STUDIES ON QUADRATIC MATRIX EQUATIONS AND
RICCATI DIFFERENTIAL EQUATIONS ASSOCIATED
WITH REGULAR $M$-MATRICES

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Abstract

The thesis is mainly about the quadratic matrix equation \( X^2 - EX - F = 0 \), where \( E \) is a diagonal matrix and \( F \) is a regular \( M \)-matrix. Quadratic matrix equations of this kind arise in noisy Wiener-Hopf problems for Markov chains. The solution of practical interest is a special \( M \)-matrix solution. The existence and uniqueness of \( M \)-matrix solutions and some numerical methods for finding the required \( M \)-matrix solution are studied by transforming the equation into a nonsymmetric algebraic Riccati equation for which the four coefficient matrices form a regular \( M \)-matrix. We also discuss the initial value problem for a matrix Riccati differential equation associated with a regular \( M \)-matrix. We show that for suitable initial matrices \( X_0 \), the initial value problem has a global solution \( X(t) \) on \([0, \infty)\). Moreover, we show that \( X(t) \) converges, as \( t \to \infty \), to the stable equilibrium solution, which is the minimal nonnegative solution of corresponding algebraic Riccati equation.
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Chapter 1

Introduction

$M$-matrix quadratic matrix equations (M-QMEs) arise in noisy Wiener-Hopf problems for Markov chains and $M$-matrix algebraic Riccati equations (MAREs) arise in applied probability and transportation theory, see [6, 9, 18, 26] for example. These equations have been studied by many mathematicians. The existence of the minimal nonnegative solutions of MAREs is proved in [6] and the existence and uniqueness of desired $M$-matrix solutions of M-QMEs is shown in [9] when these equations are associated with nonsingular or irreducible singular $M$-matrices, and the solutions can be computed by numerical methods such as fixed point iteration, Newton’s Method, Schur method, cyclic reduction, and structure-preserving doubling algorithm [6, 9, 25, 8, 11, 24, 23, 3, 16, 12, 29].

The M-QME can be studied by transforming it to an MARE. A simple transformation is used by Guo [9] to establish some theoretical results for the M-QME. The
required solution of the M-QME can be obtained by finding the minimal nonnegative solution of the MARE. That minimal solution can be found by a fixed point iteration. Recently Lu, Ahmed and Guan [25] consider the M-QME associated with a nonsingular $M$-matrix and use a more general transformation to transform the M-QME to an MARE. In doing so, they can identify a particular transformation by which the fixed point iteration can find the minimal solution of the resulting MARE much more efficiently in some situations. For the M-QME associated with a nonsingular $M$-matrix or irreducible singular $M$-matrix, it is shown in Guo [9] that the desired $M$-matrix solution of the M-QME can be found by using Newton’s method and the Schur method. For the initial value problem for a matrix Riccati differential equation associated with a nonsingular or an irreducible singular $M$-matrix, Fital and Guo [4] proved the existence of a global solution and the convergence of the global solution to the equilibrium solution, in the noncritical case. Later in [15], Guo and Yu proved the convergence for the critical case as well using the doubling procedure. The results in [4] and [15] extend the results in [19] significantly.

In this thesis we extend the study to regular singular $M$-matrices. In chapter 2 we present some basic knowledge in linear algebra, and review some useful definitions and important properties of $M$-matrix algebraic Riccati equations. In chapter 3 we show some new theoretical results for the M-QME associated with a regular singular $M$-matrix, which are then used to show the existence and uniqueness of $M$-matrix
solutions. In chapter 4 we show that some numerical methods (fixed point iteration, Newton’s method, and Schur method) will be able to get our required $M$-matrix solution. In chapter 5 we study the initial value problem (IVP) for a matrix Riccati differential equation associated with a regular singular $M$-matrix, and show that the IVP has a global solution which converges to the stable equilibrium solution which is the minimal nonnegative solution of corresponding algebraic Riccati equation. Finally, some concluding remarks are given in chapter 6.
Chapter 2

Preliminaries

Here a few important definitions and results in linear algebra will be reviewed. In particular, some results related to nonnegative matrices and $M$-matrices will be presented. After that, $M$-matrix algebraic Riccati equations (MAREs) will be introduced and some known results about these equations will be given.

2.1 Preliminaries on linear algebra

We use $\mathbb{R}$ and $\mathbb{C}$ to denote the set of all real numbers and the set of all complex numbers, respectively. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the set of all $n$-dimensional column vectors over the real and complex field, respectively. Let $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$ denote the set of all $n \times m$ real and complex matrices, respectively.

**Definition 2.1.1** Let $A \in \mathbb{C}^{n \times n}$. If a scalar $\lambda$ and a nonzero vector $x$ satisfy the
equation

\[ Ax = \lambda x, \quad x \in \mathbb{C}^n, x \neq 0, \lambda \in \mathbb{C}, \]

then \( \lambda \) is called an **eigenvalue** of \( A \) and \( x \) is called an **eigenvector** of \( A \) associated with \( \lambda \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of matrix \( A \), then its **spectral radius** \( \rho(A) \) is defined as

\[ \rho(A) = \max_{1 \leq i \leq n} |\lambda_i|. \]

**Definition 2.1.2** For \( A \in \mathbb{C}^{n \times m} \), the **kernel** of \( A \) (or the null space of \( A \)) is defined as

\[ \ker(A) = \{ x \in \mathbb{C}^m | Ax = 0 \}. \]

**Definition 2.1.3** For \( A \in \mathbb{C}^{n \times m} \) and \( B \in \mathbb{C}^{p \times q} \), the **Kronecker product** \( A \otimes B \) is the \( np \times mq \) matrix:

\[
A \otimes B = \begin{bmatrix}
    a_{11}B & \ldots & a_{1m}B \\
    \vdots & \ddots & \vdots \\
    a_{n1}B & \ldots & a_{nm}B
\end{bmatrix}.
\]

The following result can be found in [21, 22]

**Theorem 2.1.4** Let \( A \in \mathbb{C}^{m \times m} \) and \( B \in \mathbb{C}^{n \times n} \). Then the eigenvalues of \( I_n \otimes A + B \otimes I_m \) are the \( mn \) numbers \( \lambda_r + \mu_s \), where \( \lambda_r \) (\( r = 1, \ldots, m \)) and \( \mu_s \) (\( s = 1, \ldots, n \)) are the eigenvalues of \( A \) and \( B \), respectively.
Definition 2.1.5 The vectorization of an $m \times n$ matrix $A$ is given by

$$\text{vec}(A) = [a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T.$$ 

The Kronecker product can be used to rewrite linear equations in which the unknowns are matrices. For example, the matrix equation

$$AX + XB = C,$$

where $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{n \times n}$, and $C \in \mathbb{C}^{m \times n}$, is equivalent to the following system of equations:

$$[(I_n \otimes A) + (B^T \otimes I_m)] \text{vec } X = \text{vec } C.$$

Definition 2.1.6 A Jordan block $J_k(\lambda)$ is a $k \times k$ upper triangular matrix of the form

$$J_k(\lambda) = \begin{bmatrix}
\lambda & 1 \\
& \ddots & \ddots \\
& & \lambda & 1 \\
& & & \lambda
\end{bmatrix}.$$ 

A Jordan matrix $J \in \mathbb{C}^{n \times n}$ is a block diagonal matrix with each diagonal block being a Jordan block.

Theorem 2.1.7 For any $A \in \mathbb{C}^{n \times n}$, there is a nonsingular $S \in \mathbb{C}^{n \times n}$ such that

$$A = SJS^{-1},$$
where \( J_A \) is a Jordan matrix, called the **Jordan canonical form** of \( A \).

**Definition 2.1.8** Assume that \( J_A \) is the Jordan canonical form of \( A \).

1. If there is only one Jordan block of \( J_A \) associated with \( \lambda \) and it is a \( 1 \times 1 \) block, then \( \lambda \) is said to be a **simple** eigenvalue of \( A \).

2. If all Jordan blocks of \( J_A \) associated with \( \lambda \) are \( 1 \times 1 \) blocks, then \( \lambda \) is said to be a **semisimple** eigenvalue of \( A \).

**Definition 2.1.9** Let \( A \in \mathbb{C}^{n \times n} \) and a subspace \( N \subseteq \mathbb{C}^n \) is called the **invariant subspace** for the matrix \( A \) (or in short \( A \)-invariant) if \( Ax \in N \) for every \( x \in N \).

**Definition 2.1.10** A function \( \| \cdot \| : \mathbb{C}^{n \times n} \to \mathbb{R} \) is a **matrix norm** if, for all \( A, B \in \mathbb{C}^{n \times n} \), it satisfies the following:

1. \( \| A \| \geq 0, \| A \| = 0 \) if and only if \( A = 0 \).

2. \( \| cA \| = |c|\| A \| \) for all \( c \in \mathbb{C} \).

3. \( \| A + B \| \leq \| A \| + \| B \| \).

4. \( \| AB \| \leq \| A \| \| B \| \).

**Theorem 2.1.11 (Gelfand’s formula)** For any square matrix \( A \) and any matrix norm \( \| \cdot \| \), we have

\[
\rho(A) = \lim_{k \to \infty} \|A^k\|^\frac{1}{k}.
\]
We will need to use the Fréchet derivative of matrix-valued functions.

