SEMI-ANALYTICAL MODELING OF FLUID FLOW AND SOLID DEFORMATION IN HETEROGENEOUS RESERVOIRS USING UNIVERSAL BOUNDARY INTEGRAL APPROACHES

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Lei Xiao, candidate for the degree of Doctor of Philosophy in Petroleum Systems Engineering, has presented a thesis titled, *Semi-Analytical Modeling of Fluid Flow and Solid Deformation in Heterogeneous Reservoirs using Universal Boundary Integral Approaches*, in an oral examination held on November 24, 2016. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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ABSTRACT

In the Canadian oil and gas industry, heavy oil and unconventional reservoirs play a vital role in sustaining the production of crude oil and face tremendous technical challenges of enhancing recovery while reducing environmental footprints. Among various development technologies, the cold heavy oil production with sand (CHOPS) and hydraulic fracturing techniques have been widely applied in Western Canadian basins to unlock the unconsolidated heavy oil and tight formation reservoirs, respectively. Both technologies are proved to be efficient during the primary production period; however, they suffer sharp production decline and low recovery factor. Moreover, during the enhanced oil recovery (EOR) phase, the existence of wormholes and fractures will cause conformance problems and early polymer injection breakthrough. Therefore, better understanding of the wormholes distributions in CHOPS reservoirs and more reliable description of fractures in tight formations become crucial for Canadian operators to design EOR pilots and operate full field applications.

Reservoir characterization and modeling of CHOPS reservoirs with wormholes and fractured tight formations face numerous technical issues due to the existence of wormholes and fractures, which make the system no longer homogeneous and cause more complex problems in fluid flow and solid deformation. The boundary element method (BEM), which has been widely applied to solve for fluid flow and solid deformation problems, however, is limited to homogeneous systems. To inherit the merits of BEM such as near-analytical accuracy and negligible space and time subdivision, this research aims at developing boundary integral approaches, as extensions of BEM, to
heterogeneous reservoirs with arbitrary wormhole distributions, realistic fracture morphologies, and variation of geological facies. Moreover, field operators have observed that the in-situ stress state will be altered during hydraulic fracturing with associated stress shadow effects. The depletion-induced stress changes will also cause fracture closure and stress reorientation. Accordingly, the boundary integral approaches are further extended to solve for depletion-induced stress change due to poroelastic and mechanical effects in a heterogeneous reservoir with arbitrary distribution of porosity and permeability. The universal boundary integral approaches, which include the integration of various fundamental solutions along boundaries, have been proposed in this study. The developed boundary integral approaches are benchmarked by comparing with analytical solutions and numerical simulations. Representative cases are also presented to analyze complicated heterogeneous problems. Applications of the universal boundary integral approaches for heterogeneous systems are exemplified in the areas of wormhole coverage estimation, pressure and rate transient analysis of heterogeneous reservoirs, spatial-temporal stress evolution of multi-stage fractured horizontal wells, and evaluation of refracturing upside.
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DEDICATION

To Ming Lei
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CHAPTER 1 INTRODUCTION

1.1 Research Problem Statement

Three types of problems exist before, during, and maybe after the course of my PhD research, namely, reservoir engineering problems, integrated research problems, and communication problems between multiple disciplines.

In wormholed reservoirs with cold heavy oil production with sands (CHOPS) wells, regions covered by wormhole networks are of great importance to be characterized. Sanyal et al. (2011) intended to apply transient pressure analysis to detect growth of wormholes in pilot CHOPS wells. Shokri and Babadagli (2014) implemented a diffusion-limited aggregation algorithm with a dual-porosity approach into a numerical simulator to account for complex wormhole patterns and history matched production data. On the platform of commercial software, Istchenko and Gates (2014) implemented a dynamic wellbore module to model wormhole growth with multilateral wells and concluded that the reservoir heterogeneity could affect wormhole networks significantly. There is still lack of semi-analytical approaches in literature to quickly estimate wormhole coverage by analyzing the characteristics of CHOPS production data.

Heterogeneous problems become more and more critical due to increasing efforts of targeting sweet spots and better resolutions of geological models. Finite difference simulations are mostly practiced for heterogeneous reservoirs; however, they require fine gridding sizes and small time steps to obtain accurate pressure derivatives. Therefore, semi-analytical approaches become more advantageous because only boundaries need to be discretized and solutions are analytically obtained in integral forms. Accordingly,
semi-analytical approaches which can be compatible with complex geological models are desired.

Multistage fractured horizontal wells (MFHW) in unconventional reservoirs usually experience long transient flow period and sharp production decline. Poroelastic effects and mechanical loadings along propped hydraulic fractures have significant impact on the in-situ stress state. Field observations have confirmed stress evolution as pore pressure changes during production. A comprehensive review of field data from 15 oil fields was presented by Altmann et al. (2010) to show that the stress path can be as low as 0.34 and as high as 1.18. During the hydraulic fracturing process in the Barnett shale, increased minimum horizontal stress and microseismic events were reported to be related to the time-dependent fluid leak off; and the stress evolution difference of zipperfracs and simulfracs was believed to be affected by poroelasticity (Vermylen and Zoback, 2011). Nonetheless, measured data could only reveal the induced stress changes phenomenon but are unable to systematically characterize the spatio-temporal stress evolution. In the development of unconventional reservoirs with MFHW, theoretical models of predicting the induced stress by production become important to prevent fracture closure with proper completion design and optimize enhanced oil recovery (EOR) planning.

Reservoir Engineering Problems During the current low oil price environment, operators tend to reduce operating costs and invest in lower risk and higher return projects. Better analysis of existing data sets and more realistic reservoir characterization and simulations are necessary before millions of investment is approved to start up any new project. Production rates are the most available and reliable data obtained in the field. Traditional empirical type curves can only provide estimated reserves but very limited
information on reservoir characteristics. One of the most important questions for CHOPS reservoirs is: what are the sizes and coverage of wormholes developed from each CHOPS well (Chang et al., 2015)? By answering this question, I have more questions coming up. Rather than using material balance of sand production, is it possible to characterize wormholes through analysis of rate-time behavior (Fetkovich et al., 2006)? Is there a faster way than developing cumbersome numerical simulation models to match field production data with consideration of complicated physics of wormholes? What are the effects of complex wormhole configurations on the production rate signatures (Liu and Zhao, 2006)? On the other hand, due to low permeability of tight formation reservoirs, multistage fractured horizontal wells (MFHW) have relatively long transient flow period. Their rate behavior cannot be analyzed by empirical production decline type curves which are based on boundary dominated flow assumptions (Arps, 1944; Fetkovich, 1980). Therefore, some challenging questions are: can we estimate fracture characteristics from rate transient analysis (Wattenbarger et al., 1998; Nobakht et al., 2011; Zhao et al., 2016)? How complex can fractures develop and what are the effects of complex fracture networks on transient pressure and production responses (Zoback et al., 2003; Weng et al., 2011; Wu and Olson, 2016)? How much potential is left of a MFHW after tremendous pressure drop and sharp production decline (Vincent, 2011)? Are there remaining some intact reservoir regions between stages of fractures and how effectively can refracturing treatment increase production rate (Roussel and Sharma, 2013)?

**Integrated Research Problems** Hydraulic fracturing design and fracture characterization become more and more important in various projects in the oil and gas industry such as unconventional reservoir development, carbon dioxide sequestration,
caprock integrity of thermal heavy oil projects, and induced fractures by polymer injection. They require integrated teams of expertise from reservoir engineering, geomechanics, and geology to meet the technical challenges. Coupled problems of fluid flow and solid deformation are faced in the practice of hydraulic fracturing, infill well drilling, injection schemes of EOR projects, and refracturing treatments. Accordingly, several multi-disciplinary problems exist. How can hydraulic fracturing treatment change the in-situ stress state and what are the effects of stress shadow between multiple fractures (Cheng, 2012)? How do the natural fractures respond to effective stress changes and associated shear slippage (Vermylen and Zoback, 2011; Schutjens et al., 2012)? What are the controlling factors of fracture opening, fracture height, and fracture growth (Perkins and Kern, 1961; Geertsma and Deklerk, 1969; Nodgre, 1972)? How will the in-situ stress of reservoirs be altered due to production and injection activities (Warpinski and Branagan, 1989; Wolhart et al., 2007)? What is the spatio-temporal stress path change and how can it affect refracturing design and performance (Santarelli et al., 1998)? What are the mechanisms of depletion-induced fracture closure, injection-induced tensile fractures and shear-slip natural fractures (Vermylen and Zoback, 2011)?

Communication Problems between Multiple Disciplines Experts from different disciplines have their own languages of interpreting the real physics. However, seamless incorporation between disciplines has a long way to go. Geologists may desire that reservoir engineers keep the resolution of reservoir heterogeneity as high as possible during the upscaling process for reservoir simulation and they also expect feedbacks of featured flow patterns to confirm inter-well relations. Therefore, the question will be: what can reservoir engineers improve to respect detailed geology and avoid millions of
grid cells refinement? Meanwhile, between reservoir engineers and geomechanical engineers, more discussions are needed to reexamine some assumptions of modeling fluid flow and solid deformation. Is it appropriate for geomechanical engineers to assume steady state fluid flow in the estimation of stress changes due to mechanical loading and pore pressure change (Tao et al., 2010; Weng et al., 2011; Wu and Olson, 2016)? How does the transient fluid flow affect the induced stresses because of poroelasticity, given that transient fluid flow is the major flow regime for MFHW in unconventional reservoirs? Moreover, cells or grids used in geo-modeling, geomechanical models, and reservoir numerical simulations need more compatibility. Pixel-based geo-modeling uses millions of pixels to obtain high resolution of reservoir architectures (Deutsch, 2002; Strebelle, 2002). Geomechanical models usually adapt finite element and discrete element techniques while most reservoir numerical simulators are in favor of finite difference method (Marongiu-Porcu et al., 2015). Is there any approach that can describe the physics of fluid flow and solid deformation using the same gridding system which is also seamlessly compatible with geological models without losing details of geology? How can expertise from different disciplines communicate technical information in the same context of technical language and understanding?

1.2 Research Objectives

The general research objective is to seek innovative solutions for the above-stated research problems during the course of my PhD studies. Specific objectives include:

- To establish foundational knowledge from the perspective of reservoir engineering to tackle operational challenges in CHOPS reservoirs and tight formations with MFHW.
• To examine effects of complex wormhole configurations on production rate response and estimate wormhole coverage through analysis of rate-time behavior.

• To generate model-based pressure and production type curves for MFHW with consideration of fracture morphologies and local reservoir heterogeneity.

• To develop the same gridding system of boundary integral approaches to solve for reservoir engineering and geomechanics problems with compatibility to geological models.

• To solve coupled fluid flow and solid deformation problems of heterogeneous systems and calculate pore pressure and induced stress changes simultaneously by semi-analytical approaches.

• To bridge the gap among reservoir engineering, geomechanics and geology by evaluating poroelastic effects with transient fluid flow in heterogeneous systems.

• To make the theoretical models more applicable by tuning with field data. Modeling results with a higher level of confidence may be further conducted to evaluate EOR schemes for CHOPS reservoirs and tight formations.
1.3 Research Innovation

The research innovation manifests itself in the immediate and long term outcomes. Standing on the shoulders of giants, several key improvements have been achieved in this PhD study.

- The concept of effective wormhole coverage and wormhole intensity is originated by analyzing modeling results of complex wormhole networks and their effects on production rate behavior.

- The full sequence of fluid flow regimes including both transient and boundary dominated flows is implemented in the modeling of solid deformation due to poroelasticity. Stress changes due to mechanical loading and transient fluid flow are obtained by using the superposition principle.

- The universal boundary integral approaches integrating various fundamental solutions along boundaries are extended to heterogeneous systems through effective partition and coupling strategies.

- Transient pressure and production responses are correlated with the depletion-induced stress changes that render the prediction of spatio-temporal stress path change due to production and injection activities.

- Boundary integral approaches are proposed in this research as the common technical language between multiple disciplines to promote long term cross-disciplinary collaboration.
1.4 Dissertation Outline

This dissertation consists of six chapters. Chapter 1 is an introduction of the PhD research topic along with the thinking process and major objectives. Chapter 2 estimates the wormhole coverage of CHOPS wells by modeling fluid flow in wormholed reservoirs with a boundary integral approach. Chapter 3 addresses a boundary integral approach which is compatible with different geo-modeling techniques to model fluid flow in heterogeneous reservoirs. Chapter 4 investigates the coupled fluid flow and solid deformation problems due to poroelasticity in heterogeneous systems using the boundary integral approaches. Chapter 5 evaluates the refracturing upside in a Southern Saskatchewan Bakken pool with the developed analytical approaches in this work. Chapter 6 summarizes the major findings of the PhD research and outlines recommendations for field operations as well as future research.
CHAPTER 2  BOUNDARY INTEGRAL APPROACH FOR FLUID FLOW IN A CHOPS RESERVOIR WITH WORMHOLES

In wormholed reservoirs with cold heavy oil production with sand (CHOPS) wells, regions covered by wormhole networks are estimated by analyzing rate-time flow behavior along with pressure and sand production data. In this chapter, the concept of effective wormhole coverage is proposed as a proxy of the drainage region of CHOPS wells with complex wormhole networks. The wormhole intensity is defined as the total length of wormholes per unit effective wormhole coverage (Xiao and Zhao, 2012). A CHOPS flow model is developed by using the boundary integral approach (BIA) to account for various boundary conditions, wormhole morphologies, and dynamic wormhole growth (Xiao and Zhao, 2017). Transient pressure and rate responses calculated by the model are validated by comparing with analytical solutions and numerical simulations. Modeling results show that the effective wormhole coverage and wormhole intensity within the region dominate the characteristics of pressure and rate-time behavior, regardless of the detailed wormhole morphologies. Accordingly, dimensionless pressure and rate type curves are developed to match flow behavior such as wormhole-linear and transitional flow regimes to estimate effective wormhole coverage of CHOPS wells with available field data. This work extends the literature with fast type curve matching of pressure and rate-time behavior rather than generally practiced numerical simulation routines. A field case with rate and sand production data is successfully analyzed to show great potential of applying the proposed approach to characterize CHOPS wormholes.
2.1 Introduction

