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The enhanced principal rank characteristic sequence for skew-symmetric matrices

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ABSTRACT

The enhanced principal rank characteristic sequence (epr-sequence) was originally defined for an $n \times n$ real symmetric matrix or an $n \times n$ Hermitian matrix. Such a sequence is defined to be $\ell_1\ell_2 \cdots \ell_n$ where ℓ_k is $\mathtt{A}$, $\mathtt{S}$, or $\mathtt{N}$ depending on whether all, some, or none of the matrix principal minors of order k are nonzero. Here we give a complete characterization of the attainable epr-sequences for real skew-symmetric matrices. With the constraint that $\ell_k = 0$ if k is odd, we show that nearly all epr-sequences are attainable by skew-symmetric matrices, which is in contrast to the case of real symmetric or Hermitian matrices for which many epr-sequences are forbidden.

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1. Introduction

For a symmetric matrix over a field F or a complex Hermitian matrix, Brualdi et al. [2] and Barrett et al. [1] defined and studied the principal rank characteristic sequence, which records with a 1 or a 0 whether or not there is a full rank principal submatrix of each order. This concept was extended (or enhanced) in [3] by broadening the measure contained within the elements of a principal rank characteristic sequence. Accordingly this sequence considers the presence or absence of such a principal submatrix, via three possibilities in the following definition.

Definition 1.1. The *enhanced principal rank characteristic sequence* of a matrix $B \in F^{n\times n}$ is the sequence (epr-sequence) $\mathrm{epr}(B) = \ell_1\ell_2\cdots\ell_n$ where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all } k\times k \text{ principal minors of the given order are nonzero;} \\ \mathtt{S} & \text{if some but not all } k\times k \text{ principal minors are nonzero;} \\ \mathtt{N} & \text{if none of the } k\times k \text{ principal minors are nonzero, i.e., all are zero.} \end{cases}$$

We are interested in which epr-sequences are *attainable* by skew-symmetric matrices over $\mathbb{R}$, that is, are realized as the epr-sequence of some skew-symmetric matrix.

Brualdi et al. [2] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [6]. The study of epr-sequences provides additional information that may be helpful in work on the principal minor assignment problem, while remaining somewhat tractable.

For $B \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, the submatrix of B lying in rows indexed by α and columns indexed by β is denoted by $B[\alpha, \beta]$. Further, the complementary submatrix obtained from B by deleting the rows indexed by α and columns indexed by β is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while the complementary principal submatrix is denoted by $B(\alpha)$. The complement of α is denoted by α^c.

For $B \in \mathbb{R}^{n\times n}$, we let $\sigma(B)$ denote the multiset of eigenvalues of B, and we denote the rank of B by $\mathrm{rank}(B)$. Following the notation in [1], we let $\overline{\ell_i\cdots\ell_j}$ indicate that the (complete) sequence may be repeated as many times as necessary (or may be omitted entirely). All matrices considered here are skew-symmetric over $\mathbb{R}$ and of order ≥ 2 (since for $n = 1$ the only case is $\mathrm{epr}(0) = \mathtt{N}$).

1.1. Basic facts on the epr-sequences of skew-symmetric matrices

This section contains some key fundamental facts on skew-symmetric matrices that we connect to properties of their epr-sequences. The fact that a real matrix satisfies $B = -B^T$ places significant restrictions on the spectrum of B, the rank of B, and the quadratic form x^TBx, for $x \in \mathbb{R}^n$. For instance, as is well known, if $Bx = \lambda x$ and $x \neq 0$, then $\lambda = ik$ for some $k \in \mathbb{R}$, i.e., λ is pure imaginary. Hence, if $\lambda \in \sigma(B) \cap \mathbb{R}$, then

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$\lambda = 0$. Further, when n is odd, it follows that B must be singular. In particular, since B is normal, $\text{rank}(B)$ is always an even number.