**Definition 2.1.12** A function $F : \mathbb{C}^{m \times n} \to \mathbb{C}^{m \times n}$ is called Fréchet differentiable at $X \in \mathbb{C}^{m \times n}$ if there exists a bounded linear operator $F_X' : \mathbb{C}^{m \times n} \to \mathbb{C}^{m \times n}$ such that

$$\lim_{Z \to 0} \frac{\|F(X + Z) - F(X) - F_X'(Z)\|}{\|Z\|} = 0.$$ 

The operator $F_X'$ is called the Fréchet derivative of $F$ at $X$.

### 2.2 Some basic results on nonnegative matrices and $M$-matrices

Here we review some definitions and relevant results on nonnegative matrices and $M$-matrices which are going to be used in the following chapters.

**Definition 2.2.1** For any matrices $A, B \in \mathbb{R}^{n \times m}$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for all $i, j$. When $A \geq 0$, we say $A$ is a nonnegative matrix.

**Definition 2.2.2** $A \in \mathbb{R}^{n \times n}$ is called a $Z$-matrix if all its off-diagonal elements are nonpositive.

It is easily seen that any $Z$-matrix $A$ can be written as $sI - B$, where $s$ is a scalar, $I$ is the identity matrix and $B \geq 0$.

**Definition 2.2.3** A $Z$-matrix $A = sI - B$, with $B \geq 0$, is called an $M$-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is spectral radius. It is a singular $M$-matrix if $s = \rho(B)$ and a nonsingular $M$-matrix if $s > \rho(B)$. 

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Definition 2.2.4 An $n \times n$ matrix $A$ is **reducible** if for some permutation matrix $P$, $P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where $B$ and $D$ are square matrices. Otherwise, $A$ is **irreducible**.

We have the following well-known result [1, 17, 28].

**Theorem 2.2.5** For a $Z$-matrix $A$, the following are equivalent:

1. $A$ is a nonsingular $M$-matrix.

2. $A^{-1} \geq 0$.

3. $Av > 0$ for some vector $v > 0$.

4. All eigenvalues of $A$ have positive real parts.

The next result follows from the equivalence of (1) and (3) in Theorem 2.2.5.

**Lemma 2.2.6** Let $A$ be a nonsingular $M$-matrix. If $B \geq A$ is a $Z$-matrix, then $B$ is also a nonsingular $M$-matrix.

**Theorem 2.2.7** If $A$ is an irreducible nonsingular $M$-matrix, then $A^{-1} > 0$.

**Theorem 2.2.8** (Perron-Frobenius) If $A$ is an irreducible nonnegative matrix, then $\rho(A)$ is a simple eigenvalue of $A$, and $A$ has a positive eigenvector $x$ corresponding to $\rho(A)$. 

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Lemma 2.2.9 If $B$ is a nonnegative matrix, then $\rho(B)$ is an eigenvalue of $B$ and there is a nonnegative nonzero vector $x$ such that $Bx = \rho(B)x$.

It follows that, for any singular $M$-matrix $A = \rho(B)I - B$, there is a nonnegative nonzero vector $x$ such that $Ax = 0$.

The following result is summarized from [1, 28, 4].

Theorem 2.2.10 Let $A, B, C \in \mathbb{R}^{n \times n}$ with $A \geq 0$. Then

1. If $A \leq B$, then $\rho(A) \leq \rho(B)$.

2. If $A \leq B \leq C$, $A \neq B \neq C$, and $B$ is irreducible, then $\rho(A) < \rho(B) < \rho(C)$.

3. If $Av < kv$ for a positive vector $v$, then $\rho(A) < k$.

4. If $Av = kv$ for a positive vector $v$, then $\rho(A) = k$.

We will also consider regular splittings of nonsingular $M$-matrices.

Definition 2.2.11 Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix, then $A = M - N$ is called a splitting of $A$ if $M$ is nonsingular; a regular splitting if $M$ is nonsingular, $M^{-1} \geq 0$ and $N \geq 0$.

The following theorem is a basic result for regular splittings (see [28]).
Theorem 2.2.12 Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of $A$, where $A$ is nonsingular and $A^{-1} \geq 0$. If $N_2 \geq N_1 \geq 0$, then

$$1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) \geq 0.$$ 

If, moreover, $A^{-1} > 0$ and if $N_2 \geq N_1 \geq 0$, equality excluded, then

$$1 > \rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 0.$$ 

The following results are from [4].

Lemma 2.2.13 If $A$ is a Z-matrix, then $e^{-tA} \geq 0$ for $t \geq 0$.

Lemma 2.2.14 If $A$ is a nonsingular M-matrix, then $\lim_{t \to \infty} e^{-At} = 0$.

2.3 Some known results on M-matrix algebraic Riccati equations

We consider the algebraic Riccati equation (ARE)

$$XCX - XD - AX + B = 0$$

and its dual equation

$$YBY - YA - DY + C = 0,$$

where $A, B, C, D$ are real matrices of sizes $m \times m$, $m \times n$, $n \times m$, $n \times n$, respectively. These equations are called M-matrix algebraic Riccati equations (MAREs) if the
matrix
\[
K = \begin{bmatrix}
D & -C \\
-B & A
\end{bmatrix}
\]  \hspace{1cm} (2.3)

is an \(M\)-matrix.

When \(K\) is a nonsingular or irreducible singular \(M\)-matrix, the corresponding MAREs have been studied by many researchers. Recently, Guo [10] and Guo and Lu [14] extended the study to the case where \(K\) is a regular \(M\)-matrix.

**Definition 2.3.1** An \(M\)-matrix \(A\) is said to be regular if \(Av \geq 0\) for some \(v > 0\).

Note that every nonsingular \(M\)-matrix is a regular \(M\)-matrix by Theorem 2.2.5, and that every irreducible singular \(M\)-matrix is a regular \(M\)-matrix by Theorem 2.2.8.

A nonnegative solution \(\Phi\) of (2.1) is said to be minimal if \(\Phi \leq X\) for any nonnegative solution \(X\) of (2.1). When \(K\) is a nonsingular or irreducible singular \(M\)-matrix, the corresponding MAREs are known to have minimal nonnegative solutions and several efficient methods are available to find these solution (see [6, 8, 11, 2, 12, 3, 29, 16]). It is also proved in [10] that the MARE (2.1) has a minimal nonnegative solution \(\Phi\) and the dual equation (2.2) has a minimal nonnegative solution \(\Psi\) if the matrix \(K\) in (2.3) is a regular \(M\)-matrix.
Associated with the matrix $K$ in (2.3) is the matrix

$$
H = \begin{bmatrix}
I_n & 0 \\
0 & -I_m
\end{bmatrix}
\quad K = \begin{bmatrix}
D & -C \\
B & -A
\end{bmatrix}.
$$

(2.4)

It is easy to verify the following result; see [6, 10].

$$
H \begin{bmatrix}
I_n & \psi \\
\phi & I_m
\end{bmatrix} = \begin{bmatrix}
D & -C \\
B & -A
\end{bmatrix} \begin{bmatrix}
I_n & \psi \\
\phi & I_m
\end{bmatrix} = \begin{bmatrix}
I_n & \psi \\
\phi & I_m
\end{bmatrix} \begin{bmatrix}
R & 0 \\
0 & -S
\end{bmatrix},
$$

(2.5)

where $R = D - C\phi$ and $S = A - B\psi$.

We list below some basic properties of the minimal nonnegative solutions [10].

**Theorem 2.3.2** If the matrix $K$ in (2.3) is a regular $M$-matrix, then $R = D - C\phi$ and $S = A - B\psi$ are regular $M$-matrices. Moreover, $I_m - \phi\psi$ and $I_n - \psi\phi$ are both regular $M$-matrices.

**Theorem 2.3.3** Assume that the matrix $K$ in (2.3) is a regular $M$-matrix. Let all eigenvalues of $H$ in (2.4) be arranged in a descending order by their real parts, and be denoted by $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+m}$. Then the eigenvalues of $D - C\phi$ and $D - \psi B$ are $\lambda_1, \ldots, \lambda_n$, and eigenvalues of $A - B\psi$ and $A - \phi C$ are $-\lambda_{n+1}, \ldots, -\lambda_{n+m}$.

We then know that $\lambda_n$ and $\lambda_{n+1}$ are real numbers, $D - \psi B$ is nonsingular if and only if $D - C\phi$ is nonsingular, and $A - \phi C$ is nonsingular if and only if $A - B\psi$ is nonsingular.
We now assume $K$ is a regular singular $M$-matrix. Guo [10] made an assumption to ensure that the minimal nonnegative solution of (2.1) is a continuous function of $K$ on the set of regular $M$-matrices.

**Assumption 2.3.4** The matrix $H$ in (2.4) has only one linearly independent eigenvector corresponding to the zero eigenvalue of multiplicity $r \geq 1$.

Under Assumption 2.3.4, the null spaces of $H$ and $K$ are both one-dimensional. By Lemma 2.2.9, there are nonnegative nonzero vectors

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\quad\text{and}\quad
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix},
\]

where $u_1, v_1 \in \mathbb{R}^n$ and $u_2, v_2 \in \mathbb{R}^m$, such that

\[
K
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} = 0, \quad
\begin{bmatrix}
  u_1^T & u_2^T
\end{bmatrix} K = 0. \tag{2.6}
\]

Each of these vectors is unique up to a scalar multiple under the assumption.

The open left half-plane, the open right half-plane, the closed left half-plane and the closed right half-plane will be denoted by $\mathbb{C}_<$, $\mathbb{C}_>$, $\mathbb{C}_\leq$ and $\mathbb{C}_\geq$ respectively.

Then we have the following results.

