HYPOTHESIS TESTING FOR THREE MAIN RELIABILITY MODELS

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Fatemeh Shakhsi Salim, candidate for the degree of Master of Science in Statistics**,** has presented a thesis titled, ***Hypothesis Testing for Three Main Reliability Models,*** in an oral examination held on April 9, 2015. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

It is important to check what is lifetime distribution of a product in order to locate and remove the defect in future and to examine reliability of products. Here the subject of investigation is the construction of statistical tests for testing three main probability models in reliability theory consisting of model of aging and deterioration, model of weak link, and model of defect at birth. Six asymptotically locally most powerful tests will be constructed for pair wise distinction between aforementioned models and assess the tests by computer simulation.

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Chapter 1

Introduction

Nowadays we can observe a dramatic increase of productions in different areas of activities. Regardless of what is produced, it is crucial to pay more attention to the reliability of products. Reliability of a device or a product is an important indication of its quality.

The time that the manufacture works from the beginning of its operation until the first breakdown is called Lifetime. Each day companies produce manufactures that are subjects to the lifetime control for construction of a probability model of the breakdowns in order to determine its warranty services.

If the probability model of the breakdowns has the "memory less property", then it becomes meaningless to create special instructions for its replacement with a new one. In this case the lifetime follows Exponential distribution and we simply replace a product by a new one right after it breaks down. Contrary to this, if the lifetime distribution is Gamma, then the breakdown of a product is explained by its aging and deterioration.

Also a wide class of technical systems are systems with mutually independent breakdowns, when a breakdown of any element of the system is considered as the breakdown of the whole system. Under some additional assumptions the lifetime of such system can be approximated sufficiently well by the distribution function which is commonly called in mathematical statistics as the type III Extreme Value distribution or Weibull distribution.

1.1 Importance of Development and Investigation of Reliability Models

In manufacturing practice, a development of aging and deterioration processes for certain details of a device may lead to accidents or even technological disasters. Therefore, in practice it is important to remove reasons that cause aging when we operate with such details. Theoretically it is important to check what is the lifetime distribution (Generalized Exponential, or Gamma, or Weibull) in order to locate and remove a defect in the future.

In connection with this, development and investigation of Reliability models is a very important and relevant issue.

1.2 Objective of Investigation

The objective of this investigation is the construction of locally most powerful statistical tests for testing three main probability models in Reliability Theory:

1. Model of Aging and Deterioration (Gamma distribution)
2. Model of the Weak Link (Weibull distribution)
3. Model of Manufacture Defect (defect at birth or fragility), (Generalized Exponential distribution)

1.3 Outline

All aforementioned models are connected with Generalized Gamma distribution (the definition is given Chapter 2). As this is refer to in next chapter, in 1981 Volodin derived the locally most powerful test statistic to test theses models [**?**]. Using this statistic the following series of problems need to be solved:

1. Calculation of mean values, variances, and covariances of random variables that are involved in the above mentioned test statistic
2. Calculation of asymptotic mean and variance of the test statistic
3. Construction of the tests for choosing the probability Reliability Models

4. Simulation studies to assess the tests

The manuscript contains introduction, five chapters, conclusion, and references and one appendix. After introduction and background in chapters one and two, in chapter three the calculation of the main characteristics of the test statistic is presented. Chapter four is devoted to the construction of the locally most powerful tests for choosing the probability models for lifetime distribution and in chapter five. In the simulation studies is presented. In the conclusion, the summarize and review the main results of the work are presented. In appendix A, computer simulation codes for Matlab version 2014a is presented.

Chapter 2

Background

2.1 Testing Statistical Hypotheses on the Type of Distribution

Random samples of size n are represented by independent random variables $X_1, \ldots, X_n$ having a common pmf/pdf $f(x|\theta)$, $\theta \in \Theta$. Given competing hypotheses about possible values of θ, how do we decide between them?

The formal procedure, due to J. Neyman and E.S. Pearson (1920's) and extending earlier ideas of R.A. Fisher, is to identify a

Null hypothesis $H_0: \ \theta \in \Theta_0$,

which is contrasted with an

Alternative hypothesis $H_1: \ \theta \in \Theta_1$,

where Θ_0 and Θ_1 are disjoint subsets of the parameter space Θ. Call a hypothesis *simple* if it has the form $\theta = \theta'$, a known constant, and *composite* otherwise.

Deciding between the null and alternative hypotheses involves a *test statistic* $\mathfrak{T} = h(\vec{X})$ taking values in a space which is partitioned into disjoint subsets $\mathcal{A}$ and $\mathcal{R}$, called *acceptance* and *rejection* regions, and corresponding to to Θ_0 and Θ_1, respectively. If an observed value of $\mathfrak{T}$, $t = h(\vec{x}) \in \mathcal{R}$, then H_0 is *rejected* in favor of H_1, and if $t \in \mathcal{A}$ then H_0 is *accepted*. The latter term is usually taken to mean there is too little data evidence to opt decisively for H_0.

The law of $\mathfrak{T}$ depends on the unknown value of θ. A crucial role is played by *the law of* $\mathfrak{T}$ *given that* H_0 *is true.* This is well-defined only if H_0 is simple. If general, $\mathcal{A}$ and $\mathcal{R}$ are chosen so that if H_0 is true, the event $\{ t \in \mathcal{R}\}$ occurs with a small probability. Specifically, a small number α is chosen by the statistician, and then $\mathcal{R}$ such that

$$P(\mathcal{T} \in \mathcal{R}|\theta) \leq \alpha \text{ for all } \theta \in \Theta_0,$$

trying to get as close to α as possible.

If $t \in \mathcal{R}$ then we say that H_0 is *rejected at the* $100\alpha\%$ *level of significance*. Call α the *size* of the test. With these choices, we expect that $P(\theta \in \mathcal{R}|\theta) > \alpha$ if $\theta \in \Theta_1$, i.e. the probability of rejection exceeds the chosen level of significance if H_0 is false. In fact, this property cannot be inferred from the above test structure. A test which has this property is said to be *unbiased*. We have the following rationale applicable to unbiased tests for making accept/reject decisions: If $t \in \mathcal{R}$, then:

(a) *Either* H_0 is true and an event of small ($\leq \alpha$) probability has occurred: *or*

(b) H_0 is false, and an event has been observed whose probability exceeds α.

Option (b) is is the better explanation of the observed outcome; it is consistent with the intuition supporting the maximum likelihood concept. This procedure gives rise to two possible errors:

Type I error: Reject H_0 when it is true, and

Type II error: Accept H_0 when it is false.

Type I error is held to be more serious, explaining why the test is designed to control its probability of occurrence:

$$P(\text{Type I error}) = P(\mathfrak{T} \in \mathcal{R}|H_0) \leq \alpha.$$

Computing the probability of a Type II error usually is possible only if H_1 is simple. In general, we define the *power function* $\beta_{\mathfrak{T}}(\theta) = P(\mathfrak{T} \in \mathcal{R}|\theta)$, all $\theta \in \Theta$. Thus the test is designed so that $\beta_{\mathfrak{T}}(\theta) \leq \alpha$ if $\theta \in \Theta_0$. Typically the power function is close to α if $\theta \in \Theta_1$ but close to its boundary, and increasing as θ moves away from the boundary. The sensitivity of a test can be judged in terms of how quickly $\beta_{\mathfrak{T}}(\theta)$ increases above α as $\theta \in \Theta_1$ moves away from the boundary.

This (Neyman-Pearson) testing procedure is a frequentist concept: The operation meaning of the assertion 'H_0 is rejected at the $100\alpha\%$ level if significance' means that if this random experiment is independently replicated many times using the same population, then a Type I error occurs in a proportion $\leq \alpha$ of such replications.

In 'scientific' contexts H_0 represents accepted wisdom or a *status quo*, and experimental data has the express purpose of refuting rather than confirming H_0. Refutation should be compelling, beyond a reasonable doubt, thus explaining the special status accorded to Type I errors, and why α is chosen to be small. It follows that H_0 and H_1 are not inter-changeable.

On the other hand, H_0 could represent model assumptions, such as 'errors are normally distributed'. In quality control situations H_0 could be 'the process is in control', i.e. the probability p that a manufactured item is faulty is less than some very small number. For these cases, finding $t \in \mathcal{A}$ gives weight to accepting H_0 as a viable working assumption, a desirable outcome.

An important question is how to choose a test statistic? Often the choice is made on a 'common sense' basis, but there are general results which can give guidance. It seems fairly obvious that we want a test to be unbiased, and to have the property that $\beta_{\mathfrak{T}}(\theta)$ is as large as possible for all $\theta \in \Theta_1$, i.e.maximum power under H_1. Say that a test statistic is uniformly most powerful (UMP) if, for any other test statistic $\mathcal{T}'$, we have $\beta_{\mathfrak{T}}(\theta) \geq \beta_{\mathfrak{T}'}(\theta)$ for all $\theta \in \Theta_1$.

In this thesis, the test statistics $\mathfrak{T} = g(T_1, T_2, T_3)$ is a function of three pivot statistics (see Chapter 3). Asymptotic distribution of $\mathfrak{T}$ is found with the help of

Delta method, which is a procedure of stochastic representation of $\mathfrak{T}$ with the accuracy $O_P(1/\sqrt{n})$. We expand function g in Taylor series by the powers of $T_i - \mu_i, i = 1, 2, 3$, where $\mu_i = E(T_i)$ and the mathematical expectation is taken under the assumption that one of hypotheses is true (null or alternative). Generally saying, we are taking the expectation of the Generalized Gamma Distribution (see Chapter 3) because two-parameter gamma distribution $G(\lambda)$, two-parameter Weibull distribution $W(\tau)$, and generalized exponential distribution $E(\beta)$ are particular case of the Generalized Gamma distribution.

These three distributions are reduced to the ordinal Exponential distribution E^0 when parameters $\lambda = \tau = \beta = 0$. Because of that we can interpret the Exponential distribution E^0 as a boundary that separates null hypotheses and the alternative. Based on this observation we define the critical constant C for the test statistic $\mathfrak{T} > C$ when the sample is taken from the Exponential distribution E^0.

Returning to the Delta method, we expand function g into Taylor series

$$g(T_1, T_2, T_3) = g(\mu_1, \mu_2, \mu_3) + \sum_{i=1}^{3} \frac{\partial g}{\partial T_i}(\mu_1, \mu_2, \mu_3)(T_i - \mu_i) + O_P((T_i - \mu_i)(T_j - \mu_j)).$$

Since $E(T_i - \mu_i)(T_j - \mu_j) = O(1/n)$, we have that $\sqrt{n}[g(T_1, T_2, T_3) - g(\mu_1, \mu_2, \mu_3)]$ is asymptotically normal with mean zero and variance

$$nE\left[\sum_{i=1}^{3} \frac{\partial g}{\partial T_i} g(\mu_1, \mu_2, \mu_3)(T_i - \mu_i)\right]^2.$$

Therefor the test statistics $\mathfrak{T}$ is asymptotically normal with mean $g(\mu_1, \mu_2, \mu_3)$ and the variance of the form σ^2/n, where σ^2 is expressed through the elements of the covariance matrix of pivot statistics T_1, T_2, T_3 and the coefficients $\frac{\partial g}{\partial T_i} g$.

Often it is assumed that the data are coming from a specific parametric family and the rest of the analysis are done based on that model assumption. Choosing a particular model is quite difficult and the effect due to model misspecification can be quite severe.

In 1962, Cox [?] proposed the test for composite null hypothesis in discriminating between two families of distributions which are overlapping. A composite alternative hypothesis is not in the same parametric family as the null hypothesis. He also suggested the modification of the Neyman-Pearson maximum likelihood ratio test.

