

Quantum Fidelity and the Bures Metric in Operator Algebras

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Abstract

This dissertation undertakes the study of *quantum fidelity*, a distinguishability measure in the context of quantum mechanics, from the operator algebraic viewpoint. The notion of fidelity provides a quantitative measure of how close one state of a quantum system is to another state. High fidelity occurs when the two states are very close to each other. Evidently, this concept is closely related to a metric on the quantum states which is known as the *Bures metric*. In this thesis, fidelity and the Bures metric have been studied in the context of (i) unital C^* -algebras that possess a faithful positive trace functional and (ii) semifinite von Neumann algebras. In addition, these notions have been analysed in the matrix algebras in an effort to relate to the quantum information theory literature.

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This thesis is dedicated to my uncle
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Chapter 1

Introduction and Background

1.1 Motivation and thesis outline

The concepts of the Bures metric and fidelity first appeared in the work of Donald Bures [6], in an effort to establish a notion of distance on the set of normal states of a von Neumann algebra. Later Uhlmann ([54]), Alberti ([2]) and Jozsa ([32]) used these concepts in the context of quantum mechanics as a measure of closeness of quantum states. It is noteworthy that the extensive body of literature involving fidelity only utilises the fact that fidelity is defined on normal states of von Neumann algebras. Hence it has always been looked at in the Schrödinger picture using the predual of a von

Neumann algebra. When the von Neumann algebra is $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space \mathcal{H} , all the normal states can be described by the “Trace” functional. Since the definition of fidelity involves the trace functional, one can formulate the notion of fidelity on arbitrary C^* -algebras that possess faithful tracial states. This is the key motivating observation that this dissertation makes use of and analyse this notion from the Heisenberg picture rather than the Schrödinger picture. Keeping the same spirit as in the von Neumann algebra case, we concentrate on the special subset of a C^* -algebra, namely the density space that contains all positive trace 1 elements.

Chapter 1 contains the motivation of the work presented in this dissertation and background in operator algebras and information theoretic notions. The definitions of fidelity and Bures metric on a C^* -algebra are put forward in the chapter 2 and various properties have been explored. In light of the fact that a trace preserving positive linear map increases the fidelity of two density elements, Chapter 3 characterises the isometric positive linear maps on density space in full generality. It turns out that the linear maps that are strict Bures contractions, that is the maps that strictly increase fidelity, have many interesting properties and Chapter 4 contains a detailed descrip-

tions of such maps and their behaviours. Chapter 5 constitutes discussions of fidelity and quantum channels on semifinite von Neumann algebras. In Chapter 6, the various implications of results obtained for fidelity on abstract C*-algebras are analysed when the underlying space is of $d \times d$ complex matrices which is the widely known ambient for quantum information theory. The final Chapter reviews the main results and explores future avenues for research.

1.2 Operator algebras

In this section a brief introduction to some of the familiar concepts of operator algebras will be presented. For this part, see [34] and [52] for detailed descriptions.

1.2.1 C*-algebras

Definition 1.1. A C*-algebra is a complex Banach algebra \mathcal{A} with a conjugate linear operation $*$ (called involution) such that $\|xx^*\| = \|x\|^2$, for all $x \in \mathcal{A}$.

In this dissertation, we will always consider C*-algebras that are unital,

that is, C*-algebras, that possess the multiplicative identity or unit. The identity will be denoted by 1.

Definition 1.2. *Let \mathcal{A} be a C*-algebra. An element $a \in \mathcal{A}$ is called:*

- self-adjoint, if $a = a^*$,
- normal, if $aa^* = a^*a$,
- positive, if there exists an element $b \in \mathcal{A}$ such that $a = bb^*$,
- projection, if $p^2 = p = p^*$,
- unitary, if $aa^* = a^*a = 1$.

For an element $a \in \mathcal{A}$, if a is positive, then we denote it by writing $a \geq 0$.

For two self-adjoint elements a, b , we say $a \leq b$ if $b - a \geq 0$. The set of all positive elements in \mathcal{A} is denoted by \mathcal{A}_+ .

Every C*-algebra contains an *approximate identity*, that is, for every C*-algebra \mathcal{A} , there exists a net of positive elements $\{e_\lambda\}_{\lambda \in \Gamma}$ in \mathcal{A} , where Γ is an index set, such that $\|e_\lambda\| \leq 1$ and $\|e_\lambda x - x\| \rightarrow 0$ and $\|xe_\lambda - x\| \rightarrow 0$ for all $x \in \mathcal{A}$. Moreover, such an *approximate identity* can be assumed to be increasing, that is, for all $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$, implies $e_\alpha \leq e_\beta$.

A celebrated theorem of Gelfand and Naimark asserts that for every C^* -algebra \mathcal{A} , there exists a Hilbert space \mathcal{H} and an isometric $*$ -homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ denotes the bounded linear maps on \mathcal{H} . So every C^* -algebra can be realised as a norm closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$, for some \mathcal{H} .

There are ample examples of C^* -algebras in the literature. The most important ones are $\mathcal{B}(\mathcal{H})$, for some Hilbert space \mathcal{H} and $C(X)$, the algebra of continuous functions on a compact topological space X . Also the $d \times d$ complex matrices \mathcal{M}_d and for any discrete countable group G , the group C^* -algebra $C^*(G)$ are frequently discussed C^* -algebras.

1.2.2 von Neumann algebras

Before introducing the definition of von Neumann algebras we discuss some topologies that exist on $\mathcal{B}(\mathcal{H})$, for a Hilbert space \mathcal{H} .

A net of operators $\{x_\lambda\}$ in $\mathcal{B}(\mathcal{H})$ converges to an operator $x \in \mathcal{B}(\mathcal{H})$ in the *weak-operator-topology* (WOT) if

$$\lim_{\lambda} \langle x_\lambda \xi, \eta \rangle = \langle x \xi, \eta \rangle,$$

for every $\xi, \eta \in \mathcal{H}$.

A net of operators $\{x_\lambda\}$ in $\mathcal{B}(\mathcal{H})$ converges to an operator $x \in \mathcal{B}(\mathcal{H})$ in the *strong-operator-topology* (SOT) if

$$\lim_{\lambda} \|x_\lambda \xi - x \xi\| = 0, \quad \forall \xi \in \mathcal{H}.$$

Definition 1.3. A von Neumann algebra is a unital $*$ -subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, such that it is closed relative to the strong-operator-topology.

For any set $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, the commutant \mathcal{N}' of \mathcal{N} is the set defined by

$$\mathcal{N}' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx, \quad \forall y \in \mathcal{N}\}.$$

The following two theorems are very crucial in the study of von Neumann algebras. The proof of these theorems can be found in [34], [52], [19].

Theorem 1.4. [Double Commutant Theorem] A unital $*$ -subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(\mathcal{M}')' = \mathcal{M}$.

Theorem 1.5. [Kaplansky's Density Theorem] Let \mathcal{A} be a unital C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ and that $\mathcal{M} = \overline{\mathcal{A}}^{\text{SOT}}$. If

$$\mathcal{A}_1 = \{a \in \mathcal{A} : \|a\| \leq 1\}, \quad \mathcal{A}_{1,\text{sa}} = \{a \in \mathcal{A} : a = a^*, \|a\| \leq 1\},$$

and if

$$\mathcal{M}_1 = \{x \in \mathcal{M} : \|x\| \leq 1\}, \quad \mathcal{M}_{1,\text{sa}} = \{x \in \mathcal{M} : x = x^*, \|x\| \leq 1\},$$

then $\mathcal{M}_{1,\text{sa}} = \overline{\mathcal{A}_{1,\text{sa}}}^{\text{SOT}}$ and $\mathcal{M}_1 = \overline{\mathcal{A}_1}^{\text{SOT}}$.

A von Neumann algebra \mathcal{M} is called a *factor* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}1$.

Beside the weak and strong operator topologies on $\mathcal{B}(\mathcal{H})$, there is another topology on $\mathcal{B}(\mathcal{H})$ which will be used in the subsequent chapters. We note down the definition below:

Definition 1.6. *The σ -weak topology of $\mathcal{B}(\mathcal{H})$ is determined by the seminorms*

$$a \mapsto \left| \sum_{j=1}^{\infty} \langle a\xi_j, \eta_j \rangle \right|,$$

where $\{\xi_j\}$ and $\{\eta_j\}$ are two sequences in \mathcal{H} such that $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$ and

$$\sum_{j=1}^{\infty} \|\eta_j\|^2 < \infty.$$

Essentially the σ -weak topology is the weak-* topology on $\mathcal{B}(\mathcal{H})$ thought of as the dual of the pre-dual of the trace class operators. This topology is also called the *ultraweak* topology.

1.2.3 Positive linear functionals

Definition 1.7. *Let \mathcal{A} be a C^* -algebra. A linear functional $\phi : \mathcal{A} \rightarrow \mathbb{C}$ is positive if $\phi(h) \geq 0$ for every $h \in \mathcal{A}_+$. Moreover, if $\|\phi\| = 1$, then ϕ is called a state on \mathcal{A} .*

The set of all states on \mathcal{A} is denoted by $\mathcal{S}(\mathcal{A})$.

Definition 1.8. A positive linear functional ϕ on a C^* -algebra \mathcal{A} is called normal if $\phi(a) = \lim_{\lambda} \phi(a_{\lambda})$ for each increasing net $\{a_{\lambda}\}$ with least upper bound a in \mathcal{A} .

Definition 1.9. The Banach space of all normal linear functionals on a von Neumann algebra \mathcal{M} is called the predual of \mathcal{M} and is denoted by \mathcal{M}_* .

The celebrated Gelfand-Naimark-Segal (GNS) theorem shows a close relationship between states and representations of a C^* -algebra. It asserts that if ϕ is a state on a unital C^* -algebra \mathcal{A} , then there exists a unital representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ on a Hilbert space \mathcal{H} and a unit vector $\xi \in \mathcal{H}$ such that

$$\phi(x) = \langle \pi(x)\xi, \xi \rangle,$$

for every $x \in \mathcal{A}$ and also the set $\{\pi(a)\xi : a \in \mathcal{A}\}$ is dense in \mathcal{H} .

1.2.4 Positive and completely positive maps

Definition 1.10. Given two unital C^* -algebras \mathcal{A} and \mathcal{B} , a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is said to be positive if $\Phi(\mathcal{A}_+) \subseteq \mathcal{B}_+$.

If $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ are the units of \mathcal{A} and \mathcal{B} respectively, and if $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, then we say Φ is unital. Since every self-adjoint element in \mathcal{A} is the difference

of two positive elements with orthogonal supports, a positive map Φ carries self-adjoint elements to self-adjoint elements. Now, every element x in \mathcal{A} can be written as $x = a + ib$, where a, b are self-adjoint elements. Then we have

$$\Phi(x^*) = \Phi(a) - i\Phi(b) = \Phi(x)^*,$$

so Φ preserves adjoints.

According to a theorem of Russo-Dye ([48]), a positive map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is bounded and $\|\Phi\| = \|\Phi(1)\|$, hence for a unital positive map Φ , $\|\Phi\| = 1$.

Definition 1.11. *A positive linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is called normal if $\Phi(a) = \lim_{\lambda} \Phi(a_{\lambda})$ for each increasing net $\{a_{\lambda}\}$ of self-adjoint elements with least upper bound a in \mathcal{A} .*

The concept of completely positive map is a stronger one. Note that ([42], Chapter 3), for any C^* -algebra \mathcal{A} , the algebra $\mathcal{M}_k(\mathcal{A}) = \mathcal{A} \otimes \mathcal{M}_k(\mathbb{C})$ has a (unique) norm that makes it a C^* -algebra for every $k \in \mathbb{N}$, where $\mathcal{M}_k(\mathbb{C})$ is the space of $k \times k$ matrices over the complex numbers. Hence the C^* -algebra $\mathcal{M}_k(\mathcal{A})$ has the cone of positive elements. As before if \mathcal{A} and \mathcal{B} are unital C^* -algebras, then for a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$, and for each $k \in \mathbb{N}$, the map $\Phi_k : \mathcal{M}_k(\mathcal{A}) \rightarrow \mathcal{M}_k(\mathcal{B})$ is defined by

$$\Phi_k((a_{i,j})) = (\Phi(a_{i,j})).$$

Definition 1.12. We call a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ k -positive, if Φ_k is positive. Moreover, Φ is called completely positive if Φ_k is positive for every $k \in \mathbb{N}$.

With the same set up as before, we call Φ completely bounded (cb) if $\sup_k \|\Phi_k\|$ is finite, and we set

$$\|\Phi\|_{\text{cb}} = \sup_k \|\Phi_k\|.$$

Note that $\|\cdot\|_{\text{cb}}$ is a norm on the space of completely bounded maps.

The following theorem characterises completely positive maps from a C^* -algebra to $\mathcal{B}(\mathcal{H})$ in terms of $*$ -homomorphisms into $\mathcal{B}(\mathcal{K})$, for some other Hilbert space \mathcal{K} .

Theorem 1.13 (Stinespring's Dilation Theorem). *Let \mathcal{A} be a unital C^* -algebra and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space \mathcal{K} , a unital $*$ homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ and a bounded operator $V : \mathcal{H} \rightarrow \mathcal{K}$ with $\|\Phi(1)\| = \|V\|^2$ such that*

$$\Phi(a) = V^* \pi(a) V, \quad \forall a \in \mathcal{A}.$$

The following proposition reveals an important inequality for 2-positive maps on unital C^* -algebras.

Proposition 1.14 (Schwarz inequality). *Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital 2-positive map. Then*

$$\Phi(aa^*) \geq \Phi(a)\Phi(a^*), \quad (1.1)$$

for every $a \in \mathcal{A}$.

Based on the above equation, maps that satisfy the above inequality have a special name.

Definition 1.15. *A linear map that satisfies the Schwarz inequality (1.1), is called a Schwarz map.*

A Schwarz map is necessarily positive and also a contraction, that is, $\|\Phi\| \leq 1$. Proposition 1.14 shows that every 2-positive map is a Schwarz map, however the converse is not true as was shown by Choi ([10]).

For a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$, the multiplicative domain for Φ is the set

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} : \Phi(ab) = \Phi(a)\Phi(b), \Phi(ba) = \Phi(b)\Phi(a)\},$$

for all $b \in \mathcal{A}$. The following proposition relates the multiplicative domain of a map with the Schwarz inequality.

Proposition 1.16 (Choi, [9]). *Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a Schwarz map. Suppose $a \in \mathcal{A}$ satisfies $\Phi(aa^*) = \Phi(a)\Phi(a^*)$. Then $a \in \mathcal{M}_\Phi$.*

1.3 Quantum channels

For quantum channels and metrics on density operators, that are described in the remainder of this chapter, we refer to [26], [27] and [3] for further discussion.

We will give a quick introduction to a special class of completely positive maps defined on the Banach space of trace class operators $\mathcal{T}(\mathcal{H})$ on Hilbert space \mathcal{H} . Chapter 5 describes in details the sense in which one defines positive linear maps on $\mathcal{T}(\mathcal{H})$. For the introduction purpose let us assume that there is a matrix ordering in $\mathcal{T}(\mathcal{H})$ and hence one can talk about complete positivity of a map on $\mathcal{T}(\mathcal{H})$. See [18] for a self contained discussion on this topic.

Definition 1.17. *A completely positive and trace preserving bounded linear map $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ is called a quantum channel.*

Note that any linear map on $\mathcal{T}(\mathcal{H})$ induces a linear map on its dual $\mathcal{B}(\mathcal{H})$. Hence for a quantum channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$, there exists a unital normal and completely positive bounded linear map $\mathcal{E}^* : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ satisfying the relation

$$\mathrm{Tr}(\mathcal{E}(a)b) = \mathrm{Tr}(a\mathcal{E}^*(b)),$$

for all $a \in \mathcal{T}(\mathcal{H})$ and $b \in \mathcal{B}(\mathcal{H})$. In the context of quantum mechanics,

a *state* is a description of an ensemble of similarly prepared systems. An *effect*, on the other hand, is a measurement apparatus that produces either ‘yes’ or ‘no’ as an outcome. The *states* and *effects* are dual objects. The linear transformation \mathcal{E} acts on the states of a system in the Schrödinger picture. Whereas the map \mathcal{E}^* describes the same transformation of the system but in the Heisenberg picture where the effects rather than the states are transformed.

Using the Stinespring’s Dilation Theorem for the completely positive map $\mathcal{E}^* : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ and by making use of the fact that \mathcal{E}^* is normal and unital, Kraus ([36]) showed that for every channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$, there exists a (non-unique!) sequence of bounded operators $\{k_j\}_j \in \mathcal{B}(\mathcal{H})$, satisfying

$$\sum_{j=1}^{\infty} k_j^* k_j = 1, \tag{1.2}$$

such that,

$$\mathcal{E}(x) = \sum_{j=1}^{\infty} k_j x k_j^*, \tag{1.3}$$

for every $x \in \mathcal{T}(\mathcal{H})$, where the sum in Equation (1.2) converges in strong operator topology and the sum in Equation (1.3) converges in the trace norm $\|\cdot\|_1$. The representation of \mathcal{E} in Equation (1.3) is known as *Kraus representation* of \mathcal{E} .

1.3.1 Examples

(1) Fix a positive trace class operator ρ . Then define $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ by $\mathcal{E}(x) = \frac{\text{Tr}(x)}{\text{Tr}(\rho)}\rho$, for all $x \in \mathcal{T}(\mathcal{H})$. It is easy to check \mathcal{E} is trace preserving and completely positive. Also note that $\mathcal{E}^*(a) = \frac{\text{Tr}(a\rho)}{\text{Tr}(\rho)}1$, for all $a \in \mathcal{B}(\mathcal{H})$, where 1 is the identity element of $\mathcal{B}(\mathcal{H})$. Channels of this form are known as *completely depolarising channels*.

(2) Fix a unitary operator $u \in \mathcal{B}(\mathcal{H})$. Define $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ by $\mathcal{E}(x) = u x u^*$, for all $x \in \mathcal{T}(\mathcal{H})$. Then \mathcal{E} is quantum channel known as a *unitary channel*.

(3) Let \mathcal{H} be a 2-dimensional Hilbert space. Define for any $\theta \in (0, 1)$ the operators

$$k_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\theta} \end{bmatrix}, \quad k_2 = \begin{bmatrix} 0 & \sqrt{\theta} \\ 0 & 0 \end{bmatrix}.$$

Now define $\mathcal{E} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ be $\mathcal{E}(x) = k_1 x k_1^* + k_2 x k_2^*$. It is easy to see that \mathcal{E} is trace preserving and completely positive, that is, \mathcal{E} is a quantum channel. A channel of this form is known as *amplitude-damping channel*.

1.4 Metrics on density operators

1.4.1 Norms and metrics

In this section we will explore various norms and metrics that exist on the normal state space of $\mathcal{B}(\mathcal{H})$, for some Hilbert space \mathcal{H} . Recall that the predual $\mathcal{B}(\mathcal{H})_*$ of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is isometrically isomorphic to the space $\mathcal{T}(\mathcal{H})$ of all trace class operators under the map which assigns to each $a \in \mathcal{T}(\mathcal{H})$, a normal linear functional ϕ_a defined by

$$\phi_a(x) = \text{Tr}(ax) \quad \forall x \in \mathcal{B}(\mathcal{H}).$$

In particular, the map $a \mapsto \phi_a$ restricts to an affine isomorphism from the convex set of positive operators of trace one, namely $\mathcal{T}(\mathcal{H})_1^+$, onto the normal state space of $\mathcal{B}(\mathcal{H})$. The element of the set $\mathcal{T}(\mathcal{H})_1^+$ of positive trace 1 operators are called *density operators*.

We denote by $\xi\xi^*$ the rank one projection onto the subspace of \mathcal{H} spanned by the unit vector $\xi \in \mathcal{H}$. Thus $p = \xi\xi^*$ is defined by $p(\eta) = \langle \eta, \xi \rangle \xi$, for all $\eta \in \mathcal{H}$. The normal linear functional ϕ_ξ associated to a rank one projection $\xi\xi^*$ has the following form:

$$\phi_\xi(x) = \text{Tr}(x\xi\xi^*) = \langle x\xi, \xi \rangle, \quad \forall x \in \mathcal{B}(\mathcal{H}).$$

These linear functionals on $\mathcal{B}(\mathcal{H})$ are called *vector states* or *pure states*. A pure state is one that can not be expressed as a convex combination of states. The remaining density operators are called *mixed states*. In physics, a *state* represents a statistical ensemble of physical systems. The pure states correspond to the *wave functions* in quantum mechanics.

Recall that an operator $a \in \mathcal{T}(\mathcal{H})$, if $\|a\|_1 := \text{Tr}((aa^*)^{1/2}) < \infty$. Also $\mathcal{T}(\mathcal{H})$ is a two sided ideal of $\mathcal{B}(\mathcal{H})$. We call the norm $\|\cdot\|_1$ the *trace norm*. Also an operator $a \in \mathcal{B}(\mathcal{H})$ is a *Hilbert-Schmidt* operator if $\|a\|_2 := \text{Tr}(aa^*) < \infty$. The set of Hilbert-Schmidt operators is defined by $\mathcal{HS}(\mathcal{H})$. Note that for any operator $a \in \mathcal{B}(\mathcal{H})$, we have

$$\|a\| \leq \|a\|_2 \leq \|a\|_1,$$

from which we conclude that,

$$\mathcal{T}(\mathcal{H}) \subseteq \mathcal{HS}(\mathcal{H}).$$

By the above discussion, it is evident that on the density operators $\mathcal{T}(\mathcal{H})_1^+$, there are metrics naturally induced by the $\|\cdot\|_1$ or $\|\cdot\|_2$ norms, namely for two density operators a, b , the metrics are $d_1(a, b) = \|a - b\|_1$ and $d_2(a, b) = \|a - b\|_2$ respectively. However, there are other metrics that exist on $\mathcal{T}(\mathcal{H})_1^+$ which do not arise from a norm. This dissertation focuses on one

such metric called the *Bures metric*. We will study this metric in details not only on the density operators but in a more general operator algebraic framework. The next subsection contains a brief description of a metric that exists on $\mathcal{T}(\mathcal{H})_1^+$ other than the norms induced by the trace on $\mathcal{B}(\mathcal{H})$. See also the *Hilbert's Projective Metric* on the density operators in the context of quantum information theory in [46].

1.4.2 Angular metric

Let $\text{Proj}(\mathcal{H})$ be the *projective space* over a given Hilbert space \mathcal{H} . That is $\text{Proj}(\mathcal{H})$ consists of lines $[\xi] = \{\lambda\xi : \lambda \in \mathbb{C} \setminus \{0\}\}$, with ξ a unit vector in \mathcal{H} . Since two unit vectors in \mathcal{H} determine the same vector state if and only if they are equal up to a scalar, the map $\xi \mapsto \phi_\xi$ determines a bijection from $\text{Proj}(\mathcal{H})$ to pure states on $\mathcal{B}(\mathcal{H})$.

Definition 1.18. *Let $\xi, \eta \in \mathcal{H}$ be two unit vectors in \mathcal{H} , then the angle between two lines $[\xi], [\eta] \in \text{Proj}(\mathcal{H})$ has the value $\alpha \in [0, \frac{\pi}{2}]$ given by $\cos \alpha = |\langle \xi, \eta \rangle|$. The angular distance $d_a([\xi], [\eta])$ is defined by the angle α .*

Proposition 1.19 ([3]). *The angular distance d_a is a metric on the projective space $\text{Proj}(\mathcal{H})$ over a Hilbert space \mathcal{H} .*

We now pass from the projective space $\text{Proj}(\mathcal{H})$ to the pure normal states on $\mathcal{B}(\mathcal{H})$ by the canonical bijection $\xi \mapsto \phi_\xi$.

A convex subset \mathcal{F} of a convex set K is said to be a face of K if the following implication holds for $y, z \in K$ and $0 < \lambda < 1$,

$$\lambda y + (1 - \lambda)z \in \mathcal{F} \Rightarrow y, z \in \mathcal{F}.$$

We need the following lemma to arrive to the final conclusion of the discussion.

Lemma 1.20 ([3]). *The face generated by two distinct normal pure states ρ, σ of $\mathcal{B}(\mathcal{H})$ is a Euclidean 3-ball, $B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$.*

This face is called a *facial ball* and denoted by $B(\rho, \sigma)$.

Definition 1.21. *Let ρ, σ be two pure normal states of $\mathcal{B}(\mathcal{H})$. The geodesic distance $d_g(\rho, \sigma)$ between ρ, σ is the length in radians of the (shorter) great circle arc joining these points on the surface of $B(\rho, \sigma)$.*

The following theorem reveals the metric property of d_g .

Theorem 1.22. *The function d_g is a metric on the space of pure normal states on $\mathcal{B}(\mathcal{H})$.*

Although the geodesic metric is a metric defined on the pure normal states on $\mathcal{B}(\mathcal{H})$, there is no natural way to define the convex combinations of pure states namely, this metric does not extend to the whole of the density operators $\mathcal{T}(\mathcal{H})_1^+$.