**Theorem 2.3.5** [10]. Let $K$ be a regular singular $M$-matrix with Assumption 2.3.4 and let $u_1, u_2, v_1, v_2$ be as in (2.6). Then
1. If $u_1^T v_1 > u_2^T v_2$, then $D - C \Phi$ is singular and $A - \Phi C$ is nonsingular. Moreover, $H$ has $n - 1$ eigenvalues in $\mathbb{C}_>$, $m$ eigenvalues in $\mathbb{C}_<$ and one zero eigenvalue.

2. If $u_1^T v_1 < u_2^T v_2$, then $D - C \Phi$ is nonsingular and $A - \Phi C$ is singular. Moreover, $H$ has $n$ eigenvalues in $\mathbb{C}_>$, $m - 1$ eigenvalues in $\mathbb{C}_<$ and one zero eigenvalue.

3. If $u_1^T v_1 = u_2^T v_2$, then $D - C \Phi$ and $A - \Phi C$ are both singular. Moreover, $H$ has $n - 1$ eigenvalues in $\mathbb{C}_>$, $m - 1$ eigenvalues in $\mathbb{C}_<$ and two zero eigenvalues with only one linearly independent eigenvector.

The case $u_1^T v_1 = u_2^T v_2$ is called the critical case for the MAREs (2.1) and (2.2) associated with a regular singular $M$-matrix.

**Proposition 2.3.6** [10]. Let $K$ be a regular singular $M$-matrix with Assumption 2.3.4. If $u_1^T v_1 = u_2^T v_2$, then $I_m - \Phi \Psi$ and $I_n - \Psi \Phi$ are regular singular $M$-matrices.

The next result is recently proved in [14].

**Theorem 2.3.7** Let $K$ be a regular singular $M$-matrix with Assumption 2.3.4. If $u_1^T v_1 \neq u_2^T v_2$, then $I_m - \Phi \Psi$ and $I_n - \Psi \Phi$ are nonsingular $M$-matrices.

The following result is given in [14]. See also [27] for related discussions.

**Lemma 2.3.8** Let $M$ be any $n \times n$ regular singular $M$-matrix. Then 0 is a semisimple eigenvalue of $M$. 

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Assumption 2.3.4 is clarified in the next result [14].

**Lemma 2.3.9** Let $K$ of (2.3) be a regular singular M-matrix. Then Assumption 2.3.4 holds if and only if 0 is a simple eigenvalue of $K$.

We again let $\Phi$, $\Psi$ be the minimal nonnegative solutions of (2.1) and (2.2). From (2.5) we have

\[ H \begin{bmatrix} I_n \\ \Phi \end{bmatrix} = \begin{bmatrix} I_n \\ \Phi \end{bmatrix} R, \quad (2.7) \]

\[ H \begin{bmatrix} \Psi \\ I_m \end{bmatrix} = \begin{bmatrix} \Psi \\ I_m \end{bmatrix} (-S), \quad (2.8) \]

where $H$ is defined in (2.4), and $R = D - C \Phi$, $S = A - B \Psi$. 
Chapter 3

Theoretical results for $M$-matrix quadratic matrix equations

In this chapter we will present a number of new results on a quadratic matrix equation $X^2 - EX - F = 0$, mainly by applying the results in the previous chapter. We will prove the existence and uniqueness of $M$-matrix solutions of the quadratic matrix equation $X^2 - EX - F = 0$, where $E$ is a diagonal matrix and $F$ is a regular singular $M$-matrix with a simple zero eigenvalue. The equation has been studied in [9] when $F$ is a nonsingular $M$-matrix or an irreducible singular $M$-matrix. Recall that an irreducible singular $M$-matrix is always a regular singular $M$-matrix with a simple zero eigenvalue, but not the other around. So our results are extensions of those in [9].
3.1 A result for regular singular $M$-matrix AREs

We start with the following result on regular singular $M$-matrix AREs.

**Theorem 3.1.1** Let $K$ be a regular singular $M$-matrix with a simple zero eigenvalue.

Then the matrix

$$I \otimes (A - \Phi C) + (D - C \Phi)^T \otimes I$$

(3.9)

is nonsingular (singular) $M$-matrix when $u_1^T v_1 \neq u_2^T v_2$ ($u_1^T v_1 = u_2^T v_2$).

**Proof** By Lemma 2.3.9, Theorem 2.3.5 applies. For case (1), that is when $u_1^T v_1 > u_2^T v_2$, $D - C \Phi$ is singular and $A - \Phi C$ is a nonsingular $M$-matrix. Hence by Theorem 2.1.4, the matrix (3.9) is a nonsingular $M$-matrix. Similarly for case (2), that is when $u_1^T v_1 < u_2^T v_2$, $D - C \Phi$ is a nonsingular and $A - \Phi C$ is a singular $M$-matrix. Hence by Theorem 2.1.4, the matrix (3.9) is a nonsingular $M$-matrix. Finally for case (3), that is when $u_1^T v_1 = u_2^T v_2$, $D - C \Phi$ and $A - \Phi C$ are both singular $M$-matrices. Hence by Theorem 2.1.4, the matrix (3.9) is a singular $M$-matrix. \qed

3.2 Results for regular singular $M$-matrix QMEs

We now consider the $M$-matrix quadratic matrix equation (M-QME)

$$X^2 - EX - F = 0,$$

(3.10)
where $E, F \in \mathbb{R}^{n \times n}$, $E$ is a diagonal matrix and $F$ is a regular $M$-matrix. We will look for $M$-matrix solutions $X \in \mathbb{R}^{n \times n}$.

To apply the results of the previous chapter, we will change the problem of finding an $M$-matrix solution of (3.10) to that of finding a nonnegative solution of another quadratic matrix equation, as in [9] and [25].

Let

$$Y = \Delta - X,$$

(3.11)

where $\Delta = \text{diag}(\delta_1, \delta_2, \cdots, \delta_n)$ is a positive diagonal matrix. Substituting (3.11) into M-QME (3.10), we have

$$Y^2 - Y\Delta - (\Delta - E)Y + \Delta^2 - E\Delta - F = 0.$$

(3.12)

This equation has the form of equation (2.1) and the matrix $K$ in (2.3) is now

$$R = \begin{bmatrix}
\Delta & -I \\
F + E\Delta - \Delta^2 & \Delta - E
\end{bmatrix}.
$$

(3.13)

If we choose the positive diagonal entries $\delta_i (i = 1, 2, \cdots, n)$ such that

$$F + E\Delta - \Delta^2 \leq 0,$$

(3.14)

then we easily verify that $R$ of (3.13) is a $Z$-matrix. Since $F$ is an $M$-matrix, (3.14) holds if and only if $f_{ii} + e_i \delta_i - \delta_i^2 \leq 0$, for $i = 1, 2, \cdots, n$, where $e_i$ and $f_{ii}$ are diagonal entries of $E$ and $F$ respectively.
Therefore (3.14) holds if and only if
\[ \delta_i \geq \frac{e_i + \sqrt{e_i^2 + 4f_{ii}}}{2}, \text{ for } i = 1, 2, \ldots, n. \] (3.15)

We will always choose \( \delta_i \) such that (3.15) holds.

**Remark 3.2.1** If \( f_{ii} = 0 \) and \( e_i \leq 0 \), we have \( e_i + \sqrt{e_i^2 + 4f_{ii}} = 0 \). In that case, we take \( \delta_i = 1 \). Then \( \delta_i > 0 \) holds for all \( i \).

**Lemma 3.2.2** If \( F \) is a regular \( M \)-matrix then \( R \) is also a regular \( M \)-matrix.

**Proof** Since \( \delta_i \) satisfies (3.15), we already know that \( R \) of (3.13) is a \( Z \)-matrix. Since \( F \) is a regular \( M \)-matrix, by Definition 2.3.1, there is a vector \( v > 0 \) such that \( Fv \geq 0 \).

Since \( v > 0 \) and \( \Delta \) is a positive diagonal matrix, the vector \( z = \begin{bmatrix} v \\ \Delta v \end{bmatrix} > 0 \).

Now we have
\[
\begin{bmatrix}
\Delta & -I \\
F + E\Delta - \Delta^2 & \Delta - E
\end{bmatrix}
\begin{bmatrix}
v \\
\Delta v
\end{bmatrix}
= \begin{bmatrix}
\Delta v - \Delta v \\
Fv + E\Delta v - \Delta^2 v + \Delta^2 v - E\Delta v
\end{bmatrix}
= \begin{bmatrix}
0 \\
\Delta v
\end{bmatrix}
\geq 0.
\]

This shows the vector \( z > 0 \) is such that \( Rz \geq 0 \). Hence by Definition 2.3.1, \( R \) is also a regular \( M \)-matrix. \( \square \)
Lemma 3.2.3  Two non-zero vectors $w_1$ and $w_2$ are linearly independent if and only if \[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix}
\] are linearly independent vectors. Where $\Delta$ is a positive diagonal matrix.

Proof  I am going to prove it by contradiction. Suppose $w_1$ and $w_2$ are not linearly independent vectors. That is, $w_1 = kw_2$, where $k$ is a non-zero real number. Then we have
\[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix} = k
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix}.
\]
Thus \[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix}
\] are not linearly independent vectors. Hence if $w_1$ and $w_2$ are linearly independent then \[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix}
\] are also linearly independent vectors.

