[F(x_D, t_D)] = \bar{F}(x_D, s) \), and \( H(t_D - a) \) the Heaviside unit step function defined by:
$$H(t_D - a) = \begin{cases} 
0, & 0 < t_D \leq a \\
1, & t_D > a 
\end{cases} \quad (3.101)$$

In this case:

$$a = \frac{u^2 t_0}{D} \quad (3.102)$$

$$F(x_D, s) = \frac{e^{-\sqrt{s}}}{\mathcal{L}} \quad (3.103)$$

According to the **Second Shifting Property**, the following equation is obtained:

$$L^{-1} \left[ e^{-\sqrt{s} - \frac{u^2 t_0}{D}} F(x_D, s) \right] = F \left( x_D, t_D - \frac{u^2 t_0}{D} \right) H \left( t_D - \frac{u^2 t_0}{D} \right) \quad (3.104)$$

Then substituting equation (3.101) to equation (3.104) gives:

$$L^{-1} \left[ e^{-\sqrt{s}} - \frac{e^{-\frac{u^2 t_0}{D}}}{\mathcal{L}} \right] = \begin{cases} 
0, & 0 < t_D \leq \frac{u^2 t_0}{D} \\
F \left( x_D, t_D - \frac{u^2 t_0}{D} \right), & t_D > \frac{u^2 t_0}{D} 
\end{cases} \quad (3.105)$$

Apply equations (3.99) and (3.105) to equation (3.98) gives:

$$K(x_D, t_D) = \begin{cases} 
F(x_D, t_D), & 0 < t_D \leq \frac{u^2 t_0}{D} \\
F(x_D, t_D) + \frac{T_z - T_0}{T_0 - T_i} \frac{T_z - T_0}{2} e^{\frac{u^2 t_0}{D}} F \left( x_D, t_D - \frac{u^2 t_0}{D} \right), & t_D > \frac{u^2 t_0}{D} 
\end{cases} \quad (3.106)$$

Substituting equation (3.67) into equation (3.52) gives:

$$T_D(x_D, t_D) = \begin{cases} 
\frac{2x_0}{e^2} F(x_D, t_D), & 0 < t_D \leq \frac{u^2 t_0}{D} \\
\frac{2x_0}{e^2} F(x_D, t_D) + \frac{T_z - T_0}{2} \frac{e^{\frac{2x_0 - t_0}{D}}}{e^{\frac{4}{D}} F} \left( x_D, t_D - \frac{u^2 t_0}{D} \right), & t_D > \frac{u^2 t_0}{D} 
\end{cases} \quad (3.107)$$

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where

\[
F(x_D, t_D) = L^{-1}\left[ \hat{F}(x_D, s) \right] = \frac{1}{2} \left[ e^{x_D^2/4} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{-x_D^2/4} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \tag{3.108}
\]

The following term is defined to simplify the solution:

\[
M(x_D, t_D) = e^{x_D^2/4} F(x_D, t_D) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{x_D^2/4} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \tag{3.109}
\]

Then, equation (3.107) can be simplified as:

\[
T_D(x_D, t_D) = \begin{cases} 
M(x_D, t_D), & 0 < t_D \leq \frac{u^2 t_0}{D} \\
M(x_D, t_D) + \frac{T_z - T_0}{T_0 - T_t} M \left( x_D, t_D - \frac{u^2 t_0}{D} \right), & t_D > \frac{u^2 t_0}{D} \end{cases} \tag{3.110}
\]

Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[
T(x, t) = \begin{cases} 
T_t + \frac{1}{2} (T_0 - T_t) \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{D}t} \right) + e^{\frac{ux}{2\sqrt{D}t}} \text{erfc} \left( \frac{x + ut}{2\sqrt{D}t} \right) \right], & 0 < t \leq t_0 \\
T_t + \frac{1}{2} (T_0 - T_t) \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{D}t} \right) + e^{\frac{ux}{2\sqrt{D}t}} \text{erfc} \left( \frac{x + ut}{2\sqrt{D}t} \right) \right] + \frac{1}{2} (T_z - T_0) \left[ \text{erfc} \left( \frac{x - u(t - t_0)}{2\sqrt{D}(t - t_0)} \right) + e^{\frac{ux}{2\sqrt{D}(t - t_0)}} \text{erfc} \left( \frac{x + u(t - t_0)}{2\sqrt{D}(t - t_0)} \right) \right], & t > t_0 \end{cases} \tag{3.111}
\]

The final analytical solution can be simplified as follows:

\[
T(x, t) = \begin{cases} 
T_t + (T_0 - T_t) A(x, t), & 0 < t \leq t_0 \\
T_t + (T_0 - T_t) A(x, t) + (T_z - T_0) A(x, t - t_0), & t > t_0 \end{cases} \tag{3.112}
\]

\[
A(x, t) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^{\frac{ux}{2\sqrt{Dt}}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \tag{3.113}
\]
Case A-3 in **TABLE 3-1** describes a periodic fluid injection, which initially injects high-temperature fluid for a period of time and then changes to inject low-temperature fluid. The initial reservoir temperature is remained at constant, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case A-3 is given in equation (3.111) to equation (3.113). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in **FIGURE 3-8** and **FIGURE 3-9**, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-8 Temperature distribution at different times in case A-3
FIGURE 3-9 Temperature propagation at different locations in case A-3
3.5.4 Case B-1

The mathematical formulations for case B-1 are taken as follows:

\[
\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x}
\]

\[T(x,0) = T_0, \quad x \geq 0 \]

\[
\left( -D \frac{\partial T}{\partial x} + uT \right)_{x=0} = uT_0, \quad t \geq 0
\]

\[
\frac{\partial T}{\partial x} \bigg|_{x \to \infty} = 0, \quad t \geq 0
\]

3.5.4.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D}
\]

\[T_D(x_D,0) = 0 \]

\[
\left( -\frac{\partial T_D}{\partial x_D} + T_D \right)_{x_D = 0} = 1
\]

\[
\frac{\partial T_D}{\partial x_D} \bigg|_{x_D \to \infty} = 0
\]

3.5.4.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable transformation (Ogata and Banks, 1961; Farlow, 1982):

\[
T_D(x_D,t_D) = e^{-2x_D/t_D} K(x_D,t_D)
\]
Substituting equation (3.119) into equations (3.115) – (3.118) gives:

$$\frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D}$$  \hspace{1cm} (3.120)

$$K(x_D,0) = 0$$  \hspace{1cm} (3.121)

$$\left. \left( -\frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \right|_{x_D = 0} = e^{\frac{\nu_0}{4}}$$  \hspace{1cm} (3.122)

$$\left. \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \right|_{x_D \rightarrow \infty} = 0$$  \hspace{1cm} (3.123)

### 3.5.4.3 Laplace transformation

Applying Laplace transformation on equations (3.120) – (3.123), the following ordinary boundary value problem (BVP) is obtained:

$$\frac{d^2 \bar{K}}{dx_D^2} - s \bar{K}(x_D,s) = 0$$  \hspace{1cm} (3.124)

$$\left. \left( -\frac{d \bar{K}}{dx_D} + \frac{1}{2} \bar{K} \right) \right|_{x_D = 0} = \frac{1}{s - \frac{1}{4}}$$  \hspace{1cm} (3.125)

$$\left. \left( \frac{d \bar{K}}{dx_D} + \frac{1}{2} \bar{K} \right) \right|_{x_D \rightarrow \infty} = 0$$  \hspace{1cm} (3.126)

Then, the general solution (Zwillinger, 1997) to equation (3.124) is:

$$\bar{K}(x_D,s) = C_1 e^{\nu_0 \sqrt{s}} + C_2 e^{-\nu_0 \sqrt{s}}$$  \hspace{1cm} (3.127)

Applying general solution (3.127) to equations (3.125) and (3.126) gives:

$$C_1 = 0$$  \hspace{1cm} (3.128)

$$C_2 = \frac{1}{\left( s - \frac{1}{4} \right) \left( \frac{1}{2} + \sqrt{s} \right)}$$  \hspace{1cm} (3.129)

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Then, the exact solution in Laplace domain becomes:

$$K(x_D, s) = \frac{e^{-x_D \sqrt{s}}}{(s - \frac{1}{2}) \left( \frac{1}{2} + \sqrt{s} \right)}$$  \hspace{1cm} (3.130)

### 3.5.4.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformation is given:

$$L^1 \left[ \frac{e^{-x_D \sqrt{s}}}{(s - \frac{1}{2}) \left( \frac{1}{2} + \sqrt{s} \right)} \right] = \sqrt{\frac{I_D}{\pi}} e^{-x_0^2 / 4t_0 + 1} e^{\frac{t_0 - x_0^2}{4t_0}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1 + x_D + t_D}{2} e^{\frac{t_0 - x_0^2}{4t_0}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right)$$  \hspace{1cm} (3.131)

Applying the inverse Laplace transformation to equation (3.130) gives:

$$K(x_D, t_D) = \sqrt{\frac{I_D}{\pi}} e^{-x_0^2 / 4t_0} e^{t_0 - x_0^2 / 4t_0} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1 + x_D + t_D}{2} e^{x_0^2 / 4t_0} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right)$$  \hspace{1cm} (3.132)

Substituting equation (3.132) into equation (3.119) gives:

$$T_D(x_D, t_D) = \sqrt{\frac{I_D}{\pi}} e^{-(x_0^2 - t_0^2)} e^{t_0 - x_0^2 / 4t_0} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1 + x_D + t_D}{2} e^{x_0^2 / 4t_0} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right)$$  \hspace{1cm} (3.133)

Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

$$T(x, t) = T_i + (T_0 - T_i) \left[ \sqrt{\frac{u^2}{\pi D}} e^{-\frac{(x-u)^2}{4Dt}} + \frac{1}{2} \text{erfc} \left( \frac{x-ut}{2\sqrt{Dt}} \right) - \frac{1}{2} \left( 1 + \frac{ux}{D} + \frac{u^2t}{D} \right) e^{D} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) \right]$$  \hspace{1cm} (3.134)

The final analytical solution can be simplified as follows:

$$T(x, t) = T_i + (T_0 - T_i) A(x, t)$$  \hspace{1cm} (3.135)

$$A(x, t) = \sqrt{\frac{u^2}{\pi D}} e^{-\frac{(x-u)^2}{4Dt}} + \frac{1}{2} \text{erfc} \left( \frac{x-ut}{2\sqrt{Dt}} \right) - \frac{1}{2} \left( 1 + \frac{ux}{D} + \frac{u^2t}{D} \right) e^{D} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right)$$  \hspace{1cm} (3.136)
Case B-1 in TABLE 3-1 describes a continuous fluid injection at constant heat flux. The initial reservoir temperature is remained at constant, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case B-1 is given in equation (3.134) to equation (3.136). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in FIGURE 3-10 and FIGURE 3-11, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-10 Temperature distribution at different times in case B-1
FIGURE 3-11 Temperature propagation at different locations in case B-1
3.5.5 Case B-2

The mathematical formulations for case B-2 are taken as follows:

\[
\begin{align*}
\frac{\partial T}{\partial t} &= D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
T(x, 0) &= T_r, \quad x \geq 0 \\
\left( -D \frac{\partial T}{\partial x} + uT \right)_{x=0} &= u(T_0 + T_i e^{-\lambda t}), \quad t \geq 0 \\
\left. \frac{\partial T}{\partial x} \right|_{x \to \infty} &= 0, \quad t \geq 0
\end{align*}
\]

(3.137)

3.5.5.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D} 
\]

(3.138)

\[
T_D(x_D, 0) = 0
\]

(3.139)

\[
\left( -\frac{\partial T_D}{\partial x_D} + T_D \right)_{x_D = 0} = 1 + \frac{T_i}{T_0 - T_i} e^{-\lambda D / u^2}
\]

(3.140)

\[
\left. \frac{\partial T_D}{\partial x_D} \right|_{x_D \to \infty} = 0
\]

(3.141)

3.5.5.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent is introduced by the following variable
transformation (Ogata and Banks, 1961; Farlow, 1982):

\[ T_D(x_D, t_D) = e^{\frac{\lambda D}{4} t_D} K(x_D, t_D) \]  

(3.142)

Substituting equation (3.142) into equations (3.138) – (3.141) gives:

\[ \frac{\partial^2 K}{\partial x_D^2} - \frac{\partial K}{\partial t_D} = 0 \]  

(3.143)

\[ K(x_D, 0) = 0 \]  

(3.144)

\[ \left( -\frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \bigg|_{x_D = 0} = e^{\frac{\lambda D}{4}} + \frac{T_1}{T_0 - T_i} e^{\left( -\frac{\lambda D}{4} \right) T_0} \]  

(3.145)

\[ \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \bigg|_{x_D = \infty} = 0 \]  

(3.146)

### 3.5.5.3 Laplace transformation

Applying Laplace transformation on equations (3.143) – (3.146), the following ordinary boundary value problem (BVP) is obtained:

\[ \frac{d^2 \overline{K}}{d x_D^2} - s \overline{K}(x_D, s) = 0 \]  

(3.147)

\[ \left( -\frac{d \overline{K}}{d x_D} + \frac{1}{2} \overline{K} \right) \bigg|_{x_D = 0} = \frac{1}{s} - \frac{1}{4} + \frac{T_1}{T_0 - T_i} \left( s - \frac{1}{4} + \frac{\lambda D}{u^2} \right) \]  

(3.148)

\[ \left( \frac{d \overline{K}}{d x_D} + \frac{1}{2} \overline{K} \right) \bigg|_{x_D = \infty} = 0 \]  

(3.149)

Then, the general solution (Zwillinger, 1997) to equation (3.147) is:

\[ \overline{K}(x_D, s) = C_1 e^{s x_D e} + C_2 e^{-s x_D e} \]  

(3.150)
Applying general solution (3.150) to equations (3.148) and (3.149) gives:

\[ C_1 = 0 \]  
\[ C_2 = \frac{1}{\left(s - \frac{1}{4}\right) \left(\frac{1}{2} + \sqrt{s}\right)} + \frac{T_t}{T_i} \left(s - \frac{1}{4} + \frac{\lambda D}{u^2}\right) \left(\frac{1}{2} + \sqrt{s}\right) \]  

(3.151)  
(3.152)

Then, the exact solution of the ordinary boundary value problem in Laplace domain becomes:

\[ \overline{K}(x_D, s) = e^{-x_D \sqrt{s}} \left(\frac{s - \frac{1}{4}}{s - \frac{1}{4} + \lambda D} \left(\frac{1}{2} + \sqrt{s}\right) \left(\frac{1}{2} + \sqrt{s}\right)\right) + \frac{T_t}{T_i} \left(s - \frac{1}{4} + \frac{\lambda D}{u^2}\right) \left(\frac{1}{2} + \sqrt{s}\right) \]  

(3.153)

3.5.5.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformations are given:

\[ L^{-1}\left[ \frac{e^{-x_D \sqrt{s}}}{\left(s - \frac{1}{4}\right) \left(\frac{1}{2} + \sqrt{s}\right)} \right] = \sqrt{\frac{T_t}{\pi}} e^{-\frac{x_D^2}{4T_t}} + \frac{1}{2} e^{-\frac{t_D}{4}} e^{\frac{x_D^2 - 2x_D t_D}{2\sqrt{T_t}}} \]  
\[ -1 + x_D t_D e^{-\frac{x_D^2}{4}} e^{\frac{x_D^2 - 2x_D t_D}{2\sqrt{T_t}}} \]  

(3.154)

\[ L^{-1}\left[ \frac{e^{-x_D \sqrt{s}}}{\left(s - \frac{1}{4} + \frac{\lambda D}{u^2}\right) \left(\frac{1}{2} + \sqrt{s}\right)} \right] = \frac{u}{u + \omega} e^{\frac{x_D^2}{4u^2} - \frac{2u}{2u\sqrt{T_t}}} \]  
\[ u x_D - \omega t_D \]  
\[ + \frac{u}{u - \omega} e^{\frac{x_D^2}{4u^2} - \frac{2u}{2u\sqrt{T_t}}} erfc \left(\frac{x_D + t_D}{2\sqrt{T_t}}\right) \]  

(3.155)

where \( \omega = \sqrt{u^2 - 4\lambda D} \).