1.4.3 The Bures metric

For two density operators $\sigma, \rho \in \mathcal{T}(\mathcal{H})_1^+$, the Bures distance d_B is defined as:

$$d_B(\sigma, \rho) = \sqrt{1 - \text{Tr}(|\sigma^{1/2}\rho^{1/2}|)}.$$

It was proved in [26] that the Bures metric is a metric. The quantity

$$F(\sigma, \rho) = \text{Tr}(|\sigma^{1/2}\rho^{1/2}|) \tag{1.4}$$

is known as *quantum fidelity* of two elements σ, ρ . The Bures metric was introduced for normal states on a von Neumann algebra by Donald Bures in [6]. Based on Bures's work, Uhlmann in [54] introduced the concept of *transition probability* which was later named as *quantum fidelity* of two states. Jozsa in [32] gave a simple algebraic formulation of fidelity function for $d \times d$ density matrices as was given in Equation (1.4). The Bures metric on the density operators on a Hilbert space \mathcal{H} is the main motivation of this dissertation. We introduce this concept on C*-algebras with faithful trace

functionals. If \mathcal{A} is a C^* -algebra with a faithful finite trace τ , then the metric is defined on the density space $\mathcal{D}_\tau(\mathcal{A})$, that is positive elements of trace 1.

Chapter 2

Fidelity and the Bures Metric

2.1 Trace properties

In this section we will assume that we work with a unital C^* -algebra \mathcal{A} that admits a faithful trace— that is a continuous linear map $\tau : \mathcal{A} \rightarrow \mathbb{C}$ such that for all $x, y \in \mathcal{A}$, we have (i) $\tau(xy) = \tau(yx)$, (ii) $\tau(x^*x) \geq 0$, and (iii) $\tau(x^*x) = 0$ only if $x = 0$. Recall that the cone of positive elements of \mathcal{A} is denoted by \mathcal{A}_+ and the real vector space of self adjoint (or hermitian) elements of \mathcal{A} is given by \mathcal{A}_{sa} . There is a natural ordering in \mathcal{A}_{sa} ; for two elements $x, y \in \mathcal{A}_{\text{sa}}$, the notation $y \leq x$ indicates $x - y \in \mathcal{A}_+$. Also two elements $x, y \in \mathcal{A}$ are said to be *orthogonal* if $xy = yx = x^*y = xy^* = 0$.

The notation $x \perp y$ is to signify that x, y are orthogonal. We note down some useful properties of the functional τ for future reference. These properties may also be of independent interests.

Proposition 2.1. *The following statements are true for the faithful functional τ on \mathcal{A} .*

- (1) *The function $x \mapsto \tau(|x|)$ defines a norm on \mathcal{A} ,*
- (2) *$\tau(|x|) = \tau(|x^*|)$ for all $x \in \mathcal{A}$ and*
- (3) *$\tau(|ab|) \leq p^{-1}\tau(a^p) + q^{-1}\tau(b^q)$ for all $a, b \in \mathcal{A}_+$ and for positive real numbers p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the equality holds only if $b^q = a^p$.*

Proof. Because τ is faithful, we have $\tau(|x|) = 0$ only if $|x| = 0$, and so $x = 0$ when $\tau(|x|) = 0$. Moreover, the map $x \mapsto \tau(|x|)$ clearly satisfies $\alpha x \mapsto |\alpha|\tau(|x|)$, for any scalar α .

To prove the triangle inequality, let $x, y \in \mathcal{A}$. By [1, Theorem 4.2], for each $\varepsilon > 0$ there exist unitaries $u, v \in \mathcal{A}$ such that $|x + y| \leq u|x|u^* + v|y|v^* + \varepsilon 1$. Thus, $\tau(|x + y|) \leq \tau(|x|) + \tau(|y|) + \varepsilon\tau(1)$. As this is true for every $\varepsilon > 0$, we deduce that $\tau(|x + y|) \leq \tau(|x|) + \tau(|y|)$.

To prove 2, we use functional calculus. Note that since $\tau(xx^*) = \tau(x^*x)$, for any $x \in \mathcal{A}$, we have $\tau((xx^*)^k) = \tau((x^*x)^k)$ for any $k \in \mathbb{N}$. Thus by continuity of τ , we get $\tau(f(xx^*)) = \tau(f(x^*x))$, for any continuous function f . In particular choosing $f(t) = \sqrt{t}$, we get $\tau(|x|) = \tau(|x^*|)$ for every $x \in \mathcal{A}$.

Statement 3 follows from the arithmetic-geometric mean inequality on unital tracial C*-algebras. The core idea of the proof is to use the GNS representation obtained for the functional τ . Let, from the GNS representation we get $\tau(x) = \langle \pi(x)\xi, \xi \rangle$, for $x \in \mathcal{A}$, where $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a *-representation of \mathcal{A} on a Hilbert space \mathcal{H} and ξ is the unit vector in \mathcal{H} for the C*-algebra $\pi(\mathcal{A})$. If $\mathcal{M} = \pi(\mathcal{A})''$ (the double commutant), then there is a faithful normal trace $\tau_{\mathcal{M}}$ on \mathcal{M} such that $\tau(x) = \tau_{\mathcal{M}}(\pi(x))$ for all $x \in \mathcal{A}$ ([52], Proposition V.3.19). Then using the Young's inequality for von Neumann algebras we obtain the required inequality. For the detail description of Young's inequality and related discussion, see the dissertation [38]. \square

Definition 2.2. *The norm on \mathcal{A} defined in Proposition 2.1 is called the trace norm on \mathcal{A} , and is denoted by $\|\cdot\|_{1,\tau}$.*

The group of invertible elements of \mathcal{A} is denoted by $\text{GL}(\mathcal{A})$ and the set $\text{GL}(\mathcal{A})_+$ of positive invertible elements of \mathcal{A} is defined by $\text{GL}(\mathcal{A})_+ = \text{GL}(\mathcal{A}) \cap \mathcal{A}_+$. The following theorem describes some important features of the trace

functional τ . The variational characterisations of τ described in the Theorem 2.3 are inspired by the techniques given in [55].

Theorem 2.3. *If $a, \sigma, \rho \in \mathcal{A}_+$, then*

$$\tau(a) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(ay) + \tau(ay^{-1})), \quad (2.1)$$

$$\tau(|\sigma^{1/2}\rho^{1/2}|) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\rho y) + \tau(\sigma y^{-1})) = \tau(|\rho^{1/2}\sigma^{1/2}|), \quad (2.2)$$

and

$$\tau(|\sigma^{1/2}\rho^{1/2}|) = \frac{1}{4} \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\sigma y) + \tau(\sigma y^{-1}) + \tau(\rho y) + \tau(\rho y^{-1})]. \quad (2.3)$$

Proof. If $a \in \mathcal{A}_+$ and $y \in \text{GL}(\mathcal{A})_+$, then both $\tau(ay) = \tau(a^{1/2}ya^{1/2})$ and $\tau(ay^{-1}) = \tau(a^{1/2}y^{-1}a^{1/2})$ are nonnegative real numbers. Because

$$y + y^{-1} - 2 = (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 \in \mathcal{A}_+,$$

we have that

$$a^{1/2}ya^{1/2} + a^{1/2}y^{-1}a^{1/2} \geq 2a.$$

Thus,

$$\tau(a) \leq \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(ay) + \tau(ay^{-1})).$$

In taking $y = 1$, the infimum above is attained and yields

$$\tau(a) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(ay) + \tau(ay^{-1})),$$

which proves equation (2.1).

Suppose now that each of the elements σ and ρ is invertible. Thus, $b = |\rho^{1/2}\sigma^{1/2}| \in \text{GL}(\mathcal{A})_+$ and the (completely) positive linear map Ψ on \mathcal{A} defined by $\Psi(x) = b^{-1/2}\rho^{1/2}x\rho^{1/2}b^{-1/2}$, for $x \in \mathcal{A}$, is a bijection of $\text{GL}(\mathcal{A})_+$ with itself. Furthermore, if $y \in \text{GL}(\mathcal{A})_+$, then

$$\tau(\Psi(y)b) = \tau(\rho y) \quad \text{and} \quad \tau(\Psi(y)^{-1}b) = \tau(\sigma y^{-1}).$$

Hence, by equation (2.1),

$$\begin{aligned} \tau(b) &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(by) + \tau(by^{-1})) \\ &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\Psi(y)b) + \tau(\Psi(y)^{-1}b)) \\ &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\rho y) + \tau(\sigma y^{-1})). \end{aligned}$$

If it so happens that one of σ or ρ is not invertible, then they may be replaced by the positive invertible elements $\sigma_\varepsilon = \sigma + \varepsilon 1$ and $\rho_\varepsilon = \rho + \varepsilon 1$ to obtain for $b_\varepsilon = |\rho_\varepsilon^{1/2}\sigma_\varepsilon^{1/2}| \in \text{GL}(\mathcal{A})_+$ that

$$\tau(b_\varepsilon) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\rho_\varepsilon y) + \tau(\sigma_\varepsilon y^{-1})).$$

Because $\tau(b) < \tau(b_\varepsilon)$ and because $\tau(\rho_\varepsilon y) + \tau(\sigma_\varepsilon y^{-1})$ decreases to $(\tau(\rho y) +$

$\tau(\sigma y^{-1})$ as ε decreases to 0, we have that

$$\begin{aligned}\tau(b) &\leq \frac{1}{2} \inf_{\varepsilon > 0} \left[\inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(b_\varepsilon y) + \tau(b_\varepsilon y^{-1})) \right] \\ &= \frac{1}{2} \inf_{\varepsilon > 0} \left[\inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\rho_\varepsilon y) + \tau(\sigma_\varepsilon y^{-1})) \right].\end{aligned}$$

Again, using $y = 1$, we obtain

$$F_\tau(\sigma, \rho) = \tau(b) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\rho y) + \tau(\sigma y^{-1})),$$

which establishes the first part of equation (2.2). The second part of equation (2.2) follows from the first by using the fact that the map $y \mapsto y^{-1}$ is a bijection $\text{GL}(\mathcal{A})_+ \rightarrow \text{GL}(\mathcal{A})_+$.

To prove that

$$\tau(|\sigma^{1/2} \rho^{1/2}|) = \frac{1}{4} \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\sigma y) + \tau(\sigma y^{-1}) + \tau(\rho y) + \tau(\rho y^{-1})],$$

suppose once again that σ and ρ are invertible. With $b = |\sigma^{1/2} \rho^{1/2}|$ and $c = |\rho^{1/2} \sigma^{1/2}|$, the positive linear maps $\Psi, \Phi : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\Psi(x) = b^{-1/2} \rho^{1/2} x \rho^{1/2} b^{-1/2} \quad \text{and} \quad \Phi(x) = c^{-1/2} \sigma^{1/2} x \sigma^{1/2} c^{-1/2},$$

for $x \in \mathcal{A}$, are bijections. Moreover,

$$\tau(\Psi(y)b) = \tau(\rho y), \quad \tau(\Psi(y)^{-1}b) = \tau(\sigma y^{-1}),$$

$$\tau(\Phi(y)c) = \tau(\sigma y), \quad \tau(\Phi(y)^{-1}c) = \tau(\rho y^{-1}).$$

Thus, the relations above and the fact that Ψ is a positive linear bijection together imply that

$$\begin{aligned}
2\tau(b) &= \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(by) + \tau(by^{-1})) \\
&= \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(b\Psi(y)) + \tau(b\Psi(y)^{-1})) \\
&= \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\rho y) + \tau(\sigma y^{-1})).
\end{aligned}$$

Similarly,

$$2\tau(c) = \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(\sigma y) + \tau(\rho y^{-1})).$$

Hence,

$$\begin{aligned}
2\tau(b) + 2\tau(c) &= \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\rho y) + \tau(\sigma y^{-1})] + \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\sigma y) + \tau(\rho y^{-1})] \\
&\leq \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\rho y) + \tau(\sigma y^{-1}) + (\tau(\sigma y) + \tau(\rho y^{-1}))] \\
&= \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau((b+c)y) + \tau((b+c)y^{-1})) \\
&= 2\tau(b+c).
\end{aligned}$$

Therefore, the intermediate inequality above is an equality. Hence,

$$\begin{aligned}
& \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\rho y) + \tau(\sigma y^{-1})] + (\tau(\sigma y) + \tau(\rho y^{-1})) \\
&= 2(\tau(|\sigma^{1/2}\rho^{1/2}|) + \tau(|\rho^{1/2}\sigma^{1/2}|)) \\
&= 4\tau(|\sigma^{1/2}\rho^{1/2}|),
\end{aligned}$$

which proves equation (2.3) in the case where σ and ρ are invertible.

If one of σ or ρ is not invertible, then let $\sigma_\varepsilon = \sigma + \varepsilon 1$ and $\rho_\varepsilon = \rho + \varepsilon 1$, for

$$4\tau(|\sigma_\varepsilon^{1/2}\rho_\varepsilon^{1/2}|) = \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\rho_\varepsilon y) + \tau(\sigma_\varepsilon y^{-1})] + (\tau(\sigma_\varepsilon y) + \tau(\rho_\varepsilon y^{-1}))$$

Since the traces of the “ ε -elements” decrease as $\varepsilon \rightarrow 0^+$, the equation above also holds for σ and ρ , which completes the proof of equation (2.3). \square

2.2 Definition and properties of fidelity

With the useful properties of the functional τ in the previous section, we define the following set

$$\mathcal{D}_\tau(\mathcal{A}) = \{\rho \in \mathcal{A}_+ : \tau(\rho) = 1\}.$$

The set $\mathcal{D}_\tau(\mathcal{A})$ will be called *density space* of \mathcal{A} . Motivated by the use of the term “state” for density operators, we shall call the elements of $\mathcal{D}_\tau(\mathcal{A})$

τ -states or *density elements* of \mathcal{A} . Thus, in this terminology, a τ -state is a positive element of \mathcal{A} with trace 1, and is not to be confused with the traditional meaning of the word “state” in C^* -algebra theory, which refers to a positive linear functional of norm 1. We are now ready to formulate the definition of fidelity.

Definition 2.4. *The τ -fidelity between two elements $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$ is the quantity denoted by $F_\tau(\sigma, \rho)$ and defined by*

$$F_\tau(\sigma, \rho) = \tau(|\sigma^{1/2}\rho^{1/2}|).$$

The basic properties of τ -fidelity are noted below.

Proposition 2.5. *If $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$, then*

- (1) $F_\tau(\sigma, \rho) = F_\tau(\rho, \sigma)$,
- (2) $0 \leq F_\tau(\sigma, \rho) \leq 1$,
- (3) $F_\tau(\sigma, \rho) = 0$ if and only if $\sigma \perp \rho$, and
- (4) $F_\tau(\sigma, \rho) = 1$ if and only if $\sigma = \rho$.

Proof. Statement (1) is immediate from the Equation (2.2) of the Theorem 2.3. This could also be proved setting $x = \sigma^{1/2}\rho^{1/2}$ and using item (2) of the Proposition 2.1.

In the assertion (2), $0 \leq F_\tau(\sigma, \rho)$ is obvious. On the other hand, the item (3) of the Proposition 2.1 with $p = 2$, $q = 2$, we get

$$F_\tau(\sigma, \rho) = \tau(|\sigma^{1/2}\rho^{1/2}|) \leq \frac{\tau(\sigma) + \tau(\rho)}{2} = 1.$$

Thus $0 \leq F_\tau(\sigma, \rho) \leq 1$.

For the item (3), note that using the functional calculus we can prove $\sigma^{1/2} \perp \rho^{1/2}$ if and only if $\sigma \perp \rho$. Hence by the faithfulness of τ and by the definition of $F_\tau(\sigma, \rho)$, it is clear that $F_\tau(\sigma, \rho) = 0$ if and only if $\sigma \perp \rho$.

The sufficiency of the statement (4) is obvious from the definition of fidelity. For the converse part, let us assume $F_\tau(\sigma, \rho) = 1$. Then we obtain $1 = \tau(|\sigma^{1/2}\rho^{1/2}|) = \frac{\tau(\sigma) + \tau(\rho)}{2}$, which is a case of equality in the arithmetic-geometric mean inequality of the Proposition 2.1, item 3. Hence we get $\sigma = \rho$. \square

Now we enlist some more properties of τ -fidelity in the following proposition.

Proposition 2.6 (Joint Concavity). *If $a, b, c, d \in \mathcal{A}_+$, then we get*

$$F_\tau(a + b, c + d) \geq F_\tau(a, c) + F_\tau(b, d).$$

Proof. By the characterisation of fidelity given in Theorem 2.3, if $a, b \in \mathcal{A}_+$,

then

$$\tau(|a^{1/2}b^{1/2}|) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau(ay) + \tau(by^{-1})).$$

Therefore, if $a, b, c, d \in \mathcal{A}_+$, then

$$\begin{aligned} \tau(|(a+b)^{1/2}(c+d)^{1/2}|) &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} (\tau((a+b)y) + \tau((c+d)y^{-1})) \\ &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} ([\tau((ay) + \tau(cy^{-1}))] + [\tau(by) + \tau(dy^{-1})]) \\ &\geq \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} ([\tau((ay) + \tau(cy^{-1}))]) \\ &\quad + \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} ([\tau(by) + \tau(dy^{-1})]) \\ &= \tau(|a^{1/2}c^{1/2}|) + \tau(|b^{1/2}d^{1/2}|). \end{aligned}$$

The inequality above occurs from the fact that if f, g are two positive functions, then $\inf_x (f + g)(x) \geq \inf_x (f(x)) + \inf_x (g(x))$. \square

2.3 The Bures metric is a metric

Now with the help of the τ -fidelity function we define a metric in the following way:

Definition 2.7. *The function on $\mathcal{D}_\tau(\mathcal{A})$ defined by*

$$d_B^\tau(\sigma, \rho) = \sqrt{1 - F_\tau(\sigma, \rho)}, \text{ for } \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}).$$

is called the Bures metric.

Before we set out to prove that Bures metric is indeed a metric, we need to prove the following lemmas.

Lemma 2.8. *If τ is a faithful normal trace functional on a von Neumann algebra \mathcal{N} , and if $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{N})$, then*

(1) $d_B^\tau(\sigma, \rho) = d_B^\tau(\rho, \sigma)$, and

(2) if $d_B^\tau(\sigma, \rho) = 0$, then $\sigma = \rho$.

Proof. This follows from the definition of the Bures metric and Proposition 2.5. □

Lemma 2.9. *If τ is a faithful normal trace functional on a von Neumann algebra \mathcal{N} , and if $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{N})$, then*

(1) $\sqrt{2} d_B^\tau(\sigma, \rho) \leq \sqrt{\tau(|\sigma^{1/2} - \rho^{1/2}w|^2)}$, for every $w \in \mathcal{N}$ of norm $\|w\| \leq 1$,

and

(2) there exists a unitary $w \in \mathcal{N}$ for which equality in (1) holds.

Proof. If $w \in \mathcal{N}$ has norm $\|w\| \leq 1$, then $1 - w^*w \in \mathcal{N}_+$ and

$$\begin{aligned}
\tau(|\sigma^{1/2} - \rho^{1/2}w|^2) &= 2(1 - \Re[\tau(w\sigma^{1/2}\rho^{1/2})]) \\
&\geq 2(1 - |\tau(w\sigma^{1/2}\rho^{1/2})|) \\
&\geq 2(1 - \tau(|w\sigma^{1/2}\rho^{1/2}|)) \\
&\geq 2(1 - \tau(|\sigma^{1/2}\rho^{1/2}|)),
\end{aligned}$$

where the final inequality follows from $|wx|^2 = x^*(w^*w)x \leq x^*x = |x|^2$ and the monotonicity of the square root function $t \mapsto t^{1/2}$ in functional calculus. Thus, $\sqrt{2} d_B^r(\sigma, \rho) \leq \sqrt{\tau(|\sigma^{1/2} - \rho^{1/2}w|^2)}$.

Because \mathcal{N} is a finite von Neumann algebra, there exists an extreme point v of the unit ball of \mathcal{N} such that $\sigma^{1/2}\rho^{1/2} = v|\sigma^{1/2}\rho^{1/2}|$ [8]. Therefore, since the extreme points of the unit ball of a finite von Neumann algebra are necessarily unitary, $v^*v = vv^* = 1$. Thus, $v^*\sigma^{1/2}\rho^{1/2} = |\sigma^{1/2}\rho^{1/2}| = |\sigma^{1/2}\rho^{1/2}|^* = \rho^{1/2}\sigma^{1/2}v$. Hence,

$$2 - 2\tau(|\sigma^{1/2}\rho^{1/2}|) = 2 - 2\Re[\tau(v^*\sigma^{1/2}\rho^{1/2})] = \tau(|\sigma^{1/2} - \rho^{1/2}v^*|^2),$$

which yields equality in (1). □

Lemma 2.10. *If τ is a faithful normal trace functional on a von Neumann algebra \mathcal{N} , and if $\sigma, \rho, \theta \in \mathcal{D}_\tau(\mathcal{N})$, then*

$$d_B^r(\sigma, \rho) \leq d_B^r(\sigma, \theta) + d_B^r(\theta, \rho).$$

Proof. By Lemma 2.9, there are unitaries $u, w \in \mathcal{N}$ such that

$$\sqrt{2}d_B^r(\sigma, \theta) = \sqrt{\tau(|\sigma^{1/2} - \theta^{1/2}w|^2)} \text{ and } \sqrt{2}d_B^r(\theta, \rho) = \sqrt{\tau(|\theta^{1/2} - \rho^{1/2}u|^2)}.$$

Let $v = uw$. Thus,

$$\begin{aligned} \sigma^{1/2} - \rho^{1/2}v &= \sigma^{1/2} - \theta^{1/2}w + \theta^{1/2}w - \rho^{1/2}v \\ &= (\sigma^{1/2} - \theta^{1/2}w) + (\theta^{1/2} - \rho^{1/2}vw^*)w \\ &= (\sigma^{1/2} - \theta^{1/2}w) + (\theta^{1/2} - \rho^{1/2}u)w. \end{aligned}$$

Let $x = \sigma^{1/2} - \theta^{1/2}w$ and $y = -(\theta^{1/2} - \rho^{1/2}u)w$ so that $x - y = \sigma^{1/2} - \rho^{1/2}v$.

The Cauchy-Schwarz inequality for the sesquilinear form $(x, y) \mapsto \tau(xy^*)$

yields $|\tau(xy^*)| \leq \sqrt{\tau(|x|^2)\tau(|y|^2)}$, and so

$$\begin{aligned}
\left(\sqrt{\tau(|x|^2)} + \sqrt{\tau(|y|^2)}\right)^2 &= \tau(|x|^2) + \tau(|y|^2) + 2\sqrt{\tau(|x|^2)\tau(|y|^2)} \\
&\geq \tau(|x|^2) + \tau(|y|^2) + 2|\tau(xy^*)| \\
&\geq \tau(|x|^2) + \tau(|y|^2) + 2\Re[\tau(xy^*)] \\
&= \tau(|x - y|^2).
\end{aligned}$$

That is,

$$\begin{aligned}
\sqrt{\tau(|\sigma^{1/2} - \rho^{1/2}v|^2)} &\leq \sqrt{\tau(|\sigma^{1/2} - \theta^{1/2}w|^2)} + \sqrt{\tau(|(\theta^{1/2} - \rho^{1/2}u)w|^2)} \\
&= \sqrt{\tau(|\sigma^{1/2} - \theta^{1/2}w|^2)} + \sqrt{\tau(|\theta^{1/2} - \rho^{1/2}u|^2)} \\
&= \sqrt{2}d_B^r(\sigma, \theta) + \sqrt{2}d_B^r(\theta, \rho).
\end{aligned}$$

Because Lemma 2.9 asserts that $\sqrt{2}d_B^r(\sigma, \rho) \leq \sqrt{\tau(|\sigma^{1/2} - \rho^{1/2}v|^2)}$, the triangle inequality $d_B^r(\sigma, \rho) \leq d_B^r(\sigma, \theta) + d_B^r(\theta, \rho)$ follows. \square

Lemmas 2.9 and 2.10 are generalisations of the results obtained for matrices: see [26, Exercise 2.20]. Now we proceed to show that the Bures metric is indeed a metric on $\mathcal{D}_\tau(\mathcal{A})$.

Theorem 2.11. *The function d_B^τ is a metric on $\mathcal{D}_\tau(\mathcal{A})$, for every unital C^* -algebra \mathcal{A} and faithful trace functional τ on \mathcal{A} .*

Proof. If \mathcal{A} is a von Neumann algebra and if τ is a faithful normal trace functional, then Lemmas 2.8, 2.9, and 2.10 show that the function d_B^τ is a metric on $\mathcal{D}_\tau(\mathcal{A})$. If these hypotheses on \mathcal{A} and τ are not in effect, then let the GNS representation of τ be given by $\tau(x) = \langle \pi(x)\xi, \xi \rangle$, for $x \in \mathcal{A}$, where $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a unital $*$ -homomorphism and $\xi \in \mathcal{H}$ is a unit cyclic vector for the C^* -algebra $\pi(\mathcal{A})$. Let \mathcal{N} denote the double commutant of $\pi(\mathcal{A})$. By [52, Proposition V.3.19], there exists a normal trace on \mathcal{N} such that $\tau = \tau_{\mathcal{N}} \circ \pi$. It is easy to see that $d_B^\tau(\sigma, \rho) = d_B^{\tau_{\mathcal{N}}}(\pi(\sigma), \pi(\rho))$, for all $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$, and hence the metric properties of d_B^τ are inherited from the metric properties of $d_B^{\tau_{\mathcal{N}}}$. \square

2.4 Equivalence of the Bures metric and the trace-metric

Now that we have established the metric property of d_B^τ , we want to show that the metric space $\mathcal{D}_\tau(\mathcal{A})$ with respect to the metric induced by $\|\cdot\|_{1,\tau}$ and the metric d_B^τ are homeomorphic. We need the following proposition to

go forward.