Conversely, suppose \[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix}
\] are not linearly independent vectors. That is,
\[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix} = k
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix},
\] where $k$ is a non-zero real number. Now if we equate the corresponding elements then we get $w_1 = kw_2$. That implies that $w_1$ and $w_2$ are not
linearly independent vectors. Hence if \[
\begin{bmatrix}
w_1 \\
\Delta w_1
\end{bmatrix}
\]
and \[
\begin{bmatrix}
w_2 \\
\Delta w_2
\end{bmatrix}
\]
are linearly independent vectors then \(w_1\) and \(w_2\) are also linearly independent vectors. \(\square\)

**Lemma 3.2.4** If \(F\) is a regular singular \(M\)-matrix with a simple zero eigenvalue then \(R\) of (3.13) is also a regular singular \(M\)-matrix with a simple zero eigenvalue.

**Proof** I will prove it by contradiction. By Lemma 3.2.2, \(R\) is a regular \(M\)-matrix since \(F\) is a regular \(M\)-matrix. Suppose \(R\) has a zero eigenvalue with multiplicity at least two. Then, by Lemma 2.3.8, \(R\) has two linearly independent eigenvectors \(w_1\) and \(w_2\) corresponding to the zero eigenvalue. That is, \(Rw_1 = 0\) and \(Rw_2 = 0\). Let
\[
w_1 = \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix}, \text{where} \ w_{11}, w_{12} \in \mathbb{R}^n.\]
Then we have
\[
\begin{bmatrix}
\Delta & -I \\
F + E\Delta - \Delta^2 & \Delta - E
\end{bmatrix}
\begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix}
= \begin{bmatrix} \Delta w_{11} - w_{12} \\ Fw_{11} + E\Delta w_{11} - \Delta^2 w_{11} + \Delta w_{12} - Ew_{12} \end{bmatrix} = 0.
\]
Here it is clear that \(\Delta w_{11} = w_{12}\) and \(Fw_{11} + E\Delta w_{11} - \Delta^2 w_{11} + \Delta w_{12} - Ew_{12} = 0\). Then last equation implies \(Fw_{11} = 0\).

Similarly, if we assume \(w_2 = \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix}\), where \(w_{21}, w_{22} \in \mathbb{R}^n\). Then we will get \(Fw_{21} = 0\).
Now we can see that the linearly independent eigenvectors $w_1$ and $w_2$ of $R$ corresponding to the zero eigenvalue are in the form of 

\[
\begin{bmatrix}
w_{11} \\
\Delta w_{11}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
w_{21} \\
\Delta w_{22}
\end{bmatrix}
\]

respectively.

Since we assume that $w_1$ and $w_2$ are two linearly independent eigenvectors of $R$ corresponding to the zero eigenvalue then, by Lemma 3.2.3, $w_{11}$ and $w_{21}$ are two linearly independent eigenvectors of $F$ corresponding to the simple zero eigenvalue. Here is a contradiction. Hence if $F$ is a regular singular $M$-matrix with a simple zero eigenvalue then $R$ of (3.13) is also a regular singular $M$-matrix with a simple zero eigenvalue. □

For the MARE (3.12), which has the form of (2.1), the matrix $H$ in (2.4) is now

\[
L = 
\begin{bmatrix}
\Delta & -I \\
\Delta^2 - E\Delta - F & E - \Delta
\end{bmatrix}
\]

(3.16)

**Proposition 3.2.5** If $F$ is a regular singular $M$-matrix with a simple zero eigenvalue, then the matrix $L$ in (3.16) has only one linearly independent eigenvector corresponding to its zero eigenvalue of multiplicity $r \geq 1$.

**Proof** Let $F$ be a regular singular $M$-matrix with a simple zero eigenvalue. Then, by Lemma 3.2.4, $R$ of (3.13) is also a regular singular $M$-matrix with a simple zero eigenvalue...
eigenvalue. Thus, by Lemma 2.3.9, the matrix $L$ in (3.16) has only one linearly independent eigenvector corresponding to its zero eigenvalue of multiplicity $r \geq 1$. □

Note that $\Delta - Y$ is a solution of (3.10) whenever $Y$ is a solution of (3.12). First of all, (3.12) has a minimal nonnegative solution $S_\Delta$ when $F$ is a regular $M$-matrix, since $R$ of (3.13) is also a regular $M$-matrix in this case.

**Theorem 3.2.6** Let $F$ be a regular singular $M$-matrix with a simple zero eigenvalue and let $u$ and $v$ be nonnegative nonzero vectors such that $Fv = 0$ and $u^TF = 0$. Then

1. If $u^TEv = 0$, then $L$ has $n - 1$ eigenvalues in $\mathbb{C}_>$, $n - 1$ eigenvalues in $\mathbb{C}_<$ and two zero eigenvalues with only one linearly independent eigenvector, $\Delta - S_\Delta$ and $\Delta - E - S_\Delta$ are regular singular $M$-matrices.

2. If $u^TEv < 0$, then $L$ has $n - 1$ eigenvalues in $\mathbb{C}_>$, $n$ eigenvalues in $\mathbb{C}_<$ and one zero eigenvalue, $\Delta - S_\Delta$ is a regular singular $M$-matrix.

3. If $u^TEv > 0$, then $L$ has $n$ eigenvalues in $\mathbb{C}_>$, $n - 1$ eigenvalues in $\mathbb{C}_<$ and one zero eigenvalue, $\Delta - S_\Delta$ is a nonsingular $M$-matrix.

**Proof** We apply Theorem 2.3.5 to (3.12) (so $K = R$ and $H = L$). It is clear that we can take $v_1 = v$, $v_2 = \Delta v$, $u_1^T = u^T(\Delta - E)$ and $u_2^T = u^T$ in Theorem 2.3.5. Thus $u_1^Tv_1 - u_2^Tv_2 = -u^TEv)$. The conclusion follows directly from Theorem 2.3.2 and 2.3.5. □

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Theorem 3.2.7  Let $F$ be a regular singular $M$-matrix with a simple zero eigenvalue. Suppose that $u^T Ev = 0$. Then $0$ is a simple eigenvalue of $X = \Delta - S_\Delta$ and $X - E = \Delta - E - S_\Delta$, and is a double eigenvalue of $L$.

Proof Since $X$ is a solution of (3.10), we now have

$$L \begin{bmatrix} I & 0 \\ \Delta - X & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Delta - X & I \end{bmatrix} \begin{bmatrix} X & -I \\ 0 & -(X - E) \end{bmatrix}. \quad \text{(3.17)}$$

Since $u^T Ev = 0$, we know from Theorem 3.2.6 that the regular $M$-matrices $X = \Delta - S_\Delta$ and $X - E = \Delta - E - S_\Delta$ are both singular. Suppose that $0$ is not a simple eigenvalue of $X$. Then by Lemma 2.3.8 there are linearly independent vectors $w_1$ and $w_2$ such that $Xw_1 = Xw_2 = 0$. The equation (2.7) is now

$$L \begin{bmatrix} I \\ \Delta - X \end{bmatrix} = \begin{bmatrix} I \\ \Delta - X \end{bmatrix} X. \quad \text{(3.18)}$$

It follows from the equation (3.18) that

$$\begin{bmatrix} I \\ \Delta - X \end{bmatrix} w_1, \begin{bmatrix} I \\ \Delta - X \end{bmatrix} w_2$$

are two linearly independent eigenvectors of $L$ corresponding to the zero eigenvalue. Since $F$ has a simple zero eigenvalue, this is contradicting to Proposition 3.2.5. Therefore, $0$ is a simple eigenvalue of $X$. Similarly, we can show that $0$ is a simple eigenvalue of $X - E$, using the equation (2.8). It is clear from (3.17) that $0$ is a double eigenvalue of $L$. \qed
Therefore for the critical case \((u^T Ev = 0)\), we know that

1. \(X\) has a simple zero eigenvalue and \(n - 1\) eigenvalues in the open right half plane.

2. \(X - E\) has a simple zero eigenvalue and \(n - 1\) eigenvalues in the open right half plane.

3. \(L\) has one \(2 \times 2\) Jordan block associated with the zero eigenvalue of multiplicity 2.

### 3.3 The existence and uniqueness of \(M\)-matrix solutions of \(M\)-QMEs

Since

\[
\begin{bmatrix}
I & 0 \\
\Delta & I
\end{bmatrix}^{-1} L \begin{bmatrix}
I & 0 \\
\Delta & I
\end{bmatrix} = \begin{bmatrix}
0 & -I \\
-F & E
\end{bmatrix} \equiv W, 
\] (3.19)

the matrices \(L\) and \(W\) have the same eigenvalues.

**Theorem 3.3.1** If \(F\) is a regular singular \(M\)-matrix with a simple zero eigenvalue then \((3.10)\) has \(M\)-matrix solutions. For case (1) or case (2) of Theorem 3.2.6, \((3.10)\) has exactly one \(M\)-matrix as its solution and the \(M\)-matrix is singular. For case (3) of Theorem 3.2.6, \((3.10)\) has exactly one nonsingular \(M\)-matrix as its solution.
Proof For any solution $X$ of (3.10),

$$W \begin{bmatrix} I & -X \\ -X & -X \end{bmatrix} = \begin{bmatrix} I & -X \end{bmatrix} X.$$  \hspace{1cm} (3.20)

Therefore, for any $M$-matrix solution $X$ of (3.10), the column space of $(I - X^T)^T$ is an invariant subspace of $W$ associated with $n$ eigenvalues of $W$ in $\mathbb{C}_\geq$.