In 1973 Dumonceaux and Antle [?] considered the problem of selecting a model from two models with unknown location and scale parameters which the distribution of the ratio of maximum likelihoods does not depend upon the values of the nuisance location and scale parameters. Consequently, this ratio provides a convenient (and hopefully powerful) test for discriminating between two location and scale parameter models when the parameters are unknown. For any given location and scale parameter distribution one can construct tables of critical values for the Kolmogorov-Smirnov test. Lilliefors ([?], [?]) has constructed by using Monte Carlo methods a very useful set of such tables for the case in which one wishes to test the lack of fit of the Normal model or the Exponential model. They presented discriminating between the

Normal and the Cauchy; the Normal and the Exponential; or the Normal and the double Exponential. An empirical comparison of the power of this test with the power of the uniformly most powerful invariant test for discriminating between the Normal and the Cauchy is given. Some further comparisons with the Chi-square goodness of fit and the Kolmogorov-Smirnov tests for normality are also given.

Dumonceaux and Antle [?] in 1973 proposed a likelihood ratio test in discriminating between the Lognormal and Weibull distributions . The asymptotic results can also be used to obtain the critical regions of the corresponding testing of hypotheses problem. In the same year, Mann, Scheuer, and Fertig [?] derived the critical values for a test of fit of data to a two-parameter Weibull distribution with unknown parameters. Using Monte Carlo techniques, they investigated the power which is the probability of rejecting false hypotheses, of their test statistic against three-parameter Weibull and two-parameter Lognormal alternatives. Later in 1975, Mann and Fertig [?] described modifications of the two parameter Weibull goodness of fit test of Mann, Scheuer, and Fertig [?] in 1973 . It is assumed that the sole alternative of interest is any three-parameter Weibull distribution. The power of candidate test statistics is investigated, therefore, only under various three-parameter Weibull alternative hypotheses. It is found that for a fixed selection of gaps (differences of adjacent order statistics) used in the numerator and in the denominator of the approximately F distributed test statistic, nothing is gained by weighting the gaps in order to minimize the variances and thus to maximize the numbers of degrees of freedom. A test statistic which is a modified version of that of Mann, Scheuer, and Fertig is shown to have higher power under three-parameter Weibull alternatives, and a simple method for approximating critical values of the test statistic is described. The test statistic is shown to be a monotone function of an unknown threshold (location) parameter for the three-parameter Weibull model. Hence, the methodology described for testing for a zero threshold parameter can be used to obtain a confidence interval for this parameter. Methods for combining life-test data for application to progressively censored samples are also described.

Kundu and Manglick [?] in 2004 considered the problem of discriminating between Weibull and lognormal distributions . They used the ratio of the maximized likelihood in discriminating between the two distribution functions. It is observed that the asymptotic distribution is asymptotically normal and it is independent of the unknown parameters. The asymptotic distribution can be used to compute the probability of correct selection.

Dey and Kundu [?] in 2009 considered the model discrimination among the three important lifetime distributions . All these three distributions have been used quite effectively to analyze lifetime data in the Reliability analysis.

Bromideh [?] in 2012 introduced a new test statistic based on Kullback-Leibler information (distance) for model selection purposes. The Kullback-Leibler information is a measure of uncertainty between two densities.

Volodin [?] proposed testing of statistical hypotheses on the type of distribution by small samples . In view of the smallness of each sample it is hard to conclude anything about the overall population from which the sampling is alone when they are taken separately. However, it is quite reasonable to set up the hypothesis that all these samples follow the same distribution law which depends on a certain number of parameters, each of which may be different for different samples. Therefore, the problem of testing the statistical hypothesis that all samples belong to certain class of general population, all of which follow the same distribution law and distinguished from each other only by different distribution which no hypotheses is made. He proposed a new method for constructing the critical set on the basic of estimation the nuisance parameter. In certain cases this makes it possible to reduce the calculation necessary for testing the hypothesis for the the type of distribution for an arbitrary distributional hypothesis.

2.2 Locally Most Powerful Test Statistic to Distinguish Distributions Connected with Generalized Gamma Distribution

According to Volodin [?], there is a local most powerful test to distinguish two types of distributions connected with the Generalized Gamma distribution. Assume that $X_1, X_2, ...$ is a sample from a population with the Generalized Gamma distribution. That is, $X_1, X_2, ...$ is a sequence of independent identically distributed random variables with the Generalized Gamma distribution. The following statistic has been suggested by him:

$\mathfrak{T} = c_1\mathfrak{T}_1 + c_2\mathfrak{T}_2$, where

$$\mathfrak{T}_1 = \frac{1}{n}\sum \ln X_k - \ln\left(\frac{1}{n}\sum X_k\right), \quad \mathfrak{T}_2 = \ln\left(\frac{1}{n}\sum X_k\right) - \frac{\frac{1}{n}\sum X_k \ln X_k}{\frac{1}{n}\sum X_k},$$

We say that statistic $\mathfrak{T}$ is a linear combination of statistics $\mathfrak{T}_1$ and $\mathfrak{T}_2$. The rejection region of the local most powerful test mentioned above will have the usual form $\mathfrak{T} > c$. We can use statistic $\mathfrak{T}$ in order to test one of the six combinations of models related to Generalized Gamma distribution.

In continue Generalized Gamma Distribution is reviewed.

2.3 Review of the Generalized Gamma Distribution

The Generalized Gamma distribution was introduced by Stacy and Mihran [?] in 1965 in order to combine two distributions which are the Gamma distribution and the Weibull distribution . The Generalized Gamma distribution is used in many fields such as health costs which the distribution is used to examine regression modeling [?], civil engineering which used this distribution as a flood frequency analysis model [?], as well as economics which used this distribution in various income distributions modeling [?]. It is also used by many scientists for a construction of different models in Reliability Theory when a lifetime of different products is investigated. The Generalized Gamma distribution is a popular distribution because it is very flexible. This distribution is also comfortable because it includes as special cases several distributions such as the Exponential distribution and the Weibull distribution. The Generalized Gamma distribution is a three-parameter distribution. One parametrization of the Generalized Gamma distribution uses the parameters α, τ, λ. A random variable X has a Generalized Gamma distribution if its probability density function is given by:

$$f(x|\alpha,\tau,\lambda) = \frac{\tau}{\lambda\Gamma(\alpha)}(\frac{x}{\lambda})^{\alpha\tau-1}e^{-\left(\frac{x}{\lambda}\right)^{\tau}} \qquad ; x > 0, \alpha, \tau, \lambda > 0$$

According to Volodin [?], random variable X has a Generalized Gamma distribution if its density function is

$$f(x|\lambda,\beta) = \frac{1+\beta}{\Gamma\left(\frac{1+\lambda}{1+\beta}\right)}x^{\lambda}exp(-x^{1+\beta}) \qquad ; x > 0, \beta, \lambda > 0$$

If random variable X has the Generalized Gamma distribution with $\beta = 0$, then X has Gamma distribution with parameter λ, denote $X \sim G(\lambda)$. If random variable X has the Generalized Gamma distribution with $\lambda = 0$, then X has Exponential distribution with parameter β, denote $X \sim E(\beta)$. This should be noted that a more general form of the Exponential distribution is considered here. If random variable X has the Generalized Gamma distribution with $\beta = \lambda = \tau$, then X has Weibull distribution with parameter τ, denote $X \sim W(\tau)$.

The derivative of $\ln(\Gamma(x))$ is called Digamma Euler function and is denoted as $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Next, $\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2$ is called Trigamma Euler function. Hence, $\Gamma'(x) = \Gamma(x)\,\psi(x)$ and $\Gamma''(x) = \Gamma(x)\,[\psi'(x) + \psi^2(x)]$.

In the next chapter calculation of the main characteristics of the test statistic is presented.

Chapter 3

Test statistic and Calculation of the Main Characteristics

Suppose that X has a Generalized Gamma Distribution with density function:

$f(x|\lambda,\beta) = \frac{1+\beta}{\Gamma\left(\frac{1+\lambda}{1+\beta}\right)} x^{\lambda} exp(-x^{1+\beta}), \qquad x, \lambda, \beta > 0$

Note that if random variable X has the Generalized Gamma distribution with $\beta = 0$, then X has Gamma distribution with parameter λ, denote $X \sim G(\lambda)$. If random variable X has the Generalized Gamma distribution with $\lambda = 0$, then X has Exponential distribution with parameter β, denote $X \sim E(\beta)$. Here the more general form of the Exponential distribution is considered. If random variable X has the Generalized Gamma distribution with $\beta = \lambda = \tau$, then X has Weibull distribution with parameter τ, denote $X \sim W(\tau)$.

For a statistical analysis of lifetime data the following Reliability models are commonly used:

$G(\lambda)$, the model of aging and depreciation (Gamma distribution)

$W(\tau)$, the model of a weak link (Weibull distribution)

E^{β}, the model of a manufacture defect (Generalized Exponential distribution) All these distributions are particular case of the Generalized Gamma distribution and they are reduced to the general model of the lack of an aftereffect (Exponential distribution E^0) when shape parameters $\lambda = \tau = \beta = 0$.

According to Volodin [**?**], there is a locally most powerful test to distinguish two types of distributions connected with the Generalized Gamma Distribution. The test is based on the statistic: $\mathfrak{T} = c_1\mathfrak{T}_1 + c_2\mathfrak{T}_2$

where

$$\mathfrak{T}_1 = \frac{1}{n}\sum \ln X_k - \ln\left(\frac{1}{n}\sum X_k\right), \quad \mathfrak{T}_2 = \ln\left(\frac{1}{n}\sum X_k\right) - \frac{\frac{1}{n}\sum X_k \ln X_k}{\frac{1}{n}\sum X_k},$$

$X_1, \ldots, X_n$ is a random sample. The choice of constants c_1 and c_2 defines uniformly most powerful and locally most powerful tests for hypothesis that the model of the lack of an aftereffect is true with alternatives G, W, or E^{β}. In the same article [**?**] and also in [**?**] results have been obtained that are related to pairwise distinction of the above mentioned alternative models. The main goal here is to construct six asymptotically (sample size $n \to \infty$) locally most powerful tests for pairwise distinction between models G, W, and E^{β}.

All tests have the Rejection Region of the general form $\mathfrak{T} > c$ where the critical value $c = c(\alpha)$ is defined according to the given significance level α.

Statistics $\mathfrak{T}_1$ and $\mathfrak{T}_2$ depend on three pivot statistics:

$$T_1 = \frac{1}{n}\sum_1^n X_k,$$

$$T_2 = \frac{1}{n}\sum_1^n \ln(X_k) \ ,$$

$$T_3 = \frac{1}{n}\sum_1^n X_k \ln(X_k) \ .$$

Then in this notation, statistics $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be written as

$$\mathfrak{T}_1 = T_2 - \ln T_1 \ , \mathfrak{T}_2 = \ln T_1 - \frac{T_3}{T_1} \ .$$

The main test statistic takes the form

$$\mathfrak{T} = c_1(T_2 - \ln T_1) \ + c_2(\ln T_1 - \frac{T_3}{T_1} \ ).$$

3.1 Calculation of Mean Values, Variances, and Covariances of the Random Variables that Define the Test Statistic

For calculating mean values, variances, and covariances of the test statistic it is more convenient to calculate all these values for the random variables that define the test statistics. We then exploit these calculated values in Taylor Theorem and Delta Method to calculate the mean values, variances, and covariances of the test statistic.

For a random variable X denote $\mu_{k,j} = E\ (X^k \ln^j X),\ j,k = 0,1,2,\ldots$

Lemma 1.

For the calculation of the main characteristics of statistics $T_1, T_2,$ and T_3 the he following formulas are true:

Mean values of these statistics:

$$\mu_1 = ET_1 = E\ X = \mu_{10,}$$

$$\mu_2 = ET_2 = E\ \ln(X) = \mu_{01,}$$

$$\mu_3 = ET_3 = EX\ \ln(X) = \mu_{11}.$$

Variances of these statistics:

$$n\sigma_1^2 = n \cdot \text{Var } T_1 = EX^2 - \mu_{10}^2 = \mu_{20} - \mu_{10}^2,$$

$$n\sigma_2^2 = n \cdot \text{Var } T_2 = E\ln^2 X - \mu_{01}^2 = \mu_{02} - \mu_{01}^2,$$

$$n\sigma_3^2 = n \cdot \text{Var } T_3 = EX^2 \ln^2 X - \mu_{11}^2 = \mu_{22} - \mu_{11}^2.$$

Covariances of these statistic:

$$n\lambda_{12} = n \cdot Cov\,(T_1,\ T_2) = EX\ln X\ \ - \mu_{10} \cdot \mu_{01} = \mu_{11} - \mu_{10} \cdot \mu_{01},$$

$$n\lambda_{13} = n \cdot Cov\,(T_1,\ T_3) = EX^2 \ln X\ \ - \mu_{10} \cdot \mu_{11} = \mu_{21} - \mu_{10} \cdot \mu_{11},$$

$$n\lambda_{23} = n \cdot Cov\,(T_2,\ T_3) = EX\ln^2 X\ \ - \mu_{01} \cdot \mu_{11} = \mu_{12} - \mu_{01} \cdot \mu_{11}.$$

Proof.