The CHOPS technique proves to be successful around the world, especially in many heavy oil reservoirs in Alberta and Saskatchewan, Canada. As a primary recovery method, oil recovery factors of typical CHOPS wells are still under 10% OOIP (Rivero et al., 2009). To optimize the CHOPS process and seek new technology to enhance oil recovery during post-CHOPS period, it is of great importance to characterize CHOPS wells with wormholes. Tremblay et al. (1999) reviewed field tests data such as tracer and gamma ray tests, and summarized lab studies to address major mechanisms involved in cold production including foamy oil, sand production, and sand transport. Wormhole generation and foamy oil flow are two major mechanisms which contribute to the success of CHOPS (Sawatzky et al., 2002). Field observations and laboratory experiments reported in many papers have proven the existence of wormholes. Former studies including tracer tests, pressure tests, injection tests, and laboratory experiments have demonstrated the high conductivity of wormholes (Smith, 1988; Loughead and Saltularoglu, 1992; Metwally and Solanki, 1995; Elkins et al., 1972; Squires, 1993; Yeung, 1992). The presence of foamy oil was first observed by Smith (1988) who described the foamy oil as highly compressible mobile liquid phase with tiny gas bubbles. With a lower fraction of gas bubbles in foamy oil than that of conventional foams, many researchers have treated foamy oil flow as a single phase flow (Sheng et al., 1999; Maini, 1999; McCaffrey and Bowman, 1991). Sawatzky et al. (2002) extensively analyzed effects of wormholes with enhanced permeability on increased production and used numerical simulation tools to estimate wormhole footprints by history matching field data. Overall drainage area of the CHOPS wells was estimated instead of detailed wormhole
configurations. Pan et al. (2010) reviewed the most recent research related to CHOPS and summarized different models and technologies used in characterizing CHOPS wells. Different disciplines such as geostatistics, geomechanics, and seismic have been integrated with reservoir engineering to study complicated mechanisms of CHOPS (Coombe et al., 2001; Aghabarati et al., 2008). Tremblay (2009) applied a multi-well CHOPS model on the platform of a commercial reservoir simulator to optimize CHOPS spacing of infilling wells. Liu and Zhao (2005 b) used source and sink function methods to analyze transient flow behavior of a CHOPS well with fractal wormhole patterns. They successfully investigated effects of different wormhole patterns on transient pressure behavior and identified different flow regimes of a CHOPS well. Fetkovich et al. (2006) applied different decline methods to match transient linear and boundary dominated flow regimes on rate-time behavior of heavy oil produced by horizontal and multilateral wells. Sanyal et al. (2011) intended to apply transient pressure analysis to detect growth of wormholes in pilot CHOPS wells. Shokri and Babadagli (2014) implemented a diffusion-limited aggregation algorithm with a dual-porosity approach into a numerical simulator to account for complex wormhole patterns and history matched production data. On the platform of a commercial software, Istchenko and Gates (2014) implemented a dynamic wellbore module to model wormhole growth with multilateral wells and concluded that the reservoir heterogeneity could affect wormhole networks significantly.

This chapter presents modeling results to show that the overall effective wormhole coverage and wormhole intensity dominantly affect transient pressure and production rate of CHOPS wells. Based on a BIA CHOPS model considering complex wormhole morphologies and propagation, the proposed approach can effectively characterize how
far away the wormholes can grow into the reservoir and how densely they can develop around the well (Xiao and Zhao, 2017). This study, in constrast to numerical simulation efforts, focuses on using model-based type curves to analyze rate-time behavior in estimation of effective wormhole coverage of CHOPS wells.

2.2 Effective Wormhole Coverage and Wormhole Intensity

The cold production was described succinctly as “a process in which the well is transported to the oil rather than the oil transported to the well” (Sawatzky et al., 2002). Accordingly, the effective wormhole coverage, as a proxy of the drainage region of CHOPS wells, is defined as the maximum circled area in which wormholes have developed (Figure 2.1). Due to reservoir heterogeneity and anisotropy of mechanical strengths, wormholes tend to grow in asymmetric patterns as simulated in literature (Liu and Zhao, 2005 a; Istchenko and Gates, 2014). The effective wormhole coverage defined here cannot represent the exact shape of the drainage area. However, it effectively identifies the maximum region that a CHOPS well can drain through wormhole networks. Eq. [2.1] is used to calculate the effective wormhole coverage.

\[ A_{wc} = \pi R_e^2 \]  

where, \( A_{wc} \) is the effective wormhole coverage, \( R_e \) is the maximum distance that wormholes can grow from the well into reservoir.
Figure 2.1 A bird view (not to scale) of a CHOPS well (the red dot) producing from a bounded reservoirs with no-flow boundaries. The reservoir boundary is denoted as $\Gamma$, and $\Omega$ represents the reservoir domain. Wormholes grow from the well to reservoir with complex morphologies. The circled area represents the effective wormhole coverage of the CHOPS well.
The total length of wormholes grow from the well into reservoir is directly related to the total sand production. To describe how densely wormholes can develop within the effective wormhole coverage, the wormhole intensity is defined as the total wormhole length within unit effective wormhole coverage. Similar to the two dimensional fracture intensity measures \( P_{21} \) proposed by Dershowitz and Herda (1992), the wormhole intensity in this study is calculated as

\[
P_{21\text{WC}} = \frac{\int f(L) dL}{A_{\text{WC}}} = \frac{L_{\text{total}}}{\pi R_e^2}
\]  \[2.2\]

where, \( f(L) \) is a function of wormhole length with respect to space. \( L_{\text{total}} \) is the total length of wormholes developed within the effective wormhole coverage.

Intuitively, the effective wormhole coverage can approximate sizes of drainage areas, while the wormhole intensity can capture overall wormhole morphologies without details. If the distance variables are normalized by dividing \( R_e \), we can obtain the dimensionless forms of effective wormhole coverage and wormhole intensity.

Dimensionless effective wormhole coverage:

\[
A_{\text{WC}D} = \pi \left( \frac{R_e}{R_e} \right)^2 = \pi
\]  \[2.3\]

Dimensionless wormhole intensity:

\[
P_{21\text{WC}D} = \frac{L_{\text{total}} / R_e}{A_{\text{WC}D}} = \frac{L_{\text{total}}}{\pi R_e}
\]  \[2.4\]

Such dimensionless terms will be very useful in the type curve matching process, which will be addressed in the later part.
2.3 A BIA CHOPS Flow Model

To make the complicated and non-linear CHOPS process solvable, some major assumptions need to be made without losing significant physics:

2.3.1 Major assumptions

1. Wormholes created by sand erosion and dilation are mature, in which highly conductive channels exist to allow slurry flows (Tremblay et al., 1998).

2. Foamy oil, as described by Smith (1988), flows from the formation into wormholes as a continuous single phase with pressure dependent PVT properties (Sheng et al., 1999; Maini et al., 1993; McCaffrey and Bowman, 1991). If water phase is present in the reservoir, the Perrine’s method (1956) could be used reasonably to apply single phase solutions to approximate fluid flow of a multi-phase flow system.

3. The permeability, porosity, and the total compressibility in the wormhole network region can vary significantly from the region outside it according to observations and experimental studies in the literature (Tremblay et al., 1999; Tremblay, 2005). However, at the current stage of this study, the sizes and properties of the highly conductive wormhole channels are assumed to be uniform and constant for the modeling purpose.

4. The reservoir, bounded by irregular no flow boundaries, is assumed to be homogeneous and isotropic with uniform formation thickness.

5. Wormhole propagation directions and morphologies in a bird view can be complex and arbitrary.
Even though the proposed flow model has some idealized assumptions, it mainly considers effects of major mechanisms of CHOPS wells on production and pressure responses from the prospective of reservoir engineering. It is advantageous to account for dynamic growing process of effective wormhole coverage, integration of pressure dependent foamy oil PVT properties, irregularly-shaped boundaries and complex wormhole morphologies.

2.3.2 Governing equations

The diffusion equation describing the unsteady state fluid flow in porous media according to the mass conservation and Darcy’s law is:

$$
\nabla \left[ \rho(p) \cdot \frac{k(p)}{\mu(p)} \cdot \nabla p \right] = \frac{\partial \rho(p) \cdot \phi(p)}{\partial t}
$$

[2.5]

where, $\rho(p)$ and $\mu(p)$ are pressure dependent foamy oil properties. $k(p)$ and $\phi(p)$ are pressure dependent formation properties due to sand failure during depletion.

Currently, there is still a need for tremendous effort for the study of actual foamy oil phenomena and related mathematical description (Chen et al., 2015). Several researchers have investigated mechanisms of foamy oil behavior through experimental and mathematical approaches (Joseph et al., 2002; Wang et al., 2008). Foamy oil PVT properties are essentially functions of reservoir pressure if temperature remains constant (Maini et al., 1993; Chen et al., 2015). For this reason, the work of Smith (1988) on the foamy oil compressibility is applied, as indicated in Eq. [A.6] in Appendix A. Details of foamy oil mechanisms are out of scope in this study. However, from the prospective of reservoir engineering, foamy oil PVT properties are functions of reservoir pressure and
are considered in the aforementioned governing equation (Maini et al., 1993; Chen et al., 2015).

Pseudo variables of pressure and time, similar to the pseudo-pressure and pseudo-time for real gas flow (Chien and Caudle, 1994), are defined in Eq. [2.6] and [2.7] to linearize the CHOPS process. They allow proper integration of pressure dependent terms, especially the foamy oil PVT properties. Accordingly, Eq. [2.5] can be linearized in terms of pseudo variables of pressure and time as shown in Eq. [2.8] (Xiao and Zhao, 2012).

\[
M(p) = \int_{p_b}^{p} \frac{k(p)}{\mu(p)} dp 
\]

(2.6)

\[
t_a = \int_{t_0}^{t} \frac{k(p)}{\phi(p) \mu(p) c_t(p)} dt 
\]

(2.7)

\[
\nabla^2 M(p) = \frac{\partial M(p)}{\partial t_a} 
\]

(2.8)

Dimensionless variables are defined as:

\[
P_D = \frac{2\pi h}{q_b} [M(P_i) - M(p)] 
\]

(2.9)

\[
t_D = \frac{t_a}{R_e^2} 
\]

(2.10)

\[
x_D = \frac{x}{R_e}, y_D = \frac{y}{R_e}, h_D = \frac{h}{R_e} 
\]

(2.11)

\[
Q_D = \frac{Q}{q_b} 
\]

(2.12)

where, \( h \) is the formation thickness, \( P_i \) is the reservoir initial pressure, \( q_b \) is a base production rate, and \( Q \) is the production rate.
Substituted by dimensionless terms in Eq. [2.9] to Eq. [2.12], Eq. [2.8] becomes in dimensionless forms as:

$$\nabla^2 P_D = \frac{\partial P_D}{\partial t_D}$$ \[2.13\]

The system is prescribed to a zero initial condition, and the CHOPS well is producing under constant rate in a bounded reservoir with no-flow boundary condition at \( r = r_c \), described in mathematical format as:

$$P_{D(r_0, r_0, 0)} = 0$$ \[2.14\]

$$\frac{dP_{D(r_0, r_c, r_0)}}{dr_D} = 1$$ \[2.15\]

$$\frac{dP_{D(r_0, r_c, r_0)}}{dr_D} = 0$$ \[2.16\]

### 2.3.3 Boundary integral formulations

The dimensionless diffusion equation, Eq. [2.13], describes the unsteady state fluid flow in a domain \( \Omega \) with prescribed conditions along the boundary \( \Gamma \). According to Kikani and Horne (1992, 1993), boundary integral formulations are derived from the weighted residual formulations as

$$\theta P_D(x_D, y_D, t_D) = \left\{ \int_{\Gamma} \left[ (G \frac{\partial P_D}{\partial n} - P_D \frac{\partial G}{\partial n}) d\Gamma + Q_D G \right] d\tau \right\}$$ \[2.17\]

\( \theta = 2\pi \rightarrow \text{if } (x_D, y_D) \in \Omega, \)

\( \theta = \theta \rightarrow \text{if } (x_D, y_D) \in \Gamma \) \[2.18\]

where \( \theta \) are the internal angles between two adjacent boundary elements. \( G \) is the free space Green’s function of a point source, which physically represents the pressure
responses due to an instantaneous point source removed at time $\tau$ with unit strength at location $(\xi, \zeta)$ (Carslaw and Jaeger, 1959). $G_L$ is the integration of the free space Green’s function $G$ over a line source, which represents each highly conductive wormhole channel in the reservoir.

The free space Green’s function of a point source in dimensionless form is given based on the work of Carslaw and Jaeger (1959):

$$G(x_D, y_D, t_D, \xi, \zeta, \tau) = \frac{H(t_D - \tau)}{4\pi(t_D - \tau)} \exp\left[-\left(\frac{x_D - \xi}{2\sqrt{(t_D - \tau)}}\right)^2\right] \exp\left[-\left(\frac{y_D - \zeta}{2\sqrt{(t_D - \tau)}}\right)^2\right] \quad [2.19]$$

$G_L$ is obtained by integrating the free space Green’s function $G$ over line sources with total length of $L_{total}$:

$$G_L(x_D, y_D, t_D, \xi, \zeta, \tau) = \int_{L_{total}} \left\{ \frac{H(t_D - \tau)}{4\pi(t_D - \tau)} \exp\left[-\left(\frac{x_D - \xi}{2\sqrt{(t_D - \tau)}}\right)^2\right] \exp\left[-\left(\frac{y_D - \zeta}{2\sqrt{(t_D - \tau)}}\right)^2\right] \right\} dL \quad [2.20]$$

As shown in Figure 2.2, the outer boundary $\Gamma$ and wormhole channels are divided into $N_j$ constant boundary elements and $N_w$ constant wormhole elements respectively. Pseudo variables of pressure are evaluated at the center of each element. Constant elements are advantageous in this study for two reasons: first, the integral of the Green’s function along a constant element can be obtained analytically, resulting in improvement of calculating speed and accuracy; secondly, singularity at sharp corners of adjacent elements is naturally avoided. Relatively more constant elements will be used for curvature boundaries.
Figure 2.2 Discretization of reservoir boundary and wormholes into constant elements.

The boundary $\Gamma$ is divided into $N_j$ constant boundary elements. Wormholes are divided into $N_w$ constant wormhole elements.
Eq. [2.17] is subsequently written in a discrete form at the evaluating point \((\xi_{Dk}, \zeta_{Dk})\) as

\[
\theta_k P_Dk(\xi_{Dk}, \zeta_{Dk}, t_D) = \int_{t_D}^{t_f} \sum_{k=1}^{N_j} \sum_{l=1}^{N_g} \left[ G \frac{\partial P_Dk(\xi_{Dk}, \zeta_{Dk}, \tau)}{\partial n} + \sum_{i=1}^{N_w} Q_{Dk} G_{Li} \right] d\tau
\]

where, \(a_{Dk}\) is the dimensionless half-length of each reservoir boundary element.

2.3.4 Dynamic wormhole growth

According to the material balance of sand production, if the wormhole sizes are uniform and constant, the total length of wormholes can be calculated from sand production data by

\[
L_{total}(t_w) = \frac{4 \int V_s(t_w) dt}{\pi (\phi_w - \phi) d_w^2}
\]

where, \(V_s(t_w)\) is the sand production rate, \(\phi_w\) is the average porosity of wormholes, \(\phi\) is the average formation porosity, \(d_w\) is the average wormhole size in diameter, and \(t_w\) is the wormhole propagation time.