Since the property of being skew-symmetric is inherited by principal submatrices, all odd order principal submatrices of B are singular. Equivalently, $\ell_k = \mathtt{N}$ whenever k is odd. Thus the epr-sequence of any real skew-symmetric matrix begins with N and ends with either N or A depending on whether or not B is singular. Also note that if the epr-sequence of a skew-symmetric matrix begins with NA, then all of its off-diagonal entries are nonzero.

The next property discussed here is analogous to the fact that the rank of a symmetric matrix B is determined by the order of the largest invertible principal submatrix of B. This property is called the *principal rank property* in [4]. Indeed, by [4, Lemma 6.4], the same property holds for skew-symmetric matrices. That is, if the rank of a skew-symmetric matrix is k, then B contains a $k \times k$ principal submatrix of full rank. This so-called principal rank property is needed to prove the next fact on the epr-sequences of skew-symmetric matrices. The proof is essentially the same as in [1, Theorem 2.1], and is omitted here.

Theorem 1.2. *Suppose $B \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix, $\text{epr}(B) = \ell_1 \cdots \ell_n$, and $\ell_k = \ell_{k+1} = \mathtt{N}$ for some k. Then $\ell_i = \mathtt{N}$ for all $i \geq k$. (That is, if an epr-sequence of a matrix ever has NN, then it must have only Ns from that point onward.)*

It is an easy consequence of Theorem 1.2 that if $\ell_k = \mathtt{N}$ for some k with k even, then $\ell_i = \mathtt{N}$ for all $i \geq k - 1$.

Jacobi's determinantal identity is now used to relate the epr-sequence of a nonsingular matrix to that of its inverse. This implies that most epr-sequences that end in A may be grouped in natural pairings.

Theorem 1.3. *(See [3, Theorem 2.4].) If $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathtt{A}$, then $\text{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2} \cdots \ell_1\mathtt{A}$ (i.e., the sequence is reversed except for the last A).*

The following bordering result is very useful for extending existing epr-sequences.

Proposition 1.4. *If B is an $(n-1) \times (n-1)$ real skew-symmetric matrix and x is a real vector in $\mathbb{R}^{n-1}$, then*

$$M = \left[\begin{array}{c|c} B & x \\ \hline -x^T & -x^T Bx \end{array}\right] \in \mathbb{R}^{n\times n}$$

is skew-symmetric. In addition:

1. *If x is in the column space of B, then $\text{rank}(M) = \text{rank}(B)$.*
2. *If x is not in the column space of B, then $\text{rank}(M) = \text{rank}(B) + 2$.*

Proof. Observe that with B skew-symmetric, the quadratic form $x^T Bx = (x^T Bx)^T = x^T B^T x = -(x^T Bx)$. Since both x and B are real, it follows that $x^T Bx = 0$. Thus M is skew-symmetric. Let $S = \left[\begin{array}{c|c} I & 0 \\ \hline -y^T & 1 \end{array}\right]$, partitioned conformally with M, where $x = By$. Then observe that

$$SMS^T = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right],$$

which establishes statement *1*.

For statement *2*, $\text{rank}(M) \geq \text{rank}(B) + 1$, and since B is bordered by one row and column, $\text{rank}(M) \leq \text{rank}(B) + 2$. The result follows by noting that the rank of a skew-symmetric matrix must be even. □

Next we list the epr-sequences of some $n \times n$ real skew-symmetric matrices, which we first define.

- Let O_n denote the $n \times n$ zero matrix.
- Let T_n denote the $n \times n$ tridiagonal skew-symmetric matrix with every superdiagonal entry equal to 1, every subdiagonal entry equal to -1, and every other entry 0.
- Let F_n denote an $n \times n$ skew-symmetric matrix with all off-diagonal entries from the set $\{-1, 1\}$, which is sometimes known as a $(-1, 1)$ tournament matrix. Then using [5, Proposition 1], $\det(F_n) = 0$ if and only if n is odd.
- Let S_n denote the $n \times n$ skew-symmetric matrix with every entry in the first row (except the first) equal to 1, every entry in the first column (except the first) equal to -1, and every other entry 0.