The first three equalities on mean values are trivial. For the variances:

$$n\sigma_1^2 = n.VarT_1 = n.Var(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} VarX_i = \frac{1}{n}\sum_{i=1}^{n}(EX_i^2 - E^2X_i)$$

$$= EX^2 - E^2X = \mu_{20} - \mu_{10}^2.$$

$$n\sigma_2^2 = n.VarT_2 = n.Var(\frac{1}{n}\sum_{i=1}^{n}\ln X_i) = \frac{1}{n}\sum_{i=1}^{n}Var(\ln X_i) = \frac{1}{n}\sum_{i=1}^{n}\big(E(\ln X_i)^2 - E^2(\ln X_i)\big)$$

$$= E(\ln X)^2 - E^2(\ln X) = \mu_{02} - \mu_{01}^2.$$

$$n\sigma_3^2 = n.VarT_3 = n.Var(\frac{1}{n}\sum_{i=1}^{n}(X_i\ln X_i)) = \frac{1}{n}\sum_{i=1}^{n}Var(X_i\ln X_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(E(X_i\ln X_i)^2 - E^2(X_i\ln X_i) = \mu_{22} - \mu_{11}^2.$$

proof for the covariances is as follows:

$$n\lambda_{12} = n\cdot Cov(T_1,T_2) = n\cdot Cov(\frac{1}{n}\sum_{i=0}^{n}X_i, \frac{1}{n}\sum_{j=1}^{n}\ln X_j) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(X_i,\ln X_j) =$$

$$\frac{1}{n}\left(\sum_{i=1}^{n}Cov(X_i,\ln X_i) + \sum\sum_{i\neq j}Cov(X_i,\ln X_j)\right), \quad Since X_i \sim iid.$$

$$= Cov(X_i,\ln X_i) = E(X_i\ln X_i) - E(X_i)E(\ln X_i) = \mu_{11} - \mu_{10}\mu_{01},$$

$$n\lambda_{13} = n\cdot Cov(T_1,T_3) = n\cdot Cov(\frac{1}{n}\sum_{i=0}^{n}X_i, \frac{1}{n}\sum_{j=1}^{n}X_j\ln X_j) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(X_i, X_j\ln X_j) =$$

$$\frac{1}{n}\left(\sum_{i=1}^{n}Cov(X_i,X_i\ln X_i) + \sum\sum_{i\neq j}Cov(X_i,X_j\ln X_j)\right), \quad Since X_i \sim iid.$$

$$= Cov(X_i,X_i\ln X_i) = E(X_i^2\ln X_i) - E(X_i)E(X_i\ln X_i) = \mu_{21} - \mu_{10}\mu_{11},$$

$$n\lambda_{23} = n\cdot Cov(T_2,T_3) = n\cdot Cov(\frac{1}{n}\sum_{i=0}^{n}\ln X_i, \frac{1}{n}\sum_{j=1}^{n}X_j\ln X_j) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(\ln X_i, X_j\ln X_j) =$$

$$\frac{1}{n}\left(\sum_{i=1}^{n}Cov(\ln X_i,X_i\ln X_i) + \sum\sum_{i\neq j}Cov(\ln X_i,X_j\ln X_j)\right), \quad Since X_i \sim iid.$$

$$= Cov(\ln X_i,X_i\ln X_i) = E(X_i\ln^2 X_i) - E(\ln X_i)E(X_i\ln X_i) = \mu_{12} - \mu_{01}\mu_{11},$$

According to Lemma 1, in order to find the mean values, variances, and covariances of the pivot statistics $T_i,\ i = 1, 2, 3,$ for the Generalized Gamma distribution it is necessary to calculate the mean values of random variables $X^k \ln^j X, \quad k, j = 0, 1, 2.$ Therefore, first task is to calculate all moments of these random variables. The following lemma helps to succeed this task.

Lemma 2.

Let random variable X has the Generalized Gamma distribution, then

$$\mu_{k,j} = E(X^k \ln^j X) = \frac{\Gamma^{(j)}\left(\frac{\lambda+k+1}{1+\beta}\right)}{(1+\beta)^j\ \Gamma\left(\frac{1+\lambda}{1+\beta}\right)} \quad ,$$

where $\Gamma^{(j)}(a) = \frac{d^j}{dx^j}\Gamma(\ x)\Big|_{x=a}$.

Proof.

We have $\mu_{k,j} = \frac{1+\beta}{\Gamma\left(\frac{1+\lambda}{1+\beta}\right)} \int_0^\infty x^{\lambda+k} ln^j x\ e^{-x^{1+\mu}} dx$. Consider change of variables $x^{1+\beta} = t$. Then $x = t^{\frac{1}{1+\beta}}$, $dx = \frac{1}{1+\beta} t^{-\frac{\beta}{1+\beta}}\ dt$.

Hence $\mu_{k,j} = \frac{1}{\Gamma\left(\frac{1+\lambda}{1+\beta}\right)} \int_0^\infty t^{\frac{\lambda+k}{1+\beta} - \frac{\beta}{1+\beta}}\ ln^j\ t^{\frac{1}{1+\beta}}\ e^{-t} dt = \frac{(1+\beta)^{-j}}{\Gamma\left(\frac{1+\lambda}{1+\beta}\right)} \int_0^\infty t^{\frac{\lambda+k-\beta}{1+\beta}}\ ln^j\ t\ e^{-t} dt.$

The integral $\int_0^\infty t^a\ ln^j t\ e^{-t}\ dt = \frac{d^j}{da^j} \int_0^\infty t^a\ e^{-t} dt = \ \frac{d^j}{da^j}\Gamma(a+1) = = \Gamma^{(j)}(a+1)\,.$

Hence we obtain $\mu_{k,j} = \frac{\Gamma^{(j)}\left(\frac{\lambda+k+1}{1+\beta}\right)}{(1+\beta)^j\ \Gamma\left(\frac{1+\lambda}{1+\beta}\right)}$.

From the lemma above the following corollary will be concluded.

Corollary.

If $X \sim G(\lambda)$ (that is, Generalized Gamma distribution with $\beta = 0$), then

$$\mu_{k,j} = \frac{\Gamma^{(j)}(\lambda+k+1)}{\Gamma(1+\lambda)}$$

If $X \sim E(\beta)$ (that is, Generalized Gamma distribution with λ=0), then

$$\mu_{k,j} = \frac{\Gamma^{(j)}\left(\frac{k+1}{1+\beta}\right)}{(1+\beta)^j\ \Gamma\left(\frac{1}{1+\beta}\right)}$$

If $X \sim W(\tau)$ (that is, Generalized Gamma distribution with $\lambda = \beta = \tau$), then

$$\mu_{k,j} = \frac{\Gamma^{(j)}\left(\frac{\tau+k+1}{1+\tau}\right)}{(1+\tau)^j}$$

For an application in subsequent calculations, the values of μ_{kj} for *k,j=0,1,2* is written explicitly in Lemma 3. The following supporting function and formulas are used:

The derivative of $\ln(\Gamma(x))$ is called Digamma Euler function and is denoted as $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Next, $\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2$ is called Trigamma Euler function. Hence, $\Gamma'(x) = \Gamma(x)\,\psi(x)$ and $\Gamma''(x) = \Gamma(x)\,[\psi'(x) + \psi^2(x)]$.

Lemma 3.

The following formulas are true:

$$\mu_{01} = \frac{\Gamma'\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta+1)\,\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\psi\left(\frac{\lambda+1}{\beta+1}\right)}{\beta+1},$$

$$\mu_{02} = \frac{\Gamma^{(2)}\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta+1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta+1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}\left[\psi'\left(\frac{\lambda+1}{\beta+1}\right) + \psi^2\left(\frac{\lambda+1}{\beta+1}\right)\right]$$
$$= \frac{1}{(\beta+1)^2}\left[\psi'\left(\frac{\lambda+1}{\beta+1}\right) + \psi^2\left(\frac{\lambda+1}{\beta+1}\right)\right],$$

$$\mu_{10} = \frac{\Gamma\left(\frac{\lambda+2}{\beta+1}\right)}{\Gamma\left(\frac{\lambda+1}{\beta+1}\right)},$$

$$\mu_{11} = \frac{\Gamma'\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta+1)\,\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+2}{\beta+1}\right)\psi\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta+1)\,\Gamma\left(\frac{\lambda+1}{\beta+1}\right)},$$

$$\mu_{12} = \frac{\Gamma^{(2)}\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta+1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta+1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}\left[\psi'\left(\frac{\lambda+2}{\beta+1}\right) + \psi^2\left(\frac{\lambda+2}{\beta+1}\right)\right],$$

$$\mu_{20} = \frac{\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}{\Gamma\left(\frac{\lambda+1}{\beta+1}\right)},$$

$$\mu_{21} = \frac{\Gamma'\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta+1)\,\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta+1)\,\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}\psi\left(\frac{\lambda+3}{\beta+1}\right),$$

$$\mu_{22} = \frac{\Gamma^{(2)}\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta+1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta+1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}\left[\psi'\left(\frac{\lambda+3}{\beta+1}\right) + \psi^2\left(\frac{\lambda+3}{\beta+1}\right)\right].$$

Consider separately the case $X \sim E(\beta)$ (that is, the Generalized Gamma distribution with λ =0). Writing all values of μ_{kj} with *k,j=0,1,2*, for this particular case:

$$\mu_{01} = \frac{\psi(\frac{1}{1+\beta})}{1+\beta},$$

$$\mu_{02} = \frac{\psi'(\frac{1}{1+\beta}) + \psi^2(\frac{1}{1+\beta})}{(1+\beta)^2},$$

$$\mu_{10} = \frac{\Gamma(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})},$$

$$\mu_{11} = \frac{\Gamma(\frac{2}{1+\beta})\psi(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})},$$

$$\mu_{12} = \frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)^2\Gamma(\frac{1}{1+\beta})}\left[\psi'\left(\frac{2}{1+\beta}\right) + \psi^2\left(\frac{2}{1+\beta}\right)\right],$$

$$\mu_{20}{=}\frac{\Gamma\left(\frac{3}{1+\beta}\right)}{\Gamma\left(\frac{1}{1+\beta}\right)},$$

$$\mu_{21}=\frac{\Gamma(\frac{3}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})}\psi(\frac{3}{1+\beta}),$$

$$\mu_{22}=\frac{\Gamma\left(\frac{3}{1+\beta}\right)}{(1+\beta)^2\,\Gamma\left(\frac{1}{1+\beta}\right)}\left(\psi'\left(\frac{3}{1+\beta}\right)+\psi^2\left(\frac{3}{1+\beta}\right)\right)$$