To model the propagation process of growing wormholes, the wormhole propagation time \(t_w\) is subdivided into different time steps \(t_{wi}\). During each time step \(t_{wi}\), assuming wormholes grow an element with constant length \(L_{wi}\), then we can determine each time step by
\[ t_{wi} = \frac{\pi (1 - \phi_w) d_w^2 L_{wi}}{2[V_s(t_w) + V_s(t_{wi-1})]} \]  \[2.24\]

where

\[ t_w = \sum_{i=1}^{N_w} t_{wi}, t_{wi-1} = \sum_{i=1}^{N_w-1} t_{wi} \]  \[2.25\]

We also have the following relationship:

\[ L_{total}(t_w) = \sum_{i=1}^{N_w} L_{wi} = \frac{4\int V_s(t_w) dt}{\pi (\phi_w - \phi) d_w^2} \]  \[2.26\]

For the propagating wormholes, each wormhole element is a function of time step \( t_{wi} \), during which the \( i \)-th wormhole element develops (Liu and Zhao, 2006b). It has no effect on the reservoir flow until the propagation happens. At the wormhole propagation time \( t_w \), the dimensionless free space Green’s function of developed wormhole channels, as shown in Eq. [2.20], should be modified to account for the \( i \)-th developed wormhole element:

\[ G_L(x_D, y_D, t_D, \xi, \zeta, t_{mid}) = \int_{L_{mid}} \left[ \frac{H(t_D-t_{mid})}{4\pi(t_D-t_{mid})} \exp\left[-\frac{(x_D-x)^2}{2(t_D-t_{mid})}\right] \exp\left[-\frac{(y_D-y)^2}{2(t_D-t_{mid})}\right] \right] dL \]  \[2.27\]

Consequently, the process of wormhole propagation can be reasonably considered in the solution process by modifying source functions of growing wormholes in the frame of BIA. It is worth noticing that the sand production history is of paramount importance for the proposed model to represent reasonable wormhole propagations.

The major hypothesis in this study is that wormholes will not grow to infinite extent due to pressure depletion and local fluidization criterion as simulated by Istchenko and Gates (2014). Accordingly, \( R_c \) (the maximum distance that wormholes can grow
from the well into reservoir) defined in the proposed work grows as a function of time and becomes stable after certain time of growth. Meanwhile, wormholes growth will not simply stop as indicated by continuous sand production, and such growth will mainly contribute to the increase of wormhole intensity other than $R_c$.

2.3.5 BIA solution process

Laplace transformation naturally converts the convolution in Eq. [2.21] into algebraic forms in Laplace space and has been widely applied in boundary element method (BEM) and semi-analytical approaches. Moving fictitious source locations along all $N_j$ boundary elements and evaluating integrals analytically, Eq. [2.21] becomes to matrix equations in Laplace space as

$$
\sum_{k=1}^{N_j} \left( \tilde{P}_{\text{coef}}^k - 0.5 \delta_k \right) \tilde{P}_{Dk}(\xi_{Dk}, \zeta_{Dk}, s) + \sum_{k=1}^{N_j} (\tilde{P}_{\text{ncoef}}^k) \cdot \tilde{P}_{nDk}(\xi_{Dk}, \zeta_{Dk}, s) = -\frac{1}{s} \sum_{k=1}^{N_j} \left( \sum_{i=1}^{N_w} \tilde{Q}_{Dji} \cdot (\tilde{G}_L)_i \right) \quad k = 1, N_j; i = 1, N_w
$$

[2.28]

where,

$$
(\tilde{P}_{\text{coef}}^k)_k = L \left\{ \frac{\xi_{Dk}}{4t_D} \cdot \hat{\Theta}_L \left( \frac{a_{Dk} - \xi_{Dk}}{2}, \frac{a_{Dk} - \xi_{Dk}}{2}, t_D - \tau \right) \Theta_L \left( \frac{\xi_{Dk}}{2}, t_D - \tau \right) \right\}
$$

[2.29]

$$
(\tilde{P}_{\text{ncoef}}^k)_k = L \left\{ 0.5 \hat{\Theta}_L \left( \frac{a_{Dk} - \xi_{Dk}}{2}, \frac{a_{Dk} - \xi_{Dk}}{2}, t_D - \tau \right) \Theta_L \left( \frac{\xi_{Dk}}{2}, t_D - \tau \right) \right\}
$$

[2.30]

$$
(\tilde{G}_L)_i = L \left\{ 0.5 \hat{\Theta}_L \left( \frac{L_{WDi} - \xi_{Dk}}{2}, \frac{L_{WDi} - \xi_{Dk}}{2}, t_D - t_{wiD} \right) \Theta_L \left( \frac{\xi_{Dk}}{2}, t_D - t_{wiD} \right) \right\}
$$

[2.31]

$a_{Dk}$ is the dimensionless half length of the k-th boundary element, and $L_{WDi}$ is the dimensionless half length of the i-th wormhole element. $L\{\ldots\}$ represents the Laplace
transformation of functions or variables. We can also denote \( \tilde{P}_{n\tau k} = L\{\partial P_{\tau k} / \partial n\} \) and \( \tilde{P}_{\tau k} = L\{P_{\tau k}\} \). \((\xi_{\tau k}, \zeta_{\tau k})\) is the dimensionless evaluating point. Special functions in Eq. [2.29] are defined according to the work of Zhao and Thompson (2002):

\[
\begin{align*}
\hat{\Theta}_{\tau}(z1, z2, t) &= \frac{1}{2} \left[ \text{erf} \left( \frac{z2}{\sqrt{t}} \right) - \text{erf} \left( \frac{z1}{\sqrt{t}} \right) \right] \quad [2.32] \\
\Theta_{\tau}(z, t) &= \frac{1}{\sqrt{\pi t}} \exp \left( -\frac{z^2}{t} \right) \quad [2.33]
\end{align*}
\]

where,

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt, \quad \text{exp denotes the exponential function}
\]

By moving the fictitious source point along the outer boundary \( \Gamma \), we can set up \( N_j \) sets of linear equations by evaluating dimensionless pressure at \( N_j \) allocated points, a linear matrix equation system can be formed as shown in Eq. [2.34]. When boundary unknowns \( \tilde{P}_{\tau k}(\xi_{\tau k}, \zeta_{\tau k}, \xi) \) and \( \tilde{P}_{n\tau k}(\xi_{\tau k}, \zeta_{\tau k}, \xi) \) are obtained, Eq. [2.28] will be used again to evaluate dimensionless pressure at any location in the domain \( \Omega \).

\[
AX = B \quad [2.34]
\]

### 2.3.6 Variable rates under constant bottom-hole pressure

Practically, pressure and pressure derivative data are expensive to acquire comparing to daily production rate. In the proposed BIA solution process, pressure transients of the CHOPS well under constant production rate are obtained in the Laplace space. Therefore, it is straightforward to apply the Duhamel’s principle, as shown in Eq. [2.35], to calculate variable rates under constant bottom-hole pressure (BHP) in the
Laplace domain (van Everdingen and Hurst, 1949). Accordingly, by using the Stehfest inverse Laplace algorithm (Stehfest, 1970), two sets of well performance solutions with respect to time can be modeled from the proposed BIA solution process. One is variable rates under constant BHP for rate transient analysis (RTA). The other is transient pressure change under constant rate for pressure transient analysis (PTA) in drawdown well tests. The two processes are not likely to exist at the same time; however, the proposed model is able to provide modeling results for both scenarios. Both RTA and PTA can be conducted systematically depending on the availability of well tests and production data.

\[ 
\tilde{q}_{wellD}(x_D, y_D, s) = \frac{1}{\tilde{P}_{WD}(x_D, y_D, s) \cdot s^2} 
\]

where, \( \tilde{q}_{wellD}(x_D, y_D, s) \) is the dimensionless production rate of a well at \((x_D, y_D)\) under constant bottom hole pressure. \( \tilde{P}_{WD}(x_D, y_D, s) \) is the dimensionless pressure drop of a CHOPS well under constant production rate.

2.4 Model Validation

Due to the complicated mechanisms of CHOPS fluid flow, indirect approaches are used to validate the proposed BIA CHOPS flow model. I first compare the model with available analytical solutions in dimensionless forms, and then use numerical simulations to validate real-time pressure and production responses.

2.4.1 Comparison with analytical solutions

Analytical solutions of horizontal wells producing in a bounded reservoir are available in many well testing software such as Ecrin (KAPPA, 2013). If wormholes only
grow in a fixed direction without branching, a CHOPS well will behave like a horizontal well as shown by Fetkovich et al. (2006). Because the analytical solutions of a horizontal well only consider static well configurations, the wormholes are assumed to grow rapidly at the very beginning of production and then remain stable for a relatively long time. As shown in Figure 2.3, from dimensionless time 5.0E-5 to 2.0E-4, wormholes grow symmetrically in the direction of East to West, during which production rates increase to a peak value followed by a gradual decline. To the best of my knowledge, there’s no analytical solution available to validate the flow behavior of the wormhole growing period. However, field cases and laboratory studies have observed such abnormal production increment (Sawatzky et al., 2002; Istchenko and Gates, 2014), which may indirectly prove the consistency of the modeled production responses during wormhole propagation. Moreover, after dimensionless time 2.0E-4, the CHOPS well produces with static wormhole length in a bounded reservoir. After the wormhole propagation period, the modeled transient flow behavior match perfectly with the analytical solutions of a horizontal well as shown on the log-log plots of dimensionless pressure and production responses in Figure 2.3.
Figure 2.3 Comparison of the BIA CHOPS model with analytical solutions of a horizontal well (HW). The wormhole pattern is simple and has the same length with the HW.
One of the most important advantages of the proposed BIA approach is to deal with complex wormhole morphologies and boundary shapes. Liu and Zhao (2005 b) proposed an analytical approach to solve for transient pressure of a CHOPS well with fractal wormhole patterns, which were generated by the diffusion-limited aggregation (DLA) algorithm. They successfully identified transient flow regimes of CHOPS wells with fractal wormhole patterns. Due to the difficulty of obtaining closed-form Green’s functions in bounded systems, their approach limited in modeling rectangular reservoirs with wormholes parallel to boundaries. The disadvantage of using closed-form Green’s function is the expensive computing effort of summing infinite series of kernel functions. Therefore, most of their results showed pressure responses of CHOPS wells in an infinite reservoir. Figure 2.4 compares the dimensionless pressure and pressure derivatives obtained by the proposed model with Liu and Zhao’s model. The two models use the same static fractal pattern generated by DLA; however, the proposed model solves a bounded system instead of an infinite reservoir. Except for the boundary dominated flow regime, the transient pressure responses show excellent agreement with Liu and Zhao’s analytical solutions. Accordingly, the capability of modeling CHOPS wells with complex wormhole morphologies and boundaries is satisfactorily validated.

2.4.2 Comparison with numerical simulations

Tremblay (2009) used multilateral wells to represent wormholes on the platform of a commercial simulator. To validate the proposed BIA CHOPS flow model, I followed similar idea and used multilateral wells with zero wellbore storage to represent highly conductive wormholes. Since the objective of this study is to estimate effective wormhole coverage, detailed foamy oil PVT properties and sand production transport studies are out
of its scope. The black oil model (CMG IMEX, 2014) was used to generate pressure and production responses of CHOPS wells with different wormhole patterns. Because the dynamic growth of multilateral wellbores is not implemented in the numerical simulator, all wormholes patterns are assumed to be static. Table 2.1 summarizes key input parameters of the numerical simulations, whereas near wellbore block sizes are refined to obtain satisfactory transient pressure and production results.

Table 2.2 shows the effective wormhole coverage and wormhole intensities of three synthetic cases. All three cases were run for three years and the initial time step was set as 0.001 day to obtain early transient flow responses. Pressure data were obtained under the condition of constant production rate, and production data were obtained under the condition of constant bottom-hole pressure. Without detailed studies of foamy oil PVT properties in this study, PVT properties evaluated at the initial reservoir pressure were used to transform dimensionless pseudo variables to real pressure and production responses by Eq. [2.36] to Eq. [2.38]. As shown in Figures 2.5, log-log plots of pressure derivatives and production rates of all three cases are simultaneously matched with the results generated by the BIA CHOPS flow model. The mismatch of early pressure and production responses may be caused by numerical truncation errors of the finite difference approaches used in the commercial simulator. Nonetheless, the proposed model has reasonably good agreement with the numerical simulation results.

\[ \Delta P = P_i - P = \frac{q_b \mu(P_i)}{2\pi k(P_i)h} P_D \]  

\[ t = \frac{\phi(P_i) \mu(P_i) C_v(P_i) R^2}{k(P_i)} t_D \]  

\[ Q = q_b Q_D \]
Figure 2.4 Comparison of the BIA CHOPS model with analytical solutions of a horizontal well (HW). The wormhole pattern is simple and has the same length with the HW.
Table 2.1 Basic parameters used in the synthetic case studies

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
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<tbody>
<tr>
<td>Reservoir Permeability</td>
<td>1000</td>
<td>mD</td>
</tr>
<tr>
<td>Reservoir Porosity</td>
<td>0.34</td>
<td></td>
</tr>
<tr>
<td>Reservoir Temperature</td>
<td>30</td>
<td>°C</td>
</tr>
<tr>
<td>Payzone Thickness</td>
<td>7</td>
<td>m</td>
</tr>
<tr>
<td>Total Compressibility</td>
<td>$1.6 \times 10^{-6}$</td>
<td>kPa$^{-1}$</td>
</tr>
<tr>
<td>Viscosity @ 23 °C</td>
<td>2700</td>
<td>cP</td>
</tr>
<tr>
<td>Wetting</td>
<td>Water wet</td>
<td></td>
</tr>
<tr>
<td>Wormhole patterns of numerical simulations</td>
<td>Case 1</td>
<td>Case 2</td>
</tr>
<tr>
<td>------------------------------------------</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>Wormhole patterns of the BIA CHOPS flow model</td>
<td>Case 1</td>
<td>Case 2</td>
</tr>
<tr>
<td>Maximum wormhole growing distance from the well (Re), m</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>Wormhole intensity, m/m² $P_{21wc}$</td>
<td>$\frac{1}{100\pi}$</td>
<td>$\frac{1}{50\pi}$</td>
</tr>
<tr>
<td>Dimensionless wormhole intensity $P_{21wcd}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\frac{4}{\pi}$</td>
</tr>
</tbody>
</table>
Figure 2.5 Comparison of the BIA CHOPS model with numerical simulations (a) pressure responses and (b) production rates.
2.5 CHOPS Transient Pressure and Production Behavior

Since well production rates are always under various working conditions, the material balance time is used to analyze rate-time responses according to the work of Blasingame et al. (1991):

\[
T_{MB} = \int_0^t \frac{q_{well}(t)dt}{Q_{cum}(t)}
\]

The dimensionless material balance time is defined accordingly as

\[
T_{MBD} = \frac{k(P)T_{MB}}{\phi(P)\mu(P)C_i(P)R_e^2}
\]

2.5.1 Effective wormhole coverage and wormhole intensity dominate transient behavior

Wormholes may grow with various morphologies due to stress variation and reservoir heterogeneity. To investigate effect of wormhole morphology on CHOPS transient behavior, wormholes are assumed to be mature and the effective wormhole coverage is constant after the peak oil production rate. Accordingly, sand production after the wormhole growing period is mainly due to scouring effects, which will not change the effective wormhole coverage.