Applying the inverse Laplace transformation to equation (3.153) gives:
\[ K(x_D, t_D) = \sqrt{\frac{t_D}{\pi}} e^{\frac{-x_D^2}{2t_D}} + \frac{1}{2} e^{\frac{t_D - 2x_D}{4t_D}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} \frac{x_D + t_D}{2\sqrt{t_D}} e^{\frac{t_D + 2x_D}{4t_D}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \]

\[ + \frac{T_1}{T_0 - T_i} u e^{\frac{\omega^2 x_D - ax_D}{2u}} \text{erfc} \left( \frac{ux_D - \omega t_D}{2u\sqrt{t_D}} \right) \]

\[ + \frac{T_1}{T_0 - T_i} u e^{\frac{-\omega^2 x_D + ax_D}{2u}} \text{erfc} \left( \frac{ux_D + \omega t_D}{2u\sqrt{t_D}} \right) \]

\[ + \frac{T_1}{T_0 - T_i} 2u^2 e^{\frac{t_D + 2x_D}{4t_D}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \]

Substituting equation (3.156) into equation (3.142) gives:

\[ T_D(x_D, t_D) = \sqrt{\frac{t_D}{\pi}} e^{\frac{-x_D^2}{4t_D}} + \frac{1}{2} e^{\frac{x_D - t_D}{2\sqrt{t_D}}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} \frac{x_D + t_D}{2\sqrt{t_D}} e^{x_D} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \]

\[ + \frac{T_1}{T_0 - T_i} u e^{\frac{(\omega^2 - u^2)x_D}{4u}} \text{erfc} \left( \frac{ux_D - \omega t_D}{2u\sqrt{t_D}} \right) \]

\[ + \frac{T_1}{T_0 - T_i} u e^{\frac{-(\omega^2 - u^2)x_D}{4u}} \text{erfc} \left( \frac{ux_D + \omega t_D}{2u\sqrt{t_D}} \right) \]

\[ + \frac{T_1}{T_0 - T_i} 2u^2 e^{\omega^2} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \]

Recalling the dimensionless variables defined in equation (3.6) gives:

\[ T(x, t) = T_i + (T_0 - T_i) \left[ \frac{u^2}{\sqrt{\pi} D} e^{\frac{-x^2}{4Dt}} + \frac{1}{2} e^{\frac{x - ut}{2\sqrt{Dt}}} \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) - \frac{1}{2} \left( 1 + \frac{ux}{D} + \frac{u'T}{D} \right) e^{\frac{ux}{D}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \]

\[ + T_1 e^{-\lambda t} \left[ \frac{u}{u + \omega} e^{\frac{u^2x}{2D\lambda}} \text{erfc} \left( \frac{x - \omega t}{2\sqrt{Dt}} \right) + \frac{u}{u - \omega} e^{\frac{u^2x}{2D\lambda}} \text{erfc} \left( \frac{x + \omega t}{2\sqrt{Dt}} \right) \right] \]

\[ + T_1 \left[ \frac{u^2}{2\lambda D} e^{\frac{ux}{D}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \]

\[ (3.158) \]

where \( \lambda = \frac{u^2 - \omega^2}{4D} \).
The final analytical solution can be simplified as follows:

$$T(x,t) = T_i + (T_0 - T_i)A(x,t) + T_i B(x,t) \quad (3.159)$$

$$A(x,t) = \sqrt{\frac{u^2 t}{\pi D}} e^{-\left(\frac{x-ut}{4Dt}\right)^2} + \frac{1}{2} \text{erfc}\left(\frac{x-ut}{2\sqrtDt}\right) - \frac{1}{2}\left(1 + \frac{ux}{D} + \frac{u^2 t}{D}\right) e^{-\frac{x}{2\sqrtDt}} \text{erfc}\left(\frac{x+ut}{2\sqrtDt}\right) \quad (3.160)$$

$$B(x,t) = e^{-\lambda t} \left[ \frac{u}{u + \omega} e^{\frac{u-\omega}{2D} x} \text{erfc}\left(\frac{x - \omega t}{2\sqrtDt}\right) + \frac{u}{u - \omega} e^{\frac{u+\omega}{2D} x} \text{erfc}\left(\frac{x + \omega t}{2\sqrtDt}\right) \right] + \frac{u^2}{2\lambda D} e^{\frac{ux}{D}} \text{erfc}\left(\frac{x + ut}{2\sqrtDt}\right) \quad (3.161)$$

$$\omega = \sqrt{u^2 - 4\lambda D} \quad (3.162)$$

Case B-2 in **TABLE 3-1** describes a continuous fluid injection with exponentially decreasing heat flux. The initial reservoir temperature is remained at constant, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case B-2 is given in equation (3.158) to equation (3.162). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in **FIGURE 3-12** and **FIGURE 3-13**, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-12 Temperature distribution at different times in case B-2
FIGURE 3-13 Temperature propagation at different locations in case B-2
3.5.6 Case B-3

The mathematical formulations for case B-3 are taken as follows:

\[
\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x}
\]

\[T(x,0) = T_i, \quad x \geq 0\]

\[
\left. \left( -D \frac{\partial T}{\partial x} + uT \right) \right|_{x=0} = \begin{cases} uT_0, & 0 < t \leq t_0 \\ uT_0, & t > t_0 \end{cases}
\]

\[
\left. \frac{\partial T}{\partial x} \right|_{x \to \infty} = 0, \quad t \geq 0
\]

### 3.5.6.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D}
\]

\[T_D(x_D,0) = 0\]

\[
\left. \left( \frac{\partial T_D}{\partial x_D} + T_D \right) \right|_{x_D=0} = \begin{cases} 1, & 0 < t_D \leq \frac{u^2 t_0}{D} \\ \frac{T_0 - T_i}{T_0 - T_i}, & t_D > \frac{u^2 t_0}{D} \end{cases}
\]

\[
\left. \frac{\partial T_D}{\partial x_D} \right|_{x_D \to \infty} = 0
\]

### 3.5.6.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable
transformation (Ogata and Banks, 1961; Farlow, 1982):

\[ T_D(x_D, t_D) = e^{\frac{2x_D - t_D}{4}} K(x_D, t_D) \]  

(3.168)

Substituting equation (3.168) into equations (3.164) – (3.167) gives:

\[ \frac{\partial^3 K}{\partial x_D^3} = \frac{\partial K}{\partial t_D} \]  

(3.169)

\[ K(x_D, 0) = 0 \]  

(3.170)

\[ \left( -\frac{\partial K}{\partial x_D} + \frac{1}{2} K \right)_{x_D=0} = \begin{cases} t_D^{\frac{3}{4}} e^{\frac{x_D}{2}}, & 0 < t_D \leq \frac{u^2t_0}{D} \\ \frac{T_z - T_i}{T_i - T_0} e^{\frac{x_D}{2}}, & t_D > \frac{u^2t_0}{D} \end{cases} \]  

(3.171)

\[ \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right)_{x_D \rightarrow \infty} = 0 \]  

(3.172)

### 3.5.6.3 Laplace transformation

Applying Laplace transformation on equations (3.169) – (3.172), the following ordinary boundary value problem (BVP) is obtained:

\[ \frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = 0 \]  

(3.173)

\[ \left( -\frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right)_{x_D=0} = \frac{1}{s - \frac{1}{4}} + \frac{T_z - T_i}{T_i - T_0} \frac{1}{s - \frac{1}{4}} e^{-\frac{1}{4}u^2t_0} \]  

(3.174)

\[ \left( \frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right)_{x_D \rightarrow \infty} = 0 \]  

(3.175)

Then, the general solution (Zwillinger, 1997) to equation (3.173) is:

\[ \overline{K}(x_D, s) = C_1 e^{s \overline{\xi}} + C_2 e^{-s \overline{\xi}} \]  

(3.176)
Applying general solution (3.176) to equations (3.174) and (3.175) gives:

\[ C_1 = 0 \]  \hspace{1cm} (3.177)

\[ C_2 = \frac{1}{s - \frac{1}{4}} \left( \frac{1}{2} + \sqrt{s} \right) + \frac{T_z - T_0}{T_0 - T_i} \left( \frac{1}{2} + \sqrt{s} \right) e^{\left( -\frac{\sqrt{s}}{4} \right)} \]  \hspace{1cm} (3.178)

Then, the exact solution in Laplace domain becomes:

\[ \overline{K}(x_D, s) = e^{-x_D \sqrt{s}} + \frac{u^2_0}{s - \frac{1}{4}} \frac{1}{\left( \frac{1}{2} + \sqrt{s} \right)} e^{\left( -\frac{\sqrt{s}}{4} \right)} e^{-x_D \sqrt{s}} \]  \hspace{1cm} (3.179)

### 3.5.6.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformation is given:

\[
L^{-1} \left[ \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} \left( \frac{1}{2} + \sqrt{s} \right) \right] = \sqrt{\frac{t_D}{\pi}} e^{-\frac{x_D^2}{4t_D}} + \frac{1}{2} e^{-\frac{x_D^2}{4t_D}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1 + x_D + t_D}{2} e^{-\frac{x_D^2}{4t_D}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \]  \hspace{1cm} (3.180)

The *Second Shifting Property* of Laplace transformation (Debnath and Bhatta, 2007) is described as:

\[ L^{-1} \left[ e^{-ax} \overline{F}(x_D, s) \right] = F(x_D, t_D - a)H(t_D - a) \]  \hspace{1cm} (3.181)

where \( a > 0 \), \( L[F(x_D, t_D)] = \overline{F}(x_D, s) \), and \( H(t_D - a) \) is the Heaviside unit step function.

In this case:

\[ a = \frac{u^2_0}{D} \]  \hspace{1cm} (3.182)
\[ F(x_D, s) = \frac{e^{-x_0\sqrt{s}}}{\left(s - \frac{1}{4}\right)\left(\frac{1}{2} + \sqrt{s}\right)} \quad (3.183) \]

According to the Second Shifting Property, the following equation is obtained:

\[ L^{-1}\left[ e^{-\frac{u^2t_0}{D}} F(x_D, s) \right] = F\left( x_D, t_D - \frac{u^2t_0}{D} \right) H\left( t_D - \frac{u^2t_0}{D} \right) \quad (3.184) \]

Then substituting equation (3.101) to equation (3.184) gives:

\[ L^{-1}\left[ \frac{e^{-x_0\sqrt{s}}}{\left(s - \frac{1}{4}\right)\left(\frac{1}{2} + \sqrt{s}\right)} e^{-\frac{u^2t_0}{D}} \right] = \begin{cases} 0, & 0 < t_D \leq \frac{u^2t_0}{D} \\ F\left( x_D, t_D - \frac{u^2t_0}{D} \right), & t_D > \frac{u^2t_0}{D} \end{cases} \quad (3.185) \]

Applying equations (3.180) and (3.185) to equation (3.179) gives:

\[ K(x_D, t_D) = \begin{cases} F(x_D, t_D), & 0 < t_D \leq \frac{u^2t_0}{D} \\ F(x_D, t_D) + \frac{T_z - T_0}{T_0 - T_i} e^{\frac{u^2t_0}{4D}} F\left( x_D, t_D - \frac{u^2t_0}{D} \right), & t_D > \frac{u^2t_0}{D} \end{cases} \quad (3.186) \]

Substituting equation (3.186) into equation (3.168) gives:

\[ T_D(x_D, t_D) = \begin{cases} \frac{2x_0 - t_0}{e^{-\frac{t_0}{4}}} F(x_D, t_D), & 0 < t_D \leq \frac{u^2t_0}{D} \\ \frac{2x_0 - t_0}{e^{-\frac{t_0}{4}}} F(x_D, t_D) + \frac{T_z - T_0}{T_0 - T_i} \frac{u^{2\delta}}{e^{\frac{t_0}{4}}} \frac{2x_0 - t_0}{e^{\frac{t_0}{4}}} F\left( x_D, t_D - \frac{u^2t_0}{D} \right), & t_D > \frac{u^2t_0}{D} \end{cases} \quad (3.187) \]

where

\[ F(x_D, t_D) = \sqrt{\frac{t_D}{\pi}} e^{-\frac{t_0^2}{4t_D}} + \frac{1}{2} e^{-\frac{t_0^2}{4t_D}} \text{erfc}\left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} x_D + \frac{t_D}{e^{-\frac{t_0^2}{4}}} \text{erfc}\left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \quad (3.188) \]