Proposition 2.12 (Fuchs-van de Graaf Inequality). *For all $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$,*

$$1 - \frac{1}{2}\|\rho - \sigma\|_{1,\tau} \leq F_\tau(\rho, \sigma) \leq \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_{1,\tau}^2}.$$

Equivalently,

$$2 - 2F_\tau(\rho, \sigma) \leq \|\rho - \sigma\|_{1,\tau} \leq 2\sqrt{1 - F_\tau(\rho, \sigma)^2}.$$

Proof. As in the proof of Theorem 2.11, we may assume without loss of generality that \mathcal{A} is a unital C^* -subalgebra of a finite von Neumann algebra \mathcal{N} with faithful normal trace τ such that \mathcal{A} is dense in \mathcal{N} with respect to the strong operator topology.

The Powers-Størmer inequality [28, 44] asserts that $2\tau(a^{1/2}b^{1/2}) \geq \tau(a + b - |a - b|)$, for all $a, b \in \mathcal{A}_+$. Because $|\tau(x)| \leq \tau(|x|)$ for every $x \in \mathcal{A}$, we obtain $2 - \|\rho - \sigma\|_{1,\tau} \leq 2F_\tau(\rho, \sigma)$, which gives the first inequality.

For the second inequality, observe that, for any unitary $u \in \mathcal{N}$ and $\rho, \sigma \in \mathcal{D}_\tau(\mathcal{A})$,

$$2(\rho - \sigma) = (\rho^{1/2} + \sigma^{1/2}u^*)(\rho^{1/2} - u\sigma^{1/2}) + (\rho^{1/2} - \sigma^{1/2}u^*)(\rho^{1/2} + u\sigma^{1/2}).$$

By the triangle inequality for the trace norm and using the Hölder inequality [15], we see that

$$\|(\rho - \sigma)\|_{1,\tau}^2 \leq \|(\rho^{1/2} + \sigma^{1/2}u^*)\|_{2,\tau}^2 \|(\rho^{1/2} - u\sigma^{1/2})\|_{2,\tau}^2,$$

where $\|x\|_{2,\tau}$ is given by $\sqrt{\tau(x^*x)}$. Simplifying the right hand side yields

$$\begin{aligned} \|(\rho - \sigma)\|_{1,\tau}^2 &\leq (2 + 2\Re \tau(\rho^{1/2}\sigma^{1/2}u))(2 - 2\Re \tau(\rho^{1/2}\sigma^{1/2}u)) \\ &= 4 - 4(\Re \tau(\rho^{1/2}\sigma^{1/2}u))^2. \end{aligned}$$

By Lemma 2.9, $F_\tau(\rho, \sigma) = \sup_{u \in \mathcal{U}(\mathcal{N})} \Re \tau(\rho^{1/2}\sigma^{1/2}u)$, where $\mathcal{U}(\mathcal{N})$ is the unitary group of \mathcal{N} . Therefore, the inequalities above imply that $4F(\rho, \sigma)^2 \leq 4 - \|(\rho - \sigma)\|_{1,\tau}^2$. \square

If d_1^τ denotes the metric on \mathcal{A} given by $d_1^\tau(x, y) = \|x - y\|_{1,\tau}$, then we have the following corollary.

Corollary 2.13. *The metric spaces $(\mathcal{D}_\tau(\mathcal{A}), d_B^\tau)$ and $(\mathcal{D}_\tau(\mathcal{A}), d_1^\tau)$ are homeomorphic.*

2.5 Monotonicity of fidelity under positive maps

We begin with the monotonicity theorem.

Theorem 2.14. *If $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is a positive linear map such that $\tau \circ \mathcal{E} = \tau$, then*

$$F_\tau(\sigma, \rho) \leq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \quad \forall \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}). \quad (2.4)$$

Proof. Suppose that $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is a positive linear map such that $\tau \circ \mathcal{E} = \tau$.

With $y = 1$, in Theorem 2.3 equation (2.2) yields

$$F_\tau(\sigma, \rho) \leq \frac{1}{2} (\tau(\rho) + \tau(\sigma)) = \frac{1}{2} (\tau(\mathcal{E}(\rho)) + \tau(\mathcal{E}(\sigma))).$$

Equation (2.3) yields

$$\begin{aligned} 2F_\tau(\sigma, \rho) &\leq \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} [\tau(\mathcal{E}(\rho)y) + \tau(\mathcal{E}(\rho)y^{-1}) + \tau(\mathcal{E}(\sigma)y) + \tau(\mathcal{E}(\sigma)y^{-1})] \\ &= F_\tau(\mathcal{E}(\rho), \mathcal{E}(\sigma)) + F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \\ &= 2F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)). \end{aligned}$$

Hence, $F_\tau(\sigma, \rho) \leq F_\tau(\mathcal{E}(\rho), \mathcal{E}(\sigma))$, which establishes inequality (2.4) \square

Using Theorem 2.14, we get the following immediate corollary.

Corollary 2.15. *If $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is a positive linear map such that $\tau \circ \mathcal{E} = \tau$, then \mathcal{E} is a contraction on $\mathcal{D}_\tau(\mathcal{A})$ equipped with the Bures metric that is*

$$d_B^r(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq d_B^r(\sigma, \rho) \quad \forall \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}).$$

Chapter 3

Bures Isometries

In this chapter we analyse positive and τ -preserving linear maps $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ such that $F_\tau(\sigma, \rho) = F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$ for every $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$. In other words, we are interested in characterising maps that are isometries on $\mathcal{D}_\tau(\mathcal{A})$ equipped with the Bures metric. In light of the Theorem 2.14, it is natural to study such isometries.

3.1 The finite and quasi-transitive case

To begin with, we introduce the following definition.

Definition 3.1. *A positive linear map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ preserves τ -fidelity if*

(1) $\tau \circ \mathcal{E} = \mathcal{E}$, and

(2) $F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) = F_\tau(\sigma, \rho)$ for all τ -states $\sigma, \rho \in \mathcal{A}_+$.

Observe that if for a unitary element $u \in \mathcal{A}$, $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is defined by $\mathcal{E}(x) = uxu^*$ for all $x \in \mathcal{A}$, then it is evident that \mathcal{E} preserves the τ -fidelity for all pair of τ -states. Our attention in this section will be to arrive to a converse of this observation. Note that in the usual setting of quantum mechanics where a linear map acts on the trace class operators, a full characterisation of fidelity preserving maps was given by [39]. The main theorem of [39] is given below.

Theorem 3.2. [Molnár] Let $\mathcal{T}(\mathcal{H})_1^+$ be the set of density operators on a Hilbert space \mathcal{H} . Let $\mathcal{E} : \mathcal{T}(\mathcal{H})_1^+ \rightarrow \mathcal{T}(\mathcal{H})_1^+$ be a bijective transformation with the property that

$$F(\mathcal{E}(\sigma), \mathcal{E}(\rho)) = F(\sigma, \rho) \text{ for every } \sigma, \rho \in \mathcal{T}(\mathcal{H})_1^+$$

. Then there is either a unitary or anti-unitary operator $u : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\mathcal{E}(x) = uxu^*, \quad x \in \mathcal{T}(\mathcal{H})_1^+.$$

Motivated by this theorem we set out to find the analogous theorem for

τ -fidelity preserving maps on a C^* -algebra \mathcal{A} . We first prove the following lemma.

Lemma 3.3. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a positive linear map that preserves τ -fidelity, then \mathcal{E} is an injection. Moreover, if \mathcal{E} is also surjective, then \mathcal{E} is an order isomorphism.*

Proof. We first prove that \mathcal{E} is a linear injection. To this end, if $a, b \in \mathcal{A}_+$ satisfy $\mathcal{E}(a) = \mathcal{E}(b)$, then setting $\sigma = \tau(a)^{-1}a$ and $\rho = \tau(b)^{-1}b$ yields elements $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$ for which $\tau(a)\sigma = a$ and $\tau(b)\rho = b$. Thus, $\mathcal{E}(a) = \mathcal{E}(b)$ yields $\tau(a)\mathcal{E}(\sigma) = \tau(b)\mathcal{E}(\rho)$ and, by applying the trace to this last equation, $\tau(a) = \tau(b)$. Hence, $\mathcal{E}(\sigma) = \mathcal{E}(\rho)$ and, therefore,

$$1 = F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) = \tau(\sigma, \rho).$$

Proposition 2.5 implies that $\sigma = \rho$ and, consequently, that $a = b$. Therefore, $\mathcal{E}|_{\mathcal{A}_+}$ is an injective function.

Assume that $h \in \mathcal{A}$ is selfadjoint and that $\mathcal{E}(h) = 0$. There exist $h_+, h_- \in \mathcal{A}_+$ such that $h = h_+ - h_-$ and so $0 = \mathcal{E}(h_+) - \mathcal{E}(h_-)$. In other words, $\mathcal{E}(h_+) = \mathcal{E}(h_-)$ and so $h_+ = h_-$ because $\mathcal{E}|_{\mathcal{A}_+}$ is an injective function. Thus, $h = 0$. Because \mathcal{E} is real linear, this implies that $\mathcal{E}|_{\mathcal{A}_{\text{sa}}}$ is an injective function. Lastly, suppose that $x \in \mathcal{A}$ satisfies $\mathcal{E}(x) = 0$. Thus, $\mathcal{E}(x^*) = 0$ and, writing

$x = a + ib$ for some $a, b \in \mathcal{A}_{\text{sa}}$, $\mathcal{E}(a) + i\mathcal{E}(b) = \mathcal{E}(a) - i\mathcal{E}(b) = 0$. Thus, $\mathcal{E}(a) = \mathcal{E}(b) = 0$, which implies that $a = b = 0$ and $x = 0$. Hence \mathcal{E} is a linear injection.

Because \mathcal{E} is a linear injection and assumed to be surjective, \mathcal{E} admits a linear inverse \mathcal{E}^{-1} . We aim to show that \mathcal{E}^{-1} is a positive map. To this end, select $h \in \mathcal{A}_+$ and let $a = \mathcal{E}^{-1}(h)$. Therefore, because \mathcal{E} is an injection,

$$\mathcal{E}(a) = h = h^* = \mathcal{E}(a)^* = \mathcal{E}(a^*)$$

implies that $a = a^*$. Thus, there are $a_+, a_- \in \mathcal{A}_+$ such that $a = a_+ - a_-$ and $a_+ \perp a_-$. Let $b = \mathcal{E}(a_+)$ and $c = \mathcal{E}(a_-)$ to obtain $b, c \in \mathcal{A}_+$. If one of a_+ or a_- is zero, then $bc = cb = 0$. If neither a_+ nor a_- is zero, then scale them by their traces so that $\sigma = \tau(a_+)^{-1}a_+$ and $\rho = \tau(a_-)^{-1}a_-$ are τ -states. Because $\sigma \perp \rho$ and $0 = F_\tau(\sigma, \rho) = F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$, Proposition 2.5 yields $\mathcal{E}(\sigma) \perp \mathcal{E}(\rho)$. Therefore, by linearity, $\mathcal{E}(a_+) \perp \mathcal{E}(a_-)$, which shows that $bc = cb = 0$.

From $b = h + c$ we obtain

$$0 = bc = hc + c^2 \quad \text{and} \quad 0 = cb = ch + c^2.$$

Hence, $hc - ch = 0$. Because h and c commute, so do h and $c^{1/2}$. Thus,

$$0 = bc = hc + c^2 = c^{1/2}hc^{1/2} + c^2$$

expresses 0 as a sum of positive elements $c^{1/2}hc^{1/2}$ and c^2 . Therefore $c^2 = hc = 0$, and so $c = 0$ also. Hence, $0 = \mathcal{E}^{-1}(c) = a_- = 0$, which yields $a = a_+ \in \mathcal{A}_+$. Hence, \mathcal{E}^{-1} is a positive map, which implies that \mathcal{E} is an order isomorphism. \square

Corollary 3.4. *Every surjective positive linear map that preserves τ -fidelity is a Jordan isomorphism.*

Proof. By [51, Theorem 2.1.3], every order isomorphism of a unital C^* -algebra is a Jordan isomorphism. \square

When a C^* -algebra \mathcal{A} resembles a matrix algebra in terms of certain algebraic property it carries, then a concrete description of τ -fidelity preserving maps can be given. Before stating the theorem we need some more definitions.

Definition 3.5. *A unital C^* -algebra \mathcal{A} is*

- (1) *finite, if for $x, y \in \mathcal{A}$, $xy = 1$ implies $yx = 1$.*
- (2) *quasi-transitive, if $x\mathcal{A}y = \{0\}$, for some $x, y \in \mathcal{A}$, holds only if $x = 0$ or $y = 0$.*

Theorem 3.6 is one of the main theorems in this dissertation. This theorem can be seen as a generalisation of Molnár's result (Theorem 3.2) on matrix algebras.

Theorem 3.6. *Assume that \mathcal{A} is a finite and quasi-transitive C^* -algebra. If a surjective Schwarz map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ preserves τ -fidelity, then \mathcal{E} is an automorphism of \mathcal{A} .*

Proof. Note that for every $x \in \mathcal{A}$ we have that $\mathcal{E}(x^*x) \geq \mathcal{E}(x)^*\mathcal{E}(x)$, which implies that

$$\|\mathcal{E}\| \|x\|^2 = \|\mathcal{E}\| \|x^*x\| \geq \|\mathcal{E}(x^*x)\| \geq \|\mathcal{E}(x)^*\mathcal{E}(x)\| = \|\mathcal{E}(x)\|^2.$$

Since \mathcal{E} is a positive map and by Russo-Dye theorem, its norm is achieved at $1 \in \mathcal{A}$ [48], [51, Theorem 1.3.3]. With $x = 1$ and using $\|\mathcal{E}\| = \|\mathcal{E}(1)\|$, the inequality above yields $\|\mathcal{E}\|^2 \leq \|\mathcal{E}\|$, and so $\|\mathcal{E}\| \leq 1$. Because $\|\mathcal{E}\| \leq 1$, the element $\mathcal{E}(1) \in \mathcal{A}$ satisfies $\|\mathcal{E}(1)\| \leq 1$. Thus, $\mathcal{E}(1)$ is a positive contraction and, therefore, $1 - \mathcal{E}(1)$ is positive. Apply τ to obtain $0 \leq \tau(1 - \mathcal{E}(1)) = \tau(1) - \tau \circ \mathcal{E}(1) = \tau(1) - \tau(1) = 0$. Because τ is faithful and $1 - \mathcal{E}(1) \in \mathcal{A}_+$, we have $\tau(1 - \mathcal{E}(1)) = 0$ only if $1 - \mathcal{E}(1) = 0$. That is, \mathcal{E} is unital.

Since, by Lemma 3.3, \mathcal{E}^{-1} and \mathcal{E} are positive linear maps, their norms are achieved at $1 \in \mathcal{A}$. Hence, $\mathcal{E}^{-1}(1) = 1$ implies that $\|\mathcal{E}^{-1}\| = 1$. Thus, for

every $x \in \mathcal{A}$,

$$\|x\| = \|\mathcal{E}^{-1} \circ \mathcal{E}(x)\| \leq \|\mathcal{E}^{-1}\| \|\mathcal{E}(x)\| = \|\mathcal{E}(x)\| \leq \|x\|.$$

In other words, \mathcal{E} is an isometry.

Select a unitary $u \in \mathcal{A}$. Because \mathcal{E} is an isometry, we have that $\mathcal{E}(u)$ is in the closed unit ball of \mathcal{A} . Suppose that $\mathcal{E}(u) = \frac{1}{2}x + \frac{1}{2}y$ for some $x, y \in \mathcal{A}$ with $\|x\| \leq 1$ and $\|y\| \leq 1$. Then $u = \frac{1}{2}\mathcal{E}^{-1}(x) + \frac{1}{2}\mathcal{E}^{-1}(y)$. Because \mathcal{E}^{-1} is an isometry and every unitary in \mathcal{A} is an extreme point of the closed unit ball of \mathcal{A} , $u = \mathcal{E}^{-1}(x) = \mathcal{E}^{-1}(y)$, which implies that $x = y = \mathcal{E}(u)$. Hence, $\mathcal{E}(u)$ is an extreme point of the closed unit ball of \mathcal{A} . By [33, Theorem 1], the extreme point $\mathcal{E}(u)$ is necessarily a partial isometry such that, if $e = \mathcal{E}(u)^*\mathcal{E}(u)$ and $f = \mathcal{E}(u)\mathcal{E}(u^*)$, then $(1 - e)\mathcal{A}(1 - f) = \{0\}$. By the hypothesis that \mathcal{A} is quasi-transitive, we obtain $e = 1$ or $f = 1$. Thus, either $\mathcal{E}(u)^*\mathcal{E}(u) = 1$ or $\mathcal{E}(u^*)\mathcal{E}(u) = 1$. Because \mathcal{A} is finite, either of these conditions imply that $\mathcal{E}(u)$ is unitary. Hence, $\mathcal{E}(u)$ is unitary for each unitary $u \in \mathcal{A}$. Now for every unitary $u \in \mathcal{A}$, we see that

$$1 = \mathcal{E}(u^*u) = \mathcal{E}(u^*)\mathcal{E}(u) = \mathcal{E}(u)\mathcal{E}(u^*) = \mathcal{E}(uu^*).$$

Hence, every unitary in \mathcal{A} is in the multiplicative domain $\mathcal{M}_{\mathcal{E}}$ of \mathcal{E} . Because \mathcal{E} is a Schwarz map, its multiplicative domain is a C^* -subalgebra of \mathcal{A} [51,

Corollary 2.1.6]. Thus, $\mathcal{M}_{\mathcal{E}}$ contains the norm-closed span of the unitary group of \mathcal{A} , which by the Russo–Dye Theorem [48, Theorem 1] implies that $\mathcal{M}_{\mathcal{E}} = \mathcal{A}$. Hence, the bijective unital map \mathcal{E} is a homomorphism. \square

While the full matrix algebra $\mathcal{M}_d(\mathbb{C})$ of $d \times d$ complex matrices is one such example of a finite and quasi-transitive C^* -algebra, there are many other examples that are infinite-dimensional. One of the most important of such examples, because it arises as a limit of finite-dimensional algebras, is the fermion algebra $\mathcal{A} = \bigotimes_1^{\infty} \mathcal{M}_2(\mathbb{C}) = \lim_{\rightarrow} \bigotimes_1^k \mathcal{M}_2(\mathbb{C})$, which is the C^* -algebra representing the infinite canonical anticommutation relations arising from the quantum mechanical study of fermions and which has a (unique) faithful trace functional τ that arises from the normalised canonical trace on each $\mathcal{M}_{2^k}(\mathbb{C}) \cong \bigotimes_1^k \mathcal{M}_2(\mathbb{C})$ [17, §11.9].

Theorem 3.7. *The Fermion algebra $\mathcal{A} = \bigotimes_1^{\infty} \mathcal{M}_2(\mathbb{C}) = \lim_{\rightarrow} \bigotimes_1^k \mathcal{M}_2(\mathbb{C})$ is a finite and quasi-transitive C^* -algebra.*

Proof. Note that the Fermion algebra \mathcal{A} is a simple C^* -algebra which has a unique faithful trace functional τ .

Finite:

Suppose for $x, y \in \mathcal{A}$, we have $xy = 1$. We want to prove $yx = 1$

as well. From $xy = 1$, we have $\tau(xy) = 1 = \tau(yx)$. Now observe that $(yx)^2 = yxyx = yx$, and hence yx is an idempotent in \mathcal{A} . Call $a = yx$. We will show that there is a self-adjoint idempotent p such that $\tau(p) = 1$. To this end, set $z = 1 + (a - a^*)^*(a - a^*)$. Clearly z is positive and invertible. One checks that $az = za = aa^*a$. So a and a^* commute with z (and hence z^{-1}). Define $p = aa^*z^{-1}$. Then it follows that $p^2 = p$ and also p is self-adjoint. Now it is easy to check that $ap = p$ and $pa = a$.

So $1 = \tau(a) = \tau(pa) = \tau(ap) = \tau(p)$. As for any projection p , one has $p \leq 1$ in operator ordering, by faithfulness of τ we obtain $p = 1$ and hence $a = yx = 1$.

Quasi-transitive: We know that \mathcal{A} is a simple C^* -algebra. Now suppose for $x, y \in \mathcal{A}$ we have $x\mathcal{A}y = \{0\}$. Without loss of generality suppose $x \neq 0$. Now consider the (two sided) ideal

$$\mathcal{J} = \{z \in \mathcal{A} : x\mathcal{A}z = \{0\}\}.$$

Clearly $y \in \mathcal{J}$. As \mathcal{A} is simple, either $\mathcal{J} = \mathcal{A}$ or $\mathcal{J} = \{0\}$. If $\mathcal{J} = \mathcal{A}$, then since $1 \in \mathcal{A}$, we get $x = x1 = 0$, contradicting the assumption that $x \neq 0$. On the other hand, if $\mathcal{J} = \{0\}$, then $y = 0$. Hence \mathcal{A} is quasi-transitive.

□

3.2 Completely positive isometries

To study τ -fidelity preserving maps on C^* -algebras that are not finite or quasi-transitive, we need to require complete positivity of the maps than only positivity. Also we need the following definition.

Definition 3.8. *A linear map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a transformation of order zero if $\mathcal{E}(a) \perp \mathcal{E}(b)$ for all $a, b \in \mathcal{A}$ for which $a \perp b$.*

Note that Gardener in [22] has studied maps of order zero in details but we will exploit the structure theory of completely positive maps of order zero explicitly given by Winter and Zacharias in [56].

Theorem 3.9. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a completely positive linear map that preserves the τ -fidelity of all τ -states. Then there is a homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{A}$ and a positive element h in the centre of \mathcal{A} such that $\mathcal{E}(x) = \pi(x)h$, for every $x \in \mathcal{A}$. In particular, if \mathcal{A} has trivial centre, then \mathcal{E} is a unital injective homomorphism.*

Proof. Select any nonzero $a, b \in \mathcal{A}_+$ and scale them by their traces so that $\sigma = \tau(a)^{-1}a$ and $\rho = \tau(b)^{-1}b$ are τ -states. Further, assume that $a \perp b$; thus, $\sigma \perp \rho$. Hence, by Proposition 2.5, $0 = F_\tau(\sigma, \rho) = F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$ yields $\mathcal{E}(\sigma) \perp \mathcal{E}(\rho)$ and, by linearity, $\mathcal{E}(a) \perp \mathcal{E}(b)$. By Stinespring's Theorem,

$\mathcal{E}(x^*)\mathcal{E}(x) \leq \|\mathcal{E}(1)\|^2\mathcal{E}(x^*x)$ for every $x \in \mathcal{A}$. Therefore, using the proof of Remark 1.4 in [56], we see that the property “ $\mathcal{E}(a) \perp \mathcal{E}(b)$ for all positive a and b with $a \perp b$ ” implies the property “ $\mathcal{E}(x) \perp \mathcal{E}(y)$ for all $x, y \in \mathcal{A}$ for which $x \perp y$.” That is, \mathcal{E} is a completely positive linear map of order zero.

Because \mathcal{E} is a completely positive linear map of order zero, $h = \mathcal{E}(1)$ is the centre of \mathcal{A} and there is a homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{A}$ such that $\mathcal{E}(x) = \pi(x)h$, for all $x \in \mathcal{A}$ [56, Theorem 3.3]. If the centre of \mathcal{A} is trivial, then $h = \lambda 1$ for some $\lambda \in \mathbb{R}_+$. Further, $\tau(1) = \tau \circ \mathcal{E}(1) = \tau(\pi(1)h) = \lambda\tau(1)$ implies that $\lambda = 1$, and so $\mathcal{E} = \pi$. The fact that π is an injection follows from the fact that \mathcal{E} is an injection (Lemma 3.3). \square

Note that a finite von Neumann algebra is a unital C^* -algebra that possesses a faithful trace functional. Indeed, if \mathcal{M} is a finite von Neumann algebra which is a factor, then \mathcal{M} has a unique (normal) faithful trace functional τ satisfying $\tau(1) = 1$ and all other tracial functionals are positive scalar multiples of τ . Therefore, if a positive linear map $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{M}$ preserves fidelity of all pair of density operators with respect to one trace, then it does so with respect to every other trace.

Proposition 3.10. *Assume that \mathcal{M} is a finite factor. If $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{M}$ is a surjective Schwarz map that preserves fidelity, then \mathcal{E} is an automorphism*

of \mathcal{M} . If, moreover, \mathcal{M} is a finite-dimensional factor, then \mathcal{E} is a unitary channel.

Proof. Because \mathcal{M} is both finite and quasi-transitive as a C^* -algebra (see [47]), Theorem 3.6 implies that \mathcal{E} is necessarily an automorphism of \mathcal{M} (and, moreover, automatically normal). If $\mathcal{M} \cong \mathcal{M}_d(\mathbb{C})$ for some $d \in \mathbb{N}$, then every automorphism of \mathcal{M} is inner; hence, \mathcal{E} is a unitary channel. \square

Chapter 4

Bures Contractions

In this chapter we once again invoke the monotonicity of fidelity under a τ -preserving positive linear map given in Theorem 2.14. Hence by the Corollary 2.15, we know that any τ -preserving positive linear map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is a contraction on the metric space $\mathcal{D}_\tau(\mathcal{A})$ with respect to the Bures metric d_B^τ .

That is

$$d_B^\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq d_B^\tau(\sigma, \rho) \quad \forall \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}). \quad (4.1)$$

In connection with positive linear maps and Bures metric, we like to put forward the following proposition at the outset.

Proposition 4.1. *If \mathcal{E} is a τ preserving positive map and Φ is any arbitrary*

positive linear map on \mathcal{A} such that $\mathcal{E}(a) \leq \Phi(a)$, for all $a \in \mathcal{A}_+$, then

$$d_B^r(\Phi(\sigma), \Phi(\rho)) \leq d_B^r(\mathcal{E}(\sigma), \mathcal{E}(\rho)),$$

for all $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$.