With the assumption of the same total wormhole length and relatively maturely-developed wormhole networks, Table 2.3 summarizes the effective wormhole coverage, wormhole intensity, and their corresponding dimensionless forms of different wormhole patterns. Wormhole pattern A is a fractal pattern with complex wormhole structures and directions adapted from the work of Liu and Zhao (2005 b). Comparing with pattern A,
wormhole pattern B has similar wormhole intensity but relatively simpler wormhole structures. Wormhole patterns C, D, E, and F have larger effective wormhole coverage because their wormhole intensities are smaller. Though patterns C and D have different wormhole structures, their effective wormhole coverage and wormhole intensities are the same. Similar characteristics are observed in patterns E and F.

A CHOPS well producing in a bounded reservoir with different aforementioned wormhole morphologies, neglected the wormhole propagation period, is modeled by the BIA CHOPS model. Figure 2.6 (a) shows the dimensionless pressure and pressure derivative of the CHOPS well with wormhole patterns A and B, whereas the wormhole-linear flow regime is identified with a half slope and the boundary dominated flow regime has a unit slope. On the log-log plots of dimensionless production rate versus dimensionless material balance time as shown in Figure 2.6 (b), the wormhole-linear flow regime has a negative half slope while the boundary dominated flow regime has a negative unit slope. With the same total wormhole length, detailed wormhole structures and effective wormhole coverage of A and B are different. However, their log-log plots of dimensionless pressure and rate-time behavior are very similar. Despite of complex wormhole structures, patterns A and B have the same dimensionless effective wormhole coverage ($A_\text{WCD} = \pi$) and similar dimensionless wormhole intensity ($P_\text{21WCD}$).
Table 2.3 CHOPS well with various wormhole morphologies under the same total wormhole length ($L_{\text{total}} = 800 \text{ m}$)

<table>
<thead>
<tr>
<th>Wormhole Patterns</th>
<th>$R_v$ (m)</th>
<th>Dimensionless wormhole intensity ($P_{21\text{WC}}$)</th>
<th>Dimensionless wormhole coverage ($A_{\text{WCD}}$)</th>
<th>Dimensionless wormhole intensity ($P_{21\text{WCD}}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wormhole Pattern A</td>
<td>83.8</td>
<td>$\frac{0.114}{\pi}$</td>
<td>$\pi$</td>
<td>$\frac{9.55}{\pi}$</td>
</tr>
<tr>
<td>Wormhole Pattern B</td>
<td>80.0</td>
<td>$\frac{0.125}{\pi}$</td>
<td>$\pi$</td>
<td>$\frac{10}{\pi}$</td>
</tr>
<tr>
<td>Wormhole Pattern C</td>
<td>200.0</td>
<td>$\frac{0.02}{\pi}$</td>
<td>$\pi$</td>
<td>$\frac{4}{\pi}$</td>
</tr>
</tbody>
</table>
Wormhole Pattern D

Wormhole Pattern E

Wormhole Pattern F
Figure 2.6 CHOPS wells with wormhole patterns A and B (a) pressure and pressure derivative responses and (b) production rates versus material balance time.
During the wormhole-linear flow period (t_D is approximately from time 1E-5 to 1E-3), the effective wormhole coverage dominates flow behavior, resulting in same pressure and production responses. During the transitional flow period (t_D is from time 1E-3 to 0.5), detailed wormhole structures and wormhole intensities dominate flow behavior, so that the two have different responses. The boundary dominated flow regimes are mainly affected by the dimensionless reservoir sizes. The dimensionless reservoir size of wormhole pattern A is smaller than B; consequently, the boundary dominated flow of pattern A occurs earlier.

**Figure 2.7** further confirms the important finding that, similar dimensionless wormhole intensities of different wormhole morphologies will result in the same dimensionless pressure and production responses during the wormhole-linear flow regime. Wormhole-linear flow regimes of patterns C and D match each other due to the same dimensionless wormhole intensity. Similar characteristics are found in patterns E and F. Furthermore, patterns C, D, E, and F have the same effective wormhole coverage resulting in same dimensionless reservoir sizes and boundary dominated flow regimes. Differences between the transitional flow regimes of patterns C, D, E, and F are small because their detailed wormhole structures are very similar. The modeling results show that dimensionless pressure and rate-time behavior is affected primarily by dimensionless wormhole intensity and effective wormhole coverage.
Figure 2.7 CHOPS wells with wormhole patterns C, D, E and F (a) pressure and pressure derivative responses and (b) production rates versus material balance time.
2.5.2 Practical type curves

It is of great interest to use aforementioned log-log plots of pressure and production responses as diagnostic tools to characterize overall effective wormhole coverage and wormhole intensities of CHOPS wells. On the basis of the relatively simple and identical wormhole patterns B, C, and E (Table 2.3), I only need to change the number of wormhole branches in the circled area to obtain dimensionless type curves with various dimensionless wormhole intensities. Since all their dimensionless effective wormhole coverage and reservoir sizes are the same, transient pressure and production responses are only affected by dimensionless wormhole intensities. Figures 2.8 and 2.9 show the type curves of dimensionless pressure and pressure derivatives respectively. On the pressure derivative type curves, when wormhole intensities get larger, a near unit slope line occurs during the transitional flow regime due to strong interference between wormholes. Similarly on the log-log plots of dimensionless production rate versus material balance time (Figure 2.10), an apparent boundary-dominated behavior with a near negative unit slope occurs after the wormhole-linear flow. Interestingly, such closed-boundary effect during the transitional flow period has also been observed in multi-stage fractured horizontal wells with large number of stages (Zhao, 2012; Chen and Raghavan, 2013; Zhao et al., 2014).
Figure 2.8 Practical type curves of dimensionless pressure with different dimensionless wormhole intensities.
Figure 2.9 Practical type curves of dimensionless pressure derivatives with different dimensionless wormhole intensities.
Figure 2.10 Practical type curves of dimensionless production rates with different dimensionless wormhole intensities.
Figures 2.8, 2.9, and 2.10 are model-based practical pressure and rate type curves of a CHOPS well with certain wormhole propagation histories to evaluate effective wormhole coverage and wormhole intensities. It is worth noticing that such type curves shall be generated on a case-by-case basis. A procedure to use such practical type curves in this work is summarized as:

1. Use available sand production data and information of wormhole sizes to reasonably approximate wormhole propagation history.

2. Obtain adequate data of formation properties (e.g. \( \phi(p), k(p) \)) and PVT properties of foamy oil (\( \mu(p), c_i(p) \)) to calculate pseudo variables of pressure and time referring to literature (Spivey and Lee, 1986; Rahman et al., 2004).

3. If foamy oil PVT properties are not well studied, PVT properties evaluated at the initial reservoir pressure, as shown in Eq. [2.36], [2.37], and [2.38], are used to approximately transform dimensionless type curves to real pressure and production responses.

4. Once reasonable matching is systematically obtained on pressure and rate-time type curves, the dimensionless wormhole intensity is recorded. Accordingly, \( R_e \) and effective wormhole coverage will be calculated by Eq. [2.41] and Eq. [2.1] respectively.

\[
R_e = \sqrt{\frac{k(P_i)}{\phi(P_i)\mu(P_i)c_i(P_i)(\frac{t}{t_D})_{mp}}} \tag{2.41}
\]
2.6 Field Application

A CHOPS well from the Edam Waseca Sand Pool has been produced for about 100 months as shown in Figure 2.11a. Perrine (1956) proposed a simple approach to use single phase solutions to approximate fluid flow of a multi-phase flow system. To account for water production from the CHOPS well, a normalized total oil rate is calculated using Perrine’s method by Eq. [2.42].

\[ q_n = \frac{q_o B_o + q_w B_w}{B_o} \]  

[2.42]

Its oil production rate increased to about 16 m³/day within 10 months and started decreasing to as low as 2 m³/day for a long time. The water production rate of this well remains relatively stable for a long time. The normalized production rate is slightly higher than the oil production rate and follows similar trend with the oil production. In Figure 2.11b, the sand production rate is the highest at the beginning of the depletion and decreases to less than 0.1 m³/day. No additional information is available for the authors to explain the abnormal high sand production at the 45th month. Cumulative sand production volume from the well is recorded as 235.5 m³.
Figure 2.11 Field data of a CHOPS well from the Edam Waseca Sand Pool (a) oil rate, water rate, and calculated normalized production rate (b) Sand production rate.
The material balance time is calculated by using the normalized production data. Figure 2.12 shows the log-log plot of monthly normalized production rate versus material balance time. The material balance time is longer than the real producing time. On the linear scale plot of rate versus real time, a plateau is observed for an extended period of time (approximately from the 40th month to the 100th month as shown in Figure 2.11). However, on the log-log plot of production rate versus material balance time as shown in Figure 2.12, an extended period of decline trend rather than a plateau is observed, which lasts from the 10th month to the end.

Interestingly, the production decline after the peak rate follows two distinctive linear trends on the log-log plot. In Figure 2.12, the first trend lasting from the $t_{MB}$ of 10 months to the $t_{MB}$ of 100 months is characterized by a negative half slope, which is interpreted as the wormhole-linear flow regime. Following the first trend, the second trend has a negative slope which is close to one but not exact one. The near negative unit slope indicates the occurrence of transitional flow regime with strong interference between wormholes. Furthermore, the occurred plateau instead of a sharp decline on the linear-scale plot of rate versus time (Figure 2.11) indicates that the boundary dominated flow has not been reached.
Figure 2.12 Normalized production rate versus material balance time of the studied CHOPS well. Two distinctive trends with different slopes clearly show on the log-log plots.
Currently, detailed information of foamy oil PVT properties from this area is not available. Therefore, foamy oil properties evaluated at the initial reservoir pressure are used, referring to the field case from the same area in the work of Liu et al. (2006). The formation thickness is 2.2 m, average formation porosity is 0.34, average wormhole porosity is 0.54, average formation permeability is 800 mD to 1500 mD, and the oil viscosity is about 3300 cp. There is no information of the wormhole sizes. Therefore, I assume the wormhole size is 0.5 m in diameter and calculate the total wormhole length by Eq. [2.23]. The wormhole total length is calculated to be 7,231.5 m based on material balance of sand production. The calculated total wormhole length and sand production history is used as input for the BIA CHOPS flow model to generate practical type curves of rate-time behavior for the studied CHOPS well. In the model, the reservoir size is assumed to be constant and larger than effective wormhole coverage.

**Figure 2.13** shows the modeled type curves for the particular field case with different dimensionless wormhole intensities. With the same wormhole total length, wormholes with smaller wormhole intensities will have larger effective wormhole coverage. Since $R_e$ is used as the characteristic length, the dimensionless reservoir size of the CHOPS well with smaller wormhole intensities will be relatively larger according to Eq. [2.11]. Consequently, the CHOPS well with smaller wormhole intensities will produce at lower rate during the early production period and reach the boundary-dominated flow regime earlier.
Figure 2.13 Practical type curves of production decline with respective to material balance time generated by the BIA CHOPS model for the studied CHOPS well.
The late time flow characteristics are very different compared with the one in Figure 2.10, where dimensionless reservoir sizes are the same, resulting in all boundary dominated regimes merging to one stem. Two distinctive trends with negative half slope and negative unit slope are observed on all curves, which indicate the wormhole-linear and boundary-dominated flow regimes respectively. A near negative slope trend is observed on type curves with relatively large wormhole intensities during the transitional flow period due to strong interference between wormholes.

In Figure 2.14, the normalized production rate is successfully matched on the zoomed in type curves. The field rate behavior has very good agreement with the type curve with a dimensionless wormhole intensity \( P_{21WCD} \) of \( 16/\pi \). The first three production data points are not close enough to the matched type curve, and the modeled peak production rate is slightly higher than the field data. Uncertainties of wormhole propagation process and inaccurate measures of sand production may cause such mismatches. However, the identified wormhole-linear flow and transitional flow trends of the field data are reasonably matched on the type curve. By using Eq. [2.41] with the recorded matching point, \( R_e \) is interpreted as 450 m and the effective wormhole coverage of the studied CHOPS well is calculated as 635,850 m\(^3\). By type curve matching of rate-time behavior, the dimensionless wormhole intensity is recorded as \( 16/\pi \). Accordingly, the total wormhole length can be calculated with known \( R_e \) by:

\[
L_{total} = \pi R_e \cdot P_{21WCD} \tag{2.43}
\]
Figure 2.14 Type curve matching of the field data on the zoomed in practical type curves. Data are best matched on the type curve with dimensionless wormhole intensity \( P_{\text{WCD}} \) of \( 16/\pi \).
The total wormhole length of the studied CHOPS well is subsequently estimated to be 7,200 m by analyzing rate-time behavior. The estimated value is very close to the previously calculated total wormhole length based on material balance of sand production, which is approximated to be 7,231.5 m. The sand production lasts continuously during the lifetime of the CHOPS well. It may indicate consistent development of wormholes. However, the longest distance that wormholes can grow from the well to reservoir may not change drastically. Continuous sand production may be contributed by sand failures in near-wellbore regions, which may increase wormhole intensity but not the effective wormhole coverage. The realistic wormhole intensity of the studied well may be larger than the estimated value. Nonetheless, the effective wormhole coverage is reasonably estimated by analyzing rate-time behavior to characterize the overall drainage area of CHOPS wells with wormholes.

2.7 Discussion

2.7.1 Effect of dynamic wormhole growth

The unique contribution of this work is to analyze production data to estimate effective wormhole coverage and wormhole intensity from the prospective of reservoir engineering. The dynamic wormhole growth is a complicated process, which requires integrated study of fluid flow and geomechanics. Estimation of realistic wormhole dynamic growth behavior needs the effort of fully coupled stress-flow multi-physics simulators (Dusseault, 1999). Sand failure and fluidization criterion has been incorporated in commercial simulators to simulate dynamic wormhole growth; and production rate profiles are found to be significantly affected by sand production rate.
The merit of the proposed approach which does not solve for geomechanics is to provide a quick method of analyzing production data, so that effects of wormhole dynamic growth on production/pressure responses are considered in the BIA model instead of detailed sand failure geomechanics.