Therefore for the case $X\sim E(\beta)$ all mean values, variances and covariance matrix of statistics $T_{1,}\ \ T_{2,}\ \ \ T_3$ can be calculated:

$$\mu_1=\mu_{10}=\frac{\Gamma(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})},$$

$$\mu_2=\mu_{01}=\frac{\psi(\frac{1}{1+\beta})}{1+\beta},$$

$$\mu_3=\mu_{11}=\frac{\Gamma(\frac{2}{1+\beta})\psi(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})},$$

$$\sigma_1^2=\frac{1}{n}(\mu_{20}-\mu_{10}^2)=\frac{1}{n}\left[\frac{\Gamma(\frac{3}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}-\left(\frac{\Gamma(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}\right)^2\right],$$

$$\sigma_2^2=\frac{1}{n}\left(\mu_{02}-\mu_{01}^2\right)=\frac{1}{n}\left[\frac{\psi'\left(\frac{1}{1+\beta}\right)+\psi^2\left(\frac{1}{1+\beta}\right)}{(1+\beta)^2}-\left(\frac{\psi\left(\frac{1}{1+\beta}\right)}{1+\beta}\right)^2\right]$$

$$=\frac{1}{n}\left[\frac{1}{(1+\beta)^2}\psi^{'}(\frac{1}{1+\beta})\right],$$

$$\sigma_3^2=\frac{1}{n}\left(\mu_{22}-\mu_{11}^2\right)=$$

$$\frac{1}{n}\left[\frac{\Gamma\left(\frac{3}{1+\beta}\right)}{(1+\beta)^2\,\Gamma\left(\frac{1}{1+\beta}\right)}\left(\psi'\left(\frac{3}{1+\beta}\right)+\psi^2\left(\frac{3}{1+\beta}\right)\right)-\left(\frac{\Gamma\left(\frac{2}{1+\beta}\right)\psi\left(\frac{2}{1+\beta}\right)}{(1+\beta)\,\Gamma\left(\frac{1}{1+\beta}\right)}\right)^2\right]$$

$$= \frac{1}{n}\frac{1}{(1+\beta)^2\Gamma(\frac{1}{1+\beta})}\left(\Gamma\left(\frac{3}{1+\beta}\right)\left(\psi'\left(\frac{3}{1+\beta}\right)+\psi^2\left(\frac{3}{1+\beta}\right)\right)-\frac{\Gamma^2\left(\frac{2}{1+\beta}\right)\psi^2\left(\frac{2}{1+\beta}\right)}{\Gamma\left(\frac{1}{1+\beta}\right)}\right).$$

Covariance between T_1 and T_2 takes the form:

$$\lambda_{12} = \frac{1}{n}\left(\mu_{11}-\mu_{10}\mu_{01}\right) = \frac{1}{n}\left(\frac{\Gamma(\frac{2}{1+\beta})\psi(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})} - \left(\frac{\Gamma(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}\right)\left(\frac{\psi(\frac{1}{1+\beta})}{1+\beta}\right)\right)$$
$$= \frac{1}{n}\left(\frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})}\left(\psi(\frac{2}{1+\beta})-\psi(\frac{1}{1+\beta})\right)\right),$$

Covariance between T_1 and T_3 can be calculated by the following formula

$$\lambda_{13} = \frac{1}{n}\left(\mu_{21}-\mu_{10}\mu_{11}\right)$$
$$= \frac{1}{n}\left(\frac{\Gamma\left(\frac{3}{1+\beta}\right)}{(1+\beta)\,\Gamma\left(\frac{1}{1+\beta}\right)}\psi\left(\frac{3}{1+\beta}\right) - \left(\frac{\Gamma\left(\frac{2}{1+\beta}\right)}{\Gamma\left(\frac{1}{1+\beta}\right)}\right)\left(\frac{\Gamma\left(\frac{2}{1+\beta}\right)\psi\left(\frac{2}{1+\beta}\right)}{(1+\beta)\,\Gamma\left(\frac{1}{1+\beta}\right)}\right)\right)$$
$$= \frac{1}{n}\left(\frac{1}{(1+\beta)\,\Gamma\left(\frac{1}{1+\beta}\right)}\right)\left[\Gamma\left(\frac{3}{1+\beta}\right)\psi\left(\frac{3}{1+\beta}\right) - \frac{\Gamma^2\left(\frac{2}{1+\beta}\right)\psi\left(\frac{2}{1+\beta}\right)}{\Gamma\left(\frac{1}{1+\beta}\right)}\right]$$

Covariance between T_2 and T_3 can be found in the following way:

$$\lambda_{23} = \frac{1}{n}\left(\mu_{12}-\mu_{01}\mu_{11}\right)$$
$$= \frac{1}{n}\left(\frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)^2\Gamma(\frac{1}{1+\beta})}\left[\psi'\left(\frac{2}{1+\beta}\right)+\psi^2\left(\frac{2}{1+\beta}\right)\right] - \frac{\psi(\frac{1}{1+\beta})}{1+\beta}\left(\frac{\Gamma(\frac{2}{1+\beta})\psi(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})}\right)\right)$$
$$= \frac{1}{n}\left(\frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)^2\Gamma(\frac{1}{1+\beta})}\left[\psi'\left(\frac{2}{1+\beta}\right)+\psi^2\left(\frac{2}{1+\beta}\right)-\psi(\frac{1}{1+\beta})\psi(\frac{2}{1+\beta})\right]\right)$$

To summarize the following results for the case $X \sim E(\beta)$ are obtained:

Mean values:

1. $\mu_1 = \frac{\Gamma(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}$,

2. $\mu_2 = \frac{\psi(\frac{1}{1+\beta})}{1+\beta}$,

3. $\mu_3 = \frac{\Gamma(\frac{2}{1+\beta})\psi(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})}$; ;

Variances:

1. $\sigma_1^2 = \frac{1}{n}\left[\frac{\Gamma(\frac{3}{1+\beta})}{\Gamma(\frac{1}{1+\beta})} - \left(\frac{\Gamma(\frac{3}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}\right)^2\right]$,

2. $\sigma_2^2 = \frac{1}{n}\left[\frac{1}{(1+\beta)^2}\psi'\left(\frac{1}{1+\beta}\right)\right]$,

3. $\sigma_3^2 = \frac{1}{n}\frac{1}{(1+\beta)^2\Gamma(\frac{1}{1+\beta})}\left[\Gamma\left(\frac{3}{1+\beta}\right)\left(\psi'\left(\frac{3}{1+\beta}\right) + \psi^2\left(\frac{1}{1+\beta}\right)\right) - \frac{\left(\Gamma\left(\frac{2}{1+\beta}\right)\right)^2\left(\psi\left(\frac{2}{1+\beta}\right)\right)^2}{\Gamma\left(\frac{1}{1+\beta}\right)}\right]$

Covariances:

1. $\lambda_{12} = \frac{1}{n}\left(\frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})}\left(\psi(\frac{2}{1+\beta}) - \psi(\frac{1}{1+\beta})\right)\right)$,

2. $\lambda_{13} = \frac{1}{n}\left(\frac{1}{(1+\beta)\Gamma\left(\frac{1}{1+\beta}\right)}\right)\left[\Gamma\left(\frac{3}{1+\beta}\right)\psi\left(\frac{3}{1+\beta}\right) - \frac{\left(\Gamma\left(\frac{2}{1+\beta}\right)\right)^2\psi\left(\frac{2}{1+\beta}\right)}{\Gamma\left(\frac{1}{1+\beta}\right)}\right]$,

3. $\lambda_{23} = \frac{1}{n}\left(\frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)^2\Gamma(\frac{1}{1+\beta})}\left[\psi'\left(\frac{2}{1+\beta}\right) + \psi^2\left(\frac{2}{1+\beta}\right) - \psi(\frac{1}{1+\beta})\psi(\frac{2}{1+\beta})\right]\right)$.

In similar way the mean values, variances, and covariances of statistics T_1, T_2, T_3 are found for the case $X \sim W(\tau)$ (that is Generalized Gammma distribution with $\lambda = \beta = \tau$) and write all values of μ_{kj} with *k,j=0,1,2* for this particular case:

$$\mu_{01} = \frac{\psi(1)}{\tau+1},$$

$$\mu_{02} = \frac{1}{(\tau+1)^2}[\psi'(1) + \psi^2(1)],$$

$$\mu_{10} = \frac{\Gamma(\frac{1}{\tau+1})}{\tau+1},$$

$$\mu_{11} = \frac{\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2},$$

$$\mu_{12} = \frac{\Gamma(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}[\psi'(\frac{\tau+2}{\tau+1}) + \psi^2(\frac{\tau+2}{\tau+1})],$$

$$\mu_{20} = \Gamma(\frac{\tau+3}{\tau+1}),$$

$$\mu_{21} = \frac{\Gamma(\frac{\tau+3}{\tau+1})}{\tau+1} \cdot \psi(\frac{\tau+3}{\tau+1}),$$

$$\mu_{22} = \frac{\Gamma(\frac{\tau+3}{\tau+1})}{(\tau+1)^2}[\psi'(\frac{\tau+3}{\tau+1}) + \psi^2(\frac{\tau+3}{\tau+1})]$$

Therefore one can calculate all mean values, variances, and covariance matrix of statistics T_1, T_2, and T_3:

$$\mu_1 = \mu_{10} = \frac{\Gamma(\frac{1}{\tau+1})}{\tau+1},$$

$$\mu_2 = \mu_{01} = \frac{\psi(1)}{\tau+1},$$

$$\mu_3 = \mu_{11} = \frac{\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}.$$

$$\sigma_1^2 = \frac{1}{n}(\mu_{20} - \mu_{10}^2) = \frac{1}{n}\left[\Gamma(\frac{\tau+3}{\tau+1}) - \left(\frac{\Gamma(\frac{1}{\tau+1})}{\tau+1}\right)^2\right] = \frac{1}{n}\left[\frac{2\Gamma(\frac{2}{\tau+1})}{\tau+1} - \left(\frac{\Gamma(\frac{1}{\tau+1})}{\tau+1}\right)^2\right],$$

$$\sigma_2^2 = \frac{1}{n}(\mu_{02} - \mu_{01}^2) = \frac{1}{\mathrm{n}}\left[\frac{1}{(\tau+1)^2}(\psi'(1) + \psi^2(1)) - \frac{\psi^2(1)}{(\tau+1)^2}\right] = \frac{1}{\mathrm{n}}\frac{\psi'(1)}{(\tau+1)^2},$$

$$\sigma_3^2 = \frac{1}{n}(\mu_{22} - \mu_{11}^2) = \frac{1}{n}\left[\frac{2\Gamma\left(\frac{2}{\tau+1}\right)\left(\psi'\left(\frac{\tau+3}{\tau+1}\right) + \psi^2\left(\frac{\tau+3}{\tau+1}\right)\right)}{(\tau+1)^3} - \frac{\left(\Gamma\left(\frac{1}{\tau+1}\right)\psi\left(\frac{\tau+2}{\tau+1}\right)\right)^2}{(\tau+1)^4}\right].$$

$$\lambda_{12} = \frac{1}{n}(\mu_{11} - \mu_{10}\cdot\mu_{01}) = \frac{1}{n}\left[\frac{\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2} - \frac{\frac{1}{\tau+1}}{\tau+1}\cdot\frac{\psi(1)}{\tau+1}\right]$$

$$= \frac{1}{n}\left[\frac{\Gamma(\frac{1}{\tau+1})}{(\tau+1)^2}\left(\psi(\frac{\tau+2}{\tau+1}) - \psi(1)\right)\right],$$

$$\lambda_{13} = \frac{1}{n}(\mu_{21} - \mu_{10}\mu_{11}) = \frac{1}{n}\left[\frac{\Gamma(\frac{\tau+3}{\tau+1})}{\tau+1}\psi(\frac{\tau+3}{\tau+1}) - \frac{\Gamma(\frac{1}{\tau+1})}{\tau+1}\frac{\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}\right]$$

$$= \frac{1}{n}\left[\frac{2\Gamma\left(\frac{2}{\tau+1}\right)\psi\left(\frac{\tau+3}{\tau+1}\right)}{(\tau+1)^2} - \frac{\left(\Gamma\left(\frac{1}{\tau+1}\right)\right)^2\psi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau+1)^3}\right],$$

$$\lambda_{23} = \frac{1}{n}(\mu_{12} - \mu_{01}\cdot\mu_{11})$$

$$= \frac{1}{n}\left[\frac{\Gamma(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}\left(\psi'(\frac{\tau+2}{\tau+1}) + \psi^2(\frac{\tau+2}{\tau+1})\right) - \frac{\psi(1)}{\tau+1}\cdot\frac{\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}\right]$$

$$= \frac{1}{n}\left(\frac{\Gamma\left(\frac{1}{\tau+1}\right)\left[\psi'\left(\frac{\tau+2}{\tau+1}\right) + \psi^2\left(\frac{\tau+2}{\tau+1}\right)\right] - \psi(1)\Gamma\left(\frac{1}{\tau+1}\right)\psi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau+1)^3}\right).$$

To summarize, the following for the case $X \sim W(\tau)$ are obtained.