Proof. By the variational characterisation of fidelity given in Theorem 2.3,

if $\rho, \sigma \in \mathcal{D}_\tau(\mathcal{A})$, then

$$F_\tau(\rho, \sigma) = \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} \{\tau(\rho y) + \tau(\sigma y^{-1})\}.$$

Therefore, the equality above and the hypothesis that $\mathcal{E}(a) \leq \Phi(a)$, for all $a \in \mathcal{A}_+$, yield

$$\begin{aligned} F_\tau(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} \{\tau(\mathcal{E}(\rho)y) + \tau(\mathcal{E}(\sigma)y^{-1})\} \\ &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} \{\tau(y^{1/2}\mathcal{E}(\rho)y^{1/2}) + \tau(y^{-1/2}\mathcal{E}(\sigma)y^{-1/2})\} \\ &\leq \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} \{\tau(y^{1/2}\Phi(\rho)y^{1/2}) + \tau(y^{-1/2}\Phi(\sigma)y^{-1/2})\} \\ &= \frac{1}{2} \inf_{y \in \text{GL}(\mathcal{A})_+} \{\tau(\Phi(\rho)y) + \tau(\Phi(\sigma)y^{-1})\} \\ &= F_\tau(\Phi(\rho), \Phi(\sigma)) \end{aligned}$$

Hence, $d_B^r(\Phi(\rho), \Phi(\sigma)) \leq d_B^r(\mathcal{E}(\rho), \mathcal{E}(\sigma))$, for all $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$. □

Our main interest in this chapter lies in the study of the linear maps that are strict contractions in the equation (4.1), that is

$$d_B^r(\mathcal{E}(\sigma), \mathcal{E}(\rho)) < d_B^r(\sigma, \rho) \quad \forall \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}) \text{ and } \sigma \neq \rho.$$

Such maps will be called Bures contractions. Before moving further, we note that the set of all Bures contractive linear maps is denoted by \mathcal{C}_B , that is a positive linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ belongs to the set \mathcal{C}_B if

$$(1) \quad \tau \circ \Phi(x) = \tau(x) \text{ for all } x \in \mathcal{A}, \text{ and}$$

$$(2) \quad d_B^r(\Phi(\sigma), \Phi(\rho)) < d_B^r(\sigma, \rho) \text{ for all } \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}) \text{ such that } \sigma \neq \rho.$$

4.1 Abundance of Bures contractive maps

We begin this section by proving the set \mathcal{C}_B is a convex set.

Lemma 4.2. *If \mathcal{E}_1 and \mathcal{E}_2 are positive linear trace-preserving maps on \mathcal{A} , and if at least one of them is Bures contractive, then so is $\lambda\mathcal{E}_1 + (1 - \lambda)\mathcal{E}_2$, for every $\lambda \in (0, 1)$. As a consequence, the set \mathcal{C}_B is a convex set.*

Proof. Without loss of generality, assume that \mathcal{E}_1 is Bures contractive, and let $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$. By the joint concavity of fidelity in Proposition 2.6 we obtain,

$$F_\tau(\lambda\mathcal{E}_1(\sigma) + (1 - \lambda)\mathcal{E}_2(\sigma), \lambda\mathcal{E}_1(\rho) + (1 - \lambda)\mathcal{E}_2(\rho))$$

$$\begin{aligned}
&\geq \lambda F_\tau(\mathcal{E}_1(\sigma), \mathcal{E}_1(\rho)) + (1 - \lambda)F_\tau(\mathcal{E}_2(\sigma), \mathcal{E}_2(\rho)) \\
&> \lambda F_\tau(\sigma, \rho) + (1 - \lambda)F_\tau(\sigma, \rho) \\
&= F_\tau(\sigma, \rho).
\end{aligned}$$

Thus, $\lambda\mathcal{E}_1 + (1 - \lambda)\mathcal{E}_2$ is Bures contractive. \square

One of the many interesting properties of the set \mathcal{C}_B , as Proposition 4.3 suggests, is that, it is norm-dense inside the norm-closed convex set of all positive linear maps.

Proposition 4.3. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a positive, τ -preserving linear map.*

Then for every $\epsilon > 0$, there exists a map $\tilde{\mathcal{E}} \in \mathcal{C}_B$ such that $\|\mathcal{E} - \tilde{\mathcal{E}}\| < \epsilon$.

Proof. Define, for $\delta \in (0, 1)$, the linear map $\mathcal{E}_\delta = \delta\Phi + (1 - \delta)\mathcal{E}$, where Φ is the linear map on \mathcal{A} defined by $\Phi(x) = \frac{\tau(x)}{\tau(1)}1$, for $x \in \mathcal{A}$. Thus, by the triangle inequality,

$$\|\mathcal{E} - \mathcal{E}_\delta\| \leq \delta(\|\Phi\| + \|\mathcal{E}\|).$$

Thus, we have $\|\mathcal{E} - \mathcal{E}'\| < \epsilon$ for $\mathcal{E}' = \mathcal{E}_\delta$ and $\delta = \frac{\epsilon}{2(\|\Phi\| + \|\mathcal{E}\|)}$. The proof is complete once we verify that \mathcal{E}_δ is Bures contractive for arbitrary $\delta \in (0, 1)$.

By lemma 4.2, it is sufficient to prove that Φ is a Bures contraction, which is clear since Φ maps the entire metric space $\mathcal{D}_\tau(\mathcal{A})$ to the point-set $\{\frac{1}{\tau(1)}1\}$.

□

Proposition 4.3 has an analogue for the Bures metric topology on $\mathcal{D}_\tau(\mathcal{A})$:

Proposition 4.4. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a positive, τ -preserving linear map.*

Then for every $\epsilon > 0$, there exists a map $\tilde{\mathcal{E}} \in \mathcal{C}_B$ such that

$$d_B^\tau(\mathcal{E}(\rho), \tilde{\mathcal{E}}(\rho)) < \epsilon,$$

for all $\rho \in \mathcal{D}_\tau(\mathcal{A})$.

Proof. Define, for $\delta \in (0, 1)$, the linear map $\mathcal{E}_\delta = \delta\Phi + (1 - \delta)\mathcal{E}$, where Φ is the linear map on \mathcal{A} defined by $\Phi(x) = \frac{\tau(x)}{\tau(1)}1$, for $x \in \mathcal{A}$. As noted in the proof of Proposition 4.3, the map \mathcal{E}_δ is Bures contractive for every $\delta \in (0, 1)$.

If $\rho \in \mathcal{D}_\tau(\mathcal{A})$, then Proposition 2.6 yields

$$\begin{aligned} F_\tau(\mathcal{E}_\delta(\rho), \mathcal{E}(\rho)) &= F_\tau(\delta\Phi(\rho) + (1 - \delta)\mathcal{E}(\rho), \delta\mathcal{E}(\rho) + (1 - \delta)\mathcal{E}(\rho)) \\ &\geq \delta F_\tau(\Phi(\rho), \mathcal{E}(\rho)) + (1 - \delta)F_\tau(\mathcal{E}(\rho), \mathcal{E}(\rho)) \\ &\geq 1 - \delta. \end{aligned}$$

Hence, $d_B^\tau(\mathcal{E}_\delta(\rho), \mathcal{E}(\rho)) \leq \sqrt{\delta}$. Selecting \mathcal{E}' to be \mathcal{E}_δ for $\delta = \frac{\epsilon^2}{4}$ completes the proof.

□

The following lemma describes an intrinsic nature of a Bures contractive map.

Lemma 4.5. *If $\mathcal{E} \in \mathcal{C}_B$, then \mathcal{E} breaks the orthogonality of every pair of elements in \mathcal{A}_+ , that is $\mathcal{E}(a) \not\perp \mathcal{E}(b)$ whenever $a \perp b$, for all $a, b \in \mathcal{A}_+$.*

Proof. Let $a, b \in \mathcal{A}_+$ such that $a \perp b$, that is $ab = ba = 0$. Scale the elements a, b by their traces so that $\sigma = \tau(a)^{-1}a$ and $\rho = \tau(b)^{-1}b$ belong to $\mathcal{D}_\tau(\mathcal{A})$. It is evident that $\sigma \perp \rho$. By the Proposition 2.5 we get, $F_\tau(\sigma, \rho) = 0$ if and only if $\sigma \perp \rho$. Now, since $\mathcal{E} \in \mathcal{C}_B$, we must have

$$0 = F_\tau(\sigma, \rho) < F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)).$$

Hence $\mathcal{E}(\sigma) \not\perp \mathcal{E}(\rho)$. Consequently $\mathcal{E}(a) \not\perp \mathcal{E}(b)$. □

Although Lemma 4.5 shows the orthogonality breaking nature of Bures contractive maps on pairs of positive elements, Example 1 shows that such maps don't have the same action on non-positive elements.

4.2 Examples

Although Lemma 4.2 provides an easy way to generate examples of Bures contractive maps, in this section we explicitly put forward some concrete examples of such maps.

(1) Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be defined as

$$\mathcal{E}(a) = \frac{\tau(a)1}{\tau(1)}.$$

where 1 is the identity of \mathcal{A} . Then as in the proof of Proposition 4.3, it is easy to check that \mathcal{E} is Bures contractive.

We use this example to analyse the effect of orthogonality breaking maps on non positive pairs of elements of \mathcal{A} . Let $\mathcal{A} = \mathcal{M}_4$ and \mathcal{E} be as

above. Take $a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. It is easy

to see that $ab = ba = 0$ and hence $a \perp b$ but since $\mathcal{E}(a) = 0 = \mathcal{E}(b)$, we find $\mathcal{E}(a) \perp \mathcal{E}(b)$ as well. So \mathcal{E} does not break orthogonality of the hermitian (but non positive) pair a, b .

(2) Let $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ be any τ -preserving positive linear map. Then $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\mathcal{E}(x) = \lambda \frac{\tau(x)}{\tau(1)} 1 + (1 - \lambda) \Phi(x), \quad x \in \mathcal{A}$$

is a Bures Contractive linear map for all $0 < \lambda < 1$.

(3) Let $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ be any τ -preserving positive linear map and suppose $\mathcal{E} \in \mathcal{C}_B$. Then the map defined by $\tilde{\mathcal{E}} = \Phi \circ \mathcal{E}$ is a Bures contractive map. This follows from the monotonicity property of fidelity under the τ -preserving linear maps which is described in the Theorem 2.14. Indeed for $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A})$, we get

$$F_\tau(\tilde{\mathcal{E}}(\sigma), \tilde{\mathcal{E}}(\rho)) = F_\tau(\Phi \circ \mathcal{E}(\sigma), \Phi \circ \mathcal{E}(\rho)) \geq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) > F_\tau(\sigma, \rho).$$

Hence $\tilde{\mathcal{E}}$ is Bures contractive.

4.3 Multiplicative domains

Recall that for a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$, the multiplicative domain of Φ is a set on which it behaves as a homomorphism. This set is defined as

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} : \Phi(ax) = \Phi(a)\Phi(x), \Phi(xa) = \Phi(x)\Phi(a), \forall x \in \mathcal{A}\}.$$

Note that in [51] and [9], it has been proved that if \mathcal{E} is a Schwarz map, then \mathcal{M}_Φ is a C^* -algebra and equates to the following set

$$\mathcal{S}_\Phi = \{a \in \mathcal{A} : \Phi(aa^*) = \Phi(a)\Phi(a^*), \Phi(a^*a) = \Phi(a^*)\Phi(a)\}.$$

In this section we analyse the multiplicative domains of Bures contractive maps. As we will see the spectral properties of a Schwarz map are closely

related to its multiplicative domain, it is rather essential to study the multiplicative behaviour of a Schwarz map and more precisely a Bures contractive Schwarz map.

Proposition 4.6. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a τ -preserving Schwarz map. Then on the set $\mathcal{D}_\tau(\mathcal{A}) \cap \mathcal{M}_\mathcal{E}$, the map \mathcal{E} is isometric. That is,*

$$d_B^\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) = d_B^\tau(\sigma, \rho) \quad \forall \sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}) \cap \mathcal{M}_\mathcal{E}.$$

Proof. We shall prove that $F_\tau(\rho, \sigma) = F_\tau(\mathcal{E}(\rho), \mathcal{E}(\sigma))$ for all $\sigma, \rho \in \mathcal{D}_\tau(\mathcal{A}) \cap \mathcal{M}_\mathcal{E}$.

To this end, select $x \in \mathcal{A}_+ \cap \mathcal{M}_\mathcal{E}$. Because $\mathcal{M}_\mathcal{E}$ is a C^* -algebra, the element $x^{1/2}$ also belongs to $\mathcal{M}_\mathcal{E}$. Thus,

$$\mathcal{E}(x) = \mathcal{E}(x^{1/2}x^{1/2}) = \mathcal{E}(x^{1/2})\mathcal{E}(x^{1/2}) = [\mathcal{E}(x^{1/2})]^2,$$

which shows that $\mathcal{E}(x)^{1/2} = \mathcal{E}(x^{1/2})$. Now for $\rho, \sigma \in \mathcal{D}_\tau(\mathcal{A}) \cap \mathcal{M}_\mathcal{E}$, the element

$\rho^{1/2}\sigma\rho^{1/2}$ lies in $\mathcal{M}_\mathcal{E}$; thus,

$$\begin{aligned}
F_\tau(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \tau[(\mathcal{E}(\rho)^{1/2}\mathcal{E}(\sigma)\mathcal{E}(\rho)^{1/2})^{1/2}] \\
&= \tau[(\mathcal{E}(\rho^{1/2})\mathcal{E}(\sigma)\mathcal{E}(\rho^{1/2}))^{1/2}] \\
&= \tau[(\mathcal{E}(\rho^{1/2}\sigma\rho^{1/2}))^{1/2}] \\
&= \tau[\mathcal{E}((\rho^{1/2}\sigma\rho^{1/2})^{1/2})] \\
&= \tau[(\rho^{1/2}\sigma\rho^{1/2})^{1/2}] \\
&= F_\tau(\rho, \sigma).
\end{aligned}$$

Hence, \mathcal{E} is isometric on $\mathcal{D}_\tau(\mathcal{A}) \cap \mathcal{M}_\mathcal{E}$ with respect to the Bures metric. \square

The next lemma gives a necessary and sufficient criteria for a τ -density elements to be in the multiplicative domain of a positive linear map. To this end, we need the following definition.

Definition 4.7. *The centre of the convex set $\mathcal{D}_\tau(\mathcal{A})$ will be denoted by $\zeta = \tau(1)^{-1}1$.*

Lemma 4.8. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a τ -preserving Schwarz map. Then for any τ -state $\rho \in \mathcal{A}$, $\rho \in \mathcal{M}_\mathcal{E}$ if and only if $d_B^\tau(\mathcal{E}(\rho), \zeta) = d_B^\tau(\rho, \zeta)$.*

Proof. Assume that $a \in \mathcal{M}_\mathcal{E}$ and let $\rho = \tau(a)^{-1}a$. It was shown in the proof

of Proposition 4.6 that $\mathcal{E}(\rho^{1/2}) = \mathcal{E}(\rho)^{1/2}$; therefore,

$$\begin{aligned}
F_\tau(\rho, \zeta) &= F_\tau(\rho, \tau(1)^{-1}\mathbf{1}) = \frac{\tau(\rho^{1/2})}{\tau(1)^{\frac{1}{2}}} = \frac{\tau[\mathcal{E}(\rho^{1/2})]}{\tau(1)^{\frac{1}{2}}} = \frac{\tau[\mathcal{E}(\rho)^{1/2}]}{\tau(1)^{\frac{1}{2}}} \\
&= F_\tau(\mathcal{E}(\rho), \tau(1)^{-1}\mathbf{1}) \\
&= F_\tau(\mathcal{E}(\rho), \zeta).
\end{aligned}$$

Hence, $\tau(a)^{-1}a$ and $\tau(a)^{-1}\mathcal{E}(a)$ are equidistant from the centre ζ of $\mathcal{D}_\tau(\mathcal{A})$.

Conversely, assume that $d_B^\tau(\tau(a)^{-1}a, \zeta) = d_B^\tau(\tau(a)^{-1}\mathcal{E}(a), \zeta)$. Therefore, the fidelities $F_\tau((\tau(a)^{-1}a, \tau(1)^{-1}\mathbf{1}))$ and $F_\tau((\tau(a)^{-1}\mathcal{E}(a), \tau(1)^{-1}\mathbf{1}))$ coincide, which implies that $\tau(a^{1/2}) = \tau(\mathcal{E}(a))^{1/2}$. The Schwarz inequality asserts that $\mathcal{E}(a) = \mathcal{E}(a^{1/2}a^{1/2}) \geq (\mathcal{E}(a^{1/2}))^2$. Since the square root is an operator monotone function, we obtain $\mathcal{E}(a)^{1/2} \geq \mathcal{E}(a^{1/2})$. Hence,

$$\tau(a^{1/2}) = \tau[\mathcal{E}(a)^{1/2}] \geq \tau[\mathcal{E}(a^{1/2})] = \tau(a^{1/2}).$$

By the faithfulness of trace, $\mathcal{E}(a)^{1/2} = \mathcal{E}(a^{1/2})$. Therefore,

$$\mathcal{E}(a) = [\mathcal{E}(a)^{1/2}]^2 = [\mathcal{E}(a^{1/2})]^2 = \mathcal{E}(a^{1/2})\mathcal{E}(a^{1/2}),$$

which shows that $a^{1/2} \in \mathcal{S}_\mathcal{E}$. Because $\mathcal{S}_\mathcal{E} = \mathcal{M}_\mathcal{E}$ for Schwarz maps, we deduce that $a \in \mathcal{M}_\mathcal{E}$. □

The following theorem rules out the possibility of non trivial multiplicative domain of a Bures contractive positive linear map.

Theorem 4.9. *If a unital Schwarz map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is Bures contractive, then the multiplicative domain of \mathcal{E} is $\mathbb{C}1$.*

Proof. If $\rho \in \mathcal{M}_{\mathcal{E}}$, then $d_B^{\tau}(\rho, \tau(1)^{-1}1) = d_B^{\tau}(\mathcal{E}(\rho), \tau(1)^{-1}1)$ by Lemma 4.8. However, as $\mathcal{E}(1) = 1$, we have that $d_B^{\tau}(\rho, \tau(1)^{-1}1) = d_B^{\tau}(\mathcal{E}(\rho), \mathcal{E}[\tau(1)^{-1}1])$; but this can happen only if $\rho = \tau(1)^{-1}1$, since \mathcal{E} is assumed to be Bures contractive. Because positive elements of τ -value 1 span the entire algebra, each element of $\mathcal{M}_{\mathcal{E}}$ must, therefore, be a scalar multiple of $1 \in \mathcal{A}$. \square

A partial converse to Proposition 4.6 is the following result which reveals the hereditary nature of multiplicative domains.

Proposition 4.10. *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a τ -preserving Schwarz map and let $\sigma \in \mathcal{M}_{\mathcal{E}}$ be a τ -state for which there exists a $\rho \in \mathcal{D}_{\tau}(\mathcal{A})$ such that*

$$d_B^{\tau}(\mathcal{E}(\sigma), \mathcal{E}(\rho)) = d_B^{\tau}(\sigma, \rho).$$

Then $\sigma^{1/2}\rho\sigma^{1/2} \in \mathcal{M}_{\mathcal{E}}$. Moreover, if σ is invertible, then $\rho \in \mathcal{M}_{\mathcal{E}}$.

Proof. Because $\sigma \in \mathcal{M}_{\mathcal{E}}$, we have that $\mathcal{E}(\sigma)^{1/2} = \mathcal{E}(\sigma^{1/2})$ and, for every $x \in \mathcal{A}$, that $\mathcal{E}(\sigma^{1/2}x) = \mathcal{E}(\sigma^{1/2})\mathcal{E}(x)$. Thus, using the Schwarz inequality

$$\mathcal{E}(\sigma^{1/2}\rho\sigma^{1/2})^{1/2} \geq \mathcal{E}([\sigma^{1/2}\rho\sigma^{1/2}]^{1/2}),$$

we obtain

$$\begin{aligned}
F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) &= \tau([\mathcal{E}(\sigma)^{1/2}\mathcal{E}(\rho)\mathcal{E}(\sigma)^{1/2}]^{1/2}) \\
&= \tau([\mathcal{E}(\sigma^{1/2})\mathcal{E}(\rho)\mathcal{E}(\sigma^{1/2})]^{1/2}) \\
&= \tau([\mathcal{E}(\sigma^{1/2}\rho\sigma^{1/2})]^{1/2}) \\
&\geq \tau \circ \mathcal{E}[(\sigma^{1/2}\rho\sigma^{1/2})^{1/2}] \\
&= F_\tau(\sigma, \rho)
\end{aligned}$$

Hence, by the hypothesis, the inequality must be an equality, resulting in

$$[\mathcal{E}(\sigma)^{1/2}\mathcal{E}(\rho)\mathcal{E}(\sigma)^{1/2}]^{1/2} = \mathcal{E}[(\sigma^{1/2}\rho\sigma^{1/2})^{1/2}].$$

The above equation implies that $\sigma^{1/2}\rho\sigma^{1/2} \in \mathcal{M}_\mathcal{E}$.

Now assume that σ is invertible. Because $\mathcal{M}_\mathcal{E}$ is a C*-algebra, $\sigma^{-1/2} \in \mathcal{M}_\mathcal{E}$. Now the element $\rho = \sigma^{-1/2} (\sigma^{1/2}\rho\sigma^{1/2}) \sigma^{-1/2}$ also lies in $\mathcal{M}_\mathcal{E}$. \square

4.4 Fixed points

Definition 4.11. *If Φ is a linear transformation on a vector space V , then $x \in V$ is a fixed point of Φ if $\Phi(x) = x$. The vector subspace*

$$\text{Fix } \Phi = \{x \in V : \Phi(x) = x\}$$

is called the fixed point space of Φ .

Our interest here is with the fixed points of positive linear maps. The first assertion of the following proposition is widely known, while the second assertion is an algebraic variant of a theorem of Kribs [37].

The notation $[x, y]$ below denotes the commutator $[x, y] = xy - yx$.

Proposition 4.12. *The following statements hold for a unital and trace-preserving positive linear map \mathcal{E} on \mathcal{A} :*

(1) *if \mathcal{E} is a Schwarz map, then $\text{Fix } \mathcal{E}$ is a unital C^* -algebra;*

(2) *if there exist $w_1, \dots, w_n \in \mathcal{A}$ such that $\mathcal{E}(x) = \sum_{k=1}^n w_k x w_k^*$, for all $x \in \mathcal{A}$, and if the linear span of the projections in the C^* -algebra $\text{Fix } \mathcal{E}$ is dense in $\text{Fix } \mathcal{E}$, then*

$$\text{Fix } \mathcal{E} = \{x \in \mathcal{A} : [x, w_k] = [x, w_k^*] = 0, \forall k = 1, \dots, n\}.$$

Proof. Suppose that \mathcal{E} is a Schwarz map. If $x \in \text{Fix } \mathcal{E}$, then the Schwarz inequality yields $0 \leq \mathcal{E}(x^*x) - \mathcal{E}(x^*)\mathcal{E}(x) = \mathcal{E}(x^*x) - x^*x$, where the final equality is a consequence of the hypothesis $x \in \text{Fix } \mathcal{E}$. On evaluating the trace, we obtain the inequality

$$0 \leq \tau[\mathcal{E}(x^*x) - \mathcal{E}(x^*)\mathcal{E}(x)] = \tau(x^*x) - \tau(x^*x) = 0.$$

Thus, the positive element $\mathcal{E}(x^*x) - \mathcal{E}(x^*)\mathcal{E}(x)$ has zero trace, which yields $\mathcal{E}(x^*x) = \mathcal{E}(x^*)\mathcal{E}(x)$. Hence, $x \in \mathcal{S}_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$. Therefore, if $x_1, x_2 \in \text{Fix } \mathcal{E}$, then $x_1, x_2 \in \mathcal{M}_{\mathcal{E}}$ and so $\mathcal{E}(x_1x_2) = \mathcal{E}(x_1)\mathcal{E}(x_2) = x_1x_2$, which proves that $x_1x_2 \in \text{Fix } \mathcal{E}$.