The dynamic growing period of effective wormhole coverage can significantly affect the type curve matching process of the field data. The real question is that how much time it takes to have $R_e$ grown to its maximum extent. By using Eq. [2.23] and [2.24] along with the sand production profile in Figure 2.11, the time wormholes take to develop to the maximum distance ($R_e$) could be estimated and input to the BIA model however with large uncertainty. Without detailed study of sand failure geomechanics, wormhole dynamic growth with an element-wise pattern in this work is more likely a trial and error process rather than definitive quantification (Liu and Zhao, 2006b). Figure 2.15 and Figure 2.16 show the effects of various growing periods of $R_e$ on the type curve matching process with real producing time and material balance time respectively. With growing periods less than 10 months, the timing of peak production rate occurs is too early to be realistic comparing to the field data. The modeled peak production rates are also too high. The modeling results of growing period of 15 months give relatively reasonable matching on both plots. Even though the result of 20-month $R_e$ growing period has the best match on the log-log plot of production rate versus material balance time (Figure 2.16), the peak production rate on the plot versus real time (Figure 2.15) seems to be delayed comparing with the real data. Therefore, for the analyzed particular well with given production data, I narrow down the possible growing period of effective wormhole coverage to around 15 months. The available sand production data helps to
increase the confidence level of the estimation that it is neither shorter than 10 months nor longer than 20 months.

2.7.2 Effect of foamy oil compressibility constant

The proposed approach considers the effect of non-linear foamy oil PVT properties and applies the integration processes in Eq. [2.6] and [2.7] to linearize the governing equation. The BIA flow model provides means of integrating foamy oil properties as functions of pressure and time obtained from detailed experimental and mathematical investigations, which however is beyond the scope in this study.

In the field case study, values of foamy oil compressibility constant ($\beta$) are found to have significant effect on the type curve matching process. With a 15-month dynamic growing period of $R_e$ and dimensionless wormhole intensity of $16/\pi$, $\beta$ values are varied with a fixed bubble point pressure of 2500 kPa for the particular field case. Different values of $\beta$ will result in various foamy oil compressibilities (according to Appendix A) and shift the production responses with respect to material balance time (Figure 2.17). The best match obtained for the studied well is found to be using $\beta = 0.28$ with a bubble point pressure of 2500 kPa. I highly recommend that detailed investigation of foamy oil properties is carried out, as it is beneficial to accurate characterization of CHOPS well performances.
Figure 2.15 Effects of growing periods of wormholes developed to the maximum distance into the reservoir ($R_e$) on production rate with respect to real producing time. The growing periods are modeled as 2, 5, 10, 15, and 20 months respectively. The normalized production data of the field case is also provided.
Figure 2.16 Effects of growing periods of wormholes developed to the maximum distance into the reservoir ($R_w$) on production rate with respect to material balance time (log-log scale). The growing periods are modeled as 2, 5, 10, 15, and 20 months respectively. The normalized production data of the field case is also provided.
Figure 2.17 Effects of foamy oil compressibility constant ($\beta$) on the type curve matching process of the real field production data with respect to material balance time (log-log scale). The values of $\beta$ are 0.1, 0.28, 0.4, and 0.6, respectively.
2.8 Summary

CHOPS wells with wormholes are found to have distinctive pressure and production behavior which can be effectively analyzed by fast type curve matching instead of cumbersome numerical simulations. A BIA CHOPS flow model considering major mechanisms of CHOPS wells with various wormhole morphologies is developed and validated. Based on the modeling results, effective wormhole coverage and overall wormhole intensity dominantly affect transient pressure and rate-time behavior regardless of the detailed wormhole morphologies. Dimensionless pressure and production type curves are generated to characterize wormhole-linear, transitional flow, and boundary dominated flow regimes. The slope of the transitional flow regime shown on log-log plots of rate versus material balance time is close to negative one due to strong interference between wormholes. Being more informative than linear scale plots of rate versus real time, log-log plots of production rate versus material balance time show straight decline trends with negative half, near negative unit, and negative unit slopes. It is recommended to plot field production data with respect to material balance time to identify rate-time behavior and use type curves to match on characteristic flow regimes. The proposed approach is shown to be applicable to characterize the overall wormhole coverage and wormhole intensities of CHOPS wells by type curve matching the sequence of flow regimes on rate-time flow behavior.
Nomenclature

Notations

\( A_{\text{WC}} \) = effective wormhole coverage, \( \text{m}^2 \)

\( a_{\text{Dk}} \) = dimensionless half-length of each reservoir boundary element

\( B_o \) = oil formation factor, \( \text{m}^3/\text{m}^3 \)

\( B_w \) = water formation factor, \( \text{m}^3/\text{m}^3 \)

\( C_t(p) \) = pressure dependent total compressibility, \( 1/\text{Pa} \)

\( c_f \) = constant rock compressibility, \( 1/\text{Pa} \)

\( d_w \) = average wormhole size in diameter, \( \text{m} \)

\( h \) = formation thickness, \( \text{m} \)

\( k(p) \) = pressure dependent reservoir permeability, \( \text{m}^2 \)

\( L_{\text{total}} \) = total wormhole length, \( \text{m} \)

\( L_{\text{wi}} \) = wormhole propagation length in Eq. [2.24], \( \text{m} \)

\( M(p) \) = pseudo variable of pressure defined in Eq. [2.6]

\( N_j \) = number of total constant boundary elements

\( N_w \) = number of total constant wormhole elements

\( P_i \) = initial pressure, \( \text{Pa} \)

\( P_w \) = pressure drop at the wellbore, \( \text{Pa} \)

\( p \) = pressure at any location, \( \text{Pa} \)

\( p_{\text{coef}} \) = matrix coefficients for \( p_D \)

\( p_{n\text{coef}} \) = matrix coefficients for \( \partial p_D/\partial n \)
\( P_{22WC} \) = wormhole intensity, m/m²

\( Q \) = rate, m³/s

\( Q_{cum} \) = cumulative production, m³

\( q_b \) = base rate, m³/s

\( q_n \) = normalized oil production rate, m³/s

\( q_o \) = oil production rate, m³/s

\( q_{well} \) = production rate at the wellbore, m³/s

\( q_w \) = water production rate, m³/s

\( R_c \) = maximum distance that wormholes can grow from the well into reservoir, m

\( r_w \) = wellbore radius, m

\( r_e \) = radial distance from the well to the reservoir boundary, m

\( s \) = Laplace variable

\( t \) = time, s

\( t_a \) = pseudo variable of time defined in Eq. [2.7]

\( t_{MB} \) = material balance time, s

\( t_w \) = wormhole propagation time, s

\( t_{wi} \) = the i-th divided wormhole propagation time step, s

\( V_s \) = sand production rate, m³/s

\( x \) = horizontal coordinate in Cartesian coordinates system, m

\( y \) = vertical coordinate in Cartesian coordinates system, m

**Greek letters**
\( \tau \) = time when sources are removed, s

\((\xi, \zeta)\) = local coordinates, m

\( \mu(p) \) = pressure dependent foamy oil viscosity, Pa·s

\( \theta \) = internal angles between two adjacent elements

\( \phi(p) \) = pressure dependent porosity

\( \phi_w \) = average wormhole porosity

\( \phi \) = average formation porosity

\( \rho(p) \) = pressure dependent foamy oil density, kg/m\(^3\)

\( \Delta P \) = pressure change, Pa

\( \Omega \) = solution domain

\( \Gamma \) = outer boundary of the reservoir

**Functions**

\( erf \) = error function

\( exp \) = exponential function

\( f(L) \) = function of wormhole length with respect to space

\( H \) = Heaviside step function

\( \delta \) = Dirac delta function

\( G \) = free space Green’s function of a point source

\( G_L \) = free space Green’s function of a line source

\( L^{\{\ldots\}} \) = Laplace transformation

**Subscript and superscript**

\( D \) = dimensionless term
“~” = Laplace transformation of a function/variable

**SI Metric Conversion Factors**

\[
\begin{align*}
\text{bbl} & \times 1.589874 \quad \text{E-01} = \text{m}^3 \\
\text{cp} & \times 1.0* \quad \text{E-03} = \text{Pa} \cdot \text{s} \\
\text{ft} & \times 3.048* \quad \text{E-01} = \text{m} \\
\text{mD} & \times 9.869233 \quad \text{E-04} = \mu \text{m}^2 \\
\text{lbm} & \times 4.535924 \quad \text{E-01} = \text{kg} \\
\text{psi} & \times 6.894757 \quad \text{E+00} = \text{kPa}
\end{align*}
\]

*Conversion factor is exact*
CHAPTER 3  BOUNDARY INTEGRAL APPROACH FOR FLUID FLOW IN HETEROGENEOUS RESERVOIRS WITH REALISTIC GEOLOGY

Fluid flow in multi-scale and arbitrarily shaped heterogeneous systems based on geological interpretation of structures, facies, and architectures in a piece-wise manner of variation is solved by a boundary integral approach (BIA) with systematic partition and coupling strategies (Xiao et al., 2017). This technique extends the BIA to be compatible with geological models to obtain highly accurate pressure and production transients of multiple wells without finer grid sizes or smaller time steps. Comparisons to analytical solutions, existing boundary element methods (BEM), and semi-analytical methods have validated this work in aspects of subdividing strategies and modeling accuracy.

In this approach, two options are available to divide the reservoir into locally homogeneous subsystems: first is to partition along existing interface boundaries between different geo-bodies or structures and thus result in multiple irregular subsystems; the second is to subdivide the domain into identical rectangular subsystems and a few irregular subsystems as needed, and assign heterogeneity like finite difference or element simulators do. Each subsystem with uniformly average properties is then formulated in integral equations and systematically coupled with adjacent subsystems along interface boundaries to obtain solutions. Examples of calculating pressure and production transients of a fluvial system demonstrate its compatibility with both object and pixel based geologic modeling techniques. Pressure derivatives are very sensitive to geological
facies with different rock properties. This work shows a promise to develop BIA-based simulators.

3.1 Introduction

In the petroleum industry, transient pressure and production analysis is of great importance to estimate reservoir properties and provide quantitative information for enhanced oil recovery projects. The analytical solutions in homogeneous reservoir proposed by Carslaw and Jaeger (1959) are the foundation of modeling pressure and production transients. Pioneer researchers have extended these analytical solutions to bounded reservoirs with radial and rectangular boundaries, complex well-reservoir systems, multiple composite reservoirs, and commingled reservoirs (Gringarten and Ramey, 1973; Ozkan and Raghavan, 1991; Kuchuk and Wilkinson, 1991; Basquet et al., 1999a). Heterogeneous problems become more and more critical due to increasing efforts of targeting sweet spots and better resolutions of geological models. Finite difference simulations are mostly practiced for heterogeneous reservoirs; however, they require from fine gridding sizes and small time steps to obtain accurate pressure derivatives. Therefore, semi-analytical approaches become more advantageous because only boundaries need to be discretized and solutions are analytically obtained in integral forms. Zhao and Thompson (2002) used source/sink function method to calculate pressure transients of bounded complex reservoirs with heterogeneous facies. Zhao (2009) patented a reservoir simulation method to model 2D/3D heterogeneous reservoirs with complex well systems by using source/sink function and integral transformation methods. Medeiros et al. (2010) studied a horizontal well in heterogeneous formations by subdividing the reservoir into rectangular blocks and coupling analytical solutions of each
block along boundaries. Multiple scales of heterogeneities such as stimulated reservoir volumes and natural fractures in unconventional reservoirs have drawn intensive attentions recently and semi-analytical solutions were presented in a few works (Zhao, 2012; Kuchuk and Denis, 2013).

However, most of the aforementioned semi-analytical approaches are based on closed-form Green’s functions and are limited to regular-shaped heterogeneities and boundaries. Moreover, the kernel functions are often in summation forms of infinite series because of using imaging theory and are cumbersome to calculate accurately. Boundary element method (BEM) with a heritage dating back to early 18th century has taken shape in the 1970s and only required to integrate free space Green’s functions along boundaries (Cheng and Cheng, 2005). It requires less computing efforts and is suitable for reservoirs with irregular boundaries. Numbere and Tiab (1988) generated streamlines of steady state flow in homogeneous reservoirs using BEM. Kikani and Horne (1992, 1993) applied BEM to solve for unsteady state fluid flow problems in arbitrarily shaped reservoirs and later extended to composite reservoirs with multiple heterogeneous regions. Sato and Horne (1993) developed perturbation methods for steady state fluid flow in heterogeneous reservoirs. Pecher and Stanislav (1996) investigated transient pressure responses of large scale heterogeneous reservoirs and obtained solutions for vertical fractured wells with uniform flux.

In this chapter, heterogeneous systems are considered to be composed of locally homogeneous compartments with average reservoir and fluid properties, which have been shown to be consistent with effective effects of pressure transients in aforementioned literature. A boundary integral approach with systematic partition and coupling strategy,
as an extension of BEM, is proposed to deal with different types of heterogeneous systems including multi-scale composite and compartmentalized reservoirs with arbitrarily shaped boundaries (Xiao et al., 2017). Moreover, analytical integration of free space Green’s function along boundaries improves the calculating speed and accuracy. The proposed boundary integral approach shows great compatibility with complex geological models. Depending on the resolution of geological interpretation, two main subdividing strategies are also discussed.

3.2 Methodology

3.2.1 Problem definition and assumptions

A heterogeneous reservoir with five locally homogeneous regions is shown in Figure 3.1. The outer boundary of the reservoir denoted as $\Gamma_{\text{outer}}$ can be prescribed to Dirichlet condition, Neuman condition, or mixed boundary condition. To simply demonstrate the proposed methodology, I assume the pressure is initially uniform throughout the reservoir and the outer boundary is prescribed as a no-flow condition. To solve unsteady state fluid flow problems in the heterogeneous system, the following assumptions need to be addressed:

1. The fluid flow is single phase and slightly compressible, and no thermal or chemical reaction occurs.

2. Fluid properties and boundaries between locally homogeneous regions are static. Within each region, rock and fluid properties are assumed to be uniform with effective average values.
3. Wells are fully penetrating the reservoir with uniform thickness, and are producing under either constant pressure or constant rate.

According to the mass conservation and Darcy’s law, fluid flow in each locally isotropic and homogeneous region is described by the diffusivity equation as

\[ \frac{k_j}{\mu_j} \nabla^2 p_j = \phi_j c_{ij} \frac{\partial p_j}{\partial t} + \sum_{i=1}^{N_{wj}} q_{ji} \delta(x-x') \delta(y-y'), \quad j=1, N_R; \quad i=1, N_{wj} \quad [3.1] \]

where, \( \phi_j, \mu_j, k_j, c_{ij} \) are reservoir and fluid properties for each region. \( \delta \) is the Dirac delta function to define the location of each source and sink with rate \( q_{ji} \) and \( p_j \) is the pressure change in region \( j \). \( N_R \) is the total number of locally homogeneous regions of a heterogeneous reservoir, and \( N_{wj} \) is the numbers of source and sinks for each region.

### 3.2.2 Dimensionless governing equations

Dimensionless parameters not only simplify equation forms but also provide universal solutions to different reservoirs with certain physical relations. Therefore, it is essential to define the following dimensionless terms for Eq. [3.1]:

\[ t_D = \frac{0.006328 k_1}{\phi_1 \mu_1 c_{rl} l^2}, \quad x_D = \frac{x}{l}, \quad y_D = \frac{y}{l}, \quad Q_{Dj} = \frac{q_j}{q_B}; \quad p_{Dj} = \frac{k_1 h_1}{141.2 q_B \mu_1} (P_i - p_j) \quad [3.2] \]

where, \( l \) is the arbitrarily defined reference length and \( q_B \) is a known base rate with the same unit as \( q_j \), and \( P_i \) is the initial reservoir pressure.