Mean values:

1. $\mu_1 = \frac{\Gamma(\frac{1}{\tau+1})}{\tau+1}$,

2. $\mu_2 = \frac{\psi(1)}{\tau+1}$,

3. $\mu_3 = \frac{\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}$.

Variances:

1. $\sigma_1^2 = \frac{1}{n}\left[\frac{2\Gamma(\frac{2}{\tau+1})}{\tau+1} - \left(\frac{\Gamma(\frac{1}{\tau+1})}{\tau+1}\right)^2\right]$,

2. $\sigma_2^2{=}\frac{1}{n}\frac{\psi'(1)}{(\tau+1)^2}$,

3. $\sigma_3^2 = \frac{1}{n}\left[\frac{2\Gamma\left(\frac{2}{\tau+1}\right)(\psi'\left(\frac{\tau+3}{\tau+1}\right)+\psi^2\left(\frac{\tau+3}{\tau+1}\right))}{(\tau+1)^3} - \frac{\left(\Gamma\left(\frac{1}{\tau+1}\right)\psi\left(\frac{\tau+2}{\tau+1}\right)\right)^2}{(\tau+1)^4}\right]$.

Covariances:

1. $\lambda_{12} = \frac{1}{n}\left(\frac{\Gamma(\frac{1}{\tau+1})}{(\tau+1)^2}\left(\psi(\frac{\tau+2}{\tau+1}) - \psi(1)\right)\right)$,

2. $\lambda_{13} = \frac{1}{n}\left(\frac{2\Gamma\left(\frac{2}{\tau+1}\right)\psi\left(\frac{\tau+3}{\tau+1}\right)}{(\tau+1)^2} - \frac{\left(\Gamma\left(\frac{1}{\tau+1}\right)\right)^2\psi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau+1)^3}\right)$,

3. $\lambda_{23} = \frac{1}{n}\left(\frac{\Gamma\left(\frac{1}{\tau+1}\right)\left[\psi'\left(\frac{\tau+2}{\tau+1}\right)+\psi^2\left(\frac{\tau+2}{\tau+1}\right)\right]-\psi(1)\Gamma\left(\frac{1}{\tau+1}\right)\psi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau+1)^3}\right)$.

In the same way one can write out the mean values, variances, and covariances of statistics $T_{1,}$ $T_{2,}$ T_3 for the case $X \sim G(\lambda)$ (that is Generalized Gammma distribution with $\beta = 0$) and write all values of μ_{kj} with *k,j=0,1,2* for this particular case:

$$\mu_{01} = \psi\left(\lambda+1\right),$$

$$\mu_{02} = \psi'(\lambda+1) + \psi^2(\lambda+1),$$

$$\mu_{10} = \lambda+1,$$

$$\mu_{11} = \left(\lambda+1\right)\psi\left(\lambda+2\right),$$

$$\mu_{12} = (\lambda+1)[\psi'(\lambda+2) + \psi^2(\lambda+2)],$$

$$\mu_{20} = \frac{\Gamma(\lambda+3)}{\Gamma(\lambda+1)} = (\lambda+1)(\lambda+2),$$

$$\mu_{21} = (\lambda+2)(\lambda+1)\psi(\lambda+3),$$

$$\mu_{22} = (\lambda+2)(\lambda+1)[\psi'(\lambda+3) + \psi^2(\lambda+3)].$$

Therefore all mean values, variances and covariance matrix of statistics T_1, T_2, and T_3 can be calculated:

$$\mu_1 = \mu_{10} = \lambda+1,$$

$$\mu_2 = \mu_{01} = \psi\left(\lambda+1\right),$$

$$\mu_3 = \mu_{11} = \left(\lambda+1\right)\psi\left(\lambda+2\right).$$

$$\sigma_1^2 = \frac{1}{n}(\mu_{20} - \mu_{10}^2) = \frac{1}{n}\left(\lambda+1\right),$$

$$\sigma_2^2 = \frac{1}{n}(\mu_{02} - \mu_{01}^2) = \frac{1}{n}\psi'\left(\lambda+1\right),$$

$$\sigma_3^2 = \frac{1}{n}(\mu_{22} - \mu_{11}^2) = \frac{1}{n}\left(\lambda+1\right)\left[\left(\lambda+2\right)\left(\psi'\left(\lambda+3\right) + \psi^2\left(\lambda+3\right)\right) - (\lambda+1)\psi^2(\lambda+1)\right].$$

$$\lambda_{12} = \frac{1}{n}(\mu_{11} - \mu_{10}\mu_{01}) = \frac{1}{n},$$

$$\lambda_{13} = \frac{1}{n}(\mu_{21} - \mu_{10}\mu_{11}) = \frac{1}{n}(\lambda+1)(\psi(\lambda+2)+1),$$

$$\lambda_{23} = \frac{1}{n}(\mu_{12} - \mu_{01}\mu_{11}) = \frac{1}{n}(\lambda+1)\left(\psi'(\lambda+2)+\psi^2(\lambda+2)-\psi(\lambda+1)\psi(\lambda+2)\right)$$

To summarize, the following results are obtained for the case $X \sim G(\lambda)$.

Mean values:

1. $\mu_1 = \lambda + 1$,
2. $\mu_2 = \psi(\lambda+1)$,
3. $\mu_3 = (\lambda+1)\psi(\lambda+2)$.

Variances:

1. $\sigma_1^2 = \frac{1}{n}(\lambda+1)$,
2. $\sigma_2^2 = \frac{1}{n}\psi'(\lambda+1)$,
3. $\sigma_3^2 = \frac{1}{n}(\lambda+1)[(\lambda+2)(\psi'(\lambda+3)+\psi^2(\lambda+3))-(\lambda+1)\psi^2(\lambda+1)]$.

Covariances:

1. $\lambda_{12} = \frac{1}{n}$,
2. $\lambda_{13} = \frac{1}{n}(\lambda+1)(\psi(\lambda+2)+1)$,
3. $\lambda_{23} = \frac{1}{n}(\lambda+1)(\psi'(\lambda+2)+\psi^2(\lambda+2)-\psi(\lambda+1)\psi(\lambda+2))$.

3.2 Calculation of the Asymptotic Mean and Variance of the Test Statistic by Delta Method

In this section the first order Taylor polynomial from a statistical point of view and the Delta method [?] is reviewed and then they were put to use in order to calculate the asymptotic mean and variance of the test statistic.

Taylor theorem: Let $T_1, T_2, ..., T_k$ be random variables with $\mu_1, \mu_2, ..., \mu_k$, and define $T = (T_1, ..., T_k)$ and $\mu = (\mu_1, ..., \mu_k)$. Suppose $g(t)$ is a differentiable function for which we want to estimate mean and variance. We define the partial derivatives of $g(t)$ with respect to T_i at μ as $g_i'(\mu)$. The first-order Taylor series expansion of g about μ is: $g(t) = g(\mu) + \sum g_i'(\mu)(t_i - \mu_i) + Remainder$

by taking expectation and variance of both side we have:

$$\begin{aligned} &Eg(T) = g(\mu) \\ &\mathrm{var(T)} = E[(g(T) - g(\mu))^2] = E[(\sum_{i=1}^{k} g_i'(\mu)(T_i - \mu_i))^2] \\ &\qquad = \sum_{i=1}^{k} g'(\mu)^2 \mathrm{VarT_i} + 2\sum_{\mathrm{i>j}} \mathrm{g_i'}(\mu)\mathrm{g_j'}(\mu)\mathrm{Cov(T_i, T_j)} \end{aligned}$$

Delta method: Delta Method is in fact a generalized Central Limit Theorem. Let T_n be a sequence of random variables that satisfies $\sqrt{n}(T_n - \theta) \to N(0, \sigma^2)$ in distribution and a specific value of θ , suppose that $g'(\theta)$ exists and is not 0. then $\sqrt{n}(g(T_n) - g(\theta)) \to N(0, \sigma^2 g'^2(\theta))$.

Now Taylor theorem and Delta method is put into use in order to calculate the mean and variance values of the test statistic.

$\mu_{kj} = \mu_{kj}(\lambda, \beta)$ are functions of parameters λ and β. In continue the Maclaurin expansion of the statistic $\mathfrak{T}$ up to the linear terms is written. In general it has the following form:

$$\mathfrak{T} = c_1(T_2 - \ln T_1) + c_2\left(\ln T_1 - \frac{T_3}{T_1}\right) = c_1T_2 + (c_2 - c_1)\ln T_1 - c_2\frac{T_3}{T_1} =$$

$$c_1(\mu_{01} - \ln\mu_{10}) + c_2\left(\ln\mu_{10} - \frac{\mu_{11}}{\mu_{10}}\right) + (T_1 - \mu_{10})\left[(c_2 - c_1)\frac{1}{\mu_{10}} + c_2\frac{\mu_{11}}{\mu_{10}^2}\right] +$$

$$c_1(T_2 - \mu_{01}) - \frac{c_2}{\mu_{10}}(T_3 - \mu_{11}) + R_n,$$

where, because of the fact that all moments of statistics $T_i,\ i = 1, 2, 3$ are finite and nonzero, the random variable $\sqrt{n}R_n$ converges in probability to zero. This guarantees the asymptotic normality of the statistic $\mathfrak{T}$ with mean

$$\mu_{\mathfrak{T}} = E_{\mathfrak{T}} = c_1\mu_{01} + (c_2 - c_1)\ln\mu_{10} - c_2\frac{\mu_{11}}{\mu_{10}}$$

and variance $D\mathfrak{T} = \sigma_T^2/n$ where

$$\sigma_T^2 = \left[(c_2 - c_1)\frac{1}{\mu_{10}} + c_2\frac{\mu_{11}}{\mu_{10}^2}\right]^2\sigma_1^2 + c_1^2\sigma_2^2 + \frac{c_2^2}{\mu_{10}^2}\sigma_3^2 +$$

$$2\left[(c_2 - c_1)\frac{1}{\mu_{10}} + c_2\frac{\mu_{11}}{\mu_{10}^2}\right]c_1\lambda_{12} - 2\left[(c_2 - c_1)\frac{1}{\mu_{10}} + c_2\frac{\mu_{11}}{\mu_{10}^2}\right]\frac{c_2}{\mu_{10}}\lambda_{13} -$$

$$2c_1c_2\frac{1}{\mu_{10}}\lambda_{23}.$$

In Lemma 4, mean and variance values of the test statistic are calculated for each distribution separately. Here well known formulas for Gamma function and its derivatives is used: $\psi(1) = -\gamma,\ \psi(x+1) = \psi(x) + 1/x$.