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The following term is defined to simplify the solution:

\[ M(x_D, t_D) = e^{\frac{2x_D - t_D}{4}} F(x_D, t_D) \]

\[ = \sqrt{\frac{t_D}{\pi}} e^{\frac{-(x_D - t_D)^2}{4ut_D}} + \frac{1}{2} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) - \frac{1}{2} \left(1 + \frac{u^2 t_D}{D}\right) e^{\frac{u^2 t_D}{D}} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \]  (3.189)

Then equation (3.187) is simplified as:

\[ T_D(x_D, t_D) = \begin{cases} 
M(x_D, t_D), & 0 < t_D \leq \frac{u^2 t_0}{D} \\
M(x_D, t_D) + \frac{T_e - T_0}{T_0 - T_i} M\left(x_D, t_D - \frac{u^2 t_0}{D}\right), & t_D > \frac{u^2 t_0}{D} 
\end{cases} \]  (3.190)

Recalling the dimensionless variables defined in equations (3.19) – (3.21), the final analytical solution can be written as follows:

\[ T(x, t) = \begin{cases} 
T_i + (T_0 - T_i) A(x, t), & 0 < t \leq t_0 \\
T_i + (T_0 - T_i) A(x, t) + (T_e - T_0) A(x, t - t_0), & t > t_0 
\end{cases} \]  (3.191)

\[ A(x, t) = \sqrt{\frac{u^2 t}{\pi D}} e^{\frac{-(x-ut)^2}{4Dt}} + \frac{1}{2} \text{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right) - \frac{1}{2} \left(1 + \frac{u^x t}{D} + \frac{u^2 t^2}{D}\right) e^{\frac{ux t}{D}} \text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \]  (3.192)

Case B-3 in TABLE 3-1 describes a periodic fluid injection, which initially injects fluid with high heat flux for a period of time and then changes to inject fluid with low heat flux. The initial reservoir temperature is remained at constant, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case B-3 is given in equation (3.191) to equation (3.192). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in FIGURE 3-14 and FIGURE 3-15, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-14 Temperature distribution at different times in case B-3
FIGURE 3-15 Temperature propagation at different locations in case B-3
3.5.7 Case C-1

The mathematical formulations for case C-1 are taken as follows:

\[
\begin{align*}
\frac{\partial T}{\partial t} &= D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
T(x,0) &= T_i + T_2 e^{-\beta x}, \quad x \geq 0 \\
T(0,t) &= T_0, \quad t \geq 0 \\
\frac{\partial T}{\partial x} \bigg|_{x \to \infty} &= 0, \quad t \geq 0
\end{align*}
\]

(3.193)

3.5.7.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D} 
\]

(3.194)

\[
T_D(x_D,0) = \frac{T_i}{T_0 - T_i} e^{\frac{-\beta x_D}{u}}
\]

(3.195)

\[
T_D(0,t_D) = 1
\]

(3.196)

\[
\frac{\partial T_D}{\partial x_D} \bigg|_{x_D \to \infty} = 0
\]

(3.197)

3.5.7.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable transformation (Ogata and Banks, 1961; Farlow, 1982):
\[ T_D(x_D, t_D) = e^{\frac{x_D - x_0}{4}} K(x_D, t_D) \]  

(3.198)

Substituting equation (3.198) into equations (3.194) – (3.197) gives:

\[ \frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D} \]  

(3.199)

\[ K(x_D, 0) = \frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2} \frac{\rho_D}{\alpha} \right) x_0} \]  

(3.200)

\[ K(0, t_D) = e^{\frac{t_D}{4}} \]  

(3.201)

\[ \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \bigg|_{x_D \to \infty} = 0 \]  

(3.202)

### 3.5.7.3 Laplace transformation

Applying Laplace transformation on equations (3.199) – (3.202), the following ordinary boundary value problem (BVP) is obtained:

\[ \frac{d^2 \overline{K}}{d x_D^2} - s \overline{K}(x_D, s) = -\frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2} \frac{\rho_D}{\alpha} \right) x_0} \]  

(3.203)

\[ \overline{K}(0, s) = \frac{1}{s - \frac{1}{4}} \]  

(3.204)

\[ \left( \frac{d \overline{K}}{d x_D} + \frac{1}{2} \overline{K} \right) \bigg|_{x_D \to \infty} = 0 \]  

(3.205)

At first, the general solution to the homogeneous ODE needs to be derived:

\[ \frac{d^2 \overline{K}}{d x_D^2} - s \overline{K}(x_D, s) = 0 \]  

(3.206)

The general solution (Zwillinger, 1997) to homogeneous ODE (3.206) is:
Using the knowledge solving non-homogeneous ODE (Zwillinger, 1997), one particular solution to non-homogeneous ODE (3.203) is in the following form:

$$\overline{K}_p(x_D, s) = A e^{\left(\frac{1}{2} \frac{\beta D}{u}\right) s_0}$$  \hspace{1cm} (3.208)

Substituting equation (3.208) into non-homogeneous ODE (3.203) gives:

$$\frac{1}{2} \frac{\beta D}{u} T_2 s - \frac{1}{2} \frac{\beta D}{u} T_0 - T_i$$  \hspace{1cm} (3.209)

Then one particular solution is obtained:

$$\overline{K}_p(x_D, s) = \frac{1}{s - \left(\frac{1}{2} \frac{\beta D}{u}\right) T_0 - T_i} \frac{T_2}{e^{\left(\frac{1}{2} \frac{\beta D}{u}\right) s_0}}$$  \hspace{1cm} (3.210)

According to the Theorem of Superposition (Zwillinger, 1997), the general solution to the non-homogeneous ODE (3.143) is:

$$\overline{K}(x_D, s) = \overline{K}_h(x_D, s) + \overline{K}_p(x_D, s) = C_1 e^{\frac{s}{\gamma_D}} + C_2 e^{-\frac{s}{\gamma_D}} + \frac{1}{s - \left(\frac{1}{2} \frac{\beta D}{u}\right) T_0 - T_i} \frac{T_2}{e^{\left(\frac{1}{2} \frac{\beta D}{u}\right) s_0}}$$  \hspace{1cm} (3.211)

Applying general solution (3.211) to equations (3.204) and (3.205) gives:

$$C_1 = 0$$  \hspace{1cm} (3.212)

$$C_2 = \frac{1}{\frac{1}{4} T_0 - T_i} - \frac{1}{s - \left(\frac{1}{2} \frac{\beta D}{u}\right) T_0 - T_i}$$  \hspace{1cm} (3.213)

Then, the exact solution of the ordinary boundary value problem in Laplace domain becomes:
$$K(x_D, s) = \frac{e^{-x_0 \sqrt{s}}}{s - \frac{1}{4}} - \frac{T_2}{T_0 - T_i} \frac{e^{-x_0 \sqrt{s}}}{s - \left(\frac{1}{2} + \frac{BD}{u}\right)^2} + \frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2} \frac{BD}{u}\right) x_D} \frac{1}{s - \left(\frac{1}{2} + \frac{BD}{u}\right)^2} \quad (3.214)$$

### 3.5.7.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformations are given:

$$L^{-1}\left[\frac{e^{-x_0 \sqrt{s}}}{s - \frac{1}{4}}\right] = \frac{1}{2} \left[ e^{\frac{t_0 - 2x_0}{4}} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + e^{\frac{t_0 + 2x_0}{4}} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \right] \quad (3.215)$$

$$L^{-1}\left[\frac{e^{-x_0 \sqrt{s}}}{s - \left(\frac{1}{2} + \frac{BD}{u}\right)^2}\right] = \frac{1}{2} e^{\left(\frac{BD}{u} \frac{1}{2}\right)^2} \left[ e^{\left(\frac{BD}{u} \frac{1}{2}\right) t_0} \left(\frac{BD}{u} \frac{1}{2}\right)^{x_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}} + \frac{BD}{u}\sqrt{t_D}\right) \right]$$

$$+ \frac{1}{2} e^{\left(\frac{BD}{u} \frac{1}{2}\right)^2} \left[ e^{\left(\frac{BD}{u} \frac{1}{2}\right) t_0} \left(\frac{BD}{u} \frac{1}{2}\right)^{x_0} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}} + \frac{BD}{u}\sqrt{t_D}\right) \right] \quad (3.216)$$

$$L^{-1}\left[\frac{1}{s - \left(\frac{1}{2} + \frac{BD}{u}\right)^2}\right] = e^{\left(\frac{1}{2} \frac{BD}{u}\right)^2} t_0 \quad (3.217)$$

Applying the inverse Laplace transformation to equation (3.214) gives:

$$K(x_D, t_D) = \frac{1}{2} \left[ e^{\frac{t_0 - 2x_0}{4}} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + e^{\frac{t_0 + 2x_0}{4}} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \right]$$

$$- \frac{1}{2} \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u}\right)^2} \left[ e^{\left(\frac{1}{2} \frac{BD}{u}\right) t_0} \left(\frac{1}{2} \frac{BD}{u}\right)^{x_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}} + \frac{BD}{u}\sqrt{t_D}\right) \right]$$

$$- \frac{1}{2} \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u}\right)^2} \left[ e^{\left(\frac{1}{2} \frac{BD}{u}\right) t_0} \left(\frac{1}{2} \frac{BD}{u}\right)^{x_0} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}} + \frac{BD}{u}\sqrt{t_D}\right) \right]$$

$$+ \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u}\right)^2} \left[ e^{\left(\frac{1}{2} \frac{BD}{u}\right) t_0} \left(\frac{1}{2} \frac{BD}{u}\right)^{x_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}} - \frac{BD}{u}\sqrt{t_D}\right) \right]$$

$$+ \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u}\right)^2} \left[ e^{\left(\frac{1}{2} \frac{BD}{u}\right) t_0} \left(\frac{1}{2} \frac{BD}{u}\right)^{x_0} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}} - \frac{BD}{u}\sqrt{t_D}\right) \right] \quad (3.218)$$
Substituting equation (3.218) into equation (3.198) gives:

\[
T_D(x_D, t_D) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{x_D} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right]
\]

\[
- \frac{1}{2} \frac{T_2}{T_0 - T_i} e^{\left( \frac{\beta D}{u} \frac{\beta D^2}{u^2} \right) x_D} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} - \frac{\beta D}{u} \sqrt{t_D} \right)
\]

\[
- \frac{1}{2} \frac{T_2}{T_0 - T_i} e^{\left( \frac{\beta D}{u} \frac{\beta D^2}{u^2} \right) x_D} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} + \frac{\beta D}{u} \sqrt{t_D} \right)
\]

\[
+ \frac{T_2}{T_0 - T_i} e^{\left( \frac{\beta D}{u} \frac{\beta D^2}{u^2} \right) x_D} \text{erfc} \left( \frac{x_D}{2\sqrt{t_D}} \right)
\]

(3.219)

Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[
T(x, t) = T_i + \frac{1}{2} (T_0 - T_i) \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^{\frac{ux}{D}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right]
\]

\[
+ \frac{1}{2} T_2 e^{\beta D x + \beta u} \left[ 2 - \text{erfc} \left( \frac{x}{2\sqrt{Dt}} \right) \right] - e^{\frac{ux}{D} + 2\beta x} \text{erfc} \left( \frac{x + (u + 2\beta D) t}{2\sqrt{Dt}} \right)
\]

(3.220)

The final analytical solution can be simplified as follows:

\[
T(x, t) = T_i + (T_0 - T_i) A(x, t) + T_2 B(x, t)
\]

(3.221)

where

\[
A(x, t) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^{\frac{ux}{D}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right]
\]

(3.222)

\[
B(x, t) = \frac{1}{2} e^{\beta D x + \beta u} \left[ 2 - \text{erfc} \left( \frac{x - (u + 2\beta D) t}{2\sqrt{Dt}} \right) \right] - e^{\frac{ux}{D} + 2\beta x} \text{erfc} \left( \frac{x + (u + 2\beta D) t}{2\sqrt{Dt}} \right)
\]

(3.223)
Case C-1 in **TABLE 3-1** describes a continuous fluid injection at constant temperature. The initial reservoir temperature is exponentially decreasing with distance, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case C-1 is given in equation (3.220) to equation (3.223). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in **FIGURE 3-16** and **FIGURE 3-17**, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-16 Temperature distribution at different times in case C-1
FIGURE 3-17 Temperature propagation at different locations in case C-1
3.5.8 Case C-2

The mathematical formulations for case C-2 are taken as follows:

\[
\begin{align*}
\frac{\partial T}{\partial t} &= D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
T(x, 0) &= T_i + T_2 e^{-\beta x}, \quad x \geq 0 \\
T(0, t) &= T_0 + T_1 e^{-\lambda t}, \quad t \geq 0
\end{align*}
\]

(3.224)

3.5.8.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\begin{align*}
\frac{\partial T_D}{\partial \tau_D} &= \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D} \\
T_D(x_D, 0) &= \frac{T_2}{T_0 - T_i} e^{-\beta_D x_D} \\
T_D(0, \tau_D) &= 1 + \frac{T_1}{T_0 - T_i} e^{\frac{\lambda_D}{u^*} x_D} \\
\left. \frac{\partial T_D}{\partial x_D} \right|_{x_D \rightarrow \infty} &= 0
\end{align*}
\]

(3.225) (3.226) (3.227) (3.228)

3.5.8.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable transformation (Ogata and Banks, 1961; Farlow, 1982):
$T_D(x_D,t_D) = e^{\frac{2x_D - t_D}{4}} K(x_D,t_D)$  \hspace{1cm} (3.229)

Substituting equation (3.229) into equations (3.225) – (3.228) gives:

$$\frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D}$$  \hspace{1cm} (3.230)

$$K(x_D,0) = \frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2} \frac{\beta D}{u}\right) t_D}$$  \hspace{1cm} (3.231)

$$K(0,t_D) = e^{\frac{t_D}{2}} + \frac{T_i}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{\lambda D}{u}\right) t_D}$$  \hspace{1cm} (3.232)

$$\left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right)_{x_D \to \infty} = 0$$  \hspace{1cm} (3.233)