To prove the second assertion, suppose without loss of generality that \mathcal{A} is represented faithfully as a unital C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . The set $\mathcal{B}_{\mathcal{E}} = \{x \in \mathcal{A} : [x, w_k] = [x, w_k^*] = 0, \forall k = 1, \dots, n\}$ is a unital C^* -subalgebra of \mathcal{A} such that $\mathcal{B}_{\mathcal{E}} \subseteq \text{Fix } \mathcal{E}$. Conversely, choose a projection $p \in \text{Fix } \mathcal{E}$. If $\xi \in \mathcal{H}$, then,

$$\|p\xi\|^2 = \langle p\xi, \xi \rangle = \langle \mathcal{E}(p)\xi, \xi \rangle = \sum_{k=1}^n \langle pw_k^*\xi, w_k^*\xi \rangle = \sum_{k=1}^n \|pw_k^*\xi\|^2.$$

Thus, if $\xi \in \ker p$, then $w_k^*\xi \in \ker p$ for every k , which proves that $\ker p$ is invariant for each w_k^* . A similar argument, using $1 - p \in \text{Fix } \mathcal{E}$ in place of p shows that $\text{ran } p = \ker(1 - p)$ is invariant for each w_k^* . Thus, $\text{ran } p$ is a reducing subspace for each w_k^* , which proves that $pw_k = w_kp$ for each k . Hence, $p \in \mathcal{B}_{\mathcal{E}}$. Because the linear span of the projections in $\text{Fix } \mathcal{E}$ is dense in $\text{Fix } \mathcal{E}$, we deduce from the continuity of \mathcal{E} that $\text{Fix } \mathcal{E} \subseteq \mathcal{B}_{\mathcal{E}}$. \square

Note that if $\text{Fix } \mathcal{E}$ is a von Neumann algebra, then since projections are dense in every von Neumann algebra, the density condition of the above

proposition is satisfied. We utilise this observation in the following corollary:

Corollary 4.13. *If τ is a faithful normal trace functional on a von Neumann algebra \mathcal{N} , and if $\mathcal{E} : \mathcal{N} \rightarrow \mathcal{N}$ is a normal Schwarz map, then there exists a unital completely positive normal map $\Pi : \mathcal{N} \rightarrow \mathcal{N}$ such that $\Pi^2 = \Pi$ and $\text{Ran } \Pi = \text{Fix } \mathcal{E}$.*

Proof. The hypothesis on \mathcal{E} implies that $\text{Fix } \mathcal{E}$ is a unital C^* -algebra (Proposition 4.12). Furthermore, because \mathcal{E} is normal, the C^* -algebra $\text{Fix } \mathcal{E}$ is a von Neumann subalgebra of \mathcal{N} . Hence, there exists a unital completely positive normal map $\Pi : \mathcal{N} \rightarrow \mathcal{N}$ such that $\Pi^2 = \Pi$ and $\text{Ran } \Pi = \text{Fix } \mathcal{E}$ [51, Propositions 2.2.6, 2.2.1]. \square

A C^* -algebra version of Corollary 4.13 may be formulated as follows.

Definition 4.14. *A unital positive linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is uniformly ergodic if there exists a bounded linear operator $\Pi : \mathcal{A} \rightarrow \mathcal{A}$ such that*

$$\lim_{n \rightarrow \infty} \left\| \Pi - \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k \right\| = 0.$$

Proposition 4.15. ([24]) *If a unital positive linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is uniformly ergodic, then the positive idempotent operator Π in Definition 4.14 has range $\text{Fix } \Phi$.*

The final result concerning fixed points is:

Proposition 4.16. *If \mathcal{E} is a Bures contractive Schwarz map, then $\text{Fix } \mathcal{E} = \mathbb{C}1$.*

Proof. By Proposition 4.12, the fixed point subspace of a Schwarz map is a C^* -algebra; hence, $\text{Fix } \mathcal{E}$ is a subset of the multiplicative domain $\mathcal{M}_{\mathcal{E}}$ of \mathcal{E} . In addition, \mathcal{E} is a Bures contraction; therefore, $\mathcal{M}_{\mathcal{E}} = \mathbb{C}1$ by Theorem 4.9. \square

4.5 Perron-Frobenius theory

4.5.1 Irreducibility

The central idea in Frobenius theory is that of irreducibility. This notion appears as a combinatorial concept in matrix theory, but in more general contexts the notion of irreducibility is related to the absence of invariant faces in a cone.

Definition 4.17. *A nonempty subset \mathcal{F} of \mathcal{A}_+ is a face of \mathcal{A}_+ if, for $a \in \mathcal{A}_+$ and $b \in \mathcal{F}$, the inequality $a \leq b$ holds only if $a \in \mathcal{F}$.*

Definition 4.18. *Assume that $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ and $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ are positive linear maps of a C^* -algebra \mathcal{A} and a von Neumann algebra \mathcal{M} , and assume*

that Ψ is normal.

(1) If there exists a norm-closed face \mathcal{F} of \mathcal{A}_+ different from $\{0\}$ and \mathcal{A}_+ such that $\Phi(\mathcal{F}) \subseteq \mathcal{F}$, then Φ is said to be reducible.

(2) If there exists a ultraweakly-closed face \mathcal{F} of \mathcal{M}_+ different from $\{0\}$ and \mathcal{M}_+ such that $\Psi(\mathcal{F}) \subseteq \mathcal{F}$, then Ψ is said to be reducible.

A positive linear map on \mathcal{A} or \mathcal{M} that is not reducible is called irreducible.

The precise determination of the norm-closed faces and the ultraweakly-closed faces of the positive cones of, respectively, C*-algebras and von Neumann algebras is provided in the monograph [3].

The following observation is a variant of [16, Proposition 1].

Lemma 4.19. *If $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is a normal positive linear map of norm $\|\Phi\| \leq 1$ on a von Neumann algebra \mathcal{M} , then Φ is reducible if and only if there exists a nontrivial projection $p \in \mathcal{M}$ such that $\Phi(p) \leq p$.*

Proof. Assume that Φ is reducible; thus, there is a nontrivial ultraweakly-closed face \mathcal{F} of \mathcal{M}_+ such that $\Phi(\mathcal{F}) \subseteq \mathcal{F}$. Therefore, there exists an ultraweakly-closed left ideal J of \mathcal{M} such that $\mathcal{F} = J_+$, where $J_+ = J \cap \mathcal{M}_+$ [3, Theorem 3.13]. Now, if 1 were an element of \mathcal{F} , then $1 \in J_+ \subseteq J$ implies

that $x = x(1) \in J$ for every $x \in \mathcal{M}$, and so we would obtain $J = \mathcal{M}$ and $\mathcal{F} = \mathcal{M}_+$, which is impossible since \mathcal{F} is nontrivial.

Consider now $\mathcal{F}_1 = \mathcal{F} \cap \{x \in \mathcal{M} : \|x\| \leq 1\}$, which is a (weakly-closed) face of \mathcal{M}_+ . Because both \mathcal{F} and the closed unit ball of \mathcal{M} are invariant under Φ , we deduce that \mathcal{F}_1 is Φ -invariant. Furthermore, there exists a projection $p \in \mathcal{F}_1$ such that $a \leq p$ for all $a \in \mathcal{F}_1$ [3, Proposition 3.9]. By the remarks of the previous paragraph, $p \neq 1$; and since $\mathcal{F} \neq \{0\}$, \mathcal{F} has a nonzero element a of norm $\|a\| \leq 1$. Thus, p is a nontrivial projection. Because $p \in \mathcal{F}_1$ and \mathcal{F}_1 is Φ -invariant, we have that $\Phi(p) \in \mathcal{F}_1$, whence $\Phi(p) \leq p$.

Conversely, suppose that $p \in \mathcal{M}$ is a nontrivial projection such that $\Phi(p) \leq p$. Let $\mathcal{F} = \{a \in \mathcal{M}_+ : a \leq p\}$, which is a proper ultraweakly closed face of \mathcal{M}_+ . Further, if $a \in \mathcal{F}$, then $a \leq p$ implies that $\Phi(a) \leq \Phi(p) \leq p$, and so $\Phi(a) \in \mathcal{F}$. Therefore, Φ is reducible. \square

Corollary 4.20. *A contractive normal trace-preserving positive linear map \mathcal{E} on a finite von Neumann algebra \mathcal{N} is reducible if and only if there exists a nontrivial projection $p \in \mathcal{N}$ such that $\mathcal{E}(p) = p$.*

Proof. If \mathcal{E} is reducible, then Lemma 4.19 shows that $\mathcal{E}(p) \leq p$ for some

nontrivial projection $p \in \mathcal{N}$. Because $\tau \circ \mathcal{E} = \tau$,

$$0 \leq \tau(p - \mathcal{E}(p)) = \tau(p) - \tau \circ \mathcal{E}(p) = \tau(p) - \tau(p) = 0.$$

Hence, as the trace τ is faithful, $\mathcal{E}(p) = p$.

Conversely, if $\mathcal{E}(p) = p$, then \mathcal{E} is reducible by Lemma 4.19. \square

One natural occurrence of normal traces and positive linear maps on \mathcal{A} is as follows. If τ is a faithful trace functional on a unital C^* -algebra \mathcal{A} , then the bidual τ^{**} of τ is a faithful normal trace on the enveloping von Neumann algebra (i.e., on the bidual) \mathcal{A}^{**} of \mathcal{A} . Moreover, if \mathcal{E} is positive and τ -preserving on (\mathcal{A}, τ) , then \mathcal{E}^{**} is normal, positive and τ^{**} -preserving on $(\mathcal{A}^{**}, \tau^{**})$ as well. Also if \mathcal{E} satisfies the Schwarz inequality, that is if $\mathcal{E}(xx^*) \leq \mathcal{E}(x)\mathcal{E}(x^*)$, for every $x \in \mathcal{A}$, then \mathcal{E}^{**} satisfies the same inequality (Lemma 3, [22]).

In what follows, if $S \subseteq \mathcal{A}$, then S_+ shall denote $S \cap \mathcal{A}_+$.

Lemma 4.21. *The following statements are equivalent for a positive linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$:*

- (1) Φ is reducible;
- (2) Φ^{**} is reducible.

Proof. If $\pi_u : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_u)$ is the universal representation of \mathcal{A} , then $\mathcal{A} \cong \pi(\mathcal{A}) \subseteq \pi(\mathcal{A})'' = \mathcal{A}^{**}$; therefore, assume without loss of generality that \mathcal{A} is represented as a weakly dense unital C^* -subalgebra of \mathcal{A}^{**} .

If Φ^{**} is reducible, then there exists a nontrivial projection $p \in \mathcal{A}^{**}$ such that $\Phi^{**}(p) \leq p$. The set $\mathcal{F} = \{a \in \mathcal{A}_+ : a \leq p\}$ is a norm-closed proper face of \mathcal{A}_+ (as $p \notin \{0, 1\}$). Furthermore, if $a \in \mathcal{F}$, then the inequality $a \leq p$ leads to $\Phi(a) = \Phi^{**}(a) \leq \Phi^{**}(p) \leq p$, implying that $\Phi(a) \in \mathcal{F}$. Hence, Φ is reducible.

Conversely, suppose that Φ is reducible. Without loss of generality we may assume that Φ is a contraction, as the invariance of a proper norm-closed face \mathcal{F} of \mathcal{A}_+ under Φ is independent of the norm of Φ . Let J be a norm-closed left ideal such that $\mathcal{F} = J_+$, and fix an increasing right approximate identity for J – namely, an increasing net $\{e_\lambda\}_\lambda \subset J_+$ such that $\lim_\lambda \|x - xe_\lambda\| = 0$ for every $x \in J$. Considered as a subset of \mathcal{A}^{**} , the ultraweak closure of J is $\overline{J}^{\sigma\text{-wk}} = \{zq : z \in \mathcal{A}^{**}\}$, where $q \in \mathcal{A}^{**}$ is the least projection such that $aq = a$ for every $a \in J_+$ and is given by $q = \sup_\lambda e_\lambda$ [3, Proposition 3.44]. Since $\|e_\lambda\| \leq \|q\| = 1$ and Φ is contractive, the element $\Phi(e_\lambda) \in \mathcal{F}$ has norm no greater than 1; thus, $\Phi(e_\lambda) \leq q$ [3, Proposition 3.9]. Hence, using the

normality of Φ^{**} , we obtain

$$\Phi^{**}(q) = \Phi^{**}\left(\sup_{\lambda} e_{\lambda}\right) = \sup_{\lambda} \Phi^{**}(e_{\lambda}) = \sup_{\lambda} \Phi(e_{\lambda}) \leq q,$$

which implies that Φ^{**} is reducible, by Lemma 4.19. \square

The following two results establish the relationship between Bures contractiveness and irreducibility in the case of Schwarz maps.

Proposition 4.22. *A Bures contractive Schwarz map on a finite von Neumann algebra is irreducible.*

Proof. Let $\mathcal{E} : \mathcal{N} \rightarrow \mathcal{N}$ be a Bures contractive Schwarz map on a finite von Neumann algebra \mathcal{N} with faithful normal trace τ . Assume that \mathcal{E} is reducible. Thus, there is a nontrivial projection p such that $\mathcal{E}(p) = p$, by Corollary 4.20. Since $p = pp^* = p^*p = \mathcal{E}(p)^*\mathcal{E}(p)$ and $\mathcal{E}(p) = \mathcal{E}(p^*p)$, the projection p is in the multiplicative domain of \mathcal{E} . Theorem 4.9 asserts that the multiplicative domain of a Bures contractive Schwarz map is $\mathbb{C}1$. Hence, p must be 0 or 1, in contradiction to the fact that p is neither 0 nor 1. Therefore, \mathcal{E} must be irreducible. \square

Theorem 4.23. *A Bures contractive Schwarz map on a tracial C^* -algebra is irreducible.*

Proof. Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ be a Bures contractive Schwarz map. Assuming that $\mathcal{A} \subseteq \mathcal{A}^{**}$, the faithful \mathcal{E} -invariant trace τ on \mathcal{A} extends to a faithful \mathcal{E}^{**} -invariant trace τ^{**} on \mathcal{A}^{**} .

Assume, contrary to what we aim to prove, that \mathcal{E} is reducible. Therefore, by Lemma 4.21, \mathcal{E}^{**} is reducible; thus, $\mathcal{E}^{**}(p) = p$, for some nontrivial projection $p \in \mathcal{A}^{**}$. Likewise, $\mathcal{E}^{**}(q) = q$, where $q = 1 - p$. Hence, p and q are nonzero positive contractions in \mathcal{A}^{**} such that $p \perp q$ and $\mathcal{E}^{**}(p) \perp \mathcal{E}^{**}(q)$. If $a, b \in \mathcal{A}^{**}$ are positive and satisfy $a \leq p$ and $b \leq q$, then $ab = ba = 0$. To verify this, note that

$$0 \leq \tau^{**}(b^{1/2}ab^{1/2}) \leq \tau^{**}(b^{1/2}pb^{1/2}) = \tau^{**}(pbp) \leq \tau^{**}(pqp) = \tau^{**}(pq) = 0,$$

which implies that $b^{1/2}ab^{1/2} = 0$. As these are operators acting on the universal representation Hilbert space \mathcal{H}_u for \mathcal{A} , we obtain $0 = \|ab^{1/2}\xi\|^2$ for every $\xi \in \mathcal{H}_u$; hence, $ab^{1/2} = 0$, which yields $ab = 0$ and $ba = 0$.

By the Kaplansky Density Theorem, there are increasing nets $\{a_\alpha\}_\alpha$ and $\{b_\beta\}_\beta$ of positive operators in \mathcal{A} converging strongly to p and q , respectively. Thus, $p = \sup_\alpha a_\alpha$ and $q = \sup_\beta b_\beta$ yields, by the normality of \mathcal{E}^{**} , that

$$\mathcal{E}^{**}(p) = \sup_\alpha \mathcal{E}^{**}(a_\alpha) = \sup_\alpha \mathcal{E}(a_\alpha) \quad \text{and} \quad \mathcal{E}^{**}(q) = \sup_\beta \mathcal{E}^{**}(b_\beta) = \sup_\beta \mathcal{E}(b_\beta).$$

Thus, by the arguments of the previous paragraph, $a_\alpha \perp b_\beta$ and $\mathcal{E}(a_\alpha) \perp \mathcal{E}(b_\beta)$

for all α and β , which contradicts the hypothesis that \mathcal{E} is a Bures contraction.

Therefore, it must be that \mathcal{E} is irreducible. □

4.5.2 Spectral values and eigenvalues

The identity operator on \mathcal{A} will be denoted by I and the spectrum of a bounded linear operator $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is denoted by $\text{Sp } \Phi$. That is,

$$\text{Sp } \Phi = \{\lambda \in \mathbb{C} \mid \Phi - \lambda I \text{ is not an invertible operator on } \mathcal{A}\}.$$

Of special interest are the point spectrum $\text{Sp}_p \Phi$, which consists of the eigenvalues of Φ , and the approximate point spectrum $\text{Sp}_{\text{ap}} \Phi$, which consists of approximate eigenvalues of Φ . Thus, $\lambda \in \text{Sp}_{\text{ap}} \Phi$ if and only for every $\varepsilon > 0$ there exists a nonzero $x \in \mathcal{A}$ such that $\|\Phi(x) - \lambda x\| < \varepsilon \|x\|$.

If Φ is a positive linear map, then one might expect a Perron-Frobenius-type behaviour with regards to the spectrum of Φ . This has been known for several decades to be true.

Theorem 4.24. [Perron-Frobenius] If $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is a positive linear map, then the spectral radius of Φ is an element of the spectrum of Φ .

Definition 4.25. *The spectral radius of a positive linear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is called the Perron value of Φ .*

If \mathcal{E} is a τ -preserving Schwarz map on \mathcal{A} , then \mathcal{E} is unital (see the proof of Theorem 3.6) and so $1 \in \text{Sp}_p \mathcal{E}$. But as $\|\mathcal{E}\| \leq 1$, the spectrum of \mathcal{E} lies in the closed unit disc of \mathbb{C} , which implies that the spectral radius of \mathcal{E} is 1. Thus, the *spectral circle* for such \mathcal{E} is the boundary \mathbb{T} of the closed unit disc.

A theorem of Groh [23] states that if Φ is an irreducible unital Schwarz map on a unital C^* -algebra, then the peripheral point spectrum $\text{Sp}_p \Phi \cap \mathbb{T}$ is a subgroup of \mathbb{T} . In the case of Bures contractive maps, one has that this subgroup is trivial:

Proposition 4.26. *If $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ is a Bures contractive Schwarz map, then*

$$\text{Sp}_p \mathcal{E} \cap \mathbb{T} = \{1\}.$$

Proof. Suppose that $\omega \in \mathbb{T}$ is an eigenvalue of \mathcal{E} with (nonzero) eigenvector x . Because \mathcal{E} preserves selfadjointness, it is also true that $\mathcal{E}(x^*) = \bar{\omega}x^*$. Therefore,

$$\mathcal{E}(x^*x) \geq \mathcal{E}(x)^*\mathcal{E}(x) = \bar{\omega}x^*\omega x = x^*x.$$

Applying the faithful trace τ to the inequality above yields the inequality

$$0 \leq \tau[\mathcal{E}(x^*x) - \mathcal{E}(x)^*\mathcal{E}(x)] = \tau(x^*x) - \tau(x^*x) = 0.$$

Thus, the positive element $\mathcal{E}(x^*x) - \mathcal{E}(x)^*\mathcal{E}(x)$ has zero trace, which yields $\mathcal{E}(x^*x) = \mathcal{E}(x)^*\mathcal{E}(x)$. Hence, $x \in \mathcal{S}_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$. However, Theorem 4.9 asserts

that the multiplicative domain of \mathcal{E} is $\mathbb{C}1$; hence, $x \in \ker(\mathcal{E} - I) \cap \ker(\mathcal{E} - \omega I)$,
and so $\omega = 1$. This proves that $\text{Sp}_p \mathcal{E} \cap \mathbb{T} = \{1\}$. \square

Chapter 5

Fidelity and Channels in Semifinite von Neumann Algebras

Even though the results in the previous chapters like Proposition 3.10 etc. apply to a finite von Neumann algebra \mathcal{N} , the results make no reference to the predual \mathcal{N}_* . Also we have used facts related to von Neumann algebras to prove C*-algebra results, therefore all these results should be viewed as results in the Heisenberg picture. The goal of the present chapter is to develop a framework for the Schrödinger picture, and to then specialise to the setting

of semifinite von Neumann algebras for the purpose of analysing fidelity.

A useful fact about 2×2 matrices over von Neumann algebras is recorded below for later reference.

Lemma 5.1. ([52, p. 166]) *If \mathcal{M} is a von Neumann algebra acting on \mathcal{H} and if $a, b \in \mathcal{M}_+$, then the following statements are equivalent for $x \in \mathcal{M}$:*

$$(1) \quad \begin{bmatrix} a & x \\ x^* & b \end{bmatrix} \text{ is a positive operator on } \mathcal{H} \oplus \mathcal{H};$$

$$(2) \quad x = a^{1/2}yb^{1/2} \text{ for some } y \in \mathcal{M} \text{ with } \|y\| \leq 1.$$

5.1 Fidelity

Semifinite von Neumann algebras (see [52, 53]) admit a very natural analogue of the classical notion of density operator. Therefore, assume for the remainder of this chapter that \mathcal{M} is a semifinite von Neumann algebra with a faithful normal tracial weight τ , that is $\text{Span}_{\mathbb{C}}\{a \in \mathcal{M}_+ : \tau(a) < \infty\}$ is dense in \mathcal{M}_+ with respect to SOT. Because we have already encountered finite von Neumann algebras in the previous chapters, we shall also assume that \mathcal{M} is not finite. Thus, the trace of the identity $1 \in \mathcal{M}$ is infinite.

By definition, τ is a function $\mathcal{M}_+ \rightarrow [0, +\infty]$ such that, for all $a, b \in \mathcal{M}_+$,

$x \in \mathcal{M}$, and $\lambda \geq 0$ in \mathbb{R} , we have $\tau(x^*x) = \tau(xx^*)$, $\tau(a + b) = \tau(a) + \tau(b)$, $\tau(\lambda a) = \lambda\tau(a)$, $\tau(a) > 0$ if $a \neq 0$, $\tau(\sup_\alpha a_\alpha) = \sup \tau(a_\alpha)$ for every bounded increasing net $\{a_\alpha\}_\alpha$ in \mathcal{M}_+ , and for each nonzero $h \in \mathcal{M}_+$ there exists nonzero $h_0 \in \mathcal{M}_+$ with $h_0 \leq h$ and $\tau(h_0) < \infty$. As shown in [14], if $z \in \mathcal{M}$, then

$$\tau(|z|) = \int_0^\infty \mu_z(t) dt, \quad (5.1)$$

where for, each $t \in [0, \infty)$,

$$\mu_z(t) = \inf \{ \|ze\| \mid e \in \mathcal{P}(\mathcal{M}), \text{Tr}(1 - e) \leq t \},$$

and where $\mathcal{P}(\mathcal{M})$ is the projection lattice for \mathcal{M} . Moreover, $\mu_z = \mu_{z^*} = \mu_{|z|}$ and, consequently, for any $w, z \in \mathcal{M}$,

$$\mu_{|wz^*|} = \mu_{|zw^*|}. \quad (5.2)$$

Furthermore, if $h \in \mathcal{M}_+$ and if $\psi : [0, \infty) \rightarrow [0, \infty)$ is an increasing continuous function such that $\psi(0) = 0$, then

$$\mu_{\psi(h)}(t) = \psi(\mu_h(t)), \text{ for all } t \in [0, \infty). \quad (5.3)$$

Using $\psi(t) = \sqrt{t}$, equations (5.1), (5.2), (5.3) imply that

$$\tau(|h^{1/2}k^{1/2}|) = \tau(|k^{1/2}h^{1/2}|) \quad (5.4)$$

for all $h, k \in \mathcal{M}_+$.

Define:

$$\mathfrak{n}_\tau = \{x \in \mathcal{M} \mid \tau(x^*x) < \infty\}, \quad \mathfrak{m}_\tau = (\mathfrak{n}_\tau)^2 = \left\{ \sum_{j=1}^k x_j y_j \mid k \in \mathbb{N}, x_j, y_j \in \mathfrak{n}_\tau \right\}.$$

The sets \mathfrak{n}_τ and \mathfrak{m}_τ are (algebraic) ideals of \mathcal{M} . If $\mathfrak{p}_\tau \subseteq \mathcal{M}_+$ is the set of all $a \in \mathcal{M}_+$ such that $\tau(a) < \infty$, then $\mathfrak{p}_\tau = \mathfrak{m}_\tau \cap \mathcal{M}_+$ and the function $\tau|_{\mathfrak{p}_\tau}$ extends to a linear map, which we denote again by τ , on \mathfrak{m}_τ such that $\tau(x^*) = \overline{\tau(x)}$, $\tau(xy) = \tau(yx)$ for all $x \in \mathcal{M}$ and $y \in \mathfrak{m}_\tau$, and $\tau(xy) = \tau(yx)$ for all $x, y \in \mathfrak{n}_\tau$. The ideal \mathfrak{m}_τ is called the *trace ideal* of τ . If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and if τ is the canonical trace on $\mathcal{B}(\mathcal{H})$, then $\mathfrak{m}_\tau = \mathcal{T}(\mathcal{H})$. Hence, the elements of the ideal \mathfrak{m}_τ are analogues of trace-class operators.

Recall from [41] and [53, Chapter IX] that \mathcal{H} and \mathcal{M} determine a topological vector space $\mathfrak{M}(\mathcal{H})$ and a topological involutive algebra $\mathfrak{M}(\mathcal{M})$ such that \mathcal{H} and \mathcal{M} are dense (in appropriate topologies) in $\mathfrak{M}(\mathcal{H})$ and $\mathfrak{M}(\mathcal{M})$ respectively, and that $\mathfrak{M}(\mathcal{M})$ acts on $\mathfrak{M}(\mathcal{H})$ in a natural fashion. In this context, for each $z \in \mathfrak{M}(\mathcal{M})$ and $\varepsilon > 0$ there exists a projection $p \in \mathcal{M}$ such that $zp \in \mathcal{M}$ and $\tau(1-p) < \varepsilon$. The set $\mathfrak{M}(\mathcal{M})_+$ of all z^*z , for $z \in \mathfrak{M}(\mathcal{M})$, is a pointed convex cone and for each $a \in \mathfrak{M}(\mathcal{M})_+$ there is a unique $b \in \mathfrak{M}(\mathcal{M})_+$ (denoted by $a^{1/2}$) for which $b^2 = a$. Thus, the element $|z|$ given by $(z^*z)^{1/2}$ lies in $\mathfrak{M}(\mathcal{M})$ for each $z \in \mathfrak{M}(\mathcal{M})$. Furthermore, the function τ extends

to $\mathfrak{M}(\mathcal{M})_+$ via $\tau(a) = \lim_{\varepsilon \rightarrow 0^+} \tau(a(1 + \varepsilon a)^{-1})$, for $a \in \mathfrak{M}(\mathcal{M})_+$, and satisfies the usual trace properties: $\tau(z^*z) = \tau(zz^*)$, $\tau(a + b) = \tau(a) + \tau(b)$, and $\tau(\lambda a) = \lambda\tau(a)$ for all $z \in \mathfrak{M}(\mathcal{M})$, $a, b \in \mathfrak{M}(\mathcal{M})_+$, and $\lambda \in \mathbb{R}$ with $\lambda \geq 0$.