Observation 1.5. *The following epr-sequences can be easily verified, with $\bigoplus$ denoting the direct sum of matrices. For example, the descriptions for both $\text{epr}(T_n)$ and $\text{epr}(F_n)$ follow from straightforward induction arguments.*

- $\text{epr}(O_n) = \overline{\texttt{N}}$.
- $\text{epr}(T_n) = \overline{\texttt{NS}}\texttt{NA}$ *if n is even,* $\overline{\texttt{NS}}\texttt{N}$ *if n is odd.*
- $\text{epr}(F_n) = \overline{\texttt{NA}}$ *if n is even,* $\overline{\texttt{NA}}\texttt{N}$ *if n is odd.*
- $\text{epr}(S_n) = \texttt{NS}\overline{\texttt{N}}$ *for $n \geq 3$.*
- $\text{epr}(T_2 \bigoplus T_2 \bigoplus O_{n-4}) = \texttt{NSNS}\overline{\texttt{N}}$ *for $n \geq 5$.*

We now exhibit the complete list of attainable epr-sequences for skew-symmetric matrices of orders up to 6. Accompanying numerical examples are displayed in Appendix A, at the end of this paper.

Table 1
All attainable epr-sequences through $n = 6$.

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
NN	NNN	NNNN	NNNNN	NNNNNN
NA	NAN	NANN	NANNN	NANNNN
	NSN	NANA	NANAN	NANANN
		NSNN	NSNNN	NANANA
		NSNA	NSNSN	NSNNNN
			NSNAN	NSNSNN
			NANSN	NSNSNA
				NSNANN
				NSNANA
				NANSNN
				NANSNA

Table 1 suggests that every epr-sequence is attainable by a real skew-symmetric matrix provided that the sequence has every odd entry equal to N, satisfies Theorem 1.2, and ends in N or A. We now develop methods to prove this claim, which is stated precisely in Theorem 3.3 below.

2. Probabilistic methods and bordering

In this section we employ a probabilistic method (see [3, Section 4]) to establish a mechanism for bordering a given skew-symmetric matrix over $\mathbb{R}$, and predicting the derived enhanced principal rank characteristic sequence of the larger matrix based in part on the epr-sequence of the embedded matrix.

Lemma 2.1. *Suppose B is an $(n-1)\times(n-1)$ a skew-symmetric matrix. Assume* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_{n-1}$ *with no consecutive* Ns*. Then there exists a skew-symmetric matrix $M \in \mathbb{R}^{n\times n}$ such that the epr-sequence of M is $\ell'_1\ell'_2\cdots\ell'_n$, where $\ell'_n = \mathtt{N}$, $\ell'_k = \mathtt{N}$ for k odd, and for k even, with high probability, $\ell'_k = \mathtt{A}$ if $\ell_k = \mathtt{A}$ and $\ell'_k = \mathtt{S}$ if $\ell_k = \mathtt{S}$ (where $2 \le k \le n-1$).*

Proof. Given B as above, set

$$M = \left[\begin{array}{c|c} B & x \\ \hline -x^T & -x^TBx \end{array}\right] \in \mathbb{R}^{n\times n},$$

where $x = By$ for a randomly chosen vector y in $\mathbb{R}^{n-1}$. Observe that by Proposition 1.4 $\operatorname{rank}(M) = \operatorname{rank}(B)$, and since $x^TBx = 0$, it follows that M is a singular skew-symmetric matrix. Thus, $\ell'_n = \mathtt{N}$ and $\ell'_k = \mathtt{N}$ for k odd.