If $F$ is a regular singular $M$-matrix with a simple zero eigenvalue, then from Theorem 3.2.6, $\Delta - S_\Delta$ is an $M$-matrix. By the relationship between (3.10) and (3.12), $\Delta - S_\Delta$ is a solution of (3.10). For case (1) or case (2), each $M$-matrix solution must be singular, since $W$ has only $n - 1$ eigenvalues in $\mathbb{C}_>$ by Theorem 3.2.6 and (3.19). For case (2), the uniqueness is also clear since $W$ has only $n$ eigenvalues in $\mathbb{C}_\geq$. For case (1), the uniqueness is true since $W$ has $n - 1$ eigenvalues in $\mathbb{C}_>$ but the solution $X$ of (3.10) has $n - 1$ eigenvalues in $\mathbb{C}_>$ and only one zero eigenvalue by Theorem 3.2.7. Similarly for case (3), $\Delta - S_\Delta$ is a nonsingular $M$-matrix solution of (3.10). Equation (3.10) has no other nonsingular $M$-matrix solutions since $W$ has only $n$ eigenvalues in $\mathbb{C}_\geq$. \hfill \Box

For the application to noisy Wiener-Hopf problems for Markov chains, when (3.10) has both nonsingular and singular $M$-matrix solutions, the required solution is a nonsingular $M$-matrix solution.

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Now we can say that the required $M$-matrix solution of (3.10) is uniquely identifiable for all situations in Theorem 3.2.6 and will be denoted by $X_M$. Note that $X_M = \Delta - S_\Delta$.

**Proposition 3.3.2** The matrix $I \otimes (X_M - E) + (X_M)^T \otimes I$ is a singular $M$-matrix for case (1) of Theorem 3.2.6, and is a nonsingular $M$-matrix for the other two cases in Theorem 3.2.6.

**Proof** We can easily conclude the proof by applying Theorem 3.1.1. The matrix $D - C \Phi$ there is now $X_M$ and the matrix $A - \Phi C$ there is now $X_M - E$, and the number $u_1^T v_1 - u_2^T v_2$ is now $-u^T E v$. Alternatively, from Theorem 3.2.6, $I \otimes (X_M - E) + (X_M)^T \otimes I$ is a singular $M$-matrix for case (1) and it is a nonsingular $M$-matrix for case (2) and case (3) since we see from (3.17) that only one of $X_M$ and $X_M - E$ is singular. \qed
Chapter 4

Numerical methods for $M$-matrix quadratic matrix equations

In the last chapter we showed some new results and uniqueness of $M$-matrix solutions of the quadratic matrix equation (3.10). In this chapter we are going to discuss some numerical methods for finding the required $M$-matrix solution. We will consider fixed point iteration and Newton method, and examine their convergence. We will also consider the Schur method.

4.1 Fixed point iteration

Fixed point iteration has been studied in [6, 13, 9, 25] for some matrix equations. In particular, fixed point iteration has been studied in [9] for the matrix equation (3.10) with $F$ being a nonsingular $M$-matrix or an irreducible singular $M$-matrix.
Further analysis was done in [25] when $F$ is nonsingular $M$-matrix. Here we assume $F$ is a regular singular $M$-matrix with a simple zero eigenvalue.

Recall that the required $M$-matrix solution of (3.10) is $X_M = \Delta - S_\Delta$ where $\Delta$ satisfies (3.15) and $S_\Delta$ is the minimal nonnegative solution of (3.12). To find $S_\Delta$, we consider the fixed point iteration

$$Y_{k+1}\Delta + (\Delta - E)Y_{k+1} = Y_k^2 + \Delta^2 - E\Delta - F$$

(4.21)

with $Y_0 = 0$, as in [25].

Since $\Delta$ is a positive diagonal matrix and $\Delta - E$ is a nonnegative diagonal matrix, we can determine $Y_{k+1}$ from $Y_k$ by

$$Y_{k+1}(i, j) = \begin{cases} 
\frac{Y_k^2(i,i) + \Delta^2(i,i) - (E\Delta)(i,i) - F(i,i)}{2\Delta(i,i) - E(i,i)}, & i = j \\
\frac{Y_k^2(i,j) - F(i,j)}{\Delta(i,i) + \Delta(j,j) - E(i,i)}, & i \neq j
\end{cases}$$

for $1 \leq i, j \leq n$, and $k = 0, 1, \cdots$, Here $A(i, j)$ denotes $(i, j)$th entry of $A$.

**Theorem 4.1.1** Let $\{Y_k\}$ be the sequence defined by the fixed point iteration (4.21) with $Y_0 = 0$. Then the sequence $\{Y_k\}$ is well defined, $0 \leq Y_k \leq Y_{k+1}$, $Y_k \leq S_\Delta$ and $\{Y_k\}$ converges to $S_\Delta$, the minimal nonnegative solution of the MARE (3.12).
Proof Since \( S_\Delta \) is a solution of (3.12), we have

\[
S_\Delta^2 - S_\Delta \Delta - (\Delta - E)S_\Delta + \Delta^2 - E\Delta - F = 0.
\] (4.22)

For the fixed point iteration (4.21) with \( Y_0 = 0 \), we have

\[
Y_1 \Delta + (\Delta - E)Y_1 = \Delta^2 - E\Delta - F.
\]

Since \( \Delta^2 - E\Delta - F \geq 0 \) and \( Y_1(i,j) = \frac{(\Delta^2 - E\Delta - F)(i,j)}{\Delta(i,i) + \Delta(j,j) - E(i,i)} \), we have \( Y_1 \geq 0 \). Then we have \( Y_0 \leq S_\Delta, Y_0 \leq Y_1 \).

Now we will prove by induction that for any \( k \geq 0 \)

\[
0 \leq Y_k \leq Y_{k+1}, Y_k \leq S_\Delta.
\] (4.23)

It is true when \( k = 0 \). Assume that (4.23) is true for \( k = i \geq 0 \). Now by (4.21) and (4.22) we have

\[
(\Delta - E)(Y_{i+1} - S_\Delta) + (Y_{i+1} - S_\Delta)\Delta
= (\Delta - E)Y_{i+1} + Y_{i+1}\Delta - (\Delta - E)S_\Delta - S_\Delta\Delta
= Y_i^2 + \Delta^2 - E\Delta - F - (\Delta - E)S_\Delta - S_\Delta\Delta
= S_\Delta^2 - S_\Delta\Delta - (\Delta - E)S_\Delta + \Delta^2 - E\Delta - F - S_\Delta^2 + Y_i^2
= Y_i^2 - S_\Delta^2
\leq 0
\]
since $Y_i \leq S\Delta$. That implies $Y_{i+1} \leq S\Delta$. By (4.21) we have

$$(\Delta - E)(Y_{i+1} - Y_{i+2}) + (Y_{i+1} - Y_{i+2})\Delta$$

$$= -[(\Delta - E)Y_{i+2} + Y_{i+2})\Delta] + (\Delta - E)Y_{i+1} + Y_{i+1}\Delta$$

$$= -Y_{i+1}^2 - (\Delta^2 - E\Delta - F) + (\Delta - E)Y_{i+1} + Y_{i+1}\Delta$$

$$= Y_i^2 + (\Delta^2 - E\Delta - F) - Y_{i+1}^2 - (\Delta^2 - E\Delta - F)$$

$$= Y_i^2 - Y_{i+1}^2$$

$$\leq 0.$$ 

That implies $Y_{i+1} \leq Y_{i+2}$. We have thus proved that (4.23) is true for $k = i + 1$. Hence it is true for all $k \geq 0$. Therefore, the sequence $\{Y_k\}$ is well defined, monotonically increasing and bounded above by $S\Delta$. Let $\lim_{k \to \infty} Y_k = Y_*$. Then $Y_*$ is a nonnegative solution of (3.12) by (4.21). Since $Y_* \leq S\Delta$ and $S\Delta$ is minimal, that implies $Y_* = S\Delta$. \[\square\]

By using the same argument as in [13] and applying Theorem 2.1.11, we have

$$\limsup_{k \to \infty} \|Y_k - S\Delta\|^1 \leq \rho\Delta,$$  \hspace{1cm} (4.24)

where

$$\rho\Delta \equiv \rho((I \otimes (\Delta - E) + \Delta^T \otimes I)^{-1}(I \otimes S\Delta + S\Delta^T \otimes I))$$  \hspace{1cm} (4.25)
and $\otimes$ is the Kronecker product. Moreover, as in [13], equality holds in (4.24) if $S_\Delta$ is positive (which is true if $F$ is irreducible; see [7] and [9]).

The next results shows how $S_\Delta$ changes when $\Delta$ changes.

**Lemma 4.1.2** Let $E$ be a diagonal matrix and $F$ be a regular singular $M$-matrix with a simple zero eigenvalue. If $\Delta_1$ and $\Delta_2$ are diagonal matrices whose diagonal entries satisfying (3.15), and $\Delta_1 \geq \Delta_2$. Then $S_{\Delta_1} \geq S_{\Delta_2}$.

**Proof** Since $\Delta_1 - S_{\Delta_1} = \Delta_2 - S_{\Delta_2} = X_M$, we have $\Delta_1 - \Delta_2 = S_{\Delta_1} - S_{\Delta_2}$ and the result follows. \qed

We would like to choose $\Delta$ such that $\rho_\Delta$ in (4.25) is as small as possible.

**Proposition 4.1.3** For case (1) of Theorem 3.2.6, $\rho_\Delta = 1$ for all $\Delta$ satisfying (3.15).

For all other situations in Theorem 3.2.6 and $\Delta_1 \geq \Delta_2$, we have $1 > \rho_{\Delta_1} \geq \rho_{\Delta_2}$, and $1 > \rho_{\Delta_1} > \rho_{\Delta_2}$ if $F$ is irreducible and $\Delta_1 \neq \Delta_2$.