### 3.5.8.3 Laplace transformation

Applying Laplace transformation on equations (3.230) – (3.233), the following ordinary boundary value problem (BVP) is obtained:

$$\frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D,s) = -\frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2} \frac{\beta D}{u}\right) t_D}$$  \hspace{1cm} (3.234)

$$\overline{K}(0,s) = \frac{1}{s^4} + \frac{T_i}{T_0 - T_i} \frac{1}{s^4} + \frac{\lambda D}{u^2}$$  \hspace{1cm} (3.235)

$$\left( \frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right)_{x_D \to \infty} = 0$$  \hspace{1cm} (3.236)

The general solution to the equation (3.234) has already been obtained in case C-1:

$$\overline{K}(x_D,s) = C_1 e^{\frac{x_D}{s^4}} + C_2 e^{-\frac{x_D}{s^4}} + \frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2} \frac{\beta D}{u}\right) t_D} \frac{1}{s^4} + \frac{\lambda D}{u^2}$$  \hspace{1cm} (3.237)

Applying general solution (3.237) to equations (3.235) and (3.236) gives:
\[ C_1 = 0 \]  
\[ C_2 = \frac{1}{s-\frac{1}{4}} T_0 - T_t + \frac{1}{s-\frac{1}{4}} \frac{\lambda D}{u^2} - \frac{1}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \]  

Then, the exact solution in Laplace domain becomes:

\[ \overline{K}(x_D, s) = \frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\frac{1}{4}} + \frac{T_1}{T_0 - T_t} \frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\frac{1}{4}} + \frac{\lambda D}{u^2} - \frac{1}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} + \frac{T_2}{T_0 - T_t} \frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \]  

3.5.8.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformations are given:

\[ L^{-1}\left[\frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\frac{1}{4}}\right] = \frac{1}{2} \left[ e^{\frac{t_D}{4}} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + e^{-\frac{t_D}{4}} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \right] \]  

\[ L^{-1}\left[\frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\frac{1}{4}} + \frac{T_1}{T_0 - T_t} \frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\frac{1}{4}} + \frac{\lambda D}{u^2} - \frac{1}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} + \frac{T_2}{T_0 - T_t} \frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \right] \]  

where \( \omega = \sqrt{u^2 - 4\lambda D} \).

\[ L^{-1}\left[\frac{e^{-\frac{x_D}{2\sqrt{t_D}}}}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2}\right] = \frac{1}{2} e^{\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2 t_D} e^{-\frac{1}{2} \frac{\beta D}{u} x_D} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}} - \frac{\beta D}{u} \sqrt{t_D}\right) + \frac{1}{2} e^{\frac{1}{2} \frac{\beta D}{u} t_D} e^{-\frac{1}{2} \frac{\beta D}{u} x_D} \text{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}} + \frac{\beta D}{u} \sqrt{t_D}\right) \]  

\[ L^{-1}\left[\frac{1}{s-\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2}\right] = e^{\left(\frac{1}{2} + \frac{\beta D}{u}\right)^2 t_D} \]
Applying the inverse Laplace transformation to equation (3.240) gives:

\[
K(x_D, t_D) = \frac{1}{2} \left[ e^{-\frac{\lambda_0 t_D}{4}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{-\frac{\lambda_0 + 2\lambda_0}{4}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \\
+ \frac{1}{2} \frac{T_1}{T_0 - T_i} \left[ e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D - \omega t_D}{2u \sqrt{t_D}} \right) + e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u \sqrt{t_D}} \right) \right] \\
- \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ e^{\left( \frac{1}{2} \beta D \right)^t_{o} + \frac{u - \omega t_D}{2u \sqrt{t_D}}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u \sqrt{t_D}} \right) \right] \\
- \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ e^{\left( \frac{1}{2} \beta D \right)^t_{o} + \frac{u - \omega t_D}{2u \sqrt{t_D}}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) + e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u \sqrt{t_D}} \right) \right]
\]

(3.245)

Substituting equation (3.245) into equation (3.229) gives:

\[
T_D(x_D, t_D) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{\omega t_D} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \\
+ \frac{1}{2} \frac{T_1}{T_0 - T_i} \left[ e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D - \omega t_D}{2u \sqrt{t_D}} \right) + e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u \sqrt{t_D}} \right) \right] \\
- \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ e^{\left( \frac{1}{2} \beta D \right)^t_{o} + \frac{u - \omega t_D}{2u \sqrt{t_D}}} \text{erfc} \left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u \sqrt{t_D}} \right) \right] \\
- \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ e^{\left( \frac{1}{2} \beta D \right)^t_{o} + \frac{u - \omega t_D}{2u \sqrt{t_D}}} \text{erfc} \left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) + e^{\frac{\alpha^2 t_0}{4u^2}} \text{erfc} \left( \frac{u x_D + \omega t_D}{2u \sqrt{t_D}} \right) \right]
\]

(3.246)

Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[
T(x, t) = T_i + \frac{1}{2} \left( T_0 - T_i \right) \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^{\frac{ut}{2\sqrt{Dt}}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \\
+ \frac{1}{2} T_i e^{-\lambda t} \left[ e^{\frac{\lambda t}{2\sqrt{Dt}}} \text{erfc} \left( \frac{x - \omega t}{2\sqrt{Dt}} \right) + e^{\frac{\lambda t}{2\sqrt{Dt}}} \text{erfc} \left( \frac{x + \omega t}{2\sqrt{Dt}} \right) \right] \\
+ \frac{1}{2} T_2 e^{\beta D t + \beta u - \beta x} \left[ 2 - \text{erfc} \left( \frac{x - (u + 2\beta D)t}{2\sqrt{Dt}} \right) - e^{\frac{\beta u}{2\sqrt{Dt}}} \text{erfc} \left( \frac{x + (u + 2\beta D)t}{2\sqrt{Dt}} \right) \right]
\]

(3.247)
where $\lambda = \frac{u^2 - \omega^2}{4D}$.

The final analytical solution can be simplified as follows:

$$T(x, t) = T_i + (T_0 - T_i)A(x, t) + T_1B(x, t) + T_2C(x, t) \tag{3.248}$$

$$A(x, t) = \frac{1}{2} \left[ \text{erfc} \left( \frac{x - ut}{2\sqrt{Dt}} \right) + e^{\frac{-ux}{D}} \text{erfc} \left( \frac{x + ut}{2\sqrt{Dt}} \right) \right] \tag{3.249}$$

$$B(x, t) = \frac{1}{2} e^{-\lambda t} \left[ e^{\frac{-ux}{D}} \text{erfc} \left( \frac{x - \omega t}{2\sqrt{Dt}} \right) + e^{\frac{+ux}{D}} \text{erfc} \left( \frac{x + \omega t}{2\sqrt{Dt}} \right) \right] \tag{3.250}$$

$$C(x, t) = \frac{1}{2} e^{\beta^2Dt + \beta u - \beta x} \left[ 2 - \text{erfc} \left( \frac{x - (u + 2\beta D)t}{2\sqrt{Dt}} \right) - e^{\frac{-ux}{D}} \text{erfc} \left( \frac{x + (u + 2\beta D)t}{2\sqrt{Dt}} \right) \right] \tag{3.251}$$

$$\omega = \sqrt{u^2 - 4\lambda D} \tag{3.252}$$

Case C-2 in TABLE 3-1 describes a continuous fluid injection with exponentially decreasing temperature. The initial reservoir temperature is exponentially decreasing with distance, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case C-2 is given in equation (3.247) to equation (3.252). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in FIGURE 3-18 and FIGURE 3-19, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-18 Temperature distribution at different times in case C-2
FIGURE 3-19 Temperature propagation at different locations in case C-2
3.5.9 Case D-1

The mathematical formulations for case D-1 are taken as follows:

\[
\begin{align*}
\frac{\partial T}{\partial t} &= D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
T(x, 0) &= T_i + T_e e^{-\beta x}, \quad x \geq 0 \\
\left. \left(-D \frac{\partial T}{\partial x} + u T \right) \right|_{x=0} &= u T_0, \quad t \geq 0 \\
\left. \frac{\partial T}{\partial x} \right|_{x \to \infty} &= 0, \quad t \geq 0
\end{align*}
\] (3.253)

3.5.9.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D} 
\] (3.254)

\[
T_D(x_D, 0) = \frac{T_e}{T_0 - T_i} e^{\frac{\beta D}{u} x_D} 
\] (3.255)

\[
\left. \left(-\frac{\partial T_D}{\partial x_D} + T_D \right) \right|_{x_D=0} = 1 \quad \text{ (3.256)}
\]

\[
\left. \frac{\partial T_D}{\partial x_D} \right|_{x_D \to \infty} = 0 \quad \text{ (3.257)}
\]

3.5.9.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable
transformation (Ogata and Banks, 1961; Farlow, 1982):

\[
T_D(x_D, t_D) = e^{\frac{2x_D}{t_D}} K(x_D, t_D)
\]  

(3.258)

Substituting equation (3.258) into equations (3.254) – (3.257) gives:

\[
\frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D}
\]  

(3.259)

\[
K(x_D, 0) = \frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2}\frac{\beta_D}{u}\right) t_D}
\]  

(3.260)

\[
\left(-\frac{\partial K}{\partial x_D} + \frac{1}{2} K\right)_{x_D = 0} = e^{\frac{t_D}{4}}
\]  

(3.261)

\[
\left(\frac{\partial K}{\partial x_D} + \frac{1}{2} K\right)_{x_D \to \infty} = 0
\]  

(3.262)

3.5.9.3 Laplace transformation

Applying Laplace transformation on equations (3.259) – (3.262), the following ordinary boundary value problem (BVP) is obtained:

\[
\frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = -\frac{T_2}{T_0 - T_i} e^{-\left(\frac{1}{2}\frac{\beta_D}{u}\right) t_D}
\]  

(3.263)

\[
\left(-\frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K}\right)_{x_D = 0} = \frac{1}{s - \frac{1}{4}}
\]  

(3.264)

\[
\left(\frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K}\right)_{x_D \to \infty} = 0
\]  

(3.265)
The general solution to the non-homogeneous ODE (3.263) is:

\[
\bar{K}(x_D, s) = C_1 e^{-\sqrt{s} x_D} + C_2 e^{-x_D \sqrt{s}} + \frac{1}{s - \left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{\beta D}{u}\right)s} \tag{3.266}
\]

which has already been obtained in case C-1.

Applying general solution (3.266) to equations (3.264) and (3.265) gives:

\[
C_1 = 0
\tag{3.267}
\]

\[
C_2 = \frac{1}{\left(s - \frac{1}{4}\right)\left(\frac{1}{2} + \sqrt{s}\right)} - \frac{T_2}{T_0 - T_i} \frac{\beta D + u}{u} \frac{1}{s - \left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \left(\frac{1}{2} + \sqrt{s}\right) \tag{3.268}
\]

Then, the exact solution in Laplace domain becomes:

\[
\bar{K}(x_D, s) = \frac{e^{-x_D \sqrt{s}}}{\left(s - \frac{1}{4}\right)\left(\frac{1}{2} + \sqrt{s}\right)} - \frac{T_2}{T_0 - T_i} \frac{\beta D + u}{u} \frac{e^{-x_D \sqrt{s}}}{s - \left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \left(\frac{1}{2} + \sqrt{s}\right) + \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{\beta D}{u}\right)s} \frac{1}{s - \left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \tag{3.269}
\]

### 3.5.9.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformations are given:

\[
L^{-1} \left[ \frac{e^{-x_D \sqrt{s}}}{\left(s - \frac{1}{4}\right)\left(\frac{1}{2} + \sqrt{s}\right)} \right] = \sqrt{\frac{t_D}{\pi}} e^{-\frac{x_D^2}{4t_D}} + \frac{1}{2} e^{-\frac{t_D - 2x_D}{4t_D}} \text{erfc} \left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) - \frac{1 + x_D + t_D}{2} e^{-\frac{t_D - 2x_D}{4t_D}} \text{erfc} \left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \tag{3.270}
\]
Applying the inverse Laplace transformation to equation (3.269) gives:

\[
K(x_D, t_D) = \sqrt{\frac{t_D}{\pi \tau_0}} e^{- \frac{x_D^2}{\tau_0}} + \frac{1}{2} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) - \frac{1 + x_D + t_D}{2} \frac{1 + x_D + t_D}{4} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) - \frac{u}{2\beta D} \frac{T_2}{T_0 - T_i} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) - \frac{1}{2} \frac{T_2}{T_0 - T_i} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + \frac{1}{2} \frac{1 + u}{\beta D} \frac{T_2}{T_0 - T_i} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + \frac{T_2}{T_0 - T_i} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) + \frac{T_2}{T_0 - T_i} e^{\frac{t_D}{2}} e^{4 \tau_0} \text{erfc}\left(\frac{x_D - t_D}{2\sqrt{t_D}}\right) \] (3.273)

Substituting equation (3.273) into equation (3.258) gives:

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Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[ T_D(x_D, t_D) = \sqrt{\frac{t_D}{\pi}} e^{-\frac{(x_D-t_D)^2}{4t_D}} + \frac{1}{2} e^{\text{erfc} \left( \frac{x_D-t_D}{2\sqrt{t_D}} \right)} - \frac{1}{2} e^{\text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right)} \]

\[ - \frac{u}{2\beta D} \frac{T_2}{T_0-T_i} \frac{e^{\text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right)}}{2\sqrt{t_D}} \]

\[ - \frac{1}{2} \frac{T_2}{T_0-T_i} e^{\left( \frac{\beta D^2}{u^2} - \beta D \right)} t_0 \frac{\beta D}{\sqrt{t_D}} \frac{e^{\text{erfc} \left( \frac{x_D-t_D}{2\sqrt{t_D}} - \frac{\beta D}{u} \right)}}{2\sqrt{t_D}} \]

\[ + \frac{1}{2} \left( 1 + \frac{u}{\beta D} \right) \frac{T_2}{T_0-T_i} e^{\left( \frac{\beta D^2}{u^2} - \beta D \right)} t_0 \frac{\beta D}{\sqrt{t_D}} \frac{e^{\text{erfc} \left( \frac{x_D+t_D + \beta D}{2\sqrt{t_D}} \right)}}{2\sqrt{t_D}} \]

\[ + \frac{T_2}{T_0-T_i} e^{\left( \frac{\beta D^2}{u^2} - \beta D \right)} t_0 \frac{\beta D}{\sqrt{t_D}} \frac{e^{\text{erfc} \left( \frac{x_D-\beta D}{2\sqrt{t_D}} \right)}}{2\sqrt{t_D}} \]

Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[ T(x,t) = T_i + (T_0-T_i) \left[ \sqrt{\frac{u^2}{4Dt}} e^{-\frac{(x-u t)^2}{4Dt}} + \frac{1}{2} e^{\text{erfc} \left( \frac{x-u t}{2\sqrt{Dt}} \right)} - \frac{1}{2} \left( 1 + \frac{u x}{D} + \frac{u^2 t}{D} \right) \frac{e^{\text{erfc} \left( \frac{x+u t}{2\sqrt{Dt}} \right)}}{2\sqrt{Dt}} \right] \]

\[ + \frac{1}{2} T_2 e^{\beta D t + \beta u t - \beta x} \left[ 2 - \text{erfc} \left( \frac{x-(u+2\beta D)t}{2\sqrt{Dt}} \right) + \beta D + u \frac{\alpha x^2}{D} \frac{e^{\text{erfc} \left( \frac{x+(u+2\beta D)t}{2\sqrt{Dt}} \right)}}{2\sqrt{Dt}} \right] \]

\[ - T_2 \frac{u}{2\beta D} \frac{\alpha x}{e^{\text{erfc} \left( \frac{x+u t}{2\sqrt{Dt}} \right)}} \]
Case D-1 in **TABLE 3-1** describes a continuous fluid injection at constant heat flux. The initial reservoir temperature is exponentially decreasing with distance, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case D-1 is given in equation (3.275) to equation (3.278). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in **FIGURE 3-20** and **FIGURE 3-21**, which show excellent agreements between the analytical solution and COMSOL simulation.
FIGURE 3-20 Temperature distribution at different times in case D-1
FIGURE 3-21 Temperature propagation at different locations in case D-1
3.5.10 Case D-2

The mathematical formulations for case D-2 are taken as follows:

\[
\begin{align*}
\frac{\partial T}{\partial t} &= D \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \\
T(x,0) &= T_i + T_2 e^{-\beta x}, \quad x \geq 0 \\
\left(-D \frac{\partial T}{\partial x} + u T \right)_{x=0} &= u(T_0 + T_1 e^{-\lambda t}), \quad t \geq 0 \\
\frac{\partial T}{\partial x} \bigg|_{x \to \infty} &= 0, \quad t \geq 0
\end{align*}
\]

(3.279)

3.5.10.1 Dimensionless form

Using the dimensionless variables defined in equations (3.19) – (3.21), the dimensionless form of the mathematical model is obtained:

\[
\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D}
\]

(3.280)

\[
T_D(x_D,0) = \frac{T_2}{T_0 - T_i} e^{\frac{BD}{u} x_D}
\]

(3.281)

\[
\left(-\frac{\partial T_D}{\partial x_D} + T_D \right)_{x_D=0} = 1 + \frac{T_1}{T_0 - T_i} e^{\frac{BD}{u} x_D}
\]

(3.282)

\[
\frac{\partial T_D}{\partial x_D} \bigg|_{x_D \to \infty} = 0
\]

(3.283)

3.5.10.2 Transformation of dependent variable

To reduce the convection-diffusion equation to the diffusion equation which can be solved more easily, a new dependent variable is introduced by the following variable
transformation (Ogata and Banks, 1961; Farlow, 1982):

\[ T_D(x_D, t_D) = e^{\frac{2x_D - t_D}{4}} K(x_D, t_D) \]  
(3.284)

Substituting equation (3.284) into equations (3.280) – (3.283) gives:

\[ \frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D} \]  
(3.285)

\[ K(x_D, 0) = \frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u^2}\right)t_D} \]  
(3.286)

\[ \left( -\frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \bigg|_{x_D = 0} = t_D + \frac{T_1}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u^2}\right)t_D} \]  
(3.287)

\[ \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \bigg|_{x_D \to \infty} = 0 \]  
(3.288)

### 3.5.10.3 Laplace transformation

Applying Laplace transformation on equations (3.285) – (3.288), the following ordinary boundary value problem (BVP) is obtained:

\[ \frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = -\frac{T_2}{T_0 - T_i} e^{\left(\frac{1}{2} \frac{BD}{u^2}\right)t_D} \]  
(3.289)

\[ \left( -\frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right) \bigg|_{x_D = 0} = \frac{1}{s - \frac{1}{4}} + \frac{T_i}{T_0 - T_i} \frac{1}{s - \frac{1}{4} \left(\frac{BD}{u^2}\right)} \]  
(3.290)

\[ \left( \frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right) \bigg|_{x_D \to \infty} = 0 \]  
(3.291)

The general solution to the equation (3.289) has already obtained in case C-1:
\[
K(x_D,s) = C_1 e^{x_D \sqrt{s}} + C_2 e^{-x_D \sqrt{s}} + \frac{T_2}{T_0-T_i} \frac{e^{\left(\frac{1}{2}-\frac{\lambda D}{u}\right)x_0}}{s-\left(\frac{1}{2}+\frac{\lambda D}{u}\right)} \quad (3.292)
\]

Applying general solution (3.292) to equations (3.290) and (3.291) gives:

\[
C_1 = 0 \quad (3.293)
\]

\[
C_2 = \frac{1}{\left(s-\frac{1}{4}\right)\left(\frac{1}{2}+\sqrt{s}\right)} + \frac{T_1}{T_0-T_i} \frac{1}{\left(s-\frac{1}{4}+\frac{\lambda D}{u^2}\right)\left(\frac{1}{2}+\sqrt{s}\right)} - \frac{T_2}{T_0-T_i} \frac{\beta D+u}{u} \frac{1}{\left[s-\left(\frac{1}{2}+\frac{\beta D}{u}\right)^2\right]^{\frac{1}{2}+\sqrt{s}}} \quad (3.294)
\]

Then, the exact solution in Laplace domain becomes:

\[
\bar{K}(x_D,s) = \frac{e^{-x_D \sqrt{s}}}{\left(s-\frac{1}{4}\right)\left(\frac{1}{2}+\sqrt{s}\right)} + \frac{T_1}{T_0-T_i} \frac{e^{-x_D \sqrt{s}}}{\left(s-\frac{1}{4}+\frac{\lambda D}{u^2}\right)\left(\frac{1}{2}+\sqrt{s}\right)} - \frac{T_2}{T_0-T_i} \frac{\beta D+u}{u} \frac{e^{-x_D \sqrt{s}}}{\left[s-\left(\frac{1}{2}+\frac{\beta D}{u}\right)^2\right]^{\frac{1}{2}+\sqrt{s}}} + \frac{T_2}{T_0-T_i} \frac{e^{\left(\frac{1}{2}-\frac{\lambda D}{u}\right)x_0}}{s-\left(\frac{1}{2}+\frac{\beta D}{u}\right)} \quad (3.295)
\]

### 3.5.10.4 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformations are given:

\[
L^{-1} \left[ \frac{e^{-x_D \sqrt{s}}}{\left(s-\frac{1}{4}\right)\left(\frac{1}{2}+\sqrt{s}\right)} \right] = \sqrt{\frac{T_D}{\pi e^{x_D}}} \frac{-x_D}{4} + \frac{1}{2} e^{\frac{T_0-2x_D}{4}} \text{erfc} \left(\frac{x_D-t_D}{2\sqrt{T_D}}\right) - \frac{1+x_D+t_D}{2} e^{\frac{T_0+2x_D}{4}} \text{erfc} \left(\frac{x_D+t_D}{2\sqrt{T_D}}\right) \quad (3.296)
\]
\[ L^{-1}\left[ \frac{e^{-xs\sqrt{s}}}{s - \left(\frac{1}{2} + \frac{\lambda D}{u^2}\right)} \right] = \frac{u}{\omega} e^{\frac{\omega^2 t_D}{2u^2}} \operatorname{erfc}\left(\frac{ux_D - \omega t_D}{2u\sqrt{t_D}}\right) \]
\[ + \frac{u}{u - \omega} e^{\frac{\omega^2 t_D}{2u^2}} \operatorname{erfc}\left(\frac{ux_D + \omega t_D}{2u\sqrt{t_D}}\right) \]
\[ + \frac{2u^2}{\omega^2 - u^2} e^{\frac{\omega^2 t_D}{2u^2}} \operatorname{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \]  \hspace{1cm} (3.297)

where \( \omega = \sqrt{u^2 - 4\lambda D} \).

\[ L^{-1}\left[ \frac{e^{-xs\sqrt{s}}}{s - \left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \right] = \frac{1}{2} \frac{u^2}{\beta D(\beta D + u)} e^{\frac{t_D^2 + 2\beta D}{4}} \operatorname{erfc}\left(\frac{x_D + t_D}{2\sqrt{t_D}}\right) \]
\[ + \frac{1}{2} \frac{u}{\beta D + u} e^{\left(\frac{1}{2} \beta D^2\right)t_D} \left(\frac{1}{2} \beta D^2\right)^{t_D} \operatorname{erfc}\left(\frac{x_D - t_D - \beta D}{2\sqrt{t_D}u\sqrt{t_D}}\right) \]  \hspace{1cm} (3.298)
\[ - \frac{u}{2\beta D} e^{\left(\frac{1}{2} \beta D^2\right)t_D} \left(\frac{1}{2} \beta D^2\right)^{t_D} \operatorname{erfc}\left(\frac{x_D + t_D + \beta D}{2\sqrt{t_D}u\sqrt{t_D}}\right) \]

\[ L^{-1}\left[ \frac{1}{s - \left(\frac{1}{2} + \frac{\beta D}{u}\right)^2} \right] = e^{\left(\frac{1}{2} \beta D^2\right)t_D} \]  \hspace{1cm} (3.299)

Applying the inverse Laplace transformation to equation (3.295) gives:
\[ K(x_D, t_D) = \sqrt{\frac{t_D}{\pi}} e^{-\frac{(x_D-t_D)^2}{4t_D}} + \frac{1}{2} e^{\frac{t_D-x_D}{4t_D}} \text{erfc} \left( \frac{x_D-t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} e^{\frac{t_D+x_D}{4t_D}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) 
+ \frac{T_1}{T_0 - T_i} \left[ \frac{u}{u + \omega} e^{\frac{\alpha^2 t_D}{2u^2} - \frac{u^2}{2u}} \text{erfc} \left( \frac{ux_D - \omega t_D}{2u\sqrt{t_D}} \right) + \frac{\omega^2 t_D}{2u^2} e^{\frac{\alpha^2 t_D}{2u^2} - \frac{u^2}{2u}} \text{erfc} \left( \frac{ux_D + \omega t_D}{2u\sqrt{t_D}} \right) \right] 
+ \frac{T_1}{2(T_0 - T_i)} e^{\frac{t_D-x_D}{4t_D}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) - \frac{1}{u} \frac{T_2}{2(T_0 - T_i)} e^{\frac{t_D+x_D}{4t_D}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) 
\]

\[ \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ \frac{1}{2} \frac{\beta u}{u} \right]^{\frac{t_D-x_D}{2\sqrt{t_D}}} \text{erfc} \left( \frac{x_D-t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ \frac{1}{2} \frac{\beta u}{u} \right]^{\frac{t_D+x_D}{2\sqrt{t_D}}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) \]

\[ \frac{(3.300)}{2} \]

Substituting equation (3.300) into equation (3.284) gives:

\[ T_D(x_D, t_D) = \sqrt{\frac{t_D}{\pi}} e^{-\frac{(x_D-t_D)^2}{4t_D}} + \frac{1}{2} e^{\frac{t_D-x_D}{4t_D}} \text{erfc} \left( \frac{x_D-t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} e^{\frac{t_D+x_D}{4t_D}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) 
+ \frac{T_1}{T_0 - T_i} \left[ \frac{u}{u + \omega} e^{\frac{\alpha^2 t_D}{2u^2} - \frac{u^2}{2u}} \text{erfc} \left( \frac{ux_D - \omega t_D}{2u\sqrt{t_D}} \right) \right] 
+ \frac{T_1}{T_0 - T_i} \left[ \frac{u}{u + \omega} e^{\frac{\alpha^2 t_D}{2u^2} - \frac{u^2}{2u}} \text{erfc} \left( \frac{ux_D + \omega t_D}{2u\sqrt{t_D}} \right) \right] 
+ \frac{T_1}{2(T_0 - T_i)} e^{\frac{t_D-x_D}{4t_D}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) - \frac{1}{u} \frac{T_2}{2(T_0 - T_i)} e^{\frac{t_D+x_D}{4t_D}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) 
\]

\[ \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ \frac{1}{2} \frac{\beta u}{u} \right]^{\frac{t_D-x_D}{2\sqrt{t_D}}} \text{erfc} \left( \frac{x_D-t_D}{2\sqrt{t_D}} \right) - \frac{1}{2} \frac{T_2}{T_0 - T_i} \left[ \frac{1}{2} \frac{\beta u}{u} \right]^{\frac{t_D+x_D}{2\sqrt{t_D}}} \text{erfc} \left( \frac{x_D+t_D}{2\sqrt{t_D}} \right) \]

\[ \frac{(3.301)}{2} \]
Recalling the dimensionless variables defined in equations (3.19) – (3.21) gives:

\[
T(x,t) = T_i + (T_0 - T_i) e^{-\lambda t} \left[ \frac{u^2}{\pi D} e^{\frac{-4(x-ut)^2}{4Dt}} + \frac{1}{2} \text{erfc} \left( \frac{x-ut}{2\sqrt{Dt}} \right) - \frac{1}{2} \left( 1 + \frac{ux}{D} + \frac{u^2}{D} \right) e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) \right] \\
+ T_i e^{-\lambda t} \left[ \frac{u}{u+\omega} e^{\frac{u+\omega t}{2D}} \text{erfc} \left( \frac{x-\omega t}{2\sqrt{Dt}} \right) + \frac{u}{u-\omega} e^{\frac{u-\omega t}{2D}} \text{erfc} \left( \frac{x+\omega t}{2\sqrt{Dt}} \right) + \frac{u^2}{2\lambda D} T_i e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) \right] \\
- \frac{1}{2} T_2 \frac{u}{\beta D} e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) - \frac{1}{2} T_2 e^{\beta^2 D t + \beta \omega} \text{erfc} \left( \frac{x-(u+2\beta D)t}{2\sqrt{Dt}} \right) \\
+ \frac{1}{2} T_2 \frac{\beta D + u}{\beta D} e^{\frac{\beta^2 D t + \beta \omega + \beta x}{D}} \text{erfc} \left( \frac{x+(u+2\beta D)t}{2\sqrt{Dt}} \right) + T_2 e^{\beta^2 D t + \beta \omega} \\
\] (3.302)

where \( \lambda = \frac{u^2 - \omega^2}{4D} \).