The predual \mathcal{M}_* is linearly order isomorphic to set

$$\{z \in \mathfrak{M}(\mathcal{M}) \mid \tau(|z|) < \infty\},$$

and so we identify these two sets. The map $z \mapsto \tau(|z|)$ defines a norm $\|\cdot\|_1$ on \mathcal{M}_* and with respect to this norm \mathcal{M}_* is a Banach space containing \mathfrak{m}_τ as a norm-dense linear submanifold. In the case of a type I von Neumann algebra \mathcal{M} , nothing new is obtained, since $\mathfrak{M}(\mathcal{M}) = \mathcal{M}$ in this case. In particular, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then $\mathfrak{M}(\mathcal{M}) = \mathcal{M}$ and $\mathcal{M}_* = \mathfrak{m}_\tau = \mathcal{T}(\mathcal{H})$.

For each $n \in \mathbb{N}$, let $\tau^{(n)} : \mathcal{M}_n(\mathfrak{m}_\tau) \rightarrow \mathcal{M}_n(\mathbb{C})$ denote the linear map

$$\tau^{(n)}(Y) = [\tau(y_{ij})]_{i,j},$$

for $Y = [y_{ij}]_{i,j=1}^n \in \mathcal{M}_n(\mathfrak{m}_\tau)$.

Lemma 5.2. *If $Y \in \mathcal{M}_n(\mathfrak{m}_\tau)$ is a positive operator on $\bigoplus_1^n \mathcal{H}$, then $\tau^{(n)}(Y)$ is a positive operator on \mathbb{C}^n .*

Proof. Recall that $\mathfrak{p}_\tau = \mathfrak{m}_\tau \cap \mathcal{M}_+$ [52, Lemma V.2.16] and that if ω is a normal state on \mathcal{M} , then ω is a completely positive linear map of \mathcal{M} . Because the

trace $\tau : \mathcal{M}_+ \rightarrow [0, \infty]$ is normal, it is a sum of a family of normal states [52, p. 332]; hence, $\tau^{(n)}(X)$ is a (possibly nonconvergent) sum of positive elements of $\mathcal{M}_n(\mathbb{C})_+$. Therefore, because $Y \in \mathcal{M}_n(\mathfrak{m}_\tau) \cap \mathcal{M}_n(\mathcal{M})_+$, the complex matrix $\tau^{(n)}(Y)$ is a (convergent) sum of positive elements and is, hence, positive. \square

Lemma 5.3. *If $a, b \in \mathfrak{m}_\tau \cap \mathcal{M}_+$, then $|a^{1/2}b^{1/2}| \in \mathfrak{m}_\tau$.*

Proof. Recall from equation (5.1) that $\tau(|z|) = \int_0^\infty \mu_z(t) dt$, for every $z \in \mathcal{M}$. Since a and b are positive, we have, for every $s > 0$ in \mathbb{R} , that

$$\begin{aligned} \int_0^s \mu_{a^{1/2}b^{1/2}}(t) dt &\leq \int_0^s \mu_{a^{1/2}}(t)\mu_{b^{1/2}}(t) dt \\ &= \int_0^s \mu_a(t)^{1/2}\mu_b(t)^{1/2} dt \\ &\leq \left(\int_0^s \mu_a(t) dt \right)^{1/2} \left(\int_0^s \mu_b(t) dt \right)^{1/2} \\ &\leq \sqrt{\tau(a)\tau(b)}, \end{aligned}$$

where the first of the inequalities above is a consequence of [14, Corollaire 4.4]. Thus, $\tau(|a^{1/2}b^{1/2}|) \leq \sqrt{\tau(a)\tau(b)}$, which implies that $|a^{1/2}b^{1/2}| \in \mathfrak{m}_\tau$. \square

Definition 5.4. *A density operator in \mathcal{M} is a positive operator $\rho \in \mathcal{M}$ such that $\tau(\rho) = 1$.*

Let \mathfrak{s}_τ denote the set of all density operators in \mathcal{M} . By Lemma 5.3, if $\sigma, \rho \in \mathfrak{s}_\tau$, then $|\sigma^{1/2}\rho^{1/2}|$ has finite trace; thus, we may define fidelity for pairs of density operators in \mathcal{M} .

Definition 5.5. *The fidelity of a pair of density operators $\sigma, \rho \in \mathcal{M}$ is the nonnegative real number $F_\tau(\sigma, \rho)$ defined by*

$$F_\tau(\sigma, \rho) = \tau(|\sigma^{1/2}\rho^{1/2}|).$$

5.2 Duality and channels

We begin by making sense of the commonly used phrase “ ϕ is a trace-preserving completely positive linear map on $\mathcal{T}(\mathcal{H})$.” On the one hand, the phrase suggests that $\mathcal{T}(\mathcal{H})$ is treated as a matrix-ordered space; on the other hand, much of the literature interprets the phrase to mean that ϕ is a normal completely positive linear map on $\mathcal{B}(\mathcal{H})$ that preserves the trace of operators in $\mathcal{T}(\mathcal{H})$. These two interpretations are not entirely compatible.

We begin by reviewing the matrix order on the predual \mathcal{M}_* of an arbitrary von Neumann algebra \mathcal{M} . Recall from [11] that if \mathcal{S} is a matrix ordered space, and if the positive cone of $\mathcal{M}_n(\mathcal{S})$ is denoted by $\mathcal{M}_n(\mathcal{S})_+$, then a linear map $\phi : \mathcal{S} \rightarrow \mathcal{T}$ of matrix ordered spaces is k -positive if $\phi(\mathcal{M}_n(\mathcal{S})_+) \subseteq \mathcal{M}_n(\mathcal{T})_+$

for every $n = 1, \dots, k$, and ϕ is completely positive if ϕ is k -positive for all $k \in \mathbb{N}$.

The matrix order on \mathcal{M} is inherited from $\mathcal{B}(\mathcal{H})$: a matrix $X = [x_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{M})$ is positive if X is a positive operator on the n -fold direct sum $\bigoplus_1^n \mathcal{H}$.

A matrix $\Omega = [\omega_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{M}_*)$ is positive if

$$\sum_{i=1}^n \sum_{j=1}^n \omega_{ij}(x_{ij}) \geq 0, \quad (5.5)$$

for every positive $X = [x_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{M})$. Let $\mathcal{M}_n(\mathcal{M}_*)_+$ denote the positive matrices over \mathcal{M}_* . (Recall that \mathcal{M}_* consists of normal linear functionals on \mathcal{M} .) A useful criterion for membership in $\mathcal{M}_n(\mathcal{M}_*)_+$ is as follows: $\Omega = [\omega_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{M}_*)_+$ if and only if the linear map $\mathfrak{L}_\Omega : \mathcal{M} \rightarrow \mathcal{M}_n(\mathbb{C})$, defined by

$$\mathfrak{L}_\Omega(x) = [\omega_{ij}(x)]_{i,j}, \quad (5.6)$$

for $x \in \mathcal{M}$, is a completely positive map [11, Lemma 4.7].

In the case where $\mathcal{M} = \mathcal{M}_d(\mathbb{C})$, then every $\omega \in \mathcal{M}_*$ is determined uniquely by some matrix $y_\omega \in \mathcal{M}$ via the formula $\omega(x) = \text{Tr}(xy_\omega)$, for $x \in \mathcal{M}$. It can happen that a matrix $\Omega = [\omega_{ij}]_{i,j=1}^n$ of normal linear functionals is positive, yet the corresponding matrix of operators $Y_\Omega = [y_{\omega_{ij}}]_{i,j=1}^n$ fails to be positive as an operator on $\bigoplus_1^n \mathbb{C}^d$. For example, let $d = 2$ and

$\mathcal{M} = \mathcal{M}_2(\mathbb{C})$, and consider the matrix $\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \in \mathcal{M}_2(\mathcal{M}_*)$, where the corresponding operators $y_{\omega_{ij}} \in \mathcal{M}$ inducing each $\omega_{ij} \in \mathcal{M}_*$ are

$$y_{\omega_{11}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad y_{\omega_{12}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad y_{\omega_{21}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y_{\omega_{22}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix $Y_\Omega \in \mathcal{M}_2(\mathcal{M})$ is not positive, as $Y_\Omega = [1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [1]$ has one negative eigenvalue. However, the linear map $\mathfrak{L}_\Omega : \mathcal{M} \rightarrow \mathcal{M}_2(\mathbb{C})$ satisfies $\mathfrak{L}_\Omega(x) = x$, for every $x \in \mathcal{M}$, and so \mathfrak{L}_Ω is a completely positive map; by criterion (5.6), we deduce that $\Omega \in \mathcal{M}_2(\mathcal{M}_*)_+$. Thus, in identifying matrices $[\omega_{ij}]_{i,j}$ over \mathcal{M}_* with the matrices $[y_{\omega_{ij}}]_{i,j}$ over \mathcal{M} , we deduce from this example that $\mathcal{M}_2(\mathcal{M}_*)_+ \cap \mathcal{M}_2(\mathcal{M}) \not\subseteq \mathcal{M}_2(\mathcal{M})_+$. By similar reasoning, the matrix $\Delta = \begin{bmatrix} \omega_{11} & \omega_{21} \\ \omega_{12} & \omega_{22} \end{bmatrix} \in \mathcal{M}_2(\mathcal{M}_*)$ is induced by the positive operator matrix $Y_\Delta \in \mathcal{M}_2(\mathcal{M})_+$, defined by

$$Y_\Delta = \begin{bmatrix} y_{\omega_{11}} & y_{\omega_{21}} \\ y_{\omega_{12}} & y_{\omega_{22}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

but Δ is not positive in $\mathcal{M}_2(\mathcal{M}_*)$ because $\mathfrak{L}_\Delta : \mathcal{M} \rightarrow \mathcal{M}_2(\mathbb{C})$ satisfies

$\mathfrak{L}_\Delta(x) = x^t$, for every $x \in \mathcal{M}$, and it is well known that the transpose map fails to be completely positive.

Notwithstanding the discussion of the previous paragraph, it is nevertheless true that a completely positivity linear map \mathcal{E} on the matrix-ordered space $\mathcal{M}_d(\mathbb{C})_* = \mathcal{T}_d(\mathbb{C})$ (the $d \times d$ complex matrices in the trace norm) is also completely positive on the C^* -algebra $\mathcal{M}_d(\mathbb{C})$. This is a remarkably fortunate circumstance, as the matrix orders on $\mathcal{T}_d(\mathbb{C})$ and $\mathcal{M}_d(\mathbb{C})$ are distinct (as indicated in the example of the previous paragraph), and the literature (including the literature on fidelity) makes extensive use of this fortunate fact by frequently making little or no reference to the matrix order on $\mathcal{T}_d(\mathbb{C})$ and, instead, drawing entirely upon the matrix order on $\mathcal{M}_d(\mathbb{C})$. Although this fortunate circumstance may be relevant for matrix algebras, one does not expect the same situation to persist with arbitrary von Neumann algebras. Therefore, in what follows, it will be important for us to distinguish between the matricial order on the predual \mathcal{M}_* and the matricial order on \mathcal{M} in discussing completely positive linear maps on \mathcal{M}_* .

With this preface, the following definition is natural.

Definition 5.6. *If \mathcal{M} is a von Neumann algebra, then a channel, or quantum channel, is a continuous linear operator $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$ such that the*

dual map $\mathcal{E}^* : \mathcal{M} \rightarrow \mathcal{M}$ is unital, normal, and completely positive.

Not every completely positive linear map on a von Neumann algebra admits a Kraus decomposition, but those that do are called inner maps [4].

Definition 5.7. *Let \mathcal{M} be a von Neumann algebra with predual \mathcal{M}_* .*

- (1) *A completely positive linear map $\phi : \mathcal{M} \rightarrow \mathcal{M}$ is inner if there exists a sequence $\{a_k\}_{k \in \mathbb{N}}$ in \mathcal{M} such that $\phi(x) = \sum_{k=1}^{\infty} a_k^* x a_k$, for every $x \in \mathcal{M}$, where the convergence of the sum is with respect to the ultraweak topology of \mathcal{M} ; that is, for each $x \in \mathcal{M}$,*

$$\omega(\phi(x)) = \lim_{m \rightarrow \infty} \sum_{k=1}^m \omega(a_k^* x a_k),$$

for every $\omega \in \mathcal{M}_$.*

- (2) *A channel $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$ is inner if the unital completely positive linear map \mathcal{E}^* is inner.*

Note that we identify \mathfrak{m}_τ as a $\|\cdot\|_1$ -norm dense subset of \mathcal{M}_* from the discussion before Lemma 5.2. If $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$ is a channel, then \mathcal{E} is trace preserving and so \mathcal{E} maps \mathfrak{s}_τ back into itself. Because \mathfrak{s}_τ spans \mathfrak{m}_τ , we deduce that

$$\mathcal{E}(\mathfrak{m}_\tau) \subseteq \mathfrak{m}_\tau,$$

for every channel $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$.

Lemma 5.8. *If a channel $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$ is inner, then*

(1) *there exists a sequence $\{a_k\}_{k \in \mathbb{N}}$ in \mathcal{M} such that $\sum_{k=1}^{\infty} a_k^* a_k = 1$, and*

$$\mathcal{E}(\rho) = \sum_{k=1}^{\infty} a_k \rho a_k^*, \text{ for every density operator } \rho \text{ in } \mathcal{M},$$

(2) $\mathcal{E}^{(2)}(\mathcal{M}_2(\mathfrak{m}_\tau) \cap \mathcal{M}_2(\mathcal{M})_+) \subseteq \mathcal{M}_2(\mathcal{M})_+$, and

(3) $\mathcal{E}(y)^* \mathcal{E}(y) \leq \left\| \sum_{k=1}^{\infty} a_k a_k^* \right\| \mathcal{E}(y^* y)$, for every $y \in \mathfrak{m}_\tau$.

Proof. By definition of inner, there exists a sequence $\{a_k\}_{k \in \mathbb{N}}$ in \mathcal{M} such that $\sum_{k=1}^{\infty} a_k^* a_k = 1$, and $\mathcal{E}^*(x) = \sum_{k=1}^{\infty} a_k^* x a_k$, for every $x \in \mathcal{M}$, where the convergence of the sum is with respect to the ultraweak topology of \mathcal{M} .

Therefore, if $\rho \in \mathfrak{s}_\tau$ and $x \in \mathcal{M}$, then

$$\begin{aligned} \tau(\mathcal{E}(\rho)x) &= \tau(\rho \mathcal{E}^*(x)) = \lim_{m \rightarrow \infty} \sum_{k=1}^m \tau(\rho a_k^* x a_k) \\ &= \lim_{m \rightarrow \infty} \sum_{k=1}^m \tau(x a_k \rho a_k^*) \\ &= \lim_{m \rightarrow \infty} \tau \left(x \sum_{k=1}^m a_k \rho a_k^* \right), \end{aligned}$$

and so $\mathcal{E}(\rho) = \sum_{k=1}^{\infty} a_k \rho a_k^*$, which proves the first statement. The second and third statements follow immediately from the inner structure of \mathcal{E} . \square

Now that we have the appropriate set up, we are ready to state our results related to fidelity in this new set-up. The following is the main result on the fidelity of density operators.

Theorem 5.9 (Fidelity in Semifinite von Neumann Algebras).

(1) *(Basic Properties of Fidelity)* If $\sigma, \rho \in \mathfrak{s}_\tau$, then

$$(a) \quad F_\tau(\sigma, \rho) = F_\tau(\rho, \sigma),$$

$$(b) \quad 0 \leq F_\tau(\sigma, \rho) \leq 1,$$

$$(c) \quad F_\tau(\sigma, \rho) = 0 \text{ if and only if } \sigma \perp \rho, \text{ and}$$

$$(d) \quad F_\tau(\sigma, \rho) = 1 \text{ if and only if } \sigma = \rho.$$

(2) *(Monotonicity of Fidelity)* If $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$ is a linear map such that

$$\mathcal{E}^{(2)}(\mathcal{M}_2(\mathfrak{m}_\tau) \cap \mathcal{M}_2(\mathcal{M})_+) \subseteq \mathcal{M}_2(\mathcal{M})_+, \text{ then } \mathcal{E} \text{ is positive and}$$

$$F_\tau(\sigma, \rho) \leq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)), \quad (5.7)$$

for all $\sigma, \rho \in \mathfrak{s}_\tau$.

(3) *(Preservation of Fidelity)* If $\mathcal{E} : \mathcal{M}_* \rightarrow \mathcal{M}_*$ is a bijective positive linear map such that

$$F_\tau(\sigma, \rho) = F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)), \quad (5.8)$$

for all $\sigma, \rho \in \mathfrak{s}_\tau$, then \mathcal{E}^* is an automorphism of \mathcal{M} .

Proof. Equation (5.4) shows that $F_\tau(\sigma, \rho) = F_\tau(\rho, \sigma)$, whereas [20, Corollary 3.2] yields $0 \leq F_\tau(\sigma, \rho) \leq 1$. It is clear that $F_\tau(\sigma, \rho) = 0$ if and only if $\sigma \perp \rho$, while [20, Theorem 3.4] shows that $F_\tau(\sigma, \rho) = 1$ if and only if $\sigma = \rho$.

To establish monotonicity of fidelity, we draw upon a method of proof given in Watrous's monograph [55]. First, however, note that the hypothesis $\mathcal{E}^{(2)}(\mathcal{M}_2(\mathfrak{m}_\tau) \cap \mathcal{M}_2(\mathcal{M})_+) \subseteq \mathcal{M}_2(\mathcal{M})_+$ immediately yields the positivity of \mathcal{E} by considering positive 2×2 matrices of the form $\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}$, for $y \in \mathfrak{m}_\tau \cap \mathcal{M}_+$.

Suppose that $\sigma, \rho \in \mathfrak{s}_\tau$ and $x \in \mathcal{M}$, and consider the matrix $X = \begin{bmatrix} \sigma & x \\ x^* & \rho \end{bmatrix}$ in $\mathcal{M}_2(\mathcal{M})$. By Lemma 5.1, the matrix $X \in \mathcal{M}_2(\mathcal{M})_+$ if and

only if $x = \sigma^{1/2}y\rho^{1/2}$ for some $y \in \mathcal{M}$ with $\|y\| \leq 1$. Hence,

$$\begin{aligned} \sup \left\{ |\tau(x)| \mid \begin{bmatrix} \sigma & x \\ x^* & \rho \end{bmatrix} \in M_2(\mathfrak{m}_\tau)_+ \right\} &= \sup \{ |\tau(\rho^{1/2}\sigma^{1/2}y)| \mid y \in \mathcal{M}, \|y\| \leq 1 \} \\ &= \|\rho^{1/2}\sigma^{1/2}\|_1 \\ &= \tau(|\rho^{1/2}\sigma^{1/2}|) \\ &= F_\tau(\sigma, \rho). \end{aligned}$$

We shall show that a similar formula applies to the fidelity of the density operator $\mathcal{E}(\sigma)$ and $\mathcal{E}(\rho)$. To this end, fix $y \in \mathcal{M}$ with $\|y\| \leq 1$ and let $x = \sigma^{1/2}y\rho^{1/2}$. Thus, the matrix $\begin{bmatrix} \sigma & x \\ x^* & \rho \end{bmatrix}$ in $\mathcal{M}_2(\mathfrak{m}_\tau)$ belongs to the cone $\mathcal{M}_2(\mathcal{M})_+$. Therefore, by the hypothesis on \mathcal{E} , we deduce that

$$\begin{bmatrix} \mathcal{E}(\sigma) & \mathcal{E}(x) \\ \mathcal{E}(x^*) & \mathcal{E}(\rho) \end{bmatrix} \in \mathcal{M}_2(\mathcal{M})_+.$$

Hence, Lemma 5.2 yields

$$\begin{bmatrix} \tau(\mathcal{E}(\sigma)) & \tau(\mathcal{E}(x)) \\ \tau(\mathcal{E}(x^*)) & \tau(\mathcal{E}(\rho)) \end{bmatrix} = \tau^{(2)} \left(\begin{bmatrix} \mathcal{E}(\sigma) & \mathcal{E}(x) \\ \mathcal{E}(x^*) & \mathcal{E}(\rho) \end{bmatrix} \right) \in \mathcal{M}_2(\mathbb{C})_+,$$

and so $|\tau(x)| = |\tau(\mathcal{E}(x))| \leq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$. Therefore, the supremum of the real numbers $|\tau(x)|$ over all $x \in \mathfrak{m}_\tau$ for which $\begin{bmatrix} \sigma & x \\ x^* & \rho \end{bmatrix} \in \mathcal{M}_2(\mathcal{M})_+$ is also bounded above by $F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$, which implies that $F_\tau(\sigma, \rho) \leq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$.

Lastly, assume that \mathcal{E} is a positive linear bijection that preserves fidelity. Thus, \mathcal{E}^* is an invertible operator on \mathcal{M} . If $x \in \mathcal{M}_+$ and $y = (\mathcal{E}^*)^{-1}(x)$, then for every $\rho \in \mathfrak{s}_\tau$,

$$\tau(y\rho) = \tau((\mathcal{E}^*)^{-1}(x)\rho) = \tau(x\mathcal{E}^{-1}(\rho)) \geq 0.$$

That is, $\omega(y) \geq 0$ for every normal state ω on \mathcal{M} ; hence, $y \in \mathcal{M}_+$, which proves that $(\mathcal{E}^*)^{-1}$ is positive linear map. Thus, \mathcal{E}^* is a linear order isomorphism. By [51, Theorem 2.1.3], \mathcal{E}^* is a Jordan isomorphism. Because \mathcal{E}^* satisfies the Schwarz inequality $\mathcal{E}^*(x^*x) \geq \mathcal{E}^*(x^*)\mathcal{E}^*(x)$ for all $x \in \mathcal{M}$, the Jordan isomorphism \mathcal{E}^* is in fact an automorphism [50, Corollary 3.6]. \square

Corollary 5.10. *If \mathcal{E} is an inner channel, then $F_\tau(\sigma, \rho) \leq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$, for all $\sigma, \rho \in \mathfrak{s}_\tau$.*

Proof. By Lemma 5.8, \mathcal{E} satisfies $\mathcal{E}^{(2)}(\mathcal{M}_2(\mathfrak{m}_\tau) \cap \mathcal{M}_2(\mathcal{M})_+) \subseteq \mathcal{M}_2(\mathcal{M})_+$. Therefore, by Theorem 5.9, \mathcal{E} has the monotonicity property for fidelity. \square

In the classical setting of type I factors, the following known result [39] is recovered.

Corollary 5.11 (Molnár). *If $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ is a channel, then $F_\tau(\sigma, \rho) \leq F_\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho))$, for all density operators $\sigma, \rho \in \mathcal{T}(\mathcal{H})$. Furthermore, if $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ is surjective channel and preserves fidelity, then \mathcal{E} is a unitary channel.*

Proof. In this case, $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{M}_* = \mathfrak{m}_\tau = \mathcal{T}(\mathcal{H})$. By Kraus's theorem [36], every channel on $\mathcal{T}(\mathcal{H})$ is inner. Hence, \mathcal{E} has the monotonicity property for fidelity. Moreover, the proof of Lemma 3.3 is valid in $\mathcal{T}(\mathcal{H})$, and so \mathcal{E} is injective. Therefore, coupled with the hypothesis that \mathcal{E} is surjective, Theorem 5.9 indicates that \mathcal{E}^* is an automorphism of $\mathcal{B}(\mathcal{H})$. Hence, there is a unitary $u \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{E}^*(x) = u^*xu$ for all $x \in \mathcal{B}(\mathcal{H})$. That is, $\mathcal{E}(s) = usu^*$ for every $s \in \mathcal{T}(\mathcal{H})$. \square

Chapter 6

Fidelity in Quantum

Information Theory

In this chapter we study fidelity of two density operators on \mathcal{M}_d , namely the $d \times d$ matrices over the field of complex numbers. Since \mathcal{M}_d is a finite dimensional C^* -algebra, there are various different characterisations of fidelity that will be explored. Note that in this case the faithful trace τ is the usual trace functional on \mathcal{M}_d . Since quantum statistical mechanics involving finitely many particles is modeled mathematically on the $d \times d$ matrices, the fidelity function on \mathcal{M}_d is of extreme importance to distinguish two quantum states. We relate fidelity with other distinguishability measures that exist

in the context of quantum information. Since a trace preserving completely positive map, that is a quantum channel signifies a quantum operation, the effect of a quantum channel could be measured by computing fidelity of two density operators before and after applying a channel. We then study the Bures contractive channels on \mathcal{M}_d and analyse this specific class of quantum channels with respect to some of the widely developed concepts in quantum information theory, namely *quantum error correction* or *noiseless subsystems*. For information theoretic purpose consult the book [55] for an excellent introduction on fidelity. The set of density operators in \mathcal{M}_d will be denoted by $\mathcal{D}(\mathcal{M}_d)$. Elements of $\mathcal{D}(\mathcal{M}_d)$ are often called quantum states.