Now suppose $2 \le k \le n-1$ and k is even. Since $\operatorname{epr}(B)$ contains no consecutive Ns, it follows that ℓ_k is either A or S. If $\ell_k = \mathtt{S}$, then $\ell'_k = \mathtt{S}$ follows trivially. On the other hand, if $\ell_k = \mathtt{A}$, then let C be any $k\times k$ principal submatrix of M. If C lies entirely in B, then C is nonsingular. So assume C is of the form

$$C = \left[\begin{array}{c|c} B' & x' \\ \hline -(x')^T & 0 \end{array}\right],$$

where B' is a $(k-1) \times (k-1)$ principal submatrix of B and x' is the appropriately chosen sub-vector of x. Since $k-1$ is odd, it follows that $\text{rank}(B') \leq k-2$. In fact, we claim that $\text{rank}(B') = k-2$. If equality does not hold, that is $\text{rank}(B') \leq k-3$, then since $k-3$ is odd, actually $\text{rank}(B') \leq k-4$. Since $k < n$, B' is a proper principal submatrix of B, and hence B' lies in a $k \times k$ principal submatrix of B, say B''. In this case $\text{rank}(B'') \leq \text{rank}(B') + 2 \leq k-4+2 = k-2$. Hence B'' is singular, which contradicts the fact that $\ell_k = \mathtt{A}$. Therefore, $\text{rank}(B') = k-2$, so with high probability, the sub-vector x' does not lie in the column space of B'. In other words, with high probability, $\text{rank}(C) = \text{rank}(B') + 2 = k$. Hence, with high probability, $\ell'_k = \mathtt{A}$. □

For n even, the next result allows an A to be appended to an epr-sequence of an $(n-1) \times (n-1)$ skew-symmetric matrix as long as its epr-sequence does not contain consecutive Ns.

Lemma 2.2. *For n even, let B be an $(n-1) \times (n-1)$ skew-symmetric matrix with $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}$ with no consecutive* N*s. Then there exists a skew-symmetric matrix $M \in \mathbb{R}^{n\times n}$ such that the epr-sequence of M is $\ell'_1\ell'_2 \cdots \ell'_n$ where $\ell'_n = \mathtt{A}$, $\ell'_k = \mathtt{N}$ for k odd, and for k even, with high probability, $\ell'_k = \mathtt{A}$ if $\ell_k = \mathtt{A}$ and $\ell'_k = \mathtt{S}$ if $\ell_k = \mathtt{S}$ (where $2 \leq k \leq n-2$).*

Proof. Suppose n is even, $2 \leq k \leq n-2$, matrix B is given as above, and

$$M = \left[\begin{array}{c|c} B & x \\ \hline -x^T & -x^T Bx \end{array}\right] \in \mathbb{R}^{n\times n},$$

where x is a randomly chosen vector in $\mathbb{R}^{n-1}$. It follows that M is a skew-symmetric matrix and that $\ell'_k = \mathtt{N}$ for k odd, and with high probability, $\text{rank}(M) = \text{rank}(B) + 2$. Moreover, since $\text{epr}(B)$ does not contain consecutive Ns, it follows that $\text{rank}(B) = n-2$ and hence with high probability, M is of full rank. Now let k be even. If $\ell_k = \mathtt{S}$, then $\ell'_k = \mathtt{S}$ follows trivially. If $\ell_k = \mathtt{A}$, let C be any $k \times k$ principal submatrix of M. If C lies entirely in B, then C is nonsingular. So assume C is of the form

$$C = \left[\begin{array}{c|c} B' & x' \\ \hline -(x')^T & 0 \end{array}\right],$$

where B' is a $(k-1) \times (k-1)$ principal submatrix of B and x' is the appropriately chosen sub-vector of x. By a similar argument as used in the proof of Lemma 2.1, $\text{rank}(B') = k-2$. In other words, $\text{rank}(C) = \text{rank}(B') + 2 = k$. Hence, with high probability, $\ell'_k = \mathtt{A}$. □

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The two previous facts made use of the assumption that the given epr-sequence does not contain consecutive Ns. From Theorem 1.2 we know that if an epr-sequence of a matrix ever contains the subsequence NN, then it must have Ns from that point onward. Hence extending such a sequence would have to end in an additional N.