The final analytical solution can be simplified as follows:

\[
T(x,t) = T_i + (T_0 - T_i) A(x,t) + T_i B(x,t) + T_i C(x,t) \\
\] (3.303)

\[
A(x,t) = \sqrt{\frac{u^2}{\pi D}} e^{\frac{-4(x-ut)^2}{4Dt}} + \frac{1}{2} \text{erfc} \left( \frac{x-ut}{2\sqrt{Dt}} \right) - \frac{1}{2} \left( 1 + \frac{ux}{D} + \frac{u^2}{D} \right) e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) \\
\] (3.304)

\[
B(x,t) = \frac{u^2}{2\lambda D} e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) + e^{-\lambda t} \left[ \frac{u}{u+\omega} e^{\frac{u-\omega t}{2D}} \text{erfc} \left( \frac{x-\omega t}{2\sqrt{Dt}} \right) + \frac{u}{u-\omega} e^{\frac{u+\omega t}{2D}} \text{erfc} \left( \frac{x+\omega t}{2\sqrt{Dt}} \right) \right] \\
\] (3.305)

\[
C(x,t) = \frac{1}{2} e^{\beta^2 D t + \beta \omega} \left[ 2 - \text{erfc} \left( \frac{x-(u+2\beta D)t}{2\sqrt{Dt}} \right) + \left( 1 + \frac{u}{\beta D} \right) e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+(u+2\beta D)t}{2\sqrt{Dt}} \right) \right] \\
- \frac{u}{2\beta D} e^{\frac{ux}{D}} \text{erfc} \left( \frac{x+ut}{2\sqrt{Dt}} \right) \\
\] (3.306)

\[
\omega = \sqrt{u^2 - 4\lambda D} \\
\] (3.307)
Case D-2 in **TABLE 3-1** describes a continuous fluid injection with exponentially decreasing heat flux. The initial reservoir temperature is exponentially decreasing with distance, and the lower boundary condition is zero heat flow boundary.

The analytical solution to case D-2 is given in equation (3.302) to equation (3.307). The temperature profiles obtained by both analytical solution and COMSOL simulation are illustrated in **FIGURE 3-22** and **FIGURE 3-23**, which show excellent agreements between the analytical solution and COMSOL simulation.

To summary all the ten case studies under the steady fluid flow condition, the result curves of analytical solutions in this study are smoother than the result curves of COMSOL simulation. The unstable and less smooth curves of COMSOL simulation could be caused by the numerical dispersion of the finite element method used in COMSOL Multiphysics. The smooth curves generated by the analytical solutions indicate that the analytical solutions developed in this study are easier and more accurate in predicting the temperature files.
FIGURE 3-22 Temperature distribution at different times in case D-2
FIGURE 3-23 Temperature propagation at different locations in case D-2
3.6 Sensitivity Analysis

Thermal diffusivity and thermal convection velocity, which integrating the properties of the reservoir and fluids, are the two most important parameters in the mathematical models. Case A-1 in TABLE 3-1 is the base case of this study, which has the simple initial and boundary conditions.

To study the effects of the two most important coefficients in the partial differential equation, the analytical solution to the base case of this study was chosen to conduct the sensitivity analysis. The sensitivity analysis results of case A-1 can reflect how the thermal diffusivity and thermal convection velocity affect the temperature profiles.

3.6.1 Effect of thermal diffusivity

Thermal diffusivity measures the ability of a reservoir to conduct thermal energy relative to its ability to store thermal energy. This value describes how quickly a reservoir reacts to a change in temperature. Reservoirs with large thermal diffusivity rapidly adjust their temperature to that of their surroundings, because they conduct heat quickly in comparison to their volumetric heat capacity (Somerton, 1992).

The thermal convection velocities are all kept at $u=5$ cm/day. The temperature distributions and propagations with different thermal diffusivities are illustrated in FIGURE 3-24.

From FIGURE 3-24 (a), it is found that for reservoir/fluid system with larger thermal diffusivity, the heating area is larger and the slope of temperature curve is smaller, which means temperature gradient is smaller. This is because a reservoir/fluid system
with large thermal diffusivity can conduct heat quickly and adjust its temperature to that of their surroundings more rapidly.

From FIGURE 3-24 (b), the results show that for reservoir/fluid system with larger thermal diffusivity, reservoir temperature begins to increase earlier at the same location. However, the increasing rate of reservoir temperature is smaller than those with smaller thermal diffusivities. This is because reservoirs with large thermal diffusivity have poor ability to store thermal energy.

The figures of temperature profile in FIGURE 3-24 (a) and FIGURE 3-24 (b) show that there always exists a cross point at a certain distance or time. To my understanding, the cross point in FIGURE 3-24 (a) represents the middle point of the transition zone which is steam/oil mixing zone. It gives the temperature and position of this middle point. The temperature of this middle point is always kept at certain temperature. The transition zone is moving forward due to steam injection, and the position of this middle point is determined by thermal convection velocity. When the thermal convection velocity keeps at the same value, the middle points of transient zone are crossed at the same location. The cross point in FIGURE 3-24 (b) represents the time needed for the middle point of transition zone to travel to the specific position $x=40$ meter. When the thermal convection velocity keeps at the same value, the middle points of transient zone are crossed after the same time.

In FIGURE 3-24 (a) and FIGURE 3-24 (b), the temperatures of the cross points are almost the same, which proves that the crossed point is just the middle point of the transition zone. And the thermal diffusivity only affects the temperature distribution around the middle point in the transition zone.
FIGURE 3-24 (a) Temperature distribution after 3 years with different thermal diffusivities; (b) Temperature propagation at $x=40m$ with different thermal diffusivities
3.6.2 Effect of thermal convection velocity

Thermal convection velocity measures the ability of injected fluid to transport heat energy along the direction of the fluid flow. This value describes how fast the heated injection fluid can transport its heat to further distances.

The thermal diffusivities are all kept at $D=0.075 \text{ m}^2/\text{day}$. The temperature distributions and propagations with different thermal convection velocities are illustrated in FIGURE 3-25.

From FIGURE 3-25 (a), it is found that for injection fluid with higher thermal convection velocity, heat can be transported to further distance after a period of time. And the slopes of the temperature curves are nearly the same with different thermal convection velocities, which may be because the thermal diffusivities are the same.

From FIGURE 3-25 (b), the results indicate that for injection fluid with higher thermal convection velocity, reservoir temperature begins to increase earlier at the same location, and the increasing rate of reservoir temperature is larger than those with lower thermal convection velocities. This means injection fluid with higher thermal convection velocity can increase reservoir temperature to the temperature of injection fluid much faster. This is because injection fluid with high thermal convection velocity can accelerate the heat transfer in porous media.
FIGURE 3-25 (a) Temperature distribution after 3 years with different thermal convection velocities; (b) Temperature propagation at $x=40$ m with different thermal convection velocities
3.7 Results Comparison with CMG Simulation

STARS (Steam, Thermal and Advanced processes Reservoir Simulator), which is developed by CMG (Computer Modelling Group Ltd.), is the leading thermal and advanced processes reservoir simulator in the oil and gas industry. CMG STARS is ideally suited for advanced modeling of the complex oil and gas recovery processes involving the injection of steam, solvents, air and chemicals. To prove our mathematical models have practical application in the oil and gas industry, a comparison between our analytical solution and CMG STARS was conducted.

A numerical simulation model simulating transient heat transfer in a heavy oil reservoir during the SAGD process was used for comparison. The numerical simulation model was built in a three-dimensional Cartesian coordinate system by using STARS simulator, which contains 75 grids in \( x \)-direction, 4 grids in \( y \)-direction, and 25 grids in \( z \)-direction. The cell dimension is \( 2 \times 100 \times 1 \) m\(^3\). The vertical distance between the horizontal injector and horizontal producer is 5 meters.

The 3D view, X-Z view, and Y-Z view of the simulation model and grid system are displayed in FIGURE 3-26 (a), (b) and (c). The basic parameters used for simulating transient heat transfer during the SAGD process are listed in TABLE 3-3.

The transient heat transfer during the SAGD process is quite similar to our case A-1. To compare the analytical solution of case A-1 and the SAGD simulation, the two key parameters used in the mathematical model, i.e., thermal diffusivity of the reservoir/fluid system and thermal convection velocity of the fluid need to be calculated using some parameters listed in TABLE 3-3. These two parameters are estimated by using equations (3.4) – (3.6) and equations (3.9) – (3.10).
FIGURE 3-26 (a) 3D view of the grid system; (b) X-Z view of the grid system; (c) Y-Z view of the grid system
TABLE 3-3 Basic parameters used for simulating transient heat transfer in a heavy oil reservoir during the SAGD process

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial reservoir temperature, $T_i$</td>
<td>12</td>
<td>°C</td>
</tr>
<tr>
<td>Initial reservoir pressure, $P_i$</td>
<td>3100</td>
<td>kPa</td>
</tr>
<tr>
<td>Porosity, $\phi$</td>
<td>0.32</td>
<td>fraction</td>
</tr>
<tr>
<td>Permeability, $k$</td>
<td>2500</td>
<td>mD</td>
</tr>
<tr>
<td>Initial oil saturation, $s_{oi}$</td>
<td>0.7</td>
<td>fraction</td>
</tr>
<tr>
<td>Oil viscosity at reservoir temperature, $\mu_{oi}$</td>
<td>11200</td>
<td>cp</td>
</tr>
<tr>
<td>Oil viscosity at steam temperature, $\mu_o$</td>
<td>1.2</td>
<td>cp</td>
</tr>
<tr>
<td>Thermal conductivity of reservoir rock, $K_s$</td>
<td>$6.60 \times 10^5$</td>
<td>J/(m·day·°C)</td>
</tr>
<tr>
<td>Thermal conductivity of oil, $K_o$</td>
<td>$3.60 \times 10^4$</td>
<td>J/(m·day·°C)</td>
</tr>
<tr>
<td>Thermal conductivity of water, $K_w$</td>
<td>$5.35 \times 10^4$</td>
<td>J/(m·day·°C)</td>
</tr>
<tr>
<td>Volumetric heat capacity of reservoir rock, $\rho_s c_s$</td>
<td>$2.35 \times 10^6$</td>
<td>J/(m³·°C)</td>
</tr>
<tr>
<td>Specific heat capacity of oil, $c_o$</td>
<td>1675</td>
<td>J/(kg·°C)</td>
</tr>
<tr>
<td>Density of oil, $\rho_o$</td>
<td>980</td>
<td>kg/m³</td>
</tr>
<tr>
<td>Specific heat capacity of water, $c_w$</td>
<td>4184</td>
<td>J/(kg·°C)</td>
</tr>
<tr>
<td>Density of water, $\rho_w$</td>
<td>1000</td>
<td>kg/m³</td>
</tr>
<tr>
<td>Specific heat capacity of steam, $c_{st}$</td>
<td>4157</td>
<td>J/(kg·°C)</td>
</tr>
<tr>
<td>Density of steam, $\rho_{st}$</td>
<td>19.98</td>
<td>kg/m³</td>
</tr>
<tr>
<td>Steam temperature, $T_0$</td>
<td>250</td>
<td>°C</td>
</tr>
<tr>
<td>Maximum bottom hole pressure of injector, $P_0$</td>
<td>4000</td>
<td>kPa</td>
</tr>
<tr>
<td>Maximum surface water rate of injector, $q_w$</td>
<td>300</td>
<td>m³/day</td>
</tr>
<tr>
<td>Minimum bottom hole pressure of producer, $P_f$</td>
<td>3000</td>
<td>kPa</td>
</tr>
<tr>
<td>Minimum surface liquid rate of producer, $q_f$</td>
<td>600</td>
<td>m³/day</td>
</tr>
</tbody>
</table>
The overall volumetric heat capacity of the reservoir/fluid system is calculated as follows:

\[
\rho c = (1-\phi)\rho_s c_s + \phi \rho_f c_f \\
= (1-\phi)\rho_s c_s + \phi \left[ s_{o1}\rho_o c_o + (1-s_{o1})\rho_w c_w \right] \\
= 2.367 \times 10^6 
\]  

(3.308)

The overall thermal conductivity of reservoir/fluid system is calculated as follows:

\[
K = (1-\phi)K_s + \phi K_f \\
= (1-\phi)K_s + \phi \left[ s_{o1}K_o + (1-s_{o1})K_w \right] \\
= 4.62 \times 10^5 
\]  

(3.309)

Then, thermal diffusivity of the reservoir/fluid system is calculated based on equation (3.9) as follows:

\[
D = \frac{K}{\rho c} = 0.195 
\]  

(3.310)

where the units of overall volumetric heat capacity, overall thermal conductivity, and overall thermal diffusivity are J/(m^3 \cdot °C), J/(m \cdot °C \cdot day), and m^2/day, respectively.

According to equations (3.6) and (3.10), the thermal convection velocity of hot fluid is calculated as follows:

\[
u = \frac{\rho_f c_f}{\rho c} V = -\frac{\rho_w c_w}{\rho c} \frac{k}{\mu_o} \frac{dP}{dx} 
\]  

(3.311)

For the one dimensional fluid flow problem, the pressure gradient is estimated as a constant as follows:
Combining equations (3.311) and (3.312), the thermal convection velocity of hot fluid is estimated as follows:

\[
\frac{dP}{dx} = \Delta P = \frac{P_1 - P_0}{\Delta x}
\]

(3.312)

where the unit of thermal convection velocity of hot fluid is cm/day.

Based on the above calculations, the values of thermal diffusivity and thermal convection velocity are used in the case A-1 to obtain the temperature profiles generated by the analytical solution of case A-1.

To compare the results of the analytical solution with CMG simulation, the temperature profiles of the middle layer in x-direction are exported from the CMG simulation results. The comparison between the analytical solution and the CMG simulation are plotted in FIGURE 3-27 and FIGURE 3-28. It is found that the shapes and trends of temperature curves are very close to each other, which indicates relatively good agreements between the temperature profiles from this study and those from the CMG simulation.