As mentioned above, to physically relate different regions I also defined the dimensionless transmissibility ratio (\( C_T \)), storativity ratio (\( C_S \)), and diffusivity ratio (\( C_R \)) for each region.
Figure 3.1 (a) A heterogeneous reservoir with five locally homogeneous regions. Rock and fluid properties of the five regions vary between the regions, but are constant in each region; (b) Partition the reservoir to nine subsystems and name each subsystem with Arabic numbers. The outer boundary $\Gamma_{\text{outer}}$ is further divided to eight parts and belongs to eight different subsystems. 12 interface boundaries are identified between adjacent subsystems. Notice that subsystems 1 to 3 have the same properties with region I and similarly subsystems 7 to 9 share same properties with region V.
\[ C_{Tj} = \frac{k \cdot h_i}{\mu_j} / (k \cdot h_i / \mu_i) \], \[ C_{Sj} = (\phi_j \cdot c_g \cdot h_j) / (\phi_i \cdot c_i \cdot h_i) \], \[ C_{Rj} = \frac{C_{Tj}}{C_{Sj}} \], \ j = 1, N_R \quad [3.3] 

Theoretically, I can choose any region as the reference to calculate \( C_{Tj}, C_{Sj}, C_{Rj} \).

Accordingly, each homogeneous region with average rock and fluid parameters can be normalized in dimensionless forms by Eq. [3.2] and [3.3], and Eq. [3.1] becomes

\[ C_{Tj} \nabla^2 p_{Dj} = C_{Sj} \frac{\partial p_{Dj}}{\partial t_D} + \sum_{i=1}^{N_{wj}} Q_{Dji} \delta(x_D - x_{Dij}) \delta(y_D - y_{Dij}), j = 1, N_R; i = 1, N_{wj} \quad [3.4] \]

3.2.3 Boundary integral formulations

According to Kikani and Horne (1992, 1993), integral equations corresponding to Eq. [3.4] are derived from the weighted residual formulation as

\[ \theta_j p_{Dj}(x_D, y_D, C_{Rj}, t_D) = \int_{t_D}^{t_D} \int_{\Gamma_j} \left\{ \left( \frac{1}{C_{Tj}} G_j \frac{\partial p_{Dj}}{\partial n} - p_{Dj} \frac{\partial G_j}{\partial n} \right) dS + \sum_{i=1}^{N_{wj}} Q_{Dji} G_j \right\} d\tau \quad [3.5] \]

\[ \theta_j = 2\pi \rightarrow if (x_D, y_D) \in \Omega_j, \]
\[ \theta_j = \theta_j \rightarrow if (x_D, y_D) \in \Gamma_j \quad j = 1, N_R; i = 1, N_{wj} \quad [3.6] \]

where \( \theta_j \) are the internal angles between two adjacent elements on the boundaries. \( G_j \) is the free space Green’s function, which physically represents the pressure response due to an instantaneous point source removed at time \( \tau \) with unit strength at location \((\xi, \zeta)\) (Carslaw and Jaegar, 1959). The free space Green’s function in dimensionless form is

\[ G_j(x_D, y_D, t_D, \xi, \zeta, \tau) = \frac{H(t_D - \tau)}{4\pi(t_D - \tau)} \exp\left[-\frac{(x_D - \xi)^2}{2\sqrt{C_{Rj}(t_D - \tau)}}\right] \exp\left[-\frac{(y_D - \zeta)^2}{2\sqrt{C_{Rj}(t_D - \tau)}}\right] \quad [3.7] \]
3.2.4 Coupling and partition strategy of the boundary integral approach (BIA)

Kikani and Horne (1993) proposed a BEM for a two-zone composite reservoir and claimed it could be extended to multiple regions. Their work showed promise of using BEM to solve multi-scale and irregularly shaped heterogeneous systems. However, the partition strategy in their work was more convenient to deal with composite systems instead of the problem defined in Figure 3.1a. Pecher and Stanislav (1996) solved a compartmentalized problem, however, limited in a simple case with three consecutive zones. Therefore, a systematic partition and coupling strategy is proposed to solve complex heterogeneous systems, and can be applied to both composite and compartmentalized reservoirs.

It is important to differentiate the terminologies “region” and “subsystem” in this study. A region refers to an identified geological body with average rock and fluid properties bounded by physically existing boundaries. A subsystem is a locally homogeneous domain with arbitrary boundaries as a result of the partition strategy applied to a heterogeneous system. A region can be either one subsystem or divided into several subsystems. Mathematically, they have the relation as “subsystem” $\subseteq$ “region”.

Figure 3.1b shows that the heterogeneous system is partitioned into nine subsystems. Four additional subsystems result from subdivision of region I and V. Accordingly, the reservoir is fully compartmentalized and each subsystem is numbered consecutively. Notice that subsystems 1 to 3 have the same properties with region I, and similarly subsystems 7 to 9 share same properties with region V.
The partition and coupling strategy then becomes straightforward:

**Partition** the heterogeneous system into fully compartmentalized homogeneous subsystems and discretize their boundaries into elements; number the boundary elements of adjacent subsystems in opposite directions, taking subsystem 1 and 2 for example as shown in Figure 3.2.

**Couple** adjacent subsystems along the interface boundaries by continuity of pressure and flux (i.e. couple the subsystem 1 with subsystems 2 and 4 along interface boundaries $\Gamma_{1,2}$ and $\Gamma_{2,4}$ respectively).

I divide the boundaries of the $j$-th homogeneous subsystem into $N_j$ constant elements, where pressure drops are evaluated at the center of each element. Eq. [3.5] can be written in discrete forms at the dimensionless evaluating point $(\xi_{Dk}, \zeta_{Dk})$ as

$$
\theta_{jk} p_{Djk}(\xi_{Dk}, \zeta_{Dk}, C_{Rj} t_D) = \int_{t_D}^{t_D} \sum_{k=1}^{N_j} \int_{\xi_{1} - \alpha_{jk}}^{\xi_{Nj} + \alpha_{jk}} \left[ \frac{1}{C_{ij}} \frac{\partial p_{Djk}(\xi_{Dk}, \zeta_{Dk}, C_{Rj} t_D)}{\partial n} - p_{Djk}(\xi_{Dk}, \zeta_{Dk}, C_{Rj} t_D) \frac{\partial G_j}{\partial n} \right] + \sum_{i=1}^{N_{wj}} Q_{Dji} G_j d\tau \tag{3.8}
$$

where, $\alpha_{jk}$ is the dimensionless half-length of each boundary element, $N_{Subs}$ is the total number of subsystems, and $N_{wj}$ is the number of source and sinks in each subsystem.
Figure 3.2 Boundary elements numbering strategy between adjacent subsystems (i.e. subsystem 1 and 2 in Figure 3.1 b. They share the same boundary elements along the interface boundary $\Gamma_{1,2}$. 

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3.2.5 Linear matrix equations in Laplace space

Constant elements are advantageous in this study for two reasons: first, integrals of the Green’s function along the boundary in Eq. [3.8] can be obtained analytically, resulting in improvement of calculating speed and accuracy; secondly, singularity at sharp corners of adjacent elements is naturally avoided. A lot of constant elements will be used for curvature boundaries.

The first integral part of the Green’s function along the boundary becomes:

\[
\int_{\xi_n^a}^{\xi_n^b} \left[ \mathcal{G} \right] = (p_{\text{coef}}) = 0.5\Theta_L\left(\frac{-a_{Dk} - \xi_{Dk}}{2}, \frac{a_{Dk} - \xi_{Dk}}{2}, C_{Rj} t_D\right)\Theta_L\left(\frac{\xi_{Dk}}{2}, C_{Rj} t_D\right) \quad [3.10]
\]

The second integral part of the gradient of Green’s function along the boundary becomes:

\[
\int_{\xi_n^a}^{\xi_n^b} \left[ \frac{\partial \mathcal{G}}{\partial n} \right] = (p_{\text{coef}}) = \frac{\xi_{Dk}}{4C_{Rj} t_D} \Theta_L\left(\frac{-a_{Dk} - \xi_{Dk}}{2}, \frac{a_{Dk} - \xi_{Dk}}{2}, C_{Rj} t_D\right)\Theta_L\left(\frac{\xi_{Dk}}{2}, C_{Rj} t_D\right) \quad [3.11]
\]

Special functions in Eq. [3.10] and [3.11] are defined according to Zhao (2002) as

\[
\Theta_L(z_1, z_2, t) = \frac{1}{2}\left[ \text{erf}\left(\frac{z_2}{\sqrt{t}}\right) - \text{erf}\left(\frac{z_1}{\sqrt{t}}\right) \right], \quad \text{where} \ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad [3.12]
\]

\[
\Theta_L(z, t) = \frac{1}{\sqrt{\pi}} \exp\left(\frac{-z^2}{\sqrt{t}}\right), \quad \text{where} \ \text{exp}\ \text{denotes the exponential function} \quad [3.13]
\]

Laplace transformation naturally converts the convolution in Eq. [3.8] into algebraic forms in Laplace space and has been widely applied in BEM and semi-analytical approaches (Kikani and Horne, 1992 and 1993; Ozkan and Raghavan, 1991; Zhao and Thompson, 2002; Medeiros et al., 2010; Stalgorova and Mattar, 2013). Moving
fictitious source locations along all boundary elements and evaluating integrals analytically, I rewrite Eq. [3.8] to matrix equations in Laplace space as

\[
\sum_{k=1}^{N_j} \left\{ \left( \tilde{P}^{\text{coef}} \right)_{jk} - 0.5 \delta_{jk} \right\} \tilde{P}_{Djk} (\xi_{Dk}, \zeta_{Dk}, s) + \sum_{k=1}^{N_j} \left\{ \left( \frac{1}{C_{T_j}} \tilde{P}^{\text{ncoef}} \right)_{jk} \right\} \tilde{P}_{Dnjk} (\xi_{Dk}, \zeta_{Dk}, s)
\]

\[= -\frac{1}{s} \sum_{k=1}^{N_j} \{ \sum_{t=1}^{N_{nj}} Q_{Djk} (\tilde{S}_s)_{jk} \}; k = 1, N_j; j = 1, N_{\text{sub}}; i = 1, N_{wj} \]  \[\text{[3.14]}\]

where,

\[
(\tilde{S}_s)_{jk} = L \left\{ 0.25 \Theta_L \left( \frac{\xi_{Dk} - x_{Di}}{2}, C_{Rj} t_D \right) \Theta_L \left( \frac{\zeta_{Dk} - y_{Di}}{2}, C_{Rj} t_D \right) \right\}
\]

\[\text{[3.15]}\]

\(L\{\ldots\}\) is the Laplace transformation of functions with respect to \(t_D\), and I also denoted \(\tilde{p}_{Djk} = L\{\hat{\partial} p_{Djk} / \hat{\partial} n\}\) and \(\tilde{p}_{Djk} = L\{p_{Djk}\}\). \((x_{Di}, y_{Di})\) is the location of the source and sink in the subsystem \(\Omega_j\), and \((\xi_{Dk}, \zeta_{Dk})\) is the dimensionless evaluating point.

Instead of deriving analytical expressions for the Laplace transforms of the source functions documented in the work of Kikani and Horne (1992), the numerical approach of Laplace transformation of source functions of any kind proposed by Zhao and Thompson (2002) is adapted. The reason is that the source functions such as ones in Eq. [3.14] and [3.15] could become too complex to have analytical forms of Laplace transformation. The products of the functions described in Eq. [3.10] and [3.11] have been numerically transformed in Laplace space. The numerical approach detailed in Appendix A of Zhao and Thompson (2002) uniquely applied the Chebyshev polynomial with combination of early and late time approximations to evaluate the time integrals in the Laplace transformation. It is shown in their work that near-analytical accuracy could be obtained by the proposed numerical Laplace transformation.
For each subsystem $j$, I set up $N_j$ sets of linear equations by evaluating pressure drops at $N_j$ allocated points along the boundary. Due to continuity of pressure and flux at the interface boundaries of adjacent subsystems, I have additional coupling equations along the interface boundaries, taking the interface boundary between subsystem 1 and 2 for example:

$$C_{T1}p_{D1}((x_D, y_D) \in \Gamma_{12}, C_{R1}t_D) = C_{T2}p_{D2}((x_D, y_D) \in \Gamma_{12}, C_{R2}t_D)$$  \[3.16\]

$$\frac{\partial p_{D1}}{\partial n}((x_D, y_D) \in \Gamma_{12}, C_{R1}t_D) = -\frac{\partial p_{D2}}{\partial n}((x_D, y_D) \in \Gamma_{12}, C_{R2}t_D)$$  \[3.17\]

The decomposed heterogeneous system shown in Figure 3.1 b has eight sub-divided outer boundaries (each has $N_{\text{outer}}$ of constant boundary elements) and 12 interface boundaries (each has $N_{\text{inter}}$ of constant elements). The system has $8\times N_{\text{outer}}+24\times N_{\text{inter}}$ of unknown $p_{Djk}$ and $8\times N_{\text{outer}}+24\times N_{\text{inter}}$ of unknown $\partial p_{Djk} / \partial n$. A no flow boundary condition is prescribed to the eight sub-divided outer boundaries, reducing unknown $\partial p_{Djk} / \partial n$ numbers by $8\times N_{\text{outer}}$. Moreover, by coupling pressure and rate on the 12 interface boundaries, I can further reduce the numbers of unknown $p_{Djk}$ and $\partial p_{Djk} / \partial n$ by $12\times N_{\text{inter}}$ respectively. Therefore, there will be $8\times N_{\text{outer}}+12\times N_{\text{inter}}$ of unknown $p_{Djk}$ and $12\times N_{\text{inter}}$ of unknown $\partial p_{Djk} / \partial n$. By moving the fictitious sources along all boundaries in the system, total of $8\times N_{\text{outer}}+24\times N_{\text{inter}}$ of linear equations will be formed to solve for $8\times N_{\text{outer}}+12\times N_{\text{inter}}$ of unknown $p_{Djk}$ and $12\times N_{\text{inter}}$ of unknown $\partial p_{Djk} / \partial n$. Once all $p_{Djk}$ and $\partial p_{Djk} / \partial n$ along boundaries are calculated, Eq. [3.14] will be used again to evaluate pressure drop at any point in the heterogeneous system. Finally, all solutions in Laplace space are inverted to real space by using the Stehfest algorithm (Stehfest, 1970).
3.2.6 Transient production rate by Duhamel’s principle

Practically, pressure and pressure derivative data are expensive to acquire comparing to daily production rate. In the proposed BIA, pressure transients of the heterogeneous system are obtained in the Laplace space, therefore, it is straightforward to apply the Duhamel’s principle, as shown in Eq. [3.18], to calculate production responses in the Laplace domain (van Everdingen and Hurst, 1949). Consequently, transient production rate with respect to time can be obtained for complex heterogeneous systems.

\[
\tilde{q}_{WD_i}(x_{Di},y_{Di},s) = \frac{1}{\tilde{p}_{WD_i}(x_{Di},y_{Di},s) \cdot s^2}
\]  

where, \(\tilde{q}_{WD_i}(x_{Di},y_{Di},s)\) is the dimensionless production rate of a well at \((x_{Di},y_{Di})\) producing under constant bottom hole pressure. \(\tilde{p}_{WD_i}(x_{Di},y_{Di},s)\) is the dimensionless pressure drop of a producing well under constant production rate.