Proposition 2.3. *Suppose B is an $(n-1)\times(n-1)$ skew-symmetric matrix with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\mathtt{NN}$. *Then there exists a skew-symmetric matrix $M \in \mathbb{R}^{n\times n}$ such that the epr-sequence of M is $\ell_1'\ell_2'\cdots\mathtt{NNN}$ and satisfies the following conditions:*

(*i*) $\ell_k = \ell_k' = \mathtt{N}$ *for k odd,*
(*ii*) $\ell_k = \mathtt{S}$ *implies that* $\ell_k' = \mathtt{S}$ *for* $2 \le k \le n-3$,
(*iii*) $\ell_k = \mathtt{N}$ *if and only if* $\ell_k' = \mathtt{N}$ *for* $2 \le k \le n-1$, *and*
(*iv*) $\ell_k' = \mathtt{A}$ *implies that* $\ell_k = \mathtt{A}$, *and* $\ell_k = \mathtt{A}$ *implies that* $\ell_k' = \mathtt{A}$ *with high probability.*

Proof. The proof is similar to the previous lemmas in that the matrix B is bordered as follows:

$$M = \left[\begin{array}{c|c} B & x \\ \hline -x^T & -x^T Bx \end{array}\right] \in \mathbb{R}^{n\times n},$$

where $x = By$ and y is a randomly chosen vector in $\mathbb{R}^{n-1}$. Suppose $\operatorname{epr}(M) = \ell_1'\ell_2'\cdots\ell_n'$. By Proposition 1.4, $\operatorname{rank}(M) = \operatorname{rank}(B) \le n-3$, thus it follows that $\ell_{n-2}' = \ell_{n-1}' = \ell_n' = \mathtt{N}$. Since M is skew-symmetric, (*i*) and (*ii*) are immediate.

For (*iii*), we need only consider k even by (*i*). For the necessity, let $k \ge 2$ be the smallest even index such that $\ell_k = \mathtt{N}$. Then the epr-sequence of B must contain consecutive Ns from position $k-1$ through to position $n-1$. Thus the rank of B is at most $k-2$, which implies the rank of M is at most $k-2$ or that the epr-sequence of M must contain consecutive Ns from position $k-1$ through to position n, proving the necessity in (*iii*). For the sufficiency, since M contains B it follows that $\ell_k' = \mathtt{N}$ implies $\ell_k = \mathtt{N}$.

For (*iv*), since M contains B it also follows that $\ell_k' = \mathtt{A}$ implies $\ell_k = \mathtt{A}$, proving the first implication. For the second implication in (*iv*), assume that $\ell_k = \mathtt{A}$. For the sake of a contradiction, assume there exists a $k\times k$ principal submatrix M' of M that is singular. Since $\ell_k = \mathtt{A}$, M' must include the last row and column of M. That is,

$$M' = \left[\begin{array}{c|c} B' & x' \\ \hline -(x')^T & 0 \end{array}\right].$$

Then from the arguments in both of the preceding lemmas, $\operatorname{rank}(B') = k-2$, from which we conclude that M' is nonsingular with high probability. □

Note that the converse of (*ii*) in Proposition 2.3 need not hold. It is possible, though unlikely, that the bordering scheme presented in the proof of Proposition 2.3 does produce

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an epr-sequence for which the converse of (*ii*) may not hold. In fact, the following example verifies the condition "with high probability" in statement (*iv*). Consider taking the direct sum of the 4×4 matrix B that has $\text{epr}(B) = \texttt{NANN}$ (see Appendix A) with O_1, which produces a skew-symmetric matrix with epr-sequence NSNNN.

One consequence that can be drawn from the above proposition is that if an epr-sequence is observed to contain two or more consecutive Ns, then the only permissible epr-sequence that contains this given sequence must trail in Ns, as seen by Theorem 1.2. Furthermore, any such extended epr-sequence is attainable from an application of Proposition 2.3. Thus it is reasonable, when extending existing epr-sequences over real skew-symmetric matrices, to assume that such epr-sequences do not contain consecutive Ns, otherwise as already verified, such epr-sequences can only be extended by Ns.

3. Main results

In this section we derive a complete characterization of the epr-sequences of real skew-symmetric matrices. We begin by proving some results on such epr-sequences specific to the cases of n even or odd.