It is also found that numerical simulation is time-consuming due to the large number of simulation grids and small time step. The results of numerical simulations are very sensitive to grid size and time step, which are very unstable. And there must exist numerical dispersions and truncation errors in numerical results. Compared to such complex and time-consuming numerical simulations, the analytical solutions in this study are simple and efficient.
FIGURE 3-27 Temperature distribution at different times in comparison with CMG STARS simulation
FIGURE 3-28 Temperature propagation at different locations in comparison with CMG STARS simulation
CHAPTER 4 MODELING OF 1-D TRANSIENT HEAT TRANSFER COUPLED WITH UNSTEADY FLUID FLOW

4.1 Mathematical Model

4.1.1 Governing heat transfer equation

In previous investigations, thermal convection velocity and thermal diffusivity have been commonly considered as constants. However, the pressure gradient between the injected fluid zone and the crude oil zone may not be constant because of the varying operational conditions of injection wells, which results in the velocity of fluid flow in porous media being transient or unsteady.

Under unsteady flow condition, the thermal convection velocity of the fluid is time-dependent (Irani and Ghannadi, 2013), and the thermal diffusivity of reservoir/fluid system is also assumed time-dependent (Kumar, 1983; Mahato, 2012).

The partial differential equation (PDE) that describes one-dimensional transient heat transfer coupled with unsteady fluid flow is taken as:

\[ \frac{\partial T}{\partial t} = D(t) \frac{\partial^2 T}{\partial x^2} - u(t) \frac{\partial T}{\partial x} \]

(4.1)

where thermal diffusivity and thermal convection velocity are both time-dependent coefficients, and \( 0 \leq x < \infty \) and \( 0 \leq t < \infty \).

4.1.2 Initial and boundary conditions

The initial and boundary conditions are:

\[ T(x, 0) = T_i, \quad x \geq 0 \]

(4.2)
\[ T(0, t) = T_0, \quad t \geq 0 \quad (4.3) \]

\[ \frac{\partial T}{\partial x} \bigg|_{x \rightarrow \infty} = 0, \quad t \geq 0 \quad (4.4) \]

The initial reservoir temperature remains at constant. The upper boundary remains at the temperature of injection fluid, which describes a continuous fluid injection at constant temperature. The lower boundary condition is a zero heat flow boundary.

### 4.2 Analytical Solution

The objective is to characterize the temperature \( T \) as a function of \( x \) and \( t \). To solve this kind of partial differential equation with time-dependent coefficients, new space and time variables need to be introduced using integral transformation. Then, the partial differential equation can be reduced to one with constant coefficients, which has been solved in case A-1 under steady flow condition.

#### 4.2.1 Integral transformation

To simplify the partial differential equation, the time-dependent thermal diffusivity and thermal convection velocity are rewritten in these forms:

\[ D(t) = D_0 f(t) \quad (4.5) \]

\[ u(t) = u_0 g(t) \quad (4.6) \]

where \( D_0 \) and \( u_0 \) are initial thermal diffusivity and initial thermal convection velocity.

Substituting equations (4.5) and (4.6) into equation (4.1), it becomes:

\[ \frac{\partial T}{\partial t} = D_0 f(t) \frac{\partial^2 T}{\partial x^2} - u_0 g(t) \frac{\partial T}{\partial x} \quad (4.7) \]
Introducing a new space variable $X$ using the following integral transformation 
(Jaiswal et al., 2011):

$$X = \int_0^r \frac{g(t)}{f(t)} \, dx = \frac{g(t)}{f(t)} x$$  \hspace{1cm} (4.8)

Substituting equation (4.8) into equation (4.7), the PDE becomes:

$$\frac{f(t)}{g^2(t)} \frac{\partial T}{\partial t} = D_0 \frac{\partial^2 T}{\partial X^2} - u_0 \frac{\partial T}{\partial X}$$  \hspace{1cm} (4.9)

Introducing a new time variable $\tau$ using the following integral transformation  
(Crank, 1975; Jaiswal et al., 2011):

$$\tau = \int_0^r \frac{g^2(t)}{f(t)} \, dt$$  \hspace{1cm} (4.10)

Substituting equation (4.10) into equation (4.9), the PDE becomes:

$$\frac{\partial T}{\partial \tau} = D_0 \frac{\partial^2 T}{\partial X^2} - u_0 \frac{\partial T}{\partial X}$$  \hspace{1cm} (4.11)

The integral transformations in equation (4.8) and equation (4.10) are used to reduce the partial differential equation with time-dependent coefficients to one with constant coefficients. This method conducts a systematic transformation through the integral transformation strategy.

The initial and boundary conditions in equations (4.2) – (4.4) can be written in terms of the new space and time variables as:

$$T(X,0) = T_i, \hspace{0.5cm} X \geq 0$$  \hspace{1cm} (4.12)

$$T(0,\tau) = T_0, \hspace{0.5cm} \tau \geq 0$$  \hspace{1cm} (4.13)

$$\left. \frac{\partial T}{\partial X} \right|_{X \rightarrow \infty} = 0, \hspace{0.5cm} \tau \geq 0$$  \hspace{1cm} (4.14)
4.2.2 Dimensionless form

The dimensionless variables are defined as follows:

\[ T_D = \frac{T - T_i}{T_0 - T_i} \]  
\[ x_D = \frac{u_0 X}{D_0} \]  
\[ t_D = \frac{u_0^2 \tau}{D_0} \]

Rewriting the above three equations in the following forms:

\[ T = T_i + (T_0 - T_i)T_D \]  
\[ X = \frac{D_0}{u_0} x_D \]  
\[ \tau = \frac{D_0}{u_0^2} t_D \]

Performing the partial differentials, the space and time derivatives are obtained:

\[ \frac{\partial T}{\partial X} = \frac{\partial [T_i + (T_0 - T_i)T_D]}{\partial \left[ \frac{D_0}{u_0} x_D \right]} = \left( T_0 - T_i \right) \frac{\partial T_D}{\partial x_D} = \frac{u_0^2 (T_0 - T_i)}{D_0} \frac{\partial T_D}{\partial x_D} \]  
\[ \frac{\partial^2 T}{\partial X^2} = \frac{\partial}{\partial X} \left( \frac{\partial T}{\partial X} \right) = \frac{\partial}{\partial X} \left[ \frac{u_0 (T_0 - T_i)}{D_0} \frac{\partial T_D}{\partial x_D} \right] = \frac{u_0^2 (T_0 - T_i)}{D_0} \frac{\partial^2 T_D}{\partial x_D^2} \]  
\[ \frac{\partial T}{\partial \tau} = \frac{\partial [T_i + (T_0 - T_i)T_D]}{\partial \left[ \frac{D_0}{u_0^2} t_D \right]} = \left( T_0 - T_i \right) \frac{\partial T_D}{\partial t_D} = \frac{u_0^2 (T_0 - T_i)}{D_0} \frac{\partial T_D}{\partial t_D} \]
Substituting equations (4.15) – (4.23) into equations (4.11) – (4.14), the
dimensionless form of the mathematical model is obtained:

\[ \frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial x_D^2} - \frac{\partial T_D}{\partial x_D} \quad (4.24) \]

\[ T_D(x_D, 0) = 0, \quad x_D \geq 0 \quad (4.25) \]

\[ T_D(0, t_D) = 1, \quad t_D \geq 0 \quad (4.26) \]

\[ \frac{\partial T_D}{\partial x_D} \bigg|_{x_D \to \infty} = 0, \quad t_D \geq 0 \quad (4.27) \]

### 4.2.3 Transformation of dependent variable

To simplify the mathematical model, a new dependent variable is introduced by the
following variable transformation (Ogata and Banks, 1961; Farlow, 1982):

\[ T_D(x_D, t_D) = e^{\frac{2x_D - l_D}{4}} K(x_D, t_D) \quad (4.28) \]

Performing the partial differentials, the space and time derivatives are obtained:

\[ \frac{\partial T_D}{\partial t_D} = e^{\frac{2x_D - l_D}{4}} \left( \frac{\partial K}{\partial t_D} - \frac{1}{4} K \right) \quad (4.29) \]

\[ \frac{\partial T_D}{\partial x_D} = e^{\frac{2x_D - l_D}{4}} \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right) \quad (4.30) \]

\[ \frac{\partial^2 T_D}{\partial x_D^2} = e^{\frac{2x_D - l_D}{4}} \left( \frac{\partial^2 K}{\partial x_D^2} + \frac{\partial K}{\partial x_D} + \frac{1}{4} K \right) \quad (4.31) \]

Substituting equations (4.29) – (4.31) into equation (4.24) gives:

\[ \frac{\partial^2 K}{\partial x_D^2} = \frac{\partial K}{\partial t_D} \quad (4.32) \]
Substituting equations (4.28) and (4.30) into equations (4.25) – (4.27), the initial and boundary conditions are transformed to:

\[ K(x_D, 0) = 0 \] (4.33)

\[ K(0, t_D) = e^{\frac{t_D}{4}} \] (4.34)

\[ \left( \frac{\partial K}{\partial x_D} + \frac{1}{2} K \right)_{x_D \to \infty} = 0 \] (4.35)

### 4.2.4 Laplace transformation

Applying Laplace transformation on equations (4.32) – (4.35), the following ordinary boundary value problem (BVP) is obtained:

\[ \frac{d^2 \overline{K}}{dx_D^2} - s \overline{K}(x_D, s) = 0 \] (4.36)

\[ \overline{K}(0, s) = \frac{1}{s - \frac{1}{4}} \] (4.37)

\[ \left( \frac{d \overline{K}}{dx_D} + \frac{1}{2} \overline{K} \right)_{x_D \to \infty} = 0 \] (4.38)

The general solution (Zwillinger, 1997) to equation (4.36) is:

\[ \overline{K}(x_D, s) = C_1 e^{s \sqrt{s}} + C_2 e^{-s \sqrt{s}} \] (4.39)

Applying the general solution (4.39) to equations (4.37) and (4.38) gives:

\[ C_1 = 0 \] (4.40)

\[ C_2 = \frac{1}{s - \frac{1}{4}} \] (4.41)
Then, the exact solution in Laplace domain is derived:

\[ \mathcal{K}(x_D, s) = \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} \quad (4.42) \]

### 4.2.5 Inverse Laplace transformation

In Appendix A, the following inverse Laplace transformation is given:

\[
L^{-1}\left[ \frac{e^{-x_D \sqrt{s}}}{s - \frac{1}{4}} \right] = \frac{1}{2} \left[ e^{-\frac{x_D - t_D}{2\sqrt{t_D}}} \text{erfc}\left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) + e^{-\frac{x_D + 2t_D}{2\sqrt{t_D}}} \text{erfc}\left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) \right] \quad (4.43)
\]

Applying the inverse Laplace transformation to equation (4.42) gives:

\[ \mathcal{K}(x_D, t_D) = \frac{1}{2} \left[ e^{-\frac{x_D - t_D}{2\sqrt{t_D}}} \text{erfc}\left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) + e^{-\frac{x_D + 2t_D}{2\sqrt{t_D}}} \text{erfc}\left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) \right] \quad (4.44) \]

Substituting equation (4.44) into equation (4.28) gives:

\[ T_D(x_D, t_D) = \frac{1}{2} \left[ \text{erfc}\left( \frac{x_D - t_D}{2\sqrt{t_D}} \right) + e^{x_D} \text{erfc}\left( \frac{x_D + t_D}{2\sqrt{t_D}} \right) \right] \quad (4.45) \]

Recalling the defined dimensionless variables, the analytical solution is derived:

\[ T(X, \tau) = T_i + \frac{1}{2} (T_0 - T_i) \left[ \text{erfc}\left( \frac{X - u_0 \tau}{2\sqrt{D_0 \tau}} \right) + e^{\frac{u_0 X}{D_0}} \text{erfc}\left( \frac{X + u_0 \tau}{2\sqrt{D_0 \tau}} \right) \right] \quad (4.46) \]

where the new space and time variables are:

\[ X = \frac{g(t)}{f(t)} x \quad (4.47) \]

\[ \tau = \int_0^t \frac{g^2(t)}{f(t)} dt \quad (4.48) \]
4.3 Model Validation

The analytical solutions under unsteady flow condition are also validated with COMSOL Multiphysics. The simulation model, initial and boundary conditions are all the same with those in case A-1. The only difference is that the time-dependent thermal diffusivity and thermal convection velocity are used in the model.

4.4 Case Studies

Under unsteady flow condition, four cases with different time-dependent thermal diffusivity and thermal convection velocity are studied. The thermal diffusivity and thermal convection velocity are exponentially decreasing with time in case #1, exponentially increasing with time in case #2, step decreasing with time in case #3, and step increasing with time in case #4. The time-dependent functions of the four cases are:

\[ f_1(t) = g_1(t) = e^{-mt} \]  \hspace{1cm} (4.49)

\[ f_2(t) = g_2(t) = e^{mt} \]  \hspace{1cm} (4.50)

\[ f_3(t) = g_3(t) = \begin{cases} 1, & 0 \leq t < t_0 \\ \\ \frac{1}{2}, & t \geq t_0 \end{cases} \]  \hspace{1cm} (4.51)

\[ f_4(t) = g_4(t) = \begin{cases} 1, & 0 \leq t < t_0 \\ \\ 2, & t \geq t_0 \end{cases} \]  \hspace{1cm} (4.52)

where \( m \) is a unsteady parameter whose unit is s\(^{-1}\). Thus \( f(t) \) and \( g(t) \) are dimensionless functions. It is chosen such that \( f(t) = g(t) = 1 \) when either \( t = 0 \) (representing the initial stage) or \( m = 0 \) (representing the uniform diffusion along the steady flow) (Mahato, 2012).