**Proof** Let

$$T = (I \otimes (\Delta - E) + \Delta^T \otimes I) - (I \otimes S_\Delta + S_\Delta^T \otimes I) = R_1(\Delta) - R_2(\Delta),$$

where $R_1(\Delta) = (I \otimes (\Delta - E) + \Delta^T \otimes I)$ and $R_2(\Delta) = (I \otimes S_\Delta + S_\Delta^T \otimes I)$.\[33\]
Since $X_M = \Delta - S\Delta$, $T$ can be written as $T = I \otimes (X_M - E) + (X_M)^T \otimes I$. This implies $T$ is independent of $\Delta$, and $T$ can have splittings

$$T = R_1(\Delta_1) - R_2(\Delta_1) = R_1(\Delta_2) - R_2(\Delta_2). \quad (4.26)$$

For case (1) of Theorem 3.2.6, $T$ is a singular $M$-matrix by proposition 3.3.2. Thus $Tw = 0$ for some nonnegative nonzero vector $w$. It follows that

$$\left( I \otimes (\Delta - E) + \Delta^T \otimes I \right)^{-1} (I \otimes S\Delta + S^T\Delta \otimes I)w = w$$

and thus $\rho_\Delta = 1$. For case (2) and case (3) $T$ is a nonsingular $M$-matrix by proposition 3.3.2. Hence by Theorem 2.2.5, $T^{-1} \geq 0$. For $\Delta = \Delta_1$ or $\Delta_2$, since $\Delta$ is positive diagonal matrix, $\Delta - E$ is a nonnegative diagonal matrix, and $S\Delta$ is a nonnegative matrix, we have $R_1(\Delta)^{-1} \geq 0$ and $R_2(\Delta) \geq 0$. Hence

$$T = R_1(\Delta) - R_2(\Delta) = R_1(\Delta_1) - R_2(\Delta_1) = R_1(\Delta_2) - R_2(\Delta_2)$$

are regular splittings of $T$.

If $\Delta_1 \geq \Delta_2$, by Lemma 4.1.2, $S\Delta_1 \geq S\Delta_2$. We easily verify that $R_2(\Delta_1) \geq R_2(\Delta_2)$. By Theorem 2.2.12, we have $\rho(R_1(\Delta_1)^{-1} R_2(\Delta_1)) \geq \rho(R_1(\Delta_2)^{-1} R_2(\Delta_2))$, that is $\rho_{\Delta_1} \geq \rho_{\Delta_2}$.

If $F$ is irreducible, then $X_M = \Delta - S\Delta$ and $X - E = \Delta - E - S\Delta$ are irreducible $M$-matrices (see [9] and [25]). For case (2) and case (3), $T = I \otimes (X_M - E) + (X_M)^T \otimes I$
is then an irreducible nonsingular $M$-matrix and by Theorem 2.2.7, $T^{-1} > 0$. Finally since $\Delta_1 \geq \Delta_2 > 0$ and $\Delta_1 \neq \Delta_2$, by Theorem 2.2.12, we have $\rho(R_1(\Delta_1)^{-1}R_2(\Delta_1)) > \rho(R_1(\Delta_2)^{-1}R_2(\Delta_2))$, that is $\rho_{\Delta_1} > \rho_{\Delta_2}$. \hfill $\square$

This proposition implies that we should take $\Delta = \text{diag}(\delta_1, \delta_2, \cdots, \delta_n)$ in (4.21) to satisfy

$$\delta_i = (e_i + \sqrt{e_i^2 + 4f_{ii}})/2 > 0, \text{ for } i = 1, 2, \ldots, n$$

such that $\rho_\Delta$ is minimal, just like for the case when $F$ is a nonsingular $M$-matrix, studied in [25].

### 4.2 Newton’s method

The convergence of the fixed point iteration may be very slow. Other solution methods are thus needed. One efficient method for finding $X_M$ is to find $S_\Delta$ by Newton’s method.

Newton’s method has been studied in [6, 11, 13, 9] for the MARE (2.1) and M-QME (3.10). In [9] the matrix $F$ in (3.10) is assumed to be a nonsingular $M$-matrix or an irreducible singular $M$-matrix. Here we consider the case where $F$ is a regular singular $M$-matrix with a simple zero eigenvalue.

First we consider the application of Newton’s method to MARE (2.1). We know the Riccati function $\mathcal{F}$, given by
\[ \mathcal{F}(X) = XCX - AX - XD + B, \]  

(4.28) is a mapping from \( \mathbb{R}^{m \times n} \) into \( \mathbb{R}^{m \times n} \). By Definition 2.1.12 the Fréchet derivative of \( \mathcal{F} \) at a matrix \( X \) is a linear map \( \mathcal{F}'_X : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n} \) given by

\[ \mathcal{F}'_X(Z) = -((A - XC)Z + Z(D - CX)). \]  

(4.29)

The Newton method is well-known as

\[ X_{i+1} = X_i - (\mathcal{F}'_X)\mathcal{F}(X_i), \quad i = 0, 1, \ldots, \]  

(4.30) where the maps \( \mathcal{F}'_X \) all need to be invertible. In view of (4.29), the iteration (4.30) is equivalent to

\[ (A - X_iC)X_{i+1} + X_{i+1}(D - CX_i) = B - X_iCX_i, \quad i = 0, 1, \ldots \]  

(4.31)

To find \( S_{\Delta} \), we can apply (4.31) to (3.12). Set \( Y_0 = 0 \). For \( i = 0, 1, 2, \ldots \), until convergence, compute \( Y_{i+1} \) from the equation

\[ (\Delta - E - Y_i)Y_{i+1} + Y_{i+1}(\Delta - Y_i) = \Delta^2 - E\Delta - F - Y_i^2. \]  

(4.32)

**Theorem 4.2.1** Let \( F \) be a regular singular \( M \)-matrix with a simple zero eigenvalue. Assume that we have the non critical case \((u^T Ev \neq 0 \text{ in Theorem 3.2.6})\). Let \( S_{\Delta} \) be the minimal nonnegative solution of (3.12). Then for the Newton iteration (4.32) with \( Y_0 = 0 \), the sequence \( \{Y_i\} \) is well defined, \( Y_i \leq Y_{i+1} \leq S_{\Delta} \) for all \( i \geq 0 \), and

\[ \lim_{i \to \infty} Y_i = S_{\Delta}. \]
**Proof** We use the notation $M_Y = I \otimes (\Delta - E - Y) + (\Delta - Y)^T \otimes I$ for a given matrix $Y$, where $\otimes$ is Kronecker product. Since $S_\Delta$ is the minimal nonnegative solution of (3.12),

$$S_\Delta^2 - S_\Delta \Delta - (\Delta - E)S_\Delta + \Delta^2 - E\Delta - F = 0. \quad (4.33)$$

From Proposition 3.3.2 and $X_M = \Delta - S_\Delta$ it is clear that $M_{S_\Delta}$ is a nonsingular $M$-matrix.

For Newton iteration (4.32) with $Y_0 = 0$, we have

$$(\Delta - E)Y_1 + Y_1 \Delta = \Delta^2 - E\Delta - F$$

This equation is equivalent to

$$(I \otimes (\Delta - E) + \Delta^T \otimes I)\text{vec}(Y_1) = \text{vec}(\Delta^2 - E\Delta - F). \quad (4.34)$$

Where the vec operator stacks the columns of a matrix into a long vector (see Definition 2.1.5). Since $\Delta$ is positive diagonal matrix and $\Delta - S_\Delta$ is a nonnegative diagonal matrix, $I \otimes (\Delta - E) + \Delta^T \otimes I$ is a positive diagonal matrix and thus a nonsingular $M$-matrix. Recall that $\Delta^2 - E\Delta - F \geq 0$. We get from (4.34) that $Y_1 \geq 0$. Therefore the statement

$$Y_k \leq Y_{k+1}, Y_k \leq S_\Delta, M_Y \text{ is a nonsingular } M\text{-matrix} \quad (4.35)$$
is true for $k = 0$.

We now assume that (4.35) is true for $k = i \geq 0$. By (4.32) and (4.33) we have

\[
(\Delta - E - Y_i)(Y_{i+1} - S_\Delta) + (Y_{i+1} - S_\Delta)(\Delta - Y_i)
\]

\[
= (\Delta - E - Y_i)Y_{i+1} + Y_{i+1}(\Delta - Y_i) - (\Delta - E)S_\Delta + Y_iS_\Delta - S_\Delta \Delta + S_\Delta Y_i
\]

\[
= \Delta^2 - E\Delta - F - Y_i^2 - (\Delta - E)S_\Delta + Y_iS_\Delta - S_\Delta \Delta + S_\Delta Y_i
\]

\[
= \Delta^2 - E\Delta - F + S_\Delta^2 - (\Delta - E)S_\Delta - S_\Delta \Delta - S_\Delta^2 - Y_i^2 + Y_iS_\Delta + S_\Delta Y_i
\]

\[
= -S_\Delta^2 - Y_i^2 + Y_iS_\Delta + S_\Delta Y_i
\]

\[
= -(S_\Delta - Y_i)(S_\Delta - Y_i).
\]