Lemma 4

Suppose X has Generalized Gamma Distribution. The asymptotic mean and variance of the test statistic is as following for three distributions derived from generalized Gamma distribution:

Gamma distribution (**$\beta = 0$**)

Mean of the statistic:

$$\mu_{\mathfrak{T}}(\lambda, 0) = \mu(\lambda) = c_1\psi(1+\lambda) + (c_2 - c_1)\ln(1+\lambda) - c_2\psi(2+\lambda) =$$

$$\mu_0 + \lambda\left[\frac{\pi^2}{6}(c_1 - c_2) + 2c_2 - c_1\right] + o(\lambda^2)$$

Variance of the statistic:

$$\sigma^2_{\mathfrak{T}}(\lambda, 0) = \frac{1}{n}\Bigg[\frac{(c_2(\psi(\lambda+2)+1) - c_1)^2}{\lambda+1} + c_1^2\psi'(\lambda+1) +$$

$$\frac{c_2^2}{\lambda+1}(\lambda+2)\left(\psi'(\lambda+3) + \psi^2(\lambda+3)\right) - (\lambda+1)\psi^2(\lambda+1) +$$

$$2c_1\left(\frac{c_2(\psi(\lambda+2)+1) - c_1}{\lambda+1}\right) - 2c_2(\psi(\lambda+2)+1)\left(\frac{c_2(\psi(\lambda+2)+1) - c_1}{\lambda+1}\right)$$

$$-2c_1c_2\left(\psi'(\lambda+2) + \psi^2(\lambda+2) - \psi(\lambda+1)\psi(\lambda+2)\right)\Bigg]$$

Variance of $\mathfrak{T}$ *does not need to be expanded.* σ_0 *can be find simply by putting* $\lambda = 0$ *in the general formula.* $\mu_{\mathfrak{T}}$, *is expanded because it will be needed it in future calculations.*

Weibull distribution ($\lambda = \mu = \tau$)

Mean of the statistic:

$$\mu_{\mathfrak{T}}(\tau,\tau) = \mu(\tau) = c_1 \frac{\psi(1)}{1+\tau} + (c_2 - c_1)\ln\left[\frac{1}{1+\tau}\,\Gamma\left(\frac{1}{1+\tau}\right)\right] -$$

$$c_2\left[\frac{1}{1+\tau}\,\psi\left(\frac{2+\tau}{1+\tau}\right)\right] =$$

$$\mu_0 + \tau\left[c_2\left(\frac{\pi^2}{6} - 1\right) + c_1\right] + O(\tau^2).$$

Variance of the statistic:

$$\sigma^2_{\mathfrak{T}}(\tau,0) = \frac{1}{n}(p_1 + p_2 + p_3 + p_4 + p_5)$$

where

$$p_1 = \left((c_2 - c_1)\frac{\tau+1}{\Gamma(\frac{1}{\tau+1})} + c_2\frac{\psi\left(\frac{\tau+2}{\tau+1}\right)}{\Gamma\left(\frac{1}{\tau+1}\right)}\right)^2 \left(\frac{2\Gamma\left(\frac{2}{\tau+1}\right)}{\tau+1} - (\frac{\Gamma(\frac{1}{\tau+1})}{\tau+1})^2\right)$$

$$p_2 = c_1^2\frac{\psi'(1)}{(\tau+1)^2} + c_2^2\frac{(\tau+1)^2}{\Gamma^2\left(\frac{1}{\tau+1}\right)}\left(\frac{2\Gamma(\frac{2}{\tau+1})\left(\psi'(\frac{\tau+3}{\tau+1}) + \psi^2\left(\frac{\tau+3}{\tau+1}\right)\right)}{(\tau+1)^3} - \frac{\Gamma^2(\frac{1}{\tau+1})\psi^2(\frac{\tau+2}{\tau+1})}{(\tau+1)^4}\right)$$

$$p_3 = 2c_1\left(\frac{(c_2 - c_1)(\tau+1) + c_2\psi(\frac{\tau+2}{\tau+1})}{\Gamma(\frac{1}{\tau+1})}\right)\left(\frac{\Gamma(\frac{1}{\tau+1})}{(\tau+1)^2}(\psi(\frac{\tau+2}{\tau+1}) - \psi(1)\right)$$

$$p_4 = -2\left(\frac{(c_2 - c_1)(\tau+1) + c_2\psi(\frac{\tau+2}{\tau+1})}{\Gamma(\frac{1}{\tau+1})}\right)\frac{c_2(\tau+1)}{\Gamma(\frac{1}{\tau+1})}\left(\frac{2\Gamma(\frac{2}{\tau+1})\psi(\frac{\tau+3}{\tau+1})}{(\tau+1)^2} - \frac{\Gamma^2(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^3}\right)$$

$$p_5 = -2c_1c_2\frac{1}{\Gamma(\frac{1}{\tau+1})}\left(\frac{\Gamma(\frac{1}{\tau+1})(\psi'(\frac{\tau+2}{\tau+1}) + \psi^2(\frac{\tau+2}{\tau+1}) - \psi(1)\Gamma(\frac{1}{\tau+1})\psi(\frac{\tau+2}{\tau+1})}{(\tau+1)^2}\right)$$

Variance $\sigma_T^2 = \sigma^2(\tau)$ need not to be expanded.

Generalized Exponential Distribution ($\beta = 0$)

Mean of the statistic

$$\mu_{\mathfrak{T}}(0,\beta) = \mu(\beta) = c_1\frac{1}{1+\beta}\,\psi\left(\frac{1}{1+\beta}\right) +$$

$$(c_2 - c_1)\left[\ln\Gamma\left(\frac{2}{1+\beta}\right) - \ln\Gamma\left(\frac{1}{1+\beta}\right)\right] - c_2\frac{1}{1+\beta}\,\psi\left(\frac{2}{1+\beta}\right) =$$

$$\mu_0 + \beta\left[c_1\left(2 - \frac{\pi^2}{6}\right) + c_2\left(\frac{\pi^2}{3} - 3\right)\right] + O(\beta^2).$$

.

Variance of the statistic:

$$\sigma_{\mathfrak{T}}^2(\beta,0) = \frac{1}{n}(p_1 + p_2 + p_3 + p_4 + p_5 + p_6)$$

where

$$p_1 = \left(\frac{(c_2 - c_1)\Gamma(\frac{1}{1+\beta}) + c_2\frac{1}{1+\beta}\psi(\frac{2}{1+\beta})\Gamma(\frac{1}{1+\beta})}{\Gamma(\frac{2}{1+\beta})}\right)^2\left(\frac{\Gamma(\frac{3}{1+\beta})}{\Gamma(\frac{1}{1+\beta})} - (\frac{\Gamma(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})})^2\right)$$

$$p_2 = c_1^2\left(\frac{1}{(1+\beta)^2}\psi'(\frac{1}{1+\beta})\right)$$

$$p_3 = \frac{c_2^2\Gamma(\frac{1}{1+\beta})}{(1+\beta)^2\Gamma^2(\frac{2}{1+\beta})}\left(\Gamma(\frac{3}{1+\beta})(\psi'(\frac{3}{1+\beta})+\psi^2(\frac{3}{1+\beta}))-\frac{\Gamma^2(\frac{2}{1+\beta})\psi^2(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}\right)$$

$$p_4 = 2c_1\left(\frac{(c_2-c_1)\Gamma(\frac{1}{1+\beta})}{\Gamma(\frac{2}{1+\beta})}+c_2\frac{\psi(\frac{2}{1+\beta})\Gamma(\frac{1}{1+\beta})}{(1+\beta)\Gamma(\frac{2}{1+\beta})}\right)\left(\frac{\Gamma(\frac{2}{1+\beta})}{(1+\beta)\Gamma(\frac{1}{1+\beta})}(\psi(\frac{2}{1+\beta})-\psi(\frac{1}{1+\beta}))\right)$$

$$p_5 = -2\left[(c_2-c_1)\frac{\Gamma(\frac{1}{1+\beta})}{\Gamma(\frac{2}{1+\beta})}+c^2\frac{\psi(\frac{2}{1+\beta})\Gamma(\frac{1}{1+\beta})}{(1+\beta)\Gamma(\frac{2}{1+\beta})}\right]\frac{c_2}{\Gamma(\frac{2}{1+\beta})}$$
$$\frac{1}{1+\beta}\left(\Gamma(\frac{3}{1+\beta})\psi(\frac{3}{1+\beta})-\frac{\Gamma^2(\frac{2}{1+\beta})\psi(\frac{2}{1+\beta})}{\Gamma(\frac{1}{1+\beta})}\right)\Bigg]$$

$$p_6 = -2c_1c_2\frac{1}{(1+\beta)^2}\left(\psi'(\frac{2}{1+\beta})+\psi^2(\frac{2}{1+\beta})-\psi(\frac{1}{1+\beta})\psi(\frac{2}{1+\beta})\right)$$

In the next chapter all the derived results will be used to construct the actual test.

Chapter 4

Construction of Locally Most Powerful Tests for Choosing the Reliability Models

For a construction of locally most powerful tests, obviously in the class of all tests that have the form $\mathfrak{T} > c$ first mean values and Maclaurine expansions of the mean values of $\mathfrak{T}$ were found separately for each of three distributions up to the linear terms. In this chapter the tests will be specified by finding c_1 and c_2 for each one of the three Reliability models separately.

4.1 Test Rejection Region

Here the rejection region for the given significance level α should be defined. As usual, the rejection region $\mathfrak{T} > c$ where c is defined from $p(\mathfrak{T} > c) = \alpha$. From Delta method one can write:

$p(\mathfrak{T} > c) = \alpha \simeq p\left(\frac{\mathfrak{T}-\mu_0}{\frac{\sigma_0}{\sqrt{n}}} > \frac{c-\mu_0}{\frac{\sigma_0}{\sqrt{n}}}\right)$, that is $c = \mu_0 + \Phi^{-1}(1-\alpha)\frac{\sigma_0}{\sqrt{n}}$

where μ_0 and σ_0 are the mean and variance value of $\mathfrak{T}$ when the parameter of the distribution is set to zero.

Now denote θ as a generic parameter. Later, one of the parameter β, λ, or τ will be used instead of θ. Using the above derived critical value and Normal approximation:

$\alpha = p_\theta(\mathfrak{T} > c(\alpha)) = 1 - \Phi\left(\frac{c(\alpha)-\mu_\theta}{\sigma_\theta}\sqrt{n}\right) = 1 - \Phi\left(\frac{\mu_0-\mu_\theta}{\sigma_\theta}\sqrt{n} + \Phi^{-1}(1-\alpha)\frac{\sigma_0}{\sigma_\theta}\right)$

where μ_θ and σ_θ are asymptotic mean and variance that were calculated in the previous section.

4.2 Construction of the Tests

Now all left to do is specifying c_1 and c_2. These two constants should be chosen such that the probability of type I error is less than α and the power of the test is greater than α (condition of unbiasedness of the test). In the following sections these two constants will be found for testing three main Reliability models.

4.2.1 Most Locally powerful Test for Aging and Deterioration Model

Here two cases of hypothesis testing are considered.

Case 1

H_0: the sample is obtained from Gamma distribution

with the alternative

H_1: the sample is taken from Weibull distribution.

For this case the test $\mathfrak{T} > C$ with the critical constant C, the asymptotic probability of the type I error is

$$\alpha(\lambda) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\lambda)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\lambda)}{\sigma(\lambda)}\sqrt{n}\right).$$

The power of the test is

$$m(\tau) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\tau)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\tau)}{\sigma(\tau)}\sqrt{n}\right).$$

In order for the probability of type I error to be less than the given level α it is necessary that for all $\lambda > 0$ the difference $\mu_0 - \mu(\lambda)$ is positive. Asymptotic ($\lambda \to 0$) of this difference is

$$\mu_0 - \mu(\lambda) = -\lambda\left[c_1\left(\frac{\pi^2}{6} - 1\right) + c_2\left(2 - \frac{\pi^2}{6}\right)\right],$$

and if c_1 and c_2 are chosen such that the right hand side is positive, then the probability of the type I error will be less than α at least in a neighborhood of the point $\lambda = 0$. Therefore it required that

$$c_1\left(\frac{\pi^2}{6} - 1\right) + c_2\left(2 - \frac{\pi^2}{6}\right) < 0.$$

Similarly, the power of the test should be greater than α (the condition of unbiasedness of the test). Hence the difference $\mu_0 - \mu(\tau)$ should be negative. Therefore in some neighborhood of the point $\tau = 0$ the inequality

$$\mu_0 - \mu(\tau) \sim c_1 + c_2 \left(\frac{\pi^2}{6} - 1 \right) > 0$$

should be true.

One of $c_i, i = 1, 2$, can be taken as plus or minus 1. If we rewrite the inequalities obtained in the form

$$-c_1 \frac{1}{\pi^2/6 - 1} < c_2 < -c_1 \frac{\pi^2/6 - 1}{2 - \pi^2/6},$$

then it is simple to see that the only possibility to satisfy these inequalities and to make the power locally the biggest, is to let

$$c_1 = -1, \qquad c_2 = \frac{\pi^2/6 - 1}{2 - \pi^2/6} = \frac{\pi^2 - 6}{12 - \pi^2}.$$

If we substitute these constants into $\alpha(\lambda)$, then the linear term for the difference $\mu_0 - \mu(\lambda)$ becomes zero, and the whole difference behaves for $\lambda \to 0$ as $O(\lambda^2)$.