The basic parameters in the mathematical and simulation models under unsteady flow condition are listed in **TABLE 4-1**.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case #1</th>
<th>Case #2</th>
<th>Case #3</th>
<th>Case #4</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial thermal diffusivity, $D_0$</td>
<td>0.075</td>
<td>0.035</td>
<td>0.075</td>
<td>0.035</td>
<td>m$^2$/day</td>
</tr>
<tr>
<td>Initial thermal convection velocity, $u_0$</td>
<td>5.0</td>
<td>2.5</td>
<td>5.0</td>
<td>2.5</td>
<td>cm/day</td>
</tr>
<tr>
<td>Initial reservoir temperature, $T_i$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>°C</td>
</tr>
<tr>
<td>Steam temperature, $T_0$</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>°C</td>
</tr>
<tr>
<td>Unsteady parameter, $m$</td>
<td>5.0×10$^{-9}$</td>
<td>5.0×10$^{-9}$</td>
<td>N/A</td>
<td>N/A</td>
<td>s$^{-1}$</td>
</tr>
<tr>
<td>Time-dependent function, $f(t)^{[1]}$</td>
<td>$f_1(t)$</td>
<td>$f_2(t)$</td>
<td>$f_3(t)$</td>
<td>$f_4(t)$</td>
<td>dimensionless</td>
</tr>
<tr>
<td>Time-dependent function, $g(t)^{[1]}$</td>
<td>$g_1(t)$</td>
<td>$g_2(t)$</td>
<td>$g_3(t)$</td>
<td>$g_4(t)$</td>
<td>dimensionless</td>
</tr>
</tbody>
</table>

Note: [1] All time-dependent functions are given in equations (4.49) – (4.52).
### 4.4.1 Case #1

In this case, \( f_1(t) = g_1(t) = e^{mt} \), thermal diffusivity and thermal convection velocity are exponentially decreasing with time. The initial thermal diffusivity is \( D_0 = 0.075 \text{ m}^2/\text{day} \), and the initial thermal convection velocity is \( u_0 = 5.0 \text{ cm/day} \). The mathematical formulae that describe the history of thermal diffusivity and thermal convection velocity are expressed as \( D = D_0 e^{mt} \), and \( u = u_0 e^{mt} \). The unsteady parameter is \( m = 5.0 \times 10^{-9} \text{ s}^{-1} \). The history profiles of thermal diffusivity and thermal convection velocity are plotted in FIGURE 4-1 (a) and (b).

Therefore, substituting \( f_1(t) \) and \( g_1(t) \) into equations (4.47) and (4.48) gives:

\[
X = x
\]

\[
\tau = \frac{1}{m} \left( 1 - e^{-mt} \right)
\]

Substituting equations (4.53) and (4.54) into equation (4.46), the analytical solution for case #1 is obtained:

\[
T(x,t) = T_i + \frac{1}{2} (T_0 - T_i) \left[ \text{erfc} \left( \frac{x - \frac{u_0}{m} \left( 1 - e^{-mt} \right)}{2 \sqrt{\frac{D_0}{m} \left( 1 - e^{-mt} \right)}} \right) + e^{\frac{u_0x}{m}} \text{erfc} \left( \frac{x + \frac{u_0}{m} \left( 1 - e^{-mt} \right)}{2 \sqrt{\frac{D_0}{m} \left( 1 - e^{-mt} \right)}} \right) \right]
\]

The final results are illustrated in FIGURE 4-2 and FIGURE 4-3, which show excellent agreements between the analytical and numerical results.
FIGURE 4-1 (a) History profile of thermal convection velocity in case #1; (b) History profile of thermal diffusivity in case #1
FIGURE 4-2 Temperature distribution at different times in case #1
FIGURE 4-3 Temperature propagation at different locations in case #1
4.4.2 Case #2

In this case, \( f_2(t) = g_2(t) = e^{mt} \), thermal diffusivity and thermal convection velocity are exponentially increasing with time. The initial thermal diffusivity is \( D_0 = 0.035 \text{ m}^2/\text{day} \), and the initial thermal convection velocity is \( u_0 = 2.5 \text{ cm/day} \). The mathematical formulae that described the history of thermal diffusivity and thermal convection velocity are expressed as \( D = D_0e^{mt} \), and \( u = u_0e^{mt} \). The unsteady parameter is \( m = 5.0 \times 10^{-9} \text{ s}^{-1} \). The history profiles of thermal diffusivity and thermal convection velocity are plotted in FIGURE 4-4 (a) and (b).

Therefore, substituting \( f_2(t) \) and \( g_2(t) \) into equations (4.47) and (4.48) gives:

\[
X = x
\]

\[
\tau = \frac{1}{m} \left( e^{mt} - 1 \right)
\]

Substituting equations (4.56) and (4.57) into equation (4.46), the analytical solution for case #2 is obtained:

\[
T(x, t) = T_i + \frac{1}{2} (T_0 - T_i) \left[ \text{erfc} \left( \frac{x - \frac{u_0}{m} (e^{mt} - 1)}{2 \sqrt{\frac{D_0}{m} (e^{mt} - 1)}} \right) + e^{\frac{ux}{D_s}} \text{erfc} \left( \frac{x + \frac{u_0}{m} (e^{mt} - 1)}{2 \sqrt{\frac{D_0}{m} (e^{mt} - 1)}} \right) \right]
\]

The final results are illustrated in FIGURE 4-5 and FIGURE 4-6, which show excellent agreements between the analytical and numerical results.
FIGURE 4-4 (a) History profile of thermal convection velocity in case #2; (b) History profile of thermal diffusivity in case #2
FIGURE 4-5 Temperature distribution at different times in case #2
FIGURE 4-6 Temperature propagation at different locations in case #2
In this case, \( f_3(t) = g_3(t) = \begin{cases} 1, & 0 \leq t < t_0 \\ \frac{1}{2}, & t \geq t_0 \end{cases} \), thermal diffusivity and thermal convection velocity are both kept at initial values for a period of time \((t_0=2 \text{ years})\), and then decreased to half of the initial values. The initial thermal diffusivity is \( D_0=0.075 \text{ m}^2/\text{day} \), and initial thermal convection velocity is \( u_0=5.0 \text{ cm/day} \). The mathematical formulae that describe the history of thermal diffusivity and thermal convection velocity are

\[
\begin{align*}
D & = \begin{cases} D_0, & 0 \leq t < t_0 \\
\frac{1}{2}D_0, & t \geq t_0 \end{cases}, \\
u & = \begin{cases} u_0, & 0 \leq t < t_0 \\
\frac{1}{2}u_0, & t \geq t_0 \end{cases}.
\end{align*}
\]

The history profiles of thermal diffusivity and thermal convection velocity are plotted in FIGURE 4-7 (a) and (b).

Therefore, substituting \( f_3(t) \) and \( g_3(t) \) into equations (4.47) and (4.48) gives:

\[
\begin{align*}
X &= x \\
\tau &= t, \quad 0 \leq t < t_0 \\
\frac{1}{2}(t + t_0), & \quad t \geq t_0
\end{align*}
\]

Substituting the above two equations into equation (4.46), the solution becomes:

\[
T(x,t) = \begin{cases} 
T_i + \frac{1}{2}(T_0 - T_i) \left[ \text{erfc} \left( \frac{x - u_0 t}{2 \sqrt{D_0 t}} \right) + e^{\frac{u_0^2}{4D_0}} \text{erfc} \left( \frac{x + u_0 t}{2 \sqrt{D_0 t}} \right) \right], & 0 \leq t < t_0 \\
0, & \quad t \geq t_0
\end{cases}
\]

The final results are illustrated in FIGURE 4-8 and FIGURE 4-9, which show excellent agreements between the analytical and numerical results.
FIGURE 4-7 (a) History profile of thermal convection velocity in case #3; (b) History profile of thermal diffusivity in case #3
FIGURE 4-8 Temperature distribution at different times in case #3
FIGURE 4-9 Temperature propagation at different locations in case #3
4.4.4 Case #4

In this case, \( f_4(t) = g_4(t) = \begin{cases} 1, & 0 \leq t < t_0 \\ 2, & t \geq t_0 \end{cases} \), thermal diffusivity and thermal convection velocity are both kept at initial values for a period of time \((t_0=2\text{ years})\), and then increased to double of the initial values. The initial thermal diffusivity is \( D_0=0.035\text{ m}^2/\text{day} \), and initial thermal convection velocity is \( u_0=2.5\text{ cm/day} \). The mathematical formulae that describe the history of thermal diffusivity and thermal convection velocity are \( D = \begin{cases} D_0, & 0 \leq t < t_0 \\ 2D_0, & t \geq t_0 \end{cases} \), and \( u = \begin{cases} u_0, & 0 \leq t < t_0 \\ 2u_0, & t \geq t_0 \end{cases} \). The history profiles of thermal diffusivity and thermal convection velocity are plotted in FIGURE 4-10 (a) and (b).

Therefore, substituting \( f_4(t) \) and \( g_4(t) \) into equations (4.47) and (4.48) gives:

\[
X = x
\]

\[
\tau = \begin{cases} t, & 0 \leq t < t_0 \\ 2t-t_0, & t \geq t_0 \end{cases}
\]

Substituting the above two equations into equation (4.46), the solution becomes:

\[
T(x,t) = \begin{cases} T_i + \frac{1}{2}(T_o - T_i) \left[ \text{erfc} \left( \frac{x-u_0t}{2\sqrt{D_0t}} \right) + e^{\frac{u_0t}{D_0}} \text{erfc} \left( \frac{x+u_0t}{2\sqrt{D_0t}} \right) \right], & 0 \leq t < t_0 \\ T_i + \frac{1}{2}(T_o - T_i) \left[ \text{erfc} \left( \frac{x-u_0(2t-t_0)}{2\sqrt{D_0(2t-t_0)}} \right) + e^{\frac{u_0(2t-t_0)}{D_0}} \text{erfc} \left( \frac{x+u_0(2t-t_0)}{2\sqrt{D_0(2t-t_0)}} \right) \right], & t \geq t_0 \end{cases}
\]

The final results are illustrated in FIGURE 4-11 and FIGURE 4-12, which show excellent agreements between the analytical and numerical results.
FIGURE 4-10 (a) History profile of thermal convection velocity in case #4; (b) History profile of thermal diffusivity in case #4
FIGURE 4-11 Temperature distribution at different times in case #4
FIGURE 4-12 Temperature propagation at different locations in case #4
CHAPTER 5 CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The major conclusions that can be drawn from this thesis study are summarized as follows:

1) Analytical solutions to one-dimensional convection-diffusion equation with different initial and boundary conditions are developed successfully to describe one-dimensional transient heat transfer coupled with fluid flow in heavy oil reservoirs during thermal recovery processes. The analytical solutions are validated successfully with the numerical simulation results generated by COMSOL Multiphysics;

2) Heat transfer and fluid flow occur simultaneously during thermal recovery processes. Convection-conduction is a dynamic heat transfer process, which transports the energy of a moving heat source to further distances and conducts heat to the surroundings at the same time. Fluid flow motivates convective heat transfer and increases the rate of energy transfer significantly;

3) Heat conduction is dominated by temperature gradient, while heat convection is motivated by pressure gradient. The convection-diffusion equation integrates conductive and convective heat transfers, in which temperature and pressure domain can be coupled systematically;
4) For a reservoir/fluid system with larger thermal diffusivity, the heating area is larger and the slope of temperature curve is smaller, and the temperature increasing rate is smaller at the same location. For injection fluid with higher thermal convection velocity, heat can be transported to further distances, and the temperature increasing rate is larger at the same location;

5) In comparison with CMG simulation, the results of analytical solutions in this study show relatively good agreements with those from the CMG simulation. It is found that the shapes and trends of temperature curves are very close to each other. In addition, the analytical solutions in this study are simple and efficient compared to the time-consuming and less accurate numerical simulations;

6) The newly developed analytical solutions provided a theoretical guidance for temperature transient analysis (TTA) and fluid injection strategies. These analytical solutions can be used to predict temperature profiles in heavy oil reservoirs during thermal recovery processes and improve the accuracy and efficiency of temperature transient analysis in heavy oil reservoirs during the thermal recovery processes.

5.2 Recommendations

Based on this thesis study, the following recommendations can be made for future studies:
1) In this study, only one-dimensional transient heat transfer coupled with fluid flow was modeled. The mathematical models should be extended to radial flow system, two-dimensional system, and three-dimensional system. The analytical solutions in this study are point source solutions, which could be extended to line source and slab source solutions by using integration methods;

2) The thermal diffusivity and thermal convection velocity were assumed to be constant or time-dependent in this study, which may vary with other factors, such as distance, temperature, pressure, etc. In future studies, the convection-diffusion equation with variable thermal diffusivity and thermal convection velocity could be solved using analytical or semi-analytical methods;

3) In this study, the mathematical models mainly focused on the heat transfer around a single injection well. The application could be extended to multi-well systems in future work. The analytical solutions for multi-well systems could be obtained based on the developed point source solution by using the principle of superposition, which is commonly used in pressure transient analysis (PTA).
REFERENCES


Farouq Ali, S.M. Practical Heavy Oil Recovery. HOR Heavy Oil Recovery Technologies Ltd., 2006.


## APPENDIX A

### TABLE OF LAPLACE TRANSFORMATIONS

*(van Genuchten and Alves, 1982)*

<table>
<thead>
<tr>
<th></th>
<th>$F(s)$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s + a} )</td>
<td>( e^{-at} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{e^{-as}}{s - a^2} )</td>
<td>( \frac{1}{2} (B + C) )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{e^{-as}}{(s - a^2)(a + \sqrt{s})} )</td>
<td>( ta + \frac{1}{4a} B - \frac{1}{4a} (1 + 2ax + 4a^2 t) C )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{e^{-as}}{(s - a^2)(b + \sqrt{s})} )</td>
<td>( \frac{1}{2(a + b)} B - \frac{1}{2(a - b)} C + \frac{b}{a^2 - b^2} \exp(b^2 t + bx) \text{erfc}\left(\frac{x}{2\sqrt{t}} + b\sqrt{t}\right) )</td>
</tr>
</tbody>
</table>

Laplace Transformation: \( F(s) = \int_0^\infty e^{-st} f(t) \, dt \)

The following abbreviations are used in the table:

\[
A = \frac{1}{\sqrt{\pi t}} \exp\left( -\frac{x^2}{4t} \right)
\]

\[
B = \exp\left(a^2 t - ax\right) \text{erfc}\left(\frac{x}{2\sqrt{t}} - a\sqrt{t}\right)
\]

\[
C = \exp\left(a^2 t + ax\right) \text{erfc}\left(\frac{x}{2\sqrt{t}} + a\sqrt{t}\right)
\]