6.1 Finite dimensional characterisations

Using the finite dimensionality of the system, we put forward some elementary yet useful properties of fidelity. We begin with a simple formula of fidelity of density operators on 2×2 matrices that involves trace and determinant of matrices. See [32] and references therein.

Lemma 6.1. *For density operators $\sigma, \rho \in \mathcal{D}(\mathcal{M}_2)$, we have*

$$F(\sigma, \rho)^2 = \text{Tr}(\sigma\rho) + 2 \det(\sigma\rho)^{1/2}.$$

Proof. For any positive definite matrix $x \in \mathcal{M}_2$ with eigenvalues λ_1, λ_2 , we have, $\text{Tr}(x^{1/2}) = \lambda_1^{1/2} + \lambda_2^{1/2}$. Hence,

$$(\text{Tr}(x^{1/2}))^2 = \lambda_1 + \lambda_2 + 2(\lambda_1\lambda_2)^{1/2} = \text{Tr}(x) + 2\det(x)^{1/2}.$$

Now if $x = \sigma^{1/2}\rho\sigma^{1/2}$, utilising the above relation we get,

$$F(\sigma, \rho)^2 = (\text{Tr}(x^{1/2}))^2 = \text{Tr}(x) + 2\det(x)^{1/2}.$$

Note that $\text{Tr}(x) = \text{Tr}(\sigma\rho)$ and $\det(x) = \det(\sigma\rho)$. Hence the lemma. \square

The formula in Lemma 6.1 is only valid in 2×2 matrices. The next proposition draws attention upon the fidelity of *pure quantum states* which is mathematically represented by a rank one projection in \mathcal{M}_d . So a pure state is induced by a unit vector in the d -dimensional Hilbert space. So if $p \in \mathcal{D}(\mathcal{M}_d)$ is a pure state, then there exists a unit vector $\xi \in \mathbb{C}^d$ such that $p = \xi\xi^*$, where for any vector $x, y \in \mathcal{H}$, the operator $xy^* : \mathcal{H} \rightarrow \mathcal{H}$ is defined by $xy^*(a) = \langle a, y \rangle x$. A convex combination of pure states is called *mixed state*. Also, when the ambient quantum system is tensored with another auxiliary system, Proposition 6.2 provides a formula to compute the fidelity of decomposable tensors.

Proposition 6.2. *The following assertions hold for the fidelity function.*

(1) If $\sigma = \xi\xi^*$ and $\rho = \eta\eta^*$ are two rank one projections where ξ, η are unit vectors in \mathbb{C}^d , then $F(\sigma, \rho) = |\langle \xi, \eta \rangle|$.

(2) If σ_i, ρ_i are density operators in \mathcal{M}_d , for $i = 1, 2$, then

$$F(\sigma_1 \otimes \sigma_2, \rho_1 \otimes \rho_2) = F(\sigma_1, \rho_1)F(\sigma_2, \rho_2).$$

Proof. Statement 1 follows from the fact that for rank one projections $\sigma = \xi\xi^*$ and any operator a , we have

$$F(\sigma, a) = \text{Tr}((\sigma a \sigma)^{1/2}) = \text{Tr}(\xi\xi^* a \xi\xi^*)^{1/2} = \langle a\xi, \xi \rangle^{1/2} \text{Tr}(\xi\xi^*) = \langle a\xi, \xi \rangle^{1/2}.$$

Now using $a = \rho = \eta\eta^*$, we get

$$F(\sigma, \rho) = |\langle \xi, \eta \rangle|.$$

Assertion 2 follows from the general facts of tensor products of matrices. For any two matrices $x, y \in \mathcal{M}_d$, we have, $\text{Tr}(x \otimes y) = \text{Tr}(x) \text{Tr}(y)$. Also it is true for any two positive definite matrices $x, y \in \mathcal{M}_d$, that $(x \otimes y)^{1/2} = x^{1/2} \otimes y^{1/2}$. Hence statement 2 is a direct consequence of these facts. \square

Another useful characterisation of fidelity is given below.

Lemma 6.3. *For two density elements $\sigma, \rho \in \mathcal{D}(\mathcal{M}_d)$, we have*

$$F(\sigma, \rho) = \sup_{u \in \mathcal{U}_d} \{ |\operatorname{Tr}(u\sigma^{1/2}\rho^{1/2})| \}, \quad (6.1)$$

where \mathcal{U}_d is the group of $d \times d$ unitary matrices.

Proof. By the definition of fidelity of, $F(\sigma, \rho) = \|\sigma^{1/2}\rho^{1/2}\|_1$, where $\|x\|_1 = \operatorname{Tr}((x^*x)^{1/2})$, for any $x \in \mathcal{M}_d$. Now, using the duality of 1 norm $\|\cdot\|_1$ and the operator norm $\|\cdot\|$, that is,

$$\|x\|_1 = \sup_{\|a\| \leq 1} \{ |\operatorname{Tr}(xa^*)| \}.$$

Now letting $x = \sigma^{1/2}\rho^{1/2}$ and using Russo-Dye theorem [48], the above supremum is same if taken over the unitaries, that is,

$$F(\sigma, \rho) = \sup_{u \in \mathcal{U}_d} \{ |\operatorname{Tr}(u\sigma^{1/2}\rho^{1/2})| \}.$$

□

It is worth mentioning that the above formula also holds for infinite dimensional system where σ, ρ are two positive trace-class operators. Note that the supremum in the Equation (6.1) is actually a maximum as the supremum is attained due to the compactness of the set \mathcal{U}_d . The maximum is achieved on the unitary arising in the polar decomposition of $\sigma^{1/2}\rho^{1/2}$.

We finish this section with a lemma that relates to the commutant of the Kraus operators of a quantum channel with fidelity. The special structure

of completely positive maps on \mathcal{M}_d which is the Kraus decomposition, helps to extricate deeper properties of positive linear maps. We will frequently use this crucial fact that for any quantum channel that is for a completely positive and trace preserving linear maps $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$, there exists a (non-unique!) set of operators $\{a_1, \dots, a_k\}$ such that

$$\mathcal{E}(x) = \sum_{j=1}^k a_j x a_j^*, \quad (6.2)$$

for every $x \in \mathcal{M}_d$ and the operators $\{a_1, \dots, a_k\}$ satisfying $\sum_{j=1}^k a_j^* a_j = 1$, where 1 is the identity element in \mathcal{M}_d .

Lemma 6.4. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a quantum channel with the Kraus decomposition: $\mathcal{E}(x) = \sum_{k=1}^n a_k x a_k^*$, for all $x \in \mathcal{M}_d$. Let \mathcal{A} be the $*$ -algebra generated by $\{a_1, \dots, a_n\}$. If two positive elements $\sigma, \rho \in \mathcal{A}'$, then*

$$F(\sigma, \rho) \leq F(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq \|\mathcal{E}\| F(\sigma, \rho), \quad (6.3)$$

where \mathcal{A}' denotes the commutant of \mathcal{A} .

Proof. By the monotonicity property (Theorem 2.14), we know $F(\sigma, \rho) \leq F(\mathcal{E}(\sigma), \mathcal{E}(\rho))$.

For the other direction, note that if $\sigma, \rho \in \mathcal{A}'$, then

$$\mathcal{E}(\sigma) = a_1 \sigma a_1^* + \dots + a_n \sigma a_n^* = \sigma \mathcal{E}(1) = \mathcal{E}(1) \sigma.$$

Similarly, for ρ , we get $\mathcal{E}(\rho) = \rho\mathcal{E}(1) = \mathcal{E}(1)\rho$. Now, using Lemma 6.3, we get

$$\begin{aligned}
F(\mathcal{E}(\sigma), \mathcal{E}(\rho)) &= F(\sigma\mathcal{E}(1), \rho\mathcal{E}(1)) \\
&= \sup_{u \in \mathcal{U}_d} \{ |\operatorname{Tr}(u\sigma^{1/2}\mathcal{E}(1)^{1/2}\mathcal{E}(1)^{1/2}\rho^{1/2})| \} \\
&= \sup_{u \in \mathcal{U}_d} \{ |\operatorname{Tr}(u\mathcal{E}(1)\sigma^{1/2}\rho^{1/2})| \} \\
&\leq \|u\mathcal{E}(1)\| \|\sigma^{1/2}\rho^{1/2}\|_1 \\
&\leq \|\mathcal{E}(1)\| F(\sigma, \rho).
\end{aligned}$$

The proof is complete using the Russo-Dye Theorem which says that for a positive map \mathcal{E} , $\|\mathcal{E}\| = \|\mathcal{E}(1)\|$. \square

Note that, when \mathcal{E} is unital, then Proposition 4.12 asserts that $\mathcal{A}' = \operatorname{Fix} \mathcal{E}$, and hence $\sigma, \rho \in \operatorname{Fix} \mathcal{E}$. Equation (6.3) is then obvious. However, the following example is interesting in this direction.

Example 6.5. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be given by $\mathcal{E}(x) = uxu^*$, for every $x \in \mathcal{M}_d$ and a unitary element $u \in \mathcal{M}_d$. It is easily seen that $F(\sigma, \rho) = F(\mathcal{E}(\sigma), \mathcal{E}(\rho))$ for every density elements σ, ρ but not every density element is a fixed point of \mathcal{E} .*

6.2 Purification and Uhlmann's theorem

In this section we introduce a concept called *purification* of quantum state. In the context of information theory, this concept has a wide range of applications. A similar concept that is indispensable in quantum mechanics is called *partial trace* of an operator.

Let \mathcal{H}_A and \mathcal{H}_B be two Hilbert spaces associated with the system A and B respectively. The corresponding set of bounded linear maps are denoted by $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$ respectively. It is well known that $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \cong \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$. Now one may consider the linear map

$$\text{Tr}_A : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_B),$$

defined on the elementary tensors as $\text{Tr}_A(\rho_A \otimes \rho_B) = \text{Tr}(\rho_A)\rho_B$. Now extending this map linearly over all of $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, one gets a well defined map from $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ to $\mathcal{B}(\mathcal{H}_B)$. Since this map is induced by tracing out the component A of the tensor product and keeping the component B as it is, this map is called the partial trace with respect to the system A . On the similar line, one can define a linear map Tr_B from $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ to $\mathcal{B}(\mathcal{H}_A)$.

Definition 6.6 (Purification). *Let $\rho \in \mathcal{B}(\mathcal{H}_A)$ be a (mixed) state on \mathcal{H}_A . A purification of ρ is a vector ξ in an extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ such*

that $\rho = \text{Tr}_B(\xi\xi^*)$.

The following proposition guarantees the existence of purification of any state. See Theorem 2.9 in [55] for the proof.

Proposition 6.7. *Let $\rho \in \mathcal{B}(\mathcal{H}_A)$ be a density operator on \mathcal{H}_A . For any Hilbert space \mathcal{H}_B , there exists a vector $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\rho = \text{Tr}_B(\xi\xi^*)$, if and only if $\dim \mathcal{H}_B \geq \text{rank } \rho$.*

In light of the above proposition, one may choose the auxiliary Hilbert space \mathcal{H}_B to be the Hilbert space \mathcal{H}_A to search for the purifying vector in $\xi \in \mathcal{H}_A \otimes \mathcal{H}_A$.

We are now ready to relate fidelity to the concept of purification. Theorem 6.8 was first proved by Uhlmann in [54]. A simpler proof was put forward by Jozsa in [32].

Theorem 6.8. [Uhlmann's theorem] *For density operators $\sigma, \rho \in \mathcal{B}(\mathcal{H})$, we have*

$$F(\sigma, \rho) = \max \{ |\langle \xi, \eta \rangle| \}, \quad (6.4)$$

where the maximum is taken over all possible purifications ξ and η of σ and ρ respectively.

It is tacitly assumed that the purifications of σ and ρ lie on the same space. Indeed, if $\xi \in \mathcal{H} \otimes \mathcal{H}_A$ and $\eta \in \mathcal{H} \otimes \mathcal{H}_B$ are purifications of σ and ρ respectively, then we may regard both lying in a bigger Hilbert space $\mathcal{H} \otimes (\mathcal{H}_A \oplus \mathcal{H}_B)$. Actually, a stronger assertion is true (see [32]), which asserts that for $\sigma, \rho \in \mathcal{B}(\mathcal{H})$, the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ contains the purifications of σ, ρ simultaneously and moreover, the maximum in Equation (6.4) is attained in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

Corollary 6.9. *Let for the composite density operators ρ_{AB} and σ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\sigma_A = \text{Tr}_B(\sigma_{AB})$. Then we have*

$$F(\sigma_A, \rho_A) \geq F(\sigma_{AB}, \rho_{AB}).$$

Proof. Note that if $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ is any purification of σ_{AB} , where \mathcal{H}_E is a Hilbert space, then ξ is also a purification of σ_A on the space $\mathcal{H}_A \otimes (\mathcal{H}_B \otimes \mathcal{H}_E)$. Hence the assertion of the corollary follows from Theorem 6.8. \square

Exploiting the connection of fidelity with purification of states, we obtain yet another important result which we have already established in Theorem 2.14, that is, the monotonicity of fidelity under positive linear maps. Corollary 6.10 depicts same results for quantum channels. Since the proof involves a different definition of quantum channels, which is widely used in

the literature of information theory, we write it out below.

Corollary 6.10. *Let \mathcal{H} be a finite dimensional Hilbert space. If $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a quantum channel, then*

$$F(\sigma, \rho) \leq F(\mathcal{E}(\sigma), \mathcal{E}(\rho)),$$

for any density operators σ, ρ on \mathcal{H} .

Proof. Using the Stinespring dilation theorem, one can prove that (see [27], Corollary 4.19) for a channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, there exists a auxiliary Hilbert space \mathcal{H}_E (often known as the Environment), a unitary operator u on $\mathcal{H} \otimes \mathcal{H}_E$ and a fixed density operator β_E on \mathcal{H}_E such that

$$\mathcal{E}(x) = \text{Tr}_E(u(x \otimes \beta_E)u^*),$$

for every density operator x in $\mathcal{B}(\mathcal{H})$.

Now since fidelity is unchanged by unitary conjugation, using Corollary 6.9 we get,

$$\begin{aligned} F(\mathcal{E}(\sigma), \mathcal{E}(\rho)) &= F(\text{Tr}_E(u(\sigma \otimes \beta_E)u^*), \text{Tr}_E(u(\rho \otimes \beta_E)u^*)) \\ &\geq F(u(\sigma \otimes \beta_E)u^*, u(\rho \otimes \beta_E)u^*) \\ &= F(\sigma \otimes \beta_E, \rho \otimes \beta_E) \\ &= F(\sigma, \rho). \end{aligned}$$

□

6.3 Relationship with the von Neumann entropy

In this section we explore connection of fidelity with another distinguishability measure, namely the von Neumann relative entropy of two density operators. Recall that for two density operators σ, ρ , the (von Neumann) relative entropy of σ with respect ρ is defined as

$$D(\sigma, \rho) = \begin{cases} \text{Tr}(\sigma(\log \sigma - \log \rho)) & \text{if } \text{Supp}(\sigma) \subseteq \text{Supp}(\rho) \\ \infty & \text{otherwise.} \end{cases}$$

Note that relative entropy is a widely studied measure in information theory whose origin goes back to the classical Shannon entropy. Consult [55] for further discussion. The next proposition relates fidelity to the relative entropy of density operators.

Proposition 6.11. *For density operators σ, ρ , we have the following relation:*

$$D(\sigma, \rho) \geq -2 \log F(\sigma, \rho).$$

Proof. The sketch of the proof can be found in [21] but for completeness we write out the proof here.

For any $\alpha \in (0, 1) \cup (1, \infty)$ let $D_\alpha(\sigma, \rho)$ be the α -Quantum Rényi Divergence for σ, ρ , defined as:

$$D_\alpha(\sigma, \rho) = \begin{cases} \frac{1}{\alpha-1} \log(\text{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]) & \text{if } \alpha < 1 \text{ and } \sigma \not\sim \rho \\ \infty & \text{otherwise.} \end{cases}$$

It has been shown in [40] that $\lim_{\alpha \rightarrow 1} D_\alpha(\sigma, \rho) = D(\sigma, \rho)$. Also putting $\alpha = \frac{1}{2}$, we get

$$D_{\frac{1}{2}}(\sigma, \rho) = -2 \log F(\sigma, \rho).$$

It was also shown in [40], that the function $\alpha \mapsto D_\alpha$ is a monotonically non-decreasing function in α . Using this fact we have,

$$D(\sigma, \rho) = D_1(\sigma, \rho) \geq D_{\frac{1}{2}}(\sigma, \rho) = -2 \log F(\sigma, \rho).$$

□

6.4 Properties and applications of Bures contractions

In this section we revisit the Bures contractive maps. In the special case where the underlying space is of finite dimensional C^* -algebra, namely the matrix algebra \mathcal{M}_d for some $d \in \mathbb{N}$, which is the appropriate ambient space for quantum information theory, Bures contractive channels constitute a crucial class of maps in the space of quantum operations. On the similar line of study see [45] for strictly contractive quantum channels with respect to the trace metric.

Recall that a map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called a Bures contraction if

$$F(\mathcal{E}(\sigma), \mathcal{E}(\rho)) > F(\sigma, \rho) \text{ or equivalently } d_B^\tau(\mathcal{E}(\sigma), \mathcal{E}(\rho)) < d_B^\tau(\sigma, \rho),$$

for every density elements ρ, σ .

6.4.1 Primitivity and fully indecomposability

Recall that in the Section 4.5 of the Chapter 4, it has been shown that a Bures contraction on a C^* -algebra is an irreducible linear map. Using the Kraus decomposition 6.2 for a quantum channel we can prove even a stronger assertion.

To begin with the concept of fully indecomposable linear maps on \mathcal{M}_d , we first note that in a classical setting, a positive semidefinite matrix $a \in \mathcal{M}_d$ is called *fully indecomposable*, if there do not exist permutation matrices $p, q \in \mathcal{M}_d$ and square matrices a_1, a_2 without zero rows and columns, such that

$$qap = \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix}.$$

This is stronger than being *irreducible*, where we say a positive definite matrix $a \in \mathcal{M}_d$ is *irreducible*, if there does not exist a permutation matrix p and square matrices a_1, a_2 such that

$$pap^t = \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix},$$

where p^t denotes the transpose of the matrix p . Evidently, a *fully indecomposable* positive semidefinite matrix is automatically *irreducible* by setting $q = p$.

With these definitions in the classical case in mind, we set out to formulate similar definitions for positive linear maps on \mathcal{M}_d .

Definition 6.12. *A positive linear map $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called irreducible, if there exist no nontrivial projection $p \in \mathcal{M}$ such that $\Phi(p) \leq \lambda p$, for any $\lambda > 0$.*

The abstract definition of irreducibility of a linear map on a C^* -algebra given in the Chapter 4 is equivalent to the definition given above. The following definition is inspired from [30].

Definition 6.13. *A positive linear map $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called fully indecomposable, if there exist no nontrivial projections p, q of same dimension such that $\Phi(p) \leq \lambda q$, for $\lambda > 0$.*

In other words for a fully indecomposable Φ , there do not exist projections p, q of same dimension such that

$$\Phi(p\mathcal{M}_d p) \subseteq q\mathcal{M}_d q.$$

Now we are ready to state and prove the following theorem which is stronger than what we have obtained in a general C^* -algebra case in Theorem 4.23.

Theorem 6.14. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital quantum channel which is a Bures contraction. Then \mathcal{E} is fully indecomposable.*

Proof. If for two projections p, q of same dimension,

$$\mathcal{E}(p) \leq \lambda q,$$

for $\lambda > 0$, Then

$$\begin{aligned} 0 &\leq (1 - q)\mathcal{E}(p)(1 - q) \\ &\leq (1 - q)\lambda q(1 - q) \\ &= 0. \end{aligned}$$

And we obtain $(1 - q)\mathcal{E}(p)(1 - q) = 0$. Since \mathcal{E} is a channel, using the Kraus decomposition of \mathcal{E} we get

$$\begin{aligned} 0 &= (1 - q)\mathcal{E}(p)(1 - q) \\ &= \sum_{j=1}^k (1 - q)a_j p a_j^* (1 - q) \\ &= \sum_{j=1}^k [(1 - q)a_j p][(1 - q)a_j p]^*. \end{aligned}$$

Which yields

$$(1 - q)a_j p = 0 \text{ that is } a_j p = q a_j p, \forall j = 1, \dots, k.$$

Now ,

$$\begin{aligned}
\mathcal{E}(p) &= \sum_{j=1}^k a_j p a_j^* \\
&= \sum_{j=1}^k q a_j p p a_j^* q \\
&= q \mathcal{E}(p) q \\
&\leq q \|\mathcal{E}(p)\| q \\
&\leq q.
\end{aligned}$$

Since p, q are projections of same dimension, $\text{Tr}(p) = \text{Tr}(q)$ and \mathcal{E} is trace preserving, we get

$$0 \leq \text{Tr}(q - \mathcal{E}(p)) = \text{Tr}(q) - \text{Tr}(p) = 0.$$

By faithfulness of trace, we get $\mathcal{E}(p) = q$. This violates the Bures contraction property because

$$F\left(\frac{p}{\text{Tr}(p)}, \frac{1}{d}1\right) = \frac{1}{\sqrt{d}} \text{Tr}(p).$$

And also since \mathcal{E} is unital,

$$F\left(\mathcal{E}\left(\frac{p}{\text{Tr}(p)}\right), \frac{1}{d}1\right) = \frac{\text{Tr}(q)}{\sqrt{d}}.$$

And both of the above quantities are same. Hence the theorem. \square

The technique of the proof of the above theorem is inspired from [5].

Theorem 6.14 immediately implies the following corollary.

Corollary 6.15. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital quantum channel that is Bures contractive. Then \mathcal{E} is irreducible.*

The converse of the Corollary 6.15 is not true as the following example shows.

Example 6.16. Let $\mathcal{E} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ be defined as

$$\mathcal{E}(a) = \frac{1}{2}xax^* + \frac{1}{2}yay^*$$

for $a \in \mathcal{M}_2$ where $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

Clearly \mathcal{E} is a unital channel. It is not hard to see that \mathcal{E} is irreducible because no projection is a fixed point. In fact the fixed point set of \mathcal{E} is only \mathbb{C} .

Now if $\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\sigma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $F(\rho, \sigma) = 0$. As $\mathcal{E}(\rho) = \sigma$ and $\mathcal{E}(\sigma) = \rho$, $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = 0$ as well.

So we have an irreducible channel which is not Bures contractive.

What is evident though from the Theorem 6.17 below that the Cesaro sum of an irreducible positive linear map is Bures contractive. Note that,

given a unital channel $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$, the map

$$P = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k$$

exists and P is unital trace preserving and completely positive as well. Also it is well known that $\text{Ran } P = \text{Fix } \Phi$, the fixed point set of Φ i.e P is an idempotent onto the fixed point space of Φ .

Theorem 6.17. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital channel. Then \mathcal{E} is irreducible if and only if the Cesaro Sum, P is Bures contractive.*

Proof. Let us assume that \mathcal{E} is irreducible. Note that the fixed points of \mathcal{E} , $\text{Fix } \mathcal{E} = \mathbb{C}1$. Indeed, if $\mathcal{E}(a) = a$, for any $a \in \mathcal{M}_d$, then using the positivity and unitality of \mathcal{E} , without loss of generality we can assume that a is positive. Also $\mathcal{E}(a^2) \geq \mathcal{E}(a)\mathcal{E}(a)$ by Schwarz inequality of \mathcal{E} . Using the trace preservation property of \mathcal{E} and the relation $\mathcal{E}(a) = a$, we get $\mathcal{E}(a^2) = a^2$. Similarly $\mathcal{E}(a^n) = a^n = (\mathcal{E}(a))^n$ for every $n \geq 1$. So for any polynomial f , we have $\mathcal{E}(f(a)) = f(\mathcal{E}(a)) = f(a)$. Now let $a = \sum_{i=1}^k \lambda_i p_i$ be the spectral decomposition of a where p_i 's are the spectral projections onto the eigenspace corresponding the eigenvalue λ_i . Since the spectral projections are obtained from certain polynomials of a , we get $\mathcal{E}(p_i) = p_i$ for every i . But this violates the irreducibility of \mathcal{E} . Hence \mathcal{E} can not admit any non-trivial fixed point.

Hence $\text{Ran } P = \mathbb{C}$. Also by the Theorem 5.1 in [16], we have $\mathcal{M}_d = \text{Ker}(\mathcal{E} - Id) \oplus \text{Ran}(\mathcal{E} - Id) = \mathbb{C}1 \oplus \text{Ker}(P)$. Here Id is the identity operator on \mathcal{M}_d . If $y \in \text{Ran}(\mathcal{E} - Id)$ then $y = (\mathcal{E} - Id)(x)$ for some $x \in \mathcal{M}_d$. Also we have $P(y) = 0$. Since \mathcal{E} is trace preserving $\text{Tr}(y) = 0$. Hence if $y \in \text{Ran}(\mathcal{E} - Id)$ then $P(y) = 0 = \text{Tr}(y)\frac{1}{d}$. Now for $y \in \text{Ker}(\mathcal{E} - Id) = \mathbb{C}1$, we have $y = c1$ for some complex number c . Taking trace we get $\text{Tr}(y) = cd$ and hence $c = \frac{\text{Tr}(y)}{d}$. And we get $P(y) = y = \frac{\text{Tr}(y)}{d}1$. So we find for all $x \in \mathcal{M}_d$, $P(x) = \frac{\text{Tr}(x)}{d}1$ and this is a Bures contractive.