Since $Y_i \leq S_\Delta$ and $M_{Y_i}$ is a nonsingular $M$-matrix, it follows that $Y_{i+1} \leq S_\Delta$. Since $M_{Y_{i+1}} \geq M_{S_\Delta}$, by Lemma 2.2.6 we know that $M_{Y_{i+1}}$ is also a nonsingular $M$-matrix.
It follows from (4.32) that

\[(\Delta - E - Y_{i+1})(Y_{i+1} - Y_{i+2}) + (Y_{i+1} - Y_{i+2})(\Delta - Y_{i+1})\]

\[= (\Delta - E - Y_{i+1})Y_{i+1} + Y_{i+1}(\Delta - Y_{i+1}) - (\Delta - E - Y_{i+1})Y_{i+2} - Y_{i+2}(\Delta - Y_{i+1})\]

\[= (\Delta - E - Y_{i+1})Y_{i+1} + Y_{i+1}(\Delta - Y_{i}) + (Y_{i} - Y_{i+1})Y_{i+1} + Y_{i+1}(Y_{i} - Y_{i+1})\]

\[-(\Delta - E - Y_{i+1})Y_{i+2} - Y_{i+2}(\Delta - Y_{i+1})\]

\[= (\Delta^2 - E\Delta - F - Y_{i}^2) - (\Delta^2 - E\Delta - F - Y_{i+1}^2) + (Y_{i} - Y_{i+1})Y_{i+1} + Y_{i+1}(Y_{i} - Y_{i+1})\]

\[= -(Y_{i+1}^2 - Y_{i}Y_{i+1} - Y_{i+1}Y_{i} + Y_{i}^2)\]

\[= -(Y_{i+1} - Y_{i})(Y_{i+1} - Y_{i})\]

\[\leq 0.\]

Therefore, \( Y_{i+1} \leq Y_{i+2} \). We have thus proved that (4.35) is true for \( k = i + 1 \). Hence (4.35) is true for all \( k \geq 0 \) by induction.

Therefore the Newton sequence \( \{Y_K\} \) is well defined, monotonically increasing and bounded above by \( S_\Delta \). Let \( \lim_{k \to \infty} Y_k = Y^*_s \). Then \( Y^*_s \) is a nonnegative solution of (3.12) by (4.32). Since \( Y_s \leq S_\Delta \) and \( S_\Delta \) is minimal, that implies \( Y_s = S_\Delta \). \qed

Since \( M_{S_\Delta} \) is a nonsingular \( M \)-matrix, the convergence of Newton’s method is quadratic. Recall that \( \Delta = \text{diag}(\delta_1, \delta_2, \cdots, \delta_n) \) in (4.32) satisfies (3.15). As for the fixed point iteration, it is advisable to use the matrix \( \Delta \) given by (4.27).
4.3 Schur Method

Another efficient method is the Schur method, which is originated in [23]. Our presentation is an adaptation of that in [9].

Theorem 4.3.1 Let $F$ be a regular singular $M$-matrix with a simple zero eigenvalue and $W$ be the matrix in (3.19). Let $U$ be an orthogonal matrix such that

$$U^TWU = G$$

is a real Schur form of $W$, where all $1 \times 1$ or $2 \times 2$ diagonal blocks of $G$ corresponding to eigenvalues of $W$ in $\mathbb{C}_>$ appear before the $1 \times 1$ or $2 \times 2$ diagonal block of $G$ corresponding to the zero eigenvalue of $W$, which is followed by all $1 \times 1$ or $2 \times 2$ diagonal blocks of $G$ corresponding to eigenvalues of $W$ in $\mathbb{C}_<$. Let $U$ be partitioned as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

(4.36)

where $U_{11} \in \mathbb{R}^{n \times n}$. If $W$ has $n$ eigenvalues in $\mathbb{C}_>$, then $U_{11}$ is nonsingular and $-U_{21}U_{11}^{-1} = X_M$, the desired $M$-matrix solution of (3.10). If $W$ has only $n - 1$ eigenvalues in $\mathbb{C}_>$ and the $n$th column of $U$ is replaced by the eigenvector $\begin{bmatrix} v \\ 0 \end{bmatrix}$ of $W$ corresponding to the zero eigenvalue of $W$, where $v$ is the unique nonnegative vector
with \( \|v\|_2 = 1 \) such that \( Fv = 0 \), then (using the same notation after the replacement of the \( n \)th column of \( U \)) \( U_{11} \) is nonsingular and \(-U_{21}U_{11}^{-1} = X_M\).

**Proof** Here the column spaces of \((U_{11}^T \quad U_{21}^T)^T\) and \((I \quad -X^T)^T\) are the same invariant subspace of \( W \) corresponding to the \( n \) eigenvalues with the largest real parts. Note that the replacement of \( n \)th column of \( U \) is necessary when \( G \) has a \( 2 \times 2 \) diagonal block associated with the zero eigenvalue. Without the replacement, the column space of \((U_{11}^T \quad U_{21}^T)^T\) is in general not an invariant subspace of \( W \). \( \square \)
Chapter 5

A Convergence result for M-matrix Riccati differential equations

In this chapter we consider the initial value problem for a matrix Riccati differential equation, where the four coefficient matrices form an M-matrix $K$. It is already known that if $K$ is a nonsingular M-matrix or an irreducible singular M-matrix then that initial value problem has a global solution $X(t)$ on $[0, \infty)$ when $X(0)$ takes values from a suitable set of nonnegative matrices. It is also known that as $t$ goes to infinity $X(t)$ converges to the minimal nonnegative solution of the corresponding algebraic Riccati equation (see [4, 15]). In this chapter we show that for suitable initial values the Riccati differential equation has a global solution $X(t)$ on $[0, \infty)$ and $X(t)$ converges to the stable equilibrium solution as $t$ goes to infinity when $K$ is a regular singular M-matrix. Basically we drop the irreducibility assumption imposed on $K$ in [4], but the presentation here is still largely the same as in [4].
5.1 Introduction

We consider the initial value problem for the matrix Riccati differential equation (RDE)

\[ X'(t) = X(t)CX(t) - X(t)D - AX(t) + B, \quad X(0) = X_0 \]  \tag{5.37}

Where \( A, B, C, D \) are real matrices of sizes \( m \times m, m \times n, n \times m, n \times n \), respectively, such that

\[ K = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \]  \tag{5.38}

is a regular \( M \)-matrix.

We let \( \mathcal{R}(X) = XCX - XD - AX + B \), which defines a mapping from \( \mathbb{R}^{m \times n} \) into itself. The Fréchet derivative of \( \mathcal{R} \) at a matrix \( X \) is a linear operator \( \mathcal{R}'_X : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n} \) given by

\[ \mathcal{R}'_X(Z) = -((A - XC)Z + Z(D - CX)). \]

since the eigenvalues of the operator \( \mathcal{R}'_X \) are the eigenvalues of the matrix \(- (I \otimes (A - XC) + (D - CX)^T \otimes I)\), an equilibrium solution \( X \) of (5.37) is asymptotically stable if all eigenvalues of \((I \otimes (A - XC) + (D - CX)^T \otimes I)\) are in \( \mathbb{C}_{>0} \).
Note that, any eigenvalue of \((I \otimes (A - XC) + (D - CX)^T \otimes I)\) is the sum of an eigenvalue of \(A - XC\) and an eigenvalue of \(D - CX\). The following definition can be found in [4]

**Definition 5.1.1** A solution \(X\) of (2.1) is called stabilizing (almost stabilizing) if all eigenvalues of \((I \otimes (A - XC) + (D - CX)^T \otimes I)\) are in \(\mathbb{C}_>(\mathbb{C}_\geq)\).

It follows from Theorem 2.3.5 that \(\Phi\) is a stable equilibrium solution of (5.37) when \(K\) is a regular singular \(M\)-matrix with noncritical case \((u_1^Tv_1 \neq u_2^Tv_2)\). When \(K\) is a regular singular \(M\)-matrix with critical case \((u_1^Tv_1 = u_2^Tv_2)\), \(\Phi\) is not a stable equilibrium solution of (5.37) since \((I \otimes (A - \Phi C) + (D - C \Phi)^T \otimes I)\) has a zero eigenvalue by Theorem 2.3.5.

5.2 Global existence of solutions of the RDE

Since the matrices \(A\) and \(D\) in (5.37) are regular \(M\)-matrices, they can be decomposed (in many different ways) as \(A = A_1 - A_2\) and \(D = D_1 - D_2\), where \(A_2, D_2 \geq 0\) and \(A_1, D_1\) are nonsingular \(M\)-matrices. Then we have from (5.37)

\[
X' + XD_1 + A_1X = XCX + XD_2 + A_2X + B, \ X(0) = X_0 \quad (5.39)
\]
The initial value problem (5.39) can be written as its equivalent integral form (see [4]).

\[
X(t) = e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(X(s)CX(s) + X(s)D_2 + A_2X(s) + B)e^{-(t-s)D_1}ds
\]

(5.40)

Here we will show the global existence of solutions to (5.37) for suitable initial values \(X_0\) by using the Picard iteration:

\[
X^{(0)}(t) = 0,
\]

\[
X^{(m)}(t) = e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(X^{(m-1)}(s)CX^{(m-1)}(s) + X^{(m-1)}(s)D_2 + A_2X^{(m-1)}(s) + B)e^{-(t-s)D_1}ds
\]

Theorem 5.2.1 Let \(K\) in (5.38) be a regular singular \(M\)-matrix. If \(0 \leq X_0 \leq \Phi\), where \(\Phi\) is any nonnegative solution of (2.1). Then

1. \(0 \leq X^{(m)}(t) \leq X^{(m+1)}(t)\) and \(X^{(m)}(t) \leq \Phi\) for all \(t \geq 0\) and all integers \(m \geq 0\).

2. \(X^{(m)}(t)\) converges pointwise to a continuous function \(X(t)\) on \([0, \infty)\), which is a global solution to (5.37).