Theorem 3.1. *Suppose n is even. Assume B is an $(n-1) \times (n-1)$ real skew-symmetric matrix with $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_{n-1}$ and having no consecutive Ns. Then there exists a skew-symmetric matrix $M \in \mathbb{R}^{n \times n}$ such that the epr-sequence of M is $\ell_1 \ell_2 \cdots \ell_n$, where $\ell_n = \texttt{A}$ and a skew-symmetric matrix $M' \in \mathbb{R}^{n \times n}$ such that the epr-sequence of M' is $\ell_1 \ell_2 \cdots \ell_n$, where $\ell_n = \texttt{N}$.*

Proof. Given the assumed hypotheses, it follows that M exists by Lemma 2.2 and that M' exists by Lemma 2.1. Moreover, each lemma shows a constructive way to find a required matrix from a given B. □

Theorem 3.2. *Suppose n is odd. Assume B is an $(n-2) \times (n-2)$ skew-symmetric matrix with $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_{n-2}$ and having no consecutive Ns. Then there exist skew-symmetric matrices $M_1, M_2, M_3 \in \mathbb{R}^{n \times n}$ such that their epr-sequences are given by each of the following:*

(a) $\text{epr}(M_1) = \ell_1 \ell_2 \cdots \ell_{n-2}\texttt{NN}$;
(b) $\text{epr}(M_2) = \ell_1 \ell_2 \cdots \ell_{n-2}\texttt{AN}$; *and*
(c) $\text{epr}(M_3) = \ell_1 \ell_2 \cdots \ell_{n-2}\texttt{SN}$.

Proof. Consider case (a). Using Lemma 2.1, there exists an $(n-1) \times (n-1)$ skew-symmetric matrix C with $\text{epr}(C) = \ell_1 \ell_2 \cdots \ell_{n-2}\texttt{N}$. Now apply Proposition 2.3 to establish the existence of M_1.

For case (b), apply Lemma 2.2 to derive an $(n-1)\times(n-1)$ skew-symmetric matrix C with $\mathrm{epr}(C) = \ell_1\ell_2\cdots\ell_{n-2}\mathtt{A}$. Since $\mathrm{epr}(C)$ has no consecutive Ns, apply Lemma 2.1 to establish the existence of an $n\times n$ matrix M_2 with $\mathrm{epr}(M_2) = \ell_1\ell_2\cdots\ell_{n-2}\mathtt{AN}$.

For case (c), begin by using Lemma 2.1 to produce an $(n-1)\times(n-1)$ skew-symmetric matrix C with $\mathrm{epr}(C) = \ell_1\ell_2\cdots\ell_{n-2}\mathtt{N}$. Observe that, in fact, $\mathrm{epr}(C) = \ell_1\ell_2\cdots\ell_{n-3}\mathtt{NN}$, where $\ell_{n-3} = \mathtt{A}$ or $\mathtt{S}$. Hence $\mathrm{rank}(C) = n-3$. Let

$$M_3 = \left[\begin{array}{c|c} C & x \\ \hline -x^T & -x^TCx \end{array}\right] \in \mathbb{R}^{n\times n},$$

where x is a randomly chosen vector in $\mathbb{R}^{n-1}$. Then M_3 is skew symmetric and $\mathrm{epr}(M_3) = \ell'_1\ell'_2\cdots\ell'_n$, where $\ell'_k = \mathtt{N}$ whenever k is odd, and $\ell'_k = \mathtt{S}$ whenever $\ell_k = \mathtt{S}$ and $2\le k\le n-3$. Further, since x was randomly selected, it follows that $\ell'_k = \mathtt{A}$ whenever $\ell_k = \mathtt{A}$ and $2\le k\le n-3$, and $\mathrm{rank}(M_3) = \mathrm{rank}(C)+2 = n-1$. Hence there exists an $(n-1)\times(n-1)$ principal submatrix of M_3 that is nonsingular. Since C is a singular $(n-1)\times(n-1)$ principal submatrix of M_3, it follows that $\ell'_{n-1} = \mathtt{S}$. □

We are now in a position to state our main result characterizing all of the attainable epr-sequences of real skew-symmetric matrices. Here we prove that the basic necessary conditions on epr-sequences of skew-symmetric matrices are also sufficient.