Case 2

H_0: the sample is obtained from Gamma distribution

with the alternative

H_1: the sample is taken from Exponential distribution.

The type I error probability is

$$\alpha(\lambda) = 1 - \Phi \left(\frac{\sigma_0}{\sigma(\lambda)} \Phi^{-1}(1 - \alpha) + \frac{\mu_0 - \mu(\lambda)}{\sigma(\lambda)} \sqrt{n} \right).$$

and the power of the test is:

$$m(\tau) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\beta)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\beta)}{\sigma(\beta)}\sqrt{n}\right).$$

In order for the probability of the type I error to be less than the given level α and the power of the test be greater than α the following inequalities are concluded:

$\mu_0 - \mu(\lambda)$ should be positive therefore

$$c_1\left(\frac{\pi^2}{6} - 1\right) + c_2\left(2 - \frac{\pi^2}{6}\right) < 0.$$

$\mu_0 - \mu(\beta)$ should be negative. Therefore in some neighborhood of the point $\tau = 0$ the inequality

$$c_1\left(2 - \frac{\pi^2}{6}\right) + c_2\left(\frac{\pi^2}{3} - 3\right) > 0$$

should be true. One of $c_i, i = 1, 2$, can be taken as plus or minus 1. If we rewrite the inequalities obtained in the form

$$-c_1\frac{2 - \pi^2/6}{\pi^2/3 - 3} < c_2 < -c_1\frac{\pi^2/6 - 1}{2 - \pi^2/6},$$

then it is simple to see that the only possibility to satisfy these inequalities and to make the power locally the biggest, is to let

$$c_1 = -1, \qquad c_2 = \frac{\pi^2/6 - 1}{2 - \pi^2/6} = \frac{\pi^2 - 6}{12 - \pi^2}.$$

Tests for two other Reliability models can be obtained in the exactly same way and is discussed in the following.

4.2.2 Most Locally powerful Test for Defect at the Birth Model

Here two cases of hypothesis testing are considered.

Case 1

H_0: the sample is obtained from Exponential distribution

with the alternative

H_1: the sample is taken from Gamma distribution.

In this case the asymptotic ($n \to \infty$) probability of the type I error ($\lambda \to \infty$)is:

$$\alpha(\beta) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\beta)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\beta)}{\sigma(\beta)}\sqrt{n}\right)$$

and asymptotic ($n \to \infty$) power ($\lambda \to \infty$) of this test:

$$m(\lambda) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\lambda)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\lambda)}{\sigma(\lambda)}\sqrt{n}\right)$$

In order for the probability of the type I error to be less than the given level α and the power of the test be greater than α we have the following inequalities:

$\mu_0 - \mu(\beta)$ should be positive therefore

$$c_1\left(2 - \frac{\pi^2}{6}\right) + c_2\left(\frac{\pi^2}{3} - 3\right) < 0.$$

$\mu_0 - \mu(\lambda)$ should be negative. Therefore in some neighborhood of the point $\lambda = 0$ the inequality

$$c_1\left(\frac{\pi^2}{6} - 1\right) + c_2\left(2 - \frac{\pi^2}{6}\right) > 0$$

should be true. One of $c_i, i = 1, 2$, can be taken as plus or minus 1. If we rewrite the inequalities obtained in the form

$$-c_1 \frac{\pi^2/6 - 1}{2 - \pi^2/6} < c_2 < -c_1 \frac{2 - \pi^2/6}{\pi^2/3 - 3},$$

then it is simple to see that the only possibility to satisfy these inequalities and to make the power locally the biggest, is to let

$$c_1 = 1, \qquad c_2 = -\frac{2 - \pi^2/6}{\pi^2/3 - 3} = -\frac{\pi^2 - 12}{2(\pi^2 - 9)}.$$

Tests for two other Reliability models can be obtained in the exactly same way and is discussed in the following:

Case 2

H_0: the sample is obtained from Exponential distribution

with the alternative

H_1: the sample is taken from Weibull distribution.

In this case the asymptotic ($n \to \infty$) probability of the type I error ($\beta \to \infty$)is:

$$\alpha(\beta) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\beta)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\beta)}{\sigma(\beta)}\sqrt{n}\right)$$

and asymptotic ($n \to \infty$) power ($\tau \to \infty$) of this test:

$$m(\tau) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\tau)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\tau)}{\sigma(\tau)}\sqrt{n}\right)$$

In order for the probability of the type I error to be less than the given level α and the power of the test be greater than α we have the following inequalities:

$\mu_0 - \mu(\beta)$ should be positive therefore

$$c_1\left(2 - \frac{\pi^2}{6}\right) + c_2\left(\frac{\pi^2}{3} - 3\right) < 0.$$

$\mu_0 - \mu(\lambda)$ should be negative. Therefore in some neighborhood of the point $\lambda = 0$ the inequality

$$c_1 + c_2\left(\frac{\pi^2}{6} - 1\right) > 0$$

should be true. One of $c_i, i = 1, 2$, can be taken as plus or minus 1. If we rewrite the inequalities obtained in the form

$$-c_1\frac{1}{\pi^2/6 - 1} < c_2 < -c_1\frac{2 - \pi^2/6}{\pi^2/3 - 3},$$

then it is simple to see that the only possibility to satisfy these inequalities and to make the power locally the biggest, is to let

$$c_1 = 1, \qquad c_2 = \frac{2 - \pi^2/6}{\pi^2/3 - 3} = \frac{\pi^2 - 12}{2(\pi^2 - 9)}.$$

4.2.3 Most Locally powerful Test test for Weak Link Model

Here two cases of hypothesis testing are considered.

Case 1

H_0: the sample is obtained from Weibull distribution

with the alternative

H_1: the sample is taken from Gamma distribution.

In this case the asymptotic ($n \to \infty$) probability of type I error ($\tau \to \infty$) is:

$$\alpha(\tau) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\tau)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\tau)}{\sigma(\tau)}\sqrt{n}\right)$$

and asymptotic ($n \to \infty$) power ($\lambda \to \infty$) of this test:

$$m(\lambda) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\lambda)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\lambda)}{\sigma(\lambda)}\sqrt{n}\right)$$

In order for the probability of type I error to be less than the given level α and the power of the test be greater than α we have the following inequalities:

$\mu_0 - \mu(\tau)$ should be positive therefore

$$c_1 + c_2\left(\frac{\pi^2}{6} - 1\right) < 0.$$

$\mu_0 - \mu(\lambda)$ should be negative. Therefore in some neighborhood of the point $\lambda = 0$ the inequality

$$c_1\left(\frac{\pi^2}{6} - 1\right) + c_2\left(2 - \frac{\pi^2}{6}\right) > 0$$

should be true. One of $c_i, i = 1, 2$, can be taken as plus or minus 1. If we rewrite the inequalities obtained in the form

$$-c_1\frac{\pi^2/6 - 1}{2 - \pi^2/6} < c_2 < -c_1\frac{1}{\pi^2/6 - 1},$$

then it is simple to see that the only possibility to satisfy these inequalities and to make the power locally the biggest, is to let

$$c_1 = 1, \qquad c_2 = \frac{-6}{\pi^2 - 6}.$$

Case 2

H_0: the sample is obtained from Weibull distribution

with the alternative

H_1: the sample is taken from Exponential distribution.

In this case the asymptotic ($n \to \infty$) probability of type I error ($\tau \to \infty$)is:

$$\alpha(\tau) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\tau)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\tau)}{\sigma(\tau)}\sqrt{n}\right)$$

and asymptotic ($n \to \infty$) power ($\beta \to \infty$) of this test:

$$m(\beta) = 1 - \Phi\left(\frac{\sigma_0}{\sigma(\beta)}\Phi^{-1}(1-\alpha) + \frac{\mu_0 - \mu(\beta)}{\sigma(\beta)}\sqrt{n}\right)$$

In order for the probability of type I error to be less than the given level α and the power of the test be greater than α we have the following inequalities:

$\mu_0 - \mu(\tau)$ should be positive therefore

$$c_1 + c_2\left(\frac{\pi^2}{6} - 1\right) < 0.$$

$\mu_0 - \mu(\beta)$ should be negative. Therefore in some neighborhood of the point $\beta = 0$ the inequality

$$c_1\left(2 - \frac{\pi^2}{6}\right) + c_2\left(\frac{\pi^2}{3} - 3\right) > 0$$

should be true. One of $c_i, i = 1, 2$, can be taken as plus or minus 1. If we rewrite the inequalities obtained in the form

$$-c_1\frac{2 - \pi^2/6}{\pi^2/3 - 3} < c_2 < -c_1\frac{1}{\pi^2/6 - 1},$$

then it is simple to see that the only possibility to satisfy these inequalities and to make the power locally the biggest, is to let

$$c_1 = -1, \qquad c_2 = \frac{6}{\pi^2 - 6}.$$

Tests are constructed by the Delta method for the asymptotic distribution of the test statistic. Note that an application of the commonly used method of testing non nested hypothesis (see, for example, [**?**]) does not allow to calculate the constants on the test statistic in the closed form.

Chapter 5

Simulation Studies

Simulation studies use computer programming on generated data sets in order to assess the performance of different statistical methods [**?**]. In this chapter computer simulation will be used to assess the statistical tests that were presented for hypothesis testing in previous chapters.

Here sample sizes $n = 100$ and significance level $\alpha = 0.05$ are considered. There are 6 values for each of the three parameters λ, τ, and β including 0.5, 1, 1.5, 2, 2.5, and 3. Number of simulations ,N, considered $10^4 = 10,000$. Statistic $\mathfrak{T}$ is the one that is used for hypothesis testing.

Here are the three steps that are followed for the first test. The procedure for other five tests is very similar.

$H_0 : X \sim G(\lambda)$

$H_1 : X \sim W(\tau)$.

Step 1. Simulate N times a sample of size n from Exponential distribution with parameter 1. For each sample calculate the statistic $\mathfrak{T}$. Take 95th percentile for these statistic values. That is, arrange the values of the statistic from smaller to bigger $(\mathfrak{T}_{(1)} \leq \mathfrak{T}_{(2)} \leq \ldots \leq \mathfrak{T}_{(N)})$ and find the value at the place $(1-\alpha) \times N$, this will give us the constant for the rejection region

$$C(\alpha) = \mathfrak{T}_{(\alpha N)}.$$

Step 2. Fix λ and simulate N times a sample of size n from Gamma distribution with parameter λ. For each sample calculate the statistic $\mathfrak{T}$. Calculate the number $A(\lambda)$ of the statistic $\mathfrak{T}_i, 1 \leq i \leq N$ which are greater than $C(\alpha)$. For each λ, calculate type I error of the test as

$$\alpha(\lambda) = A(\lambda)/N.$$

Step 3. Fix τ and simulate N times a sample of size n from Weibull distribution with parameter τ. For each sample calculate the statistic $\mathfrak{T}$. Calculate the number $M(\tau)$ of the statistic $\mathfrak{T}_i, 1 \leq i \leq N$ which are greater than $C(\alpha)$. For each τ calculate power of the test as

$$m(\tau) = 1 - (M(\tau)/N).$$

The simulation process for other five test is very similar that the procedure are not included here and focus on the results for all 6 tests for parameters 0.5, 1, 2, 2.5, and 3. Table 5 lists all type I errors for the six tests and table 5.2 lists all power values for each of the six tests:

	Distribution parameter					
	0.5	1	1.5	2	2.5	3
Gamma vs Weibull	0.015	0.0277	0.1061	0.3185	0.6321	0.8582
Weibull vs Gamma	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Gamma vs Exponential	0.0154	0.0317	0.1187	0.3479	0.6519	0.8708
Exponential vs Gamma	0.0518	0.0545	0.0548	0.0495	0.0571	0.0508
Weibull vs Exponential	0.0015	0.0003	0.0000	0.0000	0.0000	0.0000
Exponential vs Weibull	0.0466	0.0499	0.0465	0.0488	0.0468	0.0457

Table 5.1: Simulation results, type I errors

	Distribution parameter					
	0.5	1	1.5	2	2.5	3
Gamma vs Weibull	0.8403	0.1080	0.0001	0.0000	0.0000	0.0000
Weibull vs Gamma	0.9803	0.9977	0.9998	1.0000	1.0000	1.0000
Gamma vs Exponnetial	0.9541	0.9435	0.9496	0.9468	0.9492	0.9470
Exponential vs Gamma	0.7063	0.4225	0.2534	0.189	0.1024	0.0729
Weibull vs Exponential	0.9495	0.9476	0.9480	0.9514	0.9501	0.9501
Exponential vs Weibull	0.8526	0.7795	0.8156	0.9345	0.9959	1.0000

Table 5.2: Simulation results, tests powers

From the simulation results, one can conclude table 5.3 and 5.4 when the parameter of the distribution increases. As we can see, the results of simulation shows that all six tests are locally most powerful.