3.3 Validation

This section is to validate the partition and coupling strategy and the modeling accuracy of the proposed BIA. To the author’s knowledge, there are very few published approaches to accurately model transient fluid flows in reservoirs with realistic heterogeneity. However, simplified composite reservoir models have been successfully solved by analytical, semi-analytical, and BEM approaches (KAPPA, 2013; Zhao, 2002; Kikani and Horne, 1993). Therefore, two cases of composite reservoirs shown in Figure 3.3 were solved by the proposed BIA. Figure 3.3 a is a two-region radial composite model with an outer no-flow boundary. Figure 3.3 b shows a rectangular reservoir with three locally homogeneous regions.
Figure 3.3 Two simplified heterogeneous models: (a) a two-region radial composite model with an outer no-flow boundary, where the properties of inner and outer regions are different; (b) a three-region composite rectangular model prescribed to a no-flow boundary. The two locally homogeneous regions are rectangular with different properties.
Using the BIA in this work, I partitioned the composite problem defined in Figure 3.3a to nine subsystems and set all their properties the same with the radial outer region except for the subsystem 5, which was set the same with the inner region. As shown in Figure 3.4a, the radial domain is composed of nine subsystems with irregularly shaped boundaries. This particular case shows the power of the proposed BIA by subdividing the system into multi-scale and arbitrarily-shaped subsystems. It offers more flexibility in terms of gridding system than conventional finite difference or element approaches.

Appendix B shows more details of the solution process of the radial composite system with irregular subsystems (or grids). Figure 3.4b used similar partition strategy but with rectangular-shaped subsystems to model the reservoir defined in Figure 3.3b. To differentiate from arbitrarily shaped subsystems, I call the rectangular subsystems “blocks”. Blocks 6 and 11 have same properties with region 1 and region 2 respectively, and the remaining blocks have the same properties as region 3. For all partitioned subsystems, 20 constant elements were assigned along boundaries.

A commercial software was used to obtain analytical solutions of the radial composite model (KAPPA, 2013). Figure 3.5 compares the results of this work with analytical solutions of the radial composite model. Perfect match is obtained on both dimensionless pressure and pressure derivative responses. It validates the systematic partition and coupling strategy, which converts the composite reservoir to fully compartmentalized reservoir along irregularly shaped heterogeneities.
Figure 3.4 Partition strategy of the two simplified heterogeneous model: (a) the partitioned two-region radial composite model with nine subsystems, where subsystem 5 has properties of the inner region and the rest subsystems have properties of the outer region; (b) the reservoir defined in Figure 3.3 b is partitioned with 16 rectangular subsystems (blocks). Block 6 has properties of region 1, block 11 has properties of region 2, and the rest blocks have properties of region 3.
Figure 3.5 Pressure and pressure derivative responses of a vertical well producing from the centre of a two-region radial composite model. The diffusivity of the inner region is 10 times better than the outer region. The analytical solutions were obtained from a commercial software (KAPPA, 2013).
The composite nature of the rectangular composite reservoir shown in Figure 3.3 b was solved by the BEM approach proposed by Kikani and Horne (1993). Moreover, Zhao (2009) patented a semi-analytical reservoir simulation method to solve complex heterogeneous problems based on source and sink function method. Solutions of the problem in Figure 3.3 b were also provided by his technique.

Figure 3.6 a shows good agreement of the proposed BIA with Kikani and Horne’s solutions and Zhao’s solutions on both dimensionless pressure and pressure derivatives. Figure 3.6 b compares pressure derivatives of two cases varying diffusivities in region 2. The log-log plots of pressure derivatives are sensitive to its heterogeneity. Better properties of region 2 result in lower pressure drops on derivatives. The change is subtle and lasts very shortly due to relatively long distance between the well and region 2. Only accurate modeling results can generate correct pressure derivatives to capture such subtle changes. It also indirectly shows the high accuracy of using constant boundary elements. Accordingly, the proposed BIA with systematic partition and coupling strategy is validated to solve multi-scale and irregularly shaped heterogeneous systems.
Figure 3.6 Benchmarking modeling results: (a) Pressure and pressure derivative responses of a vertical well producing from region 1, and I assigned diffusivity ratios to different regions as $C_{R_{region1}}=0.5$, $C_{R_{region2}}=100$, $C_{R_{region3}}=1$; (b) Comparison of pressure derivatives of two cases with different diffusivity ratios, Case 1: $C_{R_{region1}}=10$, $C_{R_{region2}}=1$, $C_{R_{region3}}=1$; Case 2: $C_{R_{region1}}=10$, $C_{R_{region2}}=100$, $C_{R_{region3}}=1$. 
3.4 Application to a Fluvial System

3.4.1 Conceptual geological model of a fluvial system

Seismic data, well logs, cores, and outcrop analogs always provide geologists powerful tools to interpret reservoir geometries, architectures, and properties. Reservoirs are never homogeneous in both outcrop analogs and geological models. Even in a simplified conceptual geological model as shown in Figure 3.7, a fluvial depositional system may consist of multi-scale heterogeneities with irregular shapes (Deutsch and Tran, 2002). As fundamental building blocks of a reservoir, sedimentary facies are genetic results of depositional history and may stack in particular sequence with various scales and properties.

Pioneer geologists have worked on characterizing lithology and origin of fluvial sediments, and proposed widely accepted depositional models (Allen, J.R.L., 1965; Miall, A.D., 1978). The channel fill sandstone usually provides the best porosity and permeability with sinuous shapes extending to reservoir boundaries. Levee facies are thin and silty deposits lateral to the channel facies. Crevasse splay facies were deposited discretely in the reservoir and segmented the floodplain mudstones. I need to state that no further detailed geologic modeling work has been done in this work and the heterogeneities presented in what follows are arbitrary based on the conceptual geological model. Nonetheless, my purpose is to showcase the compatibility of the proposed BIA with different geologic modeling techniques.
Figure 3.7 Bird view of a modified 2D conceptual geological model of a fluvial system (not to scale) with uniform thickness used in the object-based geo-modeling software FLUVSIM (Deutsch and Tran, 2002).
3.4.2 Compatibility with object-based geologic modeling

Object-based modeling techniques place different objects in the reservoir and simulate them in a hierarchical approach so that they can well represent geometries of different geo-bodies, and are particularly applicable in fluvial systems (Deutsch and Wang, 1996). Realizations of object-based models are usually very similar to conceptual geological models with sinuous genetic units. With the proposed BIA, I partitioned a possible realization of a fluvial system using object-based geologic modeling technique as shown in Figure 3.8. Dimensionless diffusivity ratios \( (C_R) \) of different facies are different due to heterogeneous rock properties. I chose the properties of the flood plain facie as the reference, and assumed that diffusivities of the channel, splay, and levee facies were 100, 50, and 5 times better than the flood plain facies respectively.

Unlike using finite difference and finite element method, there is no compromise to deal with the irregular boundaries of sinuous facies presented in the object-based realizations. Relatively more elements are used along the boundaries with larger curvatures. Figure 3.9 shows the pressure, pressure derivative, and production rate of a vertical well producing from the center of the fluvial system. The pressure derivative on log-log plot is very different from the homogeneous case, in which slopes evolve steeply from the zero-slope radial flow regime to the unit-slope boundary dominated flow.
**Figure 3.8** A possible realization of a fluvial system (not to scale) using object-based geologic modeling technique. It is partitioned into 16 subsystems by the proposed BIA. Different colors represent different facies with various properties (i.e. diffusivity ratios $C_R$). The diffusivity of the channel facies is 100 times better than the flood plain facies. A vertical well is producing under constant production rate from the centre of the reservoir.
Figure 3.9 Pressure, pressure derivative, and production rate of a vertical well producing from the centre of a fluvial system shown in Figure 3.8. A homogeneous case is given assuming the whole reservoir is the channel facies.
In contrast with the homogeneous case, the pressure derivative of the fluvial system shows gradual growth of slopes indicating more flow regimes. In the fluvial system, the well is located in the channel fill sandstone with the best permeability and porosity. When pressure transients reach poorer regions, the pressure derivative starts deviating from the zero slope line and approaches to pseudo-radial flow regimes before reaching the outer no-flow boundary. The transient production rate obtained by the Duhamel’s principle also shows gradual decline before reaching boundary dominated flow.

### 3.4.3 Compatibility with pixel-based geologic modeling

Pixel-based modeling techniques are practiced by assigning properties to each pixel through various approaches such as Kriging, Sequential Gaussian Simulation (SGS), and multi-point statistics (Matheron, 1963; Deutsch, 2002; Strebelle, 2002). They are very powerful to simulate multiple scales and non-stationary geology, and are able to be conditioned to various data; however, they have some difficulties to model continuous and sinuous objects. Therefore, possible realizations generated by pixel-based modeling are most likely to be consisting of regularly-shaped pixels with different properties as shown in Figure 3.10. Dimensionless diffusivity ratios ($C_R$) of different pixels are different due to heterogeneous facies they belong to. I chose the properties of pixels representing the flood plain facies as the reference, and assumed diffusivities of the channel, splay, and levee facies were 100, 50, and 5 times better than the flood plain facies respectively. The proposed BIA is also compatible with the pixel-based geological model, which is partitioned to 117 rectangular subsystems (blocks). Each block naturally represents each pixel in the geological realization.
Figure 3.10 A possible realization generated by pixel-based modeling (not to scale) with 117 pixels. I partitioned the geological model along the boundaries of all pixels with 117 blocks and solved by the proposed BIA. Four fluvial facies were simulated and the channel fill sandstone has the highest diffusivity ratio $C_R=100$. Well 1 (block 30) is located in the levee facies with poorer reservoir properties, while Well 3 (block 59) and Well 5 (block 95) are located in the channel fill facies with the best permeability and porosity. Well 2 (block 43) and Well 4 (block 79) are located in the splay facies and flood plain facies respectively.
I assigned boundary elements to each partitioned block and calculated nodal values at boundaries to obtain pressure changes at any location of the reservoir. As shown in Figure 3.10, there are five wells producing at the same time at different locations. Well 1 is located in the levee facies (block 30) with poor reservoir properties, while Well 3 (block 59) and Well 5 (block 95) are located in the channel fill facies with the best permeability and porosity. Well 2 (block 43) and Well 4 (block 79) are located in the splay facies and flood plain facies respectively. Figure 3.11 and Figure 3.12 show their pressure and production responses respectively. Generally, wells located in better facies have higher production rates. Well 3 and 5 initially have the same production rate, however, starting from around dimensionless time 0.04, Well 5 has higher production rate due to better surrounding facies. The geological model shows that the splay facies around Well 5 has ten times better diffusivity than the levee facies near Well 3. It is clearer from pressure derivative responses that Well 3 has higher pressure drops and deviates earlier from the zero slope line. It is of great interest to notice that the pressure derivative of Well 4 deviates from the zero slope line at about dimensionless time 0.1 with a negative slope, which indicates improvement of diffusivity as pressure transient travels further from the well into better facies. It is agreed with the interpretation from the geological model that the channel facies with better permeability and porosity is adjacent to Well 4. When the pressure transient has passed the channel facies, it reaches the flood plain facies with the poorest properties, and the pressure derivative increases again with positive slopes before reaching the boundary dominated flow regime.
Figure 3.11 Pressure and pressure derivative responses of five vertical wells producing under constant production rate in a fluvial system. Well 1 is located in the levee facies \((C_R=5)\), Well 2 is located in the splay facies \((C_R=50)\), Well 3 and Well 5 are located in the channel fill facies \((C_R=100)\), and Well 4 is located in the poorest flood plain facies \((C_R=1)\).
Figure 3.12 Production rates of five vertical wells producing under constant pressure in a fluvial system. Well 1 is located in the levee facies ($C_R=5$), Well 2 is located in the splay facies ($C_R=50$), Well 3 and Well 5 are located in the channel fill facies ($C_R=100$), and Well 4 is located in the poorest flood plain facies ($C_R=1$).
3.5 Discussion

3.5.1 Subdivision strategies

I have shown the accuracy and compatibility of the proposed BIA to model complex heterogeneous systems. Two options are available to subdivide the heterogeneous reservoir into locally homogeneous subsystems:

1. Partition along existing interface boundaries between different geo-bodies or structures resulting in multiple irregular subsystems (i.e. Figure 3.8). Use the partition and coupling strategy to set up different matrix equations for subsystems with various geometries. Advantage: numbers of subsystems only depend on identified geological bodies so that it can save computer memory and calculation time. Disadvantage: different realizations of geological models may have different geometries of simulated geological bodies, which result in different matrix equations for each realization. It requires efforts to re-program the whole matrix structure because geometries of partitioned subsystems have changed. Therefore, it is cumbersome to automatically update the program structure when there are hundreds of geological realizations.

2. Subdivide the domain into standard rectangular subsystems (or blocks) and a few special irregular subsystems as needed, and assign heterogeneities like finite difference or element simulators do to each block (i.e. Figure 3.10). Accordingly, a universal matrix equation system is obtained because of the identical blocks with known geometries. Users only need to input proper values of reservoir and fluid properties to each block interpreted from geological models. Block numbers only depend on the resolution of geological models (i.e. numbers of pixels of
pixel-based geological realizations), and will not affect the accuracy of modeling results due to the analytical nature of the proposed BIA. Advantage: it is easy to automatically update the program because the matrix system is universal. Also, most blocks are identical in geometry so that boundary elements can be assigned and calculated more efficiently. Disadvantage: some simple heterogeneous problems may not need to partition with so many blocks; for example, the cases in Figure 3.3 only need two or three subsystems. In such cases, relatively larger matrices will waste memory storage and affect computing speed.