Conversely, Let P be Bures contractive. If possible let us assume that \mathcal{E} be reducible. Then using the Kraus decomposition and the proof of the Theorem 6.14, one can show that there exists a non-trivial projection $q \in \mathcal{M}_d$ such that $\mathcal{E}(q) = q$. By the definition of P , we get $P(q) = q$. This is a contradiction as being a Bures contraction, P does not admit a non-trivial fixed point. \square

In the context of quantum information theory, Theorem 6.14 and the Corollary 6.15 are of important significance. Using Theorem 12 in [7] the following proposition can be put forward.

Proposition 6.18. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital quantum channel with*

the Kraus operators given by $\mathcal{E}(x) = \sum_{j=1}^n a_j x a_j^*$, for $x \in \mathcal{M}_d$. If \mathcal{E} is Bures contractive, then all of the following assertions hold.

- (1) There exist no projection $0 < p < 1$ such that $\mathcal{E}(p) = p$,
- (2) The algebra \mathcal{A} generated by the operators a_i and a_i^* is irreducible i.e \mathcal{E} has no invariant subspace $S \in \mathbb{C}^d$,
- (3) $\mathcal{A} = \mathcal{M}_d$, and
- (4) If $\rho \in \mathcal{A}'$ i.e if ρ commutes with all elements of \mathcal{A} , then $\rho = \mathbb{C}1$.

The following lemma reveals an interesting property of powers of a Bures contractive channels.

Lemma 6.19. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital Bures contractive channel. Then \mathcal{E}^n is fully indecomposable for every $n \in \mathbb{N}$.*

Proof. It is essential to note that if \mathcal{E} is Bures contractive, then \mathcal{E}^n is Bures contractive as well for every $n \in \mathbb{N}$. To see this, let $\rho, \sigma \in \mathcal{D}(\mathcal{M}_d)$. Then by the monotonicity of fidelity function given in Theorem 2.14, we have for any $n \in \mathbb{N}$,

$$F(\mathcal{E}^n(\sigma), \mathcal{E}^n(\rho)) \geq F(\mathcal{E}^{n-1}(\sigma), \mathcal{E}^{n-1}(\rho)) \geq \dots \geq F(\mathcal{E}(\sigma), \mathcal{E}(\rho)) > F(\sigma, \rho).$$

So $F(\mathcal{E}^n(\sigma), \mathcal{E}^n(\rho)) > F(\sigma, \rho)$, resulting \mathcal{E}^n to be Bures contractive for any $n \in \mathbb{N}$. Now by the Theorem 6.14, the assertion follows. \square

The next definition is well known (see [57], Theorem 6.7).

Definition 6.20. *For a positive linear map $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$, if Φ^n is irreducible for every $n \in \mathbb{N}$, then Φ is called primitive.*

Corollary 6.21. *A unital Bures contractive channel on \mathcal{M}_d is primitive.*

6.4.2 Strict positivity

A positive map $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called *strictly positive* if $\Phi(a)$ is positive and invertible for every positive element $a \in \mathcal{M}_d$. In the classical setting, that is, in the case of matrices, according to Frobenius: a matrix is strictly positive if and only if all entries are strictly positive. The non-tractable structure theory of positive maps makes it harder to characterise strictly positive linear maps on non commutative operator algebras. Consult [16], [13] for more discussions on strictly positive maps. We need a definition before we proceed to the main theorem of this section.

Definition 6.22. *Let $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a positive linear map. Then Φ is*

called ‘strictly kernel reducing’ if for every $a \geq 0$,

$$\dim \text{Ker } \Phi(a) < \dim \text{Ker } (a).$$

See [30] for various interesting properties of strictly kernel reducing maps.

We are mainly interested in the following proposition (Proposition 1.25) from [30].

Proposition 6.23. ([30]) *A positive map $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is strictly kernel reducing if and only if it is fully indecomposable.*

Now we are ready to prove the main theorem in this section.

Theorem 6.24. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital Bures contractive channel. Then \mathcal{E}^{d-1} is strictly positive.*

Proof. Given any projection q , there exists a rank one projection p such that $p \leq q$. Hence for any positive linear map Φ , $\Phi(p) \leq \Phi(q)$. So if $\Phi(p)$ is invertible, then so is $\Phi(q)$. Now if the spectral decomposition of positive element $a \in \mathcal{M}_d$ be given by

$$a = \sum_{j=1}^k \lambda_j p_j,$$

where p_j 's are spectral projections onto the the eigenspace corresponding to the eigenvalue λ_j , then $\Phi(a) = \sum_{j=1}^k \lambda_j \Phi(p_j)$. Now if for every $\xi \in \mathbb{C}^d$,

$\langle \Phi(p_j)\xi, \xi \rangle > 0$, then $\langle \Phi(a)\xi, \xi \rangle > 0$. Hence we will show that for any rank one projection p , $\mathcal{E}^{d-1}(p)$ is invertible.

In light of the Proposition 6.23 and Lemma 6.19, we know \mathcal{E}^n is strictly kernel reducing for every $n \in \mathbb{N}$. Now using the definition of strictly kernel reducing maps we get the following strict inequality for any rank one projection p :

$$\dim \text{Ker } \mathcal{E}^{d-1}(p) < \dots < \dim \text{Ker } \mathcal{E}^2(p) < \dim \text{Ker } \mathcal{E}(p) < \dim \text{Ker } (p).$$

Now since $\dim \text{Ker } (p) = d-1$ and the dimension is a non negative integral function, it is evident that $\dim \text{Ker } \mathcal{E}^{d-1}(p) = 0$. Which shows that $\mathcal{E}^{d-1}(p)$ is invertible for every rank one projection p . Hence \mathcal{E}^{d-1} is strictly positive.

□

Corollary 6.25. *Every unital Bures contraction $\mathcal{E} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ is strictly positive.*

6.4.3 Quantum error correction

Quantum error correction ([35],[31]) is one of the most widely studied topics in information theory. Given a quantum channel Φ , if there exists a subset \mathcal{S} (known as *code*) of the state space and another channel \mathcal{R} such that

$\mathcal{R}(\Phi(\rho)) = \rho, \forall \rho \in \mathcal{S}$, then Φ is called correctable on the set \mathcal{S} . It was observed by Petz in [43] that preservation of certain distinguishability measures between density operators is a sufficient condition for correctability on those operators. Quantum error correction has been a widely studied topic in the literature of quantum information theory.

The following lemma shows the impossibility of error correction on the Bures contractive channels.

Lemma 6.26. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a quantum channel. If \mathcal{E} is Bures contractive, then there is no non-empty subset \mathcal{S} of density operators for which there exists a channel \mathcal{R} such that $\mathcal{R} \circ \mathcal{E}(\rho) = \rho, \forall \rho \in \mathcal{S}$.*

Proof. If possible, suppose such a set \mathcal{S} and the corresponding channel \mathcal{R} exist. Then using the monotonicity property of fidelity under a channel we have for any $\rho, \sigma \in \mathcal{S}$,

$$F(\rho, \sigma) = F(\mathcal{R} \circ \mathcal{E}(\rho), \mathcal{R} \circ \mathcal{E}(\sigma)) \geq F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) > F(\rho, \sigma).$$

Hence all the inequality becomes equality and we have

$$F(\rho, \sigma) = F(\mathcal{E}(\rho), \mathcal{E}(\sigma)).$$

And this violates the Bures contractive property. □

Since the set of all Bures contractive maps is dense in the set of quantum channels (Proposition 4.3), the above lemma rules out the possibility of perfect error correction for majority of the quantum channels. In light of this simple lemma one can guess the importance of *approximate error correction* which has also been a topic of interest recently, see for example [49].

6.4.4 Noiseless subsystems

Given a quantum channel \mathcal{E} on $\mathcal{B}(\mathcal{H})$, where \mathcal{H} is a finite dimensional Hilbert space, the “Noiseless Subsystem” (NS) protocol (see [29],[35]) seeks subsystems \mathcal{H}^B (with $\dim \mathcal{H}^B > 1$) of the full system \mathcal{H} such that $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ and such that $\forall \rho^A, \forall \rho^B$ there exists a γ^A such that

$$\mathcal{E}(\rho^A \otimes \rho^B) = \gamma^A \otimes \rho^B.$$

Here we write ρ^A (resp. ρ^B) for operators on $\mathcal{B}(\mathcal{H}^A)$ (resp. $\mathcal{B}(\mathcal{H}^B)$).

If $\forall x \in \mathcal{B}(\mathcal{H})$, $\mathcal{E}(x) = \sum_{i=1}^n a_i x a_i^*$ is a Kraus decomposition of \mathcal{E} , then the *-algebra generated by a_i 's is called the *interaction algebra*. If \mathcal{E} is unital and \mathcal{A} is the *-algebra generated by the a_i 's, then Proposition 4.12 asserts that the fixed point set $\text{Fix } \mathcal{E}$ is equal to \mathcal{A}' , the commutant of the algebra \mathcal{A} . See also the results related to the discussion in [37]. If \mathcal{E} is not unital,

then more finer analysis is needed.

Suppose \mathcal{E} has a non-trivial fixed point ρ and let $\mathcal{K} = \text{Supp}(\rho)$. If p is a projection onto \mathcal{K} , then it is not hard to see ([7], Lemma 2) that \mathcal{K} is an invariant subspace for \mathcal{E} and hence it satisfies $\mathcal{E}(p) = p\mathcal{E}(p)p$. Now, let

$$\mathcal{A}'_p = \{x : [x, pa_i p] = 0 = [x, pa_i^* p]\} \text{ for all } i.$$

Since \mathcal{A}_p , the $*$ -algebra generated by $\{pa_1 p, \dots, pa_n p\}$, is a finite dimensional C^* -algebra, so is its commutant \mathcal{A}'_p . By Wedderburn theorem, it follows that there exists a unitary U on $p\mathcal{H}$ such that

$$U\mathcal{A}'_p U^* = \oplus_k (\mathbb{1}_{m_k} \otimes \mathcal{M}_{n_k}), \quad m_k, n_k \geq 1.$$

And by the von Neumann double commutant theorem the commutant of \mathcal{A}' has the following structure

$$\mathcal{A}_p = \mathcal{A}''_p = \oplus_k (\mathcal{M}_{m_k} \otimes \mathbb{1}_{n_k}).$$

Also we have the corresponding isomorphism

$$\mathcal{H} \cong \oplus_k \mathcal{H}^{A_k} \otimes \mathcal{H}^{B_k}, \text{ where } \dim(\mathcal{H}^{A_k})=m_k \text{ and } \dim(\mathcal{H}^{B_k})=n_k.$$

It follows that (see [12]) each of the subsystems \mathcal{H}^{B_k} is a *noiseless subsystem* for \mathcal{E} . It is also not hard to see that every noiseless subsystem arises this way that is via a fixed point of \mathcal{E} whose support is that given subsystem. Now

with these preliminary background in hand we are ready to state the main theorem in this section.

Theorem 6.27. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$, given by $\mathcal{E}(x) = \sum_{i=1}^n a_i x a_i^*$, $\forall x \in \mathcal{M}_d$ be a Bures contractive channel. Then \mathcal{E} does not admit any non-trivial noiseless subsystems.*

Proof. Let \mathcal{H}^B be a noiseless subsystem. Hence we have a decomposition of the Hilbert space $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$, for some subspace \mathcal{K} . Call \mathcal{H}^{AB} the subspace $\mathcal{H}^A \otimes \mathcal{H}^B$. Let p be the projection onto \mathcal{H}^{AB} . Then if we restrict the channel on $\mathcal{B}(\mathcal{H}^{AB})$, then it satisfies (see the discussion after Theorem 2 in [12]),

$$\mathcal{E}(p(\cdot)p) = \mathcal{E}_A \otimes Id_B, \quad (6.5)$$

where \mathcal{E}_A is a quantum channel on $\mathcal{B}(\mathcal{H}^A)$ and Id_B is the identity channel on $\mathcal{B}(\mathcal{H}^B)$.

Now we will see that a Bures contractive channel can not be of this form given in Equation (6.5). Indeed, if we choose two orthogonal positive elements $\rho^B, \sigma^B \in \mathcal{B}(\mathcal{H}^B)$, that is to say $F(\sigma^B, \rho^B) = 0$, and an element $\beta^A \in \mathcal{B}(\mathcal{H}^A)$, then it is easy to see using Proposition 6.2 that

$$F(\beta^A \otimes \sigma^B, \beta^A \otimes \rho^B) = 0.$$

Also we have,

$$\begin{aligned} F(\mathcal{E}(\beta^A \otimes \sigma^B), \mathcal{E}(\beta^A \otimes \rho^B)) &= F(\mathcal{E}_A \otimes Id_B(\beta^A \otimes \sigma^B), \mathcal{E}_A \otimes Id_B(\beta^A \otimes \rho^B)) \\ &= F(\mathcal{E}_A(\beta^A), \mathcal{E}_A(\beta^A))F(\sigma^B, \rho^B) \\ &= 0. \end{aligned}$$

This contradicts the Bures contractive property of \mathcal{E} on the subalgebra $\mathcal{B}(\mathcal{H}^{AB})$.

Hence \mathcal{E} does not admit any noiseless subsystem. \square

Chapter 7

Conclusion and Future

Research

The main achievements of this dissertation have been the development of the study of a quantum theoretic measure specifically from the operator algebraic viewpoint. The various functional analytic methods used to obtain the main results certainly enrich the existing theory of distinguishability measures in the quantum information theory. Reciprocally, taking inspirations from the quantum physics and applying them in operator algebra theory, this thesis strengthens the symbiotic relations between these two subjects.

7.1 Summary of main results

One of the main results proved in this dissertation is the following theorem:

Theorem 7.1. *Assume that \mathcal{A} is a finite and quasi-transitive C^* -algebra with a faithful trace τ . If a surjective Schwarz map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$ preserves τ -fidelity, then \mathcal{E} is an automorphism of \mathcal{A} .*

The results related to the Bures contractions are also interesting from purely operator algebraic point of view. For example the following theorem:

Theorem 7.2. *Every unital Bures contractive Schwarz map \mathcal{E} on a tracial C^* -algebra is irreducible. Moreover, the peripheral spectrum of \mathcal{E} is the trivial subgroup of the circle group \mathbb{T} , that is $\text{Sp}_p \mathcal{E} \cap \mathbb{T} = \{1\}$. Furthermore, the multiplicative domain of such a map, $\mathcal{M}_{\mathcal{E}} = \mathbb{C}1$, that is the scalar multiple of the unit 1.*

In the interests of quantum information theory, recalling that the Bures contractive channels are dense in the set of quantum channels, the following combined theorem is also a useful result.

Theorem 7.3. *Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital Bures contractive channel. Then \mathcal{E} does not admit any non-trivial noiseless subsystems and also there*

can not be any correctable codes for \mathcal{E} . Furthermore, \mathcal{E} is fully indecomposable and \mathcal{E}^{d-1} is strictly positive.

7.2 Future research

There are a number of unanswered questions that arise from the study undertaken in this dissertation. A few of these questions are recorded below.

- (1) Given any τ -preserving (completely) positive map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$, characterise the pairs of density operators σ, ρ such that

$$F_\tau(\sigma, \rho) = F_\tau(\Phi(\sigma), \Phi(\rho)).$$

That is the descriptive characterisation of the space of isometry for Φ . Also it is not clear that preservation of the other distinguishability measures like entropy, quantum f-divergence etc. implies preservation of fidelity or not.

- (2) If the two channels $\Phi, \Psi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ are Bures contractive, is it true that $\Phi \otimes \Psi$ is Bures contractive? In particular if $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is Bures contractive, then for any $n \in \mathbb{N}$, is $\mathcal{E}^{\otimes n}$ a Bures contractive channel? If not, are there examples?

(3) The minimal output entropy of a quantum channel Φ , is defined to be

$$\mathfrak{s}_{\min}(\Phi) = \min\{\mathfrak{s}(\Phi(\rho)) : \rho \in \mathcal{D}(\mathcal{M}_d)\},$$

where for any density element ρ , $\mathfrak{s}(\rho) = -\text{Tr}(\rho \ln \rho)$.

It can be observed that for a channel Φ , $\mathfrak{s}_{\min}(\Phi) = 0$ if and only if there exists a rank one projection p such that $\Phi(p)$ is again a rank one projection. Since a Bures contraction does not map rank one projections to rank one projections (Theorem 6.14), $\mathfrak{s}_{\min}(\Phi) > 0$ for any Bures contraction. So the “additivity conjecture”, which says that for any two channels Φ, Ψ ,

$$\mathfrak{s}_{\min}(\Phi \otimes \Psi) \stackrel{?}{=} \mathfrak{s}_{\min}(\Phi) + \mathfrak{s}_{\min}(\Psi),$$

comes into discussion. The additivity conjecture is known to be false (see [25]) but the counterexamples are difficult to construct.

Given a Bures contractive channel Φ , if we could select a channel Ψ such that $\Phi \otimes \Psi$ maps a rank one projection to a rank one projection, then $\mathfrak{s}_{\min}(\Phi \otimes \Psi) = 0$, however, $\mathfrak{s}_{\min}(\Phi) > 0$, giving the strict inequality of the “additivity conjecture”. Such rank one projections (preferably entangled rank one projections) might be found in the multiplicative domain of $\Phi \otimes \Psi$, for a suitable choice of Ψ .

Bibliography

- [1] Charles A. Akemann, Joel Anderson, and Gert K. Pedersen. Triangle inequalities in operator algebras. *Linear and Multilinear Algebra*, 11(2):167–178, 1982.
- [2] Peter M. Alberti. Playing with fidelities. *Rep. Math. Phys.*, 51(1):87–125, 2003.
- [3] Erik M. Alfsen and Frederic W. Shultz. *State spaces of operator algebras*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2001. Basic theory, orientations, and C^* -products.
- [4] C. Anantharaman-Delaroche and J.-F. Havet. On approximate factorizations of completely positive maps. *J. Funct. Anal.*, 90(2):411–428, 1990.

- [5] O. Bratteli, P. E. T. Jorgensen, A. Kishimoto, and R. F. Werner. Pure states on \mathcal{O}_d . *J. Operator Theory*, 43(1):97–143, 2000.
- [6] Donald Bures. An extension of Kakutani’s theorem on infinite product measures to the tensor product of semifinite w^* -algebras. *Trans. Amer. Math. Soc.*, 135:199–212, 1969.
- [7] D. Burgarth, G. Chiribella, V. Giovannetti, P. Perinotti, and K. Yuasa. Ergodic and mixing quantum channels in finite dimensions. *New J. Phys.*, 15(July):073045, 33, 2013.
- [8] Hisashi Choda. An extremal property of the polar decomposition in von Neumann algebras. *Proc. Japan Acad.*, 46:341–344, 1970.
- [9] Man Duen Choi. A Schwarz inequality for positive linear maps on C^* -algebras. *Illinois J. Math.*, 18:565–574, 1974.
- [10] Man Duen Choi. Some assorted inequalities for positive linear maps on C^* -algebras. *J. Operator Theory*, 4(2):271–285, 1980.
- [11] Man Duen Choi and Edward G. Effros. Injectivity and operator spaces. *J. Functional Analysis*, 24(2):156–209, 1977.

- [12] Man-Duen Choi and David W. Kribs. Method to find quantum noiseless subsystems. *Phys. Rev. Lett.*, 96:050501, Feb 2006.
- [13] David E. Evans and Raphael Høegh-Krohn. Spectral properties of positive maps on C^* -algebras. *J. London Math. Soc. (2)*, 17(2):345–355, 1978.
- [14] Thierry Fack. Sur la notion de valeur caractéristique. *J. Operator Theory*, 7(2):307–333, 1982.
- [15] Thierry Fack and Hideki Kosaki. Generalized s -numbers of τ -measurable operators. *Pacific J. Math.*, 123(2):269–300, 1986.
- [16] D. R. Farenick. Irreducible positive linear maps on operator algebras. *Proc. Amer. Math. Soc.*, 124(11):3381–3390, 1996.
- [17] Douglas Farenick. *Fundamentals of functional analysis*. Universitext. Springer, Cham, 2016.
- [18] Douglas Farenick, Samuel Jaques, and Mizanur Rahaman. The fidelity of density operators in an operator-algebraic framework. *J. Math. Phys.*, 57(10):102202, 15, 2016.

- [19] Douglas R. Farenick. *Algebras of linear transformations*. Universitext. Springer-Verlag, New York, 2001.
- [20] Douglas R. Farenick and S. Mahmoud Manjegani. Young's inequality in operator algebras. *J. Ramanujan Math. Soc.*, 20(2):107–124, 2005.
- [21] Omar Fawzi and Renato Renner. Quantum conditional mutual information and approximate Markov chains. *Comm. Math. Phys.*, 340(2):575–611, 2015.
- [22] L. Terrell Gardner. Linear maps of C^* -algebras preserving the absolute value. *Proc. Amer. Math. Soc.*, 76(2):271–278, 1979.
- [23] Ulrich Groh. The peripheral point spectrum of Schwarz operators on C^* -algebras. *Math. Z.*, 176(3):311–318, 1981.
- [24] Ulrich Groh. On the peripheral spectrum of uniformly ergodic positive operators on C^* -algebras. *J. Operator Theory*, 10(1):31–37, 1983.
- [25] M. B. Hastings. Superadditivity of communication capacity using entangled inputs. *Nat Phys*, 5(4):255–257, Apr 2009.
- [26] Masahito Hayashi. *Quantum information*. Springer-Verlag, Berlin, 2006. An introduction, Translated from the 2003 Japanese original.

- [27] Teiko Heinosaari and Mário Ziman. *The mathematical language of quantum theory*. Cambridge University Press, Cambridge, 2012. From uncertainty to entanglement.
- [28] Dinh Trung Hoa, Hiroyuki Osaka, and Ho Minh Toan. On generalized Powers-Størmer’s inequality. *Linear Algebra Appl.*, 438(1):242–249, 2013.
- [29] John A. Holbrook, David W. Kribs, and Raymond Laflamme. Noiseless subsystems and the structure of the commutant in quantum error correction. *Quantum Inf. Process.*, 2(5):381–419 (2004), 2003.
- [30] Martin Idel. On the structure of positive maps. Master’s thesis, Technische Universität München, 2013.
- [31] Nathaniel Johnston and David W. Kribs. Generalized multiplicative domains and quantum error correction. *Proc. Amer. Math. Soc.*, 139(2):627–639, 2011.
- [32] Richard Jozsa. Fidelity for mixed quantum states. *J. Modern Opt.*, 41(12):2315–2323, 1994.

- [33] Richard V. Kadison. Isometries of operator algebras. *Ann. Of Math.* (2), 54:325–338, 1951.
- [34] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*, volume 15 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Elementary theory, Reprint of the 1983 original.
- [35] Emanuel Knill, Raymond Laflamme, and Lorenza Viola. Theory of quantum error correction for general noise. *Phys. Rev. Lett.*, 84(11):2525–2528, 2000.
- [36] K. Kraus. General state changes in quantum theory. *Ann. Physics*, 64:311–335, 1971.
- [37] David W. Kribs. Quantum channels, wavelets, dilations and representations of \mathcal{O}_n . *Proc. Edinb. Math. Soc.* (2), 46(2):421–433, 2003.
- [38] Seyed Mahmoud Manjegani. *Inequalities in operator algebras*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–The University of Regina (Canada).

- [39] Lajos Molnár. Fidelity preserving maps on density operators. *Rep. Math. Phys.*, 48(3):299–303, 2001.
- [40] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum Rényi entropies: a new generalization and some properties. *J. Math. Phys.*, 54(12):122203, 20, 2013.
- [41] Edward Nelson. Notes on non-commutative integration. *J. Functional Analysis*, 15:103–116, 1974.
- [42] Vern Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.
- [43] Dénes Petz. Sufficiency of channels over von Neumann algebras. *Quart. J. Math. Oxford Ser. (2)*, 39(153):97–108, 1988.
- [44] Robert T. Powers and Erling Størmer. Free states of the canonical anticommutation relations. *Comm. Math. Phys.*, 16:1–33, 1970.
- [45] Maxim Raginsky. Strictly contractive quantum channels and physically realizable quantum computers. *Phys. Rev. A*, 65:032306, 2002.

- [46] David Reeb, Michael J. Kastoryano, and Michael M. Wolf. Hilbert's projective metric in quantum information theory. *J. Math. Phys.*, 52(8):082201, 33, 2011.
- [47] C. E. Rickart. Banach algebras with an adjoint operation. *Ann. of Math. (2)*, 47:528–550, 1946.
- [48] B. Russo and H. A. Dye. A note on unitary operators in C^* -algebras. *Duke Math. J.*, 33:413–416, 1966.
- [49] Benjamin Schumacher and Michael D. Westmoreland. Approximate quantum error correction. *Quantum Inf. Process.*, 1(1-2):5–12, 2002.
- [50] Erling Størmer. On the Jordan structure of C^* -algebras. *Trans. Amer. Math. Soc.*, 120:438–447, 1965.
- [51] Erling Størmer. *Positive linear maps of operator algebras*. Springer Monographs in Mathematics. Springer, Heidelberg, 2013.
- [52] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, *Operator Algebras and Non-commutative Geometry*, 5.

- [53] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. , Operator Algebras and Non-commutative Geometry, 6.
- [54] A. Uhlmann. The “transition probability” in the state space of a $*$ -algebra. *Rep. Mathematical Phys.*, 9(2):273–279, 1976.
- [55] J. Watrous. *Theory of quantum information*. draft manuscript, 2015.
- [56] Wilhelm Winter and Joachim Zacharias. Completely positive maps of order zero. *Münster J. Math.*, 2:311–324, 2009.
- [57] M. M. Wolf. *Quantum Channels and Operations- Guided Tour*. Online Lecture Notes, 2012.