Proof We will prove part (1) by induction. For \(m = 1\) we have by Lemma 2.2.13,

\[
X^{(1)}(t) = e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}B e^{-(t-s)D_1}ds \geq 0
\]
since $X^{(0)}(t) = 0$, $X_0 \geq 0$, and $B \geq 0$.

Then we can easily see that $0 \leq X^{(1)}(t)$ and $X^{(0)}(t) \leq \Phi$. So the inequalities in part (1) are true for $m = 0$. Now we assume that the inequalities in part (1) are true for $m = k$, and we will show $X^{(k+1)}(t) \leq X^{(k+2)}(t)$. Indeed,

$$X^{(k+1)}(t) \leq e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(X^{(k+1)}(s)C)X^{(k+1)}(s) + X^{(k+1)}(s)D_2$$

$$+ A_2X^{(k+1)}(s) + B)e^{-(t-s)D_1}ds = X^{(k+2)}(t).$$

Similarly,

$$X^{(k+1)}(t) \leq e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(\Phi C \Phi + \Phi D_2 + A_2 \Phi + B)e^{-(t-s)D_1}ds$$

$$= e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(A_1 \Phi + \Phi D_1)e^{-(t-s)D_1}ds$$

$$= e^{-tA_1}X_0e^{-tD_1} + \Phi - e^{-tA_1} \Phi e^{-tD_1}$$

$$= \Phi - e^{-tA_1}(\Phi - X_0)e^{-tD_1}$$

$$\leq \Phi.$$  

We have thus proved that the inequalities in part (1) are true for $m = k + 1$. Hence they are true for all $m \geq 0$. This completes the proof of part (1).

For the proof of part (2), it follows from part (1) that $X^{(m)}(t)$ converges pointwise to a function $X(t)$ on $[0, \infty)$, and $X(t) \leq \Phi$. Assuming $m \to \infty$ in the Picard iteration and applying the monotone convergence theorem for Lebesgue integrals, we conclude
that the limit function $X(t)$ satisfies (5.40). It then follows from the boundness of $X(t)$ that $X(t)$ is also continuous, which in turns implies that $X(t)$ is differentiable and is a solution to (5.37).

\[ \square \]

### 5.3 Convergence to the stable equilibrium solution

Here we will show that the convergence of $X(t)$ to $\Phi$ can be guaranteed for suitable initial values $X_0$ when $K$ in (5.38) is a regular singular $M$-matrix with noncritical case ($u_1^T v_1 \neq u_2^T v_2$). The initial value problem (5.37) is related to the initial value problem for the corresponding linear system:

\[
\begin{bmatrix}
Y'(t) \\
Z'(t)
\end{bmatrix} = \begin{bmatrix}
D & -C \\
B & -A
\end{bmatrix} \begin{bmatrix}
Y(t) \\
Z(t)
\end{bmatrix}, \quad \begin{bmatrix}
Y \\
Z
\end{bmatrix}(0) = \begin{bmatrix}
I \\
X_0
\end{bmatrix},
\]

(5.41)
as described in the following Lemma (see [4] and [5], for example).

**Lemma 5.3.1** The initial value problem (5.37) has a solution $X(t)$ on $[0, \infty)$ if and only if $Y(t)$ is nonsingular on $[0, \infty)$ for the solution \(\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix}\) to (5.41). In this case, $X(t) = Z(t)Y^{-1}(t)$.

Before we proceed, we prove the following simple result.

**Lemma 5.3.2** If $A$ is a regular singular $M$-matrix with a simple zero eigenvalue, then $e^{-At}$ is bounded on $[0, \infty)$.  

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Proof Let the Jordan canonical form of $A$ be $A = PJP^{-1}$ and let $f(x) = e^{-tx}$. Then for the $k \times k$ Jordan block

$$J_k(\lambda) = \begin{bmatrix}
\lambda & 1 \\
& \ddots & \ddots \\
& & \ddots & 1 \\
& & & \lambda
\end{bmatrix}$$

we have

$$f(J_k(\lambda)) = \begin{bmatrix}
f(\lambda) & \frac{f'(\lambda)}{1!} & \cdots & \frac{f^{r-1}(\lambda)}{(r-1)!} \\
0 & f(\lambda) & \cdots & \vdots \\
& \vdots & \ddots & \frac{f'(\lambda)}{1!} \\
0 & \cdots & 0 & f(\lambda)
\end{bmatrix}.$$  

Note that 0 is a simple eigenvalue of $A$ and the remaining eigenvalues of $A$ are in $\mathbb{C}_>$. We have $\lim_{t \to \infty} f(J_k(\lambda)) = 0$ if $\lambda$ is in $\mathbb{C}_>$. When $\lambda = 0$, we have $k = 1$ and $\lim_{t \to \infty} f(J_k(\lambda)) = [1]$.

It follows that

$$\lim_{t \to \infty} e^{-At} = \lim_{t \to \infty} f(A) = P \lim_{t \to \infty} f(J) P^{-1}$$

exists, and thus $e^{-At}$ is bounded on $[0, \infty)$.

Theorem 5.3.3 If $K$ in (5.38) is a regular singular $M$-matrix with a simple 0 eigenvalue with $u_1^T v_1 \neq u_2^T v_2$, $0 \leq X_0 \leq \Phi$ and $X(t)$ is the solution of (5.37). Then $X(t) \to \Phi$ as $t \to \infty$. 

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Proof From Theorem 5.2.1 it is clear that $X(t)$ exists on $[0, \infty)$, and from Lemma 5.3.1 we have,

$$
\begin{bmatrix}
I \\
X(t)
\end{bmatrix}
= 
\begin{bmatrix}
Y(t) \\
Z(t)
\end{bmatrix}

Y^{-1}(t) = e^{Ht}

\begin{bmatrix}
I \\
X_0
\end{bmatrix}
Y^{-1}(t)
$$

(5.42)

where $H$ is the matrix in (2.4). From (2.5) and Corollary 2.3.7 we have

$$
H = U
\begin{bmatrix}
R & 0 \\
0 & -S
\end{bmatrix}
U^{-1},
$$

where $R = D - C \Phi$, $S = A - B \Psi$, and

$$
U = 
\begin{bmatrix}
I_n & \Psi \\
\Phi & I_m
\end{bmatrix}.
$$

By (5.42) and the power series expansion of the exponential function, we have

$$
\begin{bmatrix}
I \\
X(t)
\end{bmatrix}
= U
\begin{bmatrix}
e^{Rt} & 0 \\
0 & e^{-St}
\end{bmatrix}
U^{-1}
\begin{bmatrix}
I \\
X_0
\end{bmatrix}
Y^{-1}(t).
$$

(5.43)
Let

\[
U^{-1} \begin{bmatrix} I \\ X_0 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.
\] (5.44)

Then we have

\[ V_1 + \Psi V_2 = I, \]

and

\[ \Phi V_1 + V_2 = X_0. \]

It follows that

\[
I - \Psi X_0 = V_1 + \Psi V_2 - \Psi (\Phi V_1 + V_2)
= (I - \Psi \Phi)V_1.
\]

Now we will show that \( V_1 \) is nonsingular. Here we have

\[
0 \leq X_0 \leq \Phi
\]

\[ \Rightarrow 0 \leq \Psi X_0 \leq \Psi \Phi. \]

By Theorem 2.2.10 and Corollary 2.3.7, we have

\[
\rho(\Psi X_0) \leq \rho(\Psi \Phi) < 1.
\]

Hence \( V_1 \) is invertible.
By (5.43) and (5.44)

\[
\begin{bmatrix}
I \\
X(t)
\end{bmatrix} = U
\begin{bmatrix}
I \\
W_1(t)
\end{bmatrix} W_2(t)
\] (5.45)

where

\[
W_1(t) = e^{-St}V_2 V_1^{-1} e^{-Rt},
\]

and

\[
W_2(t) = e^{Rt} V_1 Y(t)^{-1}.
\]

Since \( R \) and \( S \) are both regular \( M \)-matrices and one of them is nonsingular and the other one is singular with a simple zero eigenvalue. That implies, by Lemma 2.2.14 and Lemma 5.3.2,

\[
\lim_{t \to \infty} W_1(t) = 0.
\]

From (5.45) we have

\[
W_2(t) = (I + \Psi W_1(t))^{-1},
\]

\[
X(t) = (\Phi + W_1(t))W_2(t).
\]

Thus,

\[
\lim_{t \to \infty} W_2(t) = I \text{ and } \lim_{t \to \infty} X(t) = \Phi.
\]
Conclusion

In this thesis, we have proved some new results on the $M$-matrix quadratic matrix equation (M-QME) $X^2 - EX - F = 0$, and using these results we have proved that, if $F$ is a regular singular $M$-matrix with a simple zero eigenvalue then the M-QME always has $M$-matrix solutions. Moreover, we have shown that the M-QME has a unique nonsingular $M$-matrix solution when it has nonsingular $M$-matrix solutions, and has a unique singular $M$-matrix solution when it does not have nonsingular $M$-matrix solutions. This uniquely identified solution is the desired solution in practice. Then we have discussed a few numerical methods to get the desired $M$-matrix solution of the M-QME. Finally we have shown that the initial value problem for the $M$-matrix Riccati differential equation (M-RDE) has a global solution which converges to the stable equilibrium solution which is the minimal nonnegative solution of corresponding algebraic Riccati equation. Through the study in this thesis, we have
gained significant new information about the M-QMEs and M-RDEs.
Bibliography