Theorem 3.3. *Suppose $\ell_1\ell_2\cdots\ell_n$ is a given sequence of length n from $\{\mathtt{A},\mathtt{S},\mathtt{N}\}$. Then $\ell_1\ell_2\cdots\ell_n$ is the epr-sequence of a real skew-symmetric matrix if and only if the following conditions hold:*

(*i*) $\ell_k = \mathtt{N}$ *for k odd,*
(*ii*) *if $\ell_s = \ell_{s+1} = \mathtt{N}$, then $\ell_t = \mathtt{N}$ for all $t\ge s$, and*
(*iii*) $\ell_n \in \{\mathtt{A},\mathtt{N}\}$.

Proof. We have already established that conditions (*i*)–(*iii*) are necessary. To establish sufficiency, assume $\ell_1\ell_2\cdots\ell_n$ is a given sequence of length n from $\{\mathtt{A},\mathtt{S},\mathtt{N}\}$ that satisfies conditions (*i*)–(*iii*).

If this sequence contains consecutive Ns, consider the smallest k for which $\ell_k = \ell_{k+1} = \mathtt{N}$. Then applying Proposition 2.3 as often as needed will produce a real skew-symmetric matrix M such that $\mathrm{epr}(M) = \ell_1\ell_2\cdots\ell_{k-1}\mathtt{N}\cdots\mathtt{N}$.

Otherwise, we consider a proof by induction on n. From Table 1 in Section 1.1 and the examples in Appendix A, the result holds for sequences of length up to 6. So assume that (*i*), (*ii*), (*iii*) hold for all such sequences of length at most $n-1$. Let $\ell_1\ell_2\cdots\ell_n$ be a given sequence of length n from $\{\mathtt{A},\mathtt{S},\mathtt{N}\}$ that satisfies conditions (*i*)–(*iii*), where the possible only occurrence of NN may be at the end of the sequence $\ell_1\ell_2\cdots\ell_n$, i.e., $\ell_{n-1} = \ell_n = \mathtt{N}$.

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Case 1: Assume n is even. Then by the induction hypothesis, the sequence $\ell_1\ell_2\cdots\ell_{n-1}$ of length $n-1$ from $\{\mathtt{A},\mathtt{S},\mathtt{N}\}$ can be realized as the epr-sequence of an $(n-1)\times(n-1)$ real skew-symmetric M. Using Theorem 3.1, there exist $n\times n$ skew-symmetric matrices M_1 and M_2 such that $\operatorname{epr}(M_1) = \ell_1\ell_2\cdots\ell_{n-1}\mathtt{N}$ and $\operatorname{epr}(M_2) = \ell_1\ell_2\cdots\ell_{n-1}\mathtt{A}$, which completes the proof for n even.

Case 2: Assume n is odd. Then by the induction hypothesis, the sequence $\ell_1\ell_2\cdots\ell_{n-2}$ of length $n-2$ from $\{\mathtt{A},\mathtt{S},\mathtt{N}\}$ can be realized as the epr-sequence of an $(n-2)\times(n-2)$ real skew-symmetric M. Since the sequence $\ell_1\ell_2\cdots\ell_n$ satisfies conditions (i)–(iii), it follows that

$$\ell_1\ell_2\cdots\ell_n = \begin{cases} \ell_1\ell_2\cdots\ell_{n-2}\mathtt{NN} & \text{or} \\ \ell_1\ell_2\cdots\ell_{n-2}\mathtt{SN} & \text{or} \\ \ell_1\ell_2\cdots\ell_{n-2}\mathtt{AN}. & \end{cases}$$

By Theorem 3.2, all three such sequences are realizable by an $n\times n$ skew-symmetric matrix, which completes the proof in this case.