### 3.5.2 Effects of boundary elements on modeling accuracy

Because of the analytical evaluation of integrals used in the source functions (Eq. [3.10] and [3.11]) and analytic nature of solving partial differential equations using free space Green’s function (Cheng and Cheng, 2005), the calculated pressure derivatives of heterogeneous cases using the proposed BIA approach show great accuracy by using very few boundary elements. For the radial composite model shown Figure 3.3 a, boundary elements have been assigned to all subsystems as illustrated in Appendix B. Ideally, each subsystem should have optimized numbers of boundary elements due to different morphologies of all subsystems. However, at the preliminary stage of testing the concept of the proposed BIA approach, it is much easier in programming to assign the same number of boundary elements to all subsystems. Therefore, the same numbers of boundary elements have been assigned to all the subsystems in Figure 3.4 a and the sensitivity of boundary elements numbers on the accuracy of pressure derivative responses has been evaluated in this work. To better capture the effects of property contrast between the inner and outer regions on pressure derivative, I input the diffusivity
ratio \((C_R)\) of the inner region as 0.1 and the \(C_R\) of the outer region as 1000. **Figure 3.13** shows that when the numbers of boundary elements of each subsystem are less than 12 the calculated responses will show slightly less accuracy in this particular heterogeneous case. The disagreement starts from the concave feature on the pressure derivatives, during which the pressure transients diffuse from the inner region to the outer region and feel the heterogeneity. It indicates that local heterogeneity requires higher accuracy in terms of reaching by increasing boundary elements number.

The MATLAB code based on Eq. [3.14] for the proposed model has been run on a PC with Intel i3 2.1GHz CPU. **Table 3.1** summarizes the CPU run time of using different numbers of boundary elements for the entire curve of each case shown in **Figure 3.13**. Computation time will increase significantly with more boundary elements. Also, the computation time spent on the numerical Laplace transformation according to Zhao and Thompson (2002) is relatively large. Nonetheless, the numerical Laplace transformation is capable of calculating complex source functions in the proposed BIA approach. Future work needs to focus on improving the efficiency of numerical computation of Laplace transformation.

The curvature features of subsystems generally require more boundary elements to represent the curved boundaries. On the other hand, the heterogeneous case shown in **Figure 3.10** has used different strategies to subdivide the system into identical blocks. On each side of the block, the same number of elements is used to subdivide the boundary. 15, 10, 5, 2, and 1 boundary elements on each side have been evaluated. Due to the difference of surrounding heterogeneity at each well, the sensitivities of boundary element numbers on pressure derivative responses are various at different wells.
Figure 3.13 Pressure derivative responses of a vertical well producing from the center of a two-region radial composite model in Figure 3.3 a. The diffusivity ratio of the inner region is 0.1 and the diffusivity ratio of the outer region is 1000. Boundary element numbers of each subsystem ($Nb$) evaluated in the sensitivity study are 4, 8, 12, 40, and 80 respectively. The curve of $Nb=40$ has been overlapped by the curve of $Nb=80$. 
Table 3.1 Summary of CPU run time (on the platform of 2.1 GHz Intel i3 Core CPU) of the proposed BIA approach using different boundary element numbers for the entire curve of each case shown in Figure 3.13

<table>
<thead>
<tr>
<th>Boundary Element Number</th>
<th>( Nb=4 )</th>
<th>( Nb=8 )</th>
<th>( Nb=12 )</th>
<th>( Nb=40 )</th>
<th>( Nb=80 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Total CPU Run Time (s)</strong></td>
<td>234</td>
<td>288</td>
<td>318</td>
<td>984</td>
<td>2701</td>
</tr>
<tr>
<td><strong>Total Percentage of Numerical Laplace Transform Computation</strong></td>
<td>93.5%</td>
<td>92.7%</td>
<td>92.5%</td>
<td>93.5%</td>
<td>91.2%</td>
</tr>
</tbody>
</table>
As shown in Figure 3.14, when the boundary element numbers of each block side is greater than 2 the calculated pressure derivative responses of Well 1 will be very accurate. Figure 3.15 shows that, even though there is only one boundary element for each block side, the calculated pressure derivative of Well 3 will be almost the same as the one using 15 elements on each block side. The local heterogeneities surrounding the wells would definitely affect the choice of boundary element numbers to obtain high accuracy of modeling results. The optimized selection of boundary elements numbers for various heterogeneous geo-bodies should be considered in future development of BIA-based simulators.

3.6 Summary

A boundary integral approach (BIA) with systematic partition and coupling strategy is presented to solve transient fluid flow in multi-scale and arbitrarily shaped heterogeneous systems, and is compatible with both object-based and pixel-based geologic modeling techniques. The proposed BIA, exemplified with a fluvial system, is aware of realistic geology and only needs to discretize boundaries of partitioned subsystems. The modeling results are independent of gridding size and time steps because of the analytical nature of BIA and Laplace transformation. Transient pressure and production responses of multiple wells producing in the fluvial system are obtained simultaneously. Log-log plots of pressure derivatives, which have altered pressure transient behavior and flow regimes, are shown to be very sensitive to surrounding facies. This work, with two discussed partition options, has shown a promise of developing BIA-based reservoir simulators, which will be accurate to model transient fluid flow and compatible with realistic geology.
Figure 3.14 Pressure derivative responses of Well 1 located in the levee facies ($C_R=5$), producing under constant production rate in a fluvial system. Boundary element numbers ($Nb$) of each side of the identical blocks shown in Figure 3.10 evaluated in the sensitivity study are 1, 2, 5, 10, and 15 respectively. The curves of $Nb=5$, $Nb=10$, and $Nb=15$ are overlapped.
Figure 3.15 Pressure derivative responses of Well 3 located in the channel fill facies ($C_r=100$), producing under constant production rate in a fluvial system. Boundary element numbers ($Nb$) of each side of the identical blocks shown in Figure 3.10 evaluated in the sensitivity study are 1, 2, 5, 10, and 15 respectively. The curves of $Nb=2$, $Nb=5$, $Nb=10$, and $Nb=15$ are mostly overlapped.
Nomenclature

Notations

\( a_{Dk} \) = dimensionless half-length of the \( k \)-th constant boundary element

\( C_{Rj} \) = diffusivity ratio between the \( j \)-th and the reference subsystem

\( C_{Sj} \) = storativity ratio between the \( j \)-th and the reference subsystem

\( C_{Tj} \) = transmissibility ratio between the \( j \)-th and the reference subsystem

\( C_{ij} \) = total compressibility of the \( j \)-th subsystem, \(1/\text{psi}\)

\( dp_{WD} \) = dimensionless pressure derivative at the wellbore

\( h \) = formation thickness, ft

\( k \) = permeability of the reservoir, md

\( l \) = reference length, ft

\( Nb \) = identical boundary element numbers of each subsystem in the sensitivity study

\( N_j \) = number of total boundary elements of the \( j \)-th subsystem

\( N_R \) = number of locally homogeneous regions

\( N_{subs} \) = number of subdivided subsystems

\( N_{outer} \) = number of constant elements of sub-divided outer boundaries

\( N_{inter} \) = number of constant elements of each inter boundaries

\( N_{wj} \) = number of total source/sinks in the \( j \)-th subsystem

\( p_{WD} \) = dimensionless pressure drop at the wellbore

\( P_i \) = initial pressure, psi

\( p \) = pressure at any location, psi

\( p_{coef} \) = matrix coefficients for pressure variable \( p_{Dj} \) in the \( j \)-th subsystem
\( p_{ncoef} \) = matrix coefficients for flux variable \( p_{Dnjk} \) in the j-th subsystem

\( q \) = rate, bbl/day

\( q_B \) = base rate, bbl/day

\( q_{WD} \) = dimensionless production rate at the wellbore

\( Q_{Dj} \) = dimensionless rate of source/sink in the j-th subsystem

\( s \) = Laplace variable

\( S_s \) = strength coefficients for the source and sink \( Q_{Dj} \)

\( t \) = time variable, days

\( x \) = horizontal coordinate in Cartesian coordinates system, ft

\( y \) = vertical coordinate in Cartesian coordinates system, ft

**Greek letters**

\((\xi, \zeta)\) = local coordinates, ft

\( \mu \) = fluid viscosity, cp

\( \theta_j \) = internal angles between two adjacent elements

\( \phi \) = porosity, fraction

\( \Omega \) = solution domain

\( \Gamma_{outer} \) = outer boundary of the reservoir

\( \Gamma_j \) = boundaries of the j-th subsystem

\( \Gamma_{i,j} \) = interface boundaries between the i-th and j-th subsystem

**Functions**

\( \text{erf} \) = error function

\( \exp \) = exponential function
\( H \) = Heaviside step function

\( G_j \) = free space Green’s function for the jth subsystem defined in Eq. [3.7]

\( L^{\{\ldots\}} \) = Laplace transformation

\( \delta \) = Dirac delta function

**Subscript and superscript**

D = dimensionless term

i = variable of the i-th source/sink in the j-th subsystem

j = variable of the j-th subsystem

k = variable of the k-th constant element

“~” = Laplace transformation of a function/variable

**SI Metric Conversion Factors**

\[ \text{bbl} \times 1.589874 \times 10^{-1} = \text{m}^3 \]

\[ \text{cp} \times 1.0* \times 10^{-3} = \text{Pa}\cdot\text{s} \]

\[ \text{ft} \times 3.048* \times 10^{-1} = \text{m} \]

\[ \text{mD} \times 9.869233 \times 10^{-4} = \mu\text{m}^2 \]

\[ \text{lb}_m \times 4.535924 \times 10^{-1} = \text{kg} \]

\[ \text{psi} \times 6.894757 \times 10^0 = \text{kPa} \]

*Conversion factor is exact*
CHAPTER 4  BOUNDARY INTEGRAL APPROACH FOR TRANSIENT PRESSURE RESPONSES AND INDUCED STRESS CHANGE IN HETEROGENEOUS RESERVOIRS

Multistage fractured horizontal wells (MFHW) in unconventional reservoirs usually experience long transient flow period and sharp production decline. Poroelastic effects and mechanical loadings along propped hydraulic fractures have significant impact on the in-situ stress state. Field observations have confirmed stress evolution as pore pressure changes during production. However, measured data could only reveal the induced stress change phenomenon but are unable to systematically characterize the spatio-temporal stress evolution. In the development of unconventional reservoirs with MFHW, theoretical models of predicting the induced stress by production become important to prevent fracture closure with proper completion design and optimize enhanced oil recovery (EOR) planning. This chapter proposes a boundary integral approach to simultaneously model transient fluid flow and spatio-temporal stress evolution. Innovatively, the proposed approach explicitly considers hydraulic fractures with realistic morphology and local heterogeneities around the MFHW with various permeability and porosity. The boundary integral approach (BIA) and displacement discontinuity method (DDM) are effectively integrated to solve coupled fluid flow and solid deformation problems. Benchmarking is performed by comparing this work with analytical solutions available in linear poroelasticity, fracture mechanics and well-known commercial
software in well testing. Modeling results show that propped fractures increase the total compressive stress which varies with fracture widths and net pressure. Meanwhile, production from the MFHW reduces pore pressure and the total compressive stress. The depletion-induced stress components on fractures show transient characteristics which are consistent with the linear, transitional, pseudo-radial and boundary dominated flow regimes commonly identified in well testing (Xiao et al., 2017). Effects of fracture properties and reservoir heterogeneity on the spatio-temporal stress evolution are clearly shown on the log-log plots of derivative responses of depletion-induced stress components versus time. The stress path $\Delta \sigma_h / \Delta \rho$, which describes the induced minimum horizontal stress per unit pore pressure change monitored at the well, can be quantified with informative trends over time by the proposed modeling approach.

### 4.1 Introduction

Total stress fields in general are changing with time and space according to observations in different basins over various geological timescales, whereas the mini-fracture tests from the Gulf Coast, Lake Maracaibo and Brunei have shown coupled relationships between induced total stress and pore pressure change (Breckels and van Eekelen, 1982). Minimum horizontal stresses recorded from many fields around the world have changed due to injection and production activities (Salz, 1977; Teufel et al., 1991; Whitehead et al., 1987). Addis (1997) analyzed datasets from six fields and found that linear trends exist when the minimum horizontal stress decreases with declining reservoir pore pressure. Field observations indicate that the induced stress evolution by production should be considered to be a transient process as the pore pressures decrease
non-uniformly without reaching the equilibrium state (Addis, 1997). Santarelli et al. (1998) pointed out that the ratio of induced minimum horizontal stress and pore pressure change ($\Delta \sigma_h / \Delta p$), namely the typical stress path, was non-uniform and could not be predicted by simple models. Zoback and Zinke (2002) also found that the stress path within the same formation may vary with wells penetrating to different structures of the reservoir. A comprehensive review of field data from 15 oil fields was presented by Altmann et al. (2010) to show that the stress path could be as low as 0.34 and as high as 1.18. During the hydraulic fracturing process in the Barnett shale, increased minimum horizontal stress and microseismic events were reported to be related to the time-dependent fluid leak off; and the stress evolution difference of zipperfracs and simulfracs was believed to be affected by poroelasticity (Vermylen and Zoback, 2011).

The spatio-temporal stress evolution can significantly affect the fluid drainage, seismic velocities, the lifetime of horizontal wells, and fracture conductivity (Holt et al., 2004; Addis, 1997). Because the seismic wave velocity is sensitive to stress path change, it is possible to use the time-lapse seismic to monitor stress paths as proposed by Sayers (2004). In the risk assessments for CO$_2$ sequestration in depleted hydrocarbon reservoirs, stress path evaluation including the magnitudes and directions of maximum and minimum horizontal stresses are crucial to mitigate drilling risks (Fang and Khaksar, 2010). Therefore, it is of great importance to develop predictive models to evaluate the induced stress evolution by fluid flow. The simple passive basin response method with constant values failed to match complicated field data of stress paths (Addis, 1997). Several analytical and numerical models have been introduced in the literature to predict the spatio-temporal characteristics of stresses induced by fluid flow. Rudnicki (1999)
presented an analytical model to analyze stress evolution of an ellipsoidal reservoir with contrast elastic properties such as shear modulus and Poisson’s ratio between the reservoir and surrounding formations. Chen and Teufel (2001) used a 2D poroelastic plane strain model to show that the depletion-induced stress path is time dependent and anisotropic even in an infinite homogeneous reservoir. They also concluded that the stress path values tend to be steady after the wells being produced for sufficiently long time. Holt et al. (2004) applied analytical models to evaluate the effects of stress path change on geomechanical behavior such as reservoir compaction, permeability change, and dynamic processes of rock failure. Soltanzadeh and Hawkes (2007) summarized the stress paths from 13 fields with different lithology and stress regimes. They also reviewed analytical and numerical modeling approaches to predict depletion-induced stress changes, however, few models were found to investigate reservoir heterogeneities with different permeability and porosity (Soltanzadeh and Hawkes, 2007). A numerical model of a two-zone composite reservoir with different permeability was developed by Altmann et al. (2010) to show that reservoir heterogeneity could significantly affect the spatio-temporal stress evolution. Schutjens et al. (2012) proposed an integrated workflow to estimate depletion-induced stress change and found the stress change in sediment is time-dependent. In their geomechanical model, constant stress path values were input to the 3D geomechanical simulators to account for depletion-induced stress changes. Shahri and Miska (2013) applied an analytical model derived from linear poroelasticity to predict reservoir stress changes in space and