Next chapter,the results and conclusions will be summarized.

Test	Simulation Result	Conclusion
Gamma vs Weibull	• Type I error increases monotonically (for parameters < 1.5 , type error I is less than the assigned significance level) • Power decreases monotonically (for all parameters $>$ 1, power is less than 0.1)	• For parameters around 0.5, the test is powerful with first type error less than the assigned significance level
Weibull vs Gamma	• Type I error for all parameters is less than the assigned significance level • Power of the test increases monotonically (for all parameters > 0.98)	• For all parameters, the test is powerful with type I error less than the assigned significance level
Gamma vs Exponential	• Type I error increases monotonically (for parameters less than 1.5, type I error is less than the assigned significance level) • Power of the test slowly decrease and increase (for all parameters greater than 0.9541)	• For parameters around 1, the test is powerful with type I error less than the assigned significance level

Table 5.3: Conclusions from simulation studies

Test	Simulation Result	Conclusion
Exponential vs Gamma	• Type I error slowly increases and decreases (for all parameters stays less than the assigned significance level) • Power of the test decreases (for all parameters less than 0.7063)	• For parameters around 0.5, the test is powerful with type I error less than the assigned significance level
Weibull vs Exponential	• Type I error decreases monotonically (for all parameters less than 0.0015) • Power of the test increase monotonically (for all parameters greater than 0.9495)	• For all parameters, the test is powerful with type I error less than the assigned significance level
Exponential vs Weibull	• Type I error slowly increases and decreases (for all parameters is less than 0.0488) • Power of the test decreases and increase slowly (for all parameters greater than 0.8526)	• For all parameters, the test is power full with type I error less than the assigned significance level

Table 5.4: Conclusion from simulation studies - continue

Chapter 6

Conclusions

In this thesis, locally powerful statistical tests for testing three main probability models in Reliability theory that are related to Generalized Gamma Distribution (model of aging and deterioration, model of the weak link, and model of manufacture defect) were constructed . In this regard considering $X_1, X_2, ...$ is a sequence of independent identically distributed random variables with Generalized Gamma distribution. The following test statistic was used. $\mathfrak{T} = c_1\mathfrak{T}_1 + c_2\mathfrak{T}_2$, where

$$\mathfrak{T}_1 = \frac{1}{n}\sum \ln X_k - \ln\left(\frac{1}{n}\sum X_k\right), \quad \mathfrak{T}_2 = \ln\left(\frac{1}{n}\sum X_k\right) - \frac{\frac{1}{n}\sum X_k \ln X_k}{\frac{1}{n}\sum X_k},$$

Which according to Volodin [?], is a local most powerful test statistic to distinguish two types of distribution connected with Generalized Gamma distribution.

Statistics $\mathfrak{T}_1$ and $\mathfrak{T}_2$ depends on three pivot statistics:

$$T_1 = \frac{1}{n}\sum_1^n X_k,$$

$$T_2 = \frac{1}{n}\sum_1^n \ln(X_k)\ ,$$

$$T_3 = \frac{1}{n}\sum_1^n X_k \ln(X_k)\ .$$

With this notation, statistics $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be written as

$$\mathfrak{T}_1 = T_2 - \ln T_1\ , \mathfrak{T}_2 = \ln T_1 - \frac{T_3}{T_1}\ .$$

Therefore the main test statistic takes the form

$$\mathfrak{T} = c_1(T_2 - \ln T_1)\ + c_2(\ln T_1 - \frac{T_3}{T_1}\ ).$$

In chapter three of this thesis Lemma 1, Lemma 2, and Lemma 3 calculates mean values, variances, and covariances of pivotal statistics that are involved in the main test statistic (T_1, T_2, and T_3), for the three Reliability models.

Then using the Taylor theorem and Delta method in Lemma 4, asymptotic mean and variance for three Reliability models are calculated for the main test statistic ($\mathfrak{T}$). The mean values obtained are as following for three distribution:

Gamma Distribution

$$\mu_{\mathfrak{T}}(\lambda, 0) = \mu(\lambda) = c_1\psi(1+\lambda) + (c_2 - c_1)\ln(1+\lambda) - c_2\psi(2+\lambda) =$$

$$\mu_0 + \lambda\left[\frac{\pi^2}{6}(c_1 - c_2) + 2c_2 - c_1\right] + o(\lambda^2)$$

Weibull distribution:

$$\mu_{\mathfrak{T}}(\tau,\tau) = \mu(\tau) = c_1 \frac{\psi(1)}{1+\tau} + (c_2 - c_1)\ln\left[\frac{1}{1+\tau}\Gamma\left(\frac{1}{1+\tau}\right)\right] -$$
$$c_2\left[\frac{1}{1+\tau}\psi\left(\frac{2+\tau}{1+\tau}\right)\right] =$$
$$\mu_0 + \tau\left[c_2\left(\frac{\pi^2}{6}-1\right)+c_1\right] + O(\tau^2).$$

Generalized Exponential distribution:

$$\mu_{\mathfrak{T}}(0,\beta) = \mu(\beta) = c_1 \frac{1}{1+\beta}\psi\left(\frac{1}{1+\beta}\right) +$$
$$(c_2 - c_1)\left[\ln\Gamma\left(\frac{2}{1+\beta}\right) - \ln\Gamma\left(\frac{1}{1+\beta}\right)\right] - c_2 \frac{1}{1+\beta}\psi\left(\frac{2}{1+\beta}\right) =$$
$$\mu_0 + \beta\left[c_1\left(2-\frac{\pi^2}{6}\right) + c_2\left(\frac{\pi^2}{3}-3\right)\right] + O(\beta^2).$$

Then in chapter four the above results were used to construct the locally most powerful tests for choosing the probability Reliability models. All tests have the rejection region of the general form $\mathfrak{T} > c$ where the critical value $c = c(\alpha)$ is defined according to the given significance level α, by the equality $c(\alpha) = \mu_0 + \sigma_0\Phi^{-1}(1-\alpha)/\sqrt{n}$ where $\Phi^{-1}(1-\alpha)$ is the quantile of the standard normal distribution and μ_0 and σ_0 are the mean and variance values of $\mathfrak{T}$ where the parameter of the distribution is set to zero.

Using the above derived critical value and Normal approximation we have:

$\alpha = p_\theta(\mathfrak{T} > c(\alpha)) = 1 - \Phi\left(\frac{c(\alpha)-\mu_\theta}{\sigma_\theta}\sqrt{n}\right) = 1 - \Phi\left(\frac{\mu_0-\mu_\theta}{\sigma_\theta}\sqrt{n} + \Phi^{-1}(1-\alpha)\frac{\sigma_0}{\sigma_\theta}\right)$

where μ_θ and σ_θ are asymptotic mean and variance of the main test statistic with generic parameter θ that were calculated in chapter three.

The final step in order to construct the test was specifying c_1 and c_2 . These two constants should be chosen such that the probability of type I error is less than α and the power of the test is greater than α (condition of unbiasedness of the test).

The values found for these two constants that provide the locally most powerful tests for null hypothesis H_0 with alternatives H_1 for testing the three main Reliability models comes in table 6.1.

H_0	H_1	c_1	c_2
G	W	-1	$\frac{\pi^2-6}{12-\pi^2}$
W	G	1	$\frac{-6}{\pi^2-6}$
G	E^μ	-1	$\frac{\pi^2-6}{12-\pi^2}$
E^μ	G	1	$\frac{\pi^2-12}{2(\pi^2-9)}$
W	E^μ	-1	$\frac{6}{\pi^2-6}$
E^μ	W	1	$\frac{\pi^2-12}{2(\pi^2-9)}$

Table 6.1: c_1 and c_2 constants for the locally most powerful tests

In chapter 5, computer simulation was used to assess the proposed tests. 10^4 samples of size 100 was generated and the statistics $\mathfrak{T}$ calculated for each simulation for every individual test. Type I error and power of each test calculated and it was observed form the simulation results that all constructed six test are powerful.

Appendix A

Computer Simulation Codes

The software is used for simulation is Matlab version 2014a.

```
% Main function

N=10000;

n=100;

alpha=0.05;

samples=zeros(N,n);

zetas=zeros(N,1);

distparams=[.5,1,1.5,2,2.5,3];

al=zeros(6,6);

m =zeros(6,6);


for testnum=1:6
```

```
    expdist=makedist('Exponential');
[c1,c2]=testfactors(testnum);
for i=1:N
    r=random(expdist,n,1);
    samples(i,:)=r;
    zetas(i,1)=fstatistic(r,c1,c2);
end
sortzetas=sort(zetas);
critical=sortzetas(round((alpha)*N));

for p=1:6
    switch testnum
        case 1
            al(testnum,p)=
            testresult('G',distparams(p),N,n,critical,c1,c2);
            m(testnum,p)=
            1-testresult('W',distparams(p),N,n,critical,c1,c2);

        case 2
            al(testnum,p)=
```

```
    testresult('W',distparams(p),N,n,critical,c1,c2);
    m(testnum,p)=
    1-testresult('G',distparams(p),N,n,critical,c1,c2);

case 3
    al(testnum,p)=
    testresult('G',distparams(p),N,n,critical,c1,c2);
    m(testnum,p)=
    1-testresult('E',distparams(p),N,n,critical,c1,c2);


case 4
    al(testnum,p)=
    testresult('E',distparams(p),N,n,critical,c1,c2);
    m(testnum,p)=
    1-testresult('G',distparams(p),N,n,critical,c1,c2);


case 5
    al(testnum,p)=
```

```
                testresult('W',distparams(p),N,n,critical,c1,c2);
                m(testnum,p)=
                 1-testresult('E',distparams(p),N,n,critical,c1,c2);


            case 6
                al(testnum,p)=
                testresult('E',distparams(p),N,n,critical,c1,c2);
                m(testnum,p)=
                1-testresult('W',distparams(p),N,n,critical,c1,c2);


        end
    end
end

function result=testresult(distname,parameter, N,n,c,c1,c2)
if(distname=='G')
    dist=makedist('Gamma', 'a',1+parameter,'b',1);
elseif(distname=='E')
    dist=makedist('Exponential','mu',1/(1+parameter));
```

```
else
    dist=makedist('Weibull',  'a',1,'b',1+parameter);
end

count=0;
for  i=1:N
    r=random(dist,n,1);
    if(distname=='E')
       r=r/gamma(1/(1+parameter));
    end
    if(fstatistic(r,c1,c2)>=c)
      count=count+1;
    end
end
result=count/N;
end

        function  [c1,c2]=testfactors(testNum)
switch(testNum)
     case 1
         c1=-1;
```

```
        c2=(pi^2-6)/(12-pi^2);
    case 2
        c1=1;
        c2=(-6)/(pi^2-6);


     case 3
        c1=-1;
        c2=(pi^2-6)/(12-pi^2);
     case 4
        c1=1;
        c2=(pi^2-12)/(2*(pi^2-9));
     case 5
        c1=-1;
        c2=(6)/(pi^2-6);
     otherwise
        c1=+1;
        c2=(pi^2-12)/(2*(pi^2-9));
 end
end

       function z=fstatistic(x,c1,c2)
```

```
lnx=log(x);

xlnx=x.*lnx;

t1=mean(x);

t2=mean(lnx);

t3=mean(xlnx);


  z=c1*(t2-log(t1))+c2*(log(t1)-t3/t1);


end
```