Hence, by induction, the sequence $\ell_1\ell_2\cdots\ell_n$ is the epr-sequence of an $n\times n$ real skew-symmetric matrix. □

It follows from the above theorem that for real skew-symmetric matrices, any epr-sequence that satisfies the three stated conditions is attainable. This is in contrast to the case of real symmetric and Hermitian matrices for which many epr-sequences are forbidden [3, Section 2]. For example, no such matrix can have NSA in its epr-sequence, and the sequence NAN is not attainable by any 3×3 real symmetric matrix but is attainable by the skew-symmetric matrix F_3. In fact there is no known characterization of all attainable epr-sequences for real symmetric or Hermitian matrices, whereas the result of Theorem 3.3 gives such a characterization for real skew-symmetric matrices. In addition, it might be interesting to consider if Theorem 3.3 holds over the field of rational numbers. A similar open question comparing principal rank characteristic sequences over the reals versus over the rationals is posed in [1].

Recall that a complex matrix C is *skew-Hermitian* if $C = -C^*$. Here we note that the skew-Hermitian epr-problem is equivalent to the complex Hermitian epr-problem as considered in [3]. To see this, observe that if C is skew-Hermitian, then $B = iC$ is Hermitian. Moreover, since $\operatorname{epr}(C) = \operatorname{epr}(B)$, it follows that any attainable epr-sequence over complex Hermitian matrices is an attainable epr-sequence over complex skew-Hermitian matrices and conversely.

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Appendix A

Examples illustrating the claims in Table 1 follow (the zero matrix is omitted).

$n = 2$: $\text{epr}(T_2) = \texttt{NA}$.
$n = 3$: $\text{epr}(F_3) = \texttt{NAN}$, $\text{epr}(T_3) = \texttt{NSN}$.
$n = 4$: $\text{epr}(F_4) = \texttt{NANA}$, $\text{epr}(S_4) = \texttt{NSNN}$, $\text{epr}(T_4) = \texttt{NSNA}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ -2 & -1 & 1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANN}$. Note that B is obtained by bordering F_3 with a vector x in the column space of F_3.

$n = 5$: $\text{epr}(F_5) = \texttt{NANAN}$, $\text{epr}(S_5) = \texttt{NSNNN}$, $\text{epr}(T_5) = \texttt{NSNSN}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 & 2 \\ -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & 1 & 0 & 1 \\ -3 & -2 & 1 & -1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANNN}$. Note that B is obtained by bordering the last example for $n = 4$ with a vector x in its column space.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NSNAN}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ -2 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANSN}$.

$n = 6$: $\text{epr}(F_6) = \texttt{NANANA}$, $\text{epr}(S_6) = \texttt{NSNNNN}$, $\text{epr}(T_2 \bigoplus T_2 \bigoplus O_2) = \texttt{NSNSNN}$, $\text{epr}(T_6) = \texttt{NSNSNA}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & 0 & -2 & 1 \\ -1 & -1 & 1 & 2 & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANSNN}$.

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Let $B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -2 & 1 \\ -1 & -1 & 1 & 2 & 0 & 1 \\ 1 & -1 & 1 & -1 & -1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANSNA}$.

Let B be the inverse of the previous matrix. Then, by Theorem 1.3, $\text{epr}(B) = \texttt{NSNANA}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 8 \\ -1 & -1 & 0 & 1 & 1 & 4 \\ -1 & -1 & -1 & 0 & 1 & 2 \\ -1 & -1 & -1 & -1 & 0 & 5 \\ -1 & -8 & -4 & -2 & -5 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANANN}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 3 \\ -1 & -1 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & 0 & -4 \\ -1 & -3 & -4 & 2 & 4 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NSNANN}$.

Let $B = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 5 \\ -1 & 0 & 1 & 1 & 2 & 3 \\ -1 & -1 & 0 & -1 & -1 & -2 \\ -2 & -1 & 1 & 0 & 1 & 1 \\ -3 & -2 & 1 & -1 & 0 & -1 \\ -5 & -3 & 2 & -1 & 1 & 0 \end{bmatrix}$. Then $\text{epr}(B) = \texttt{NANNNN}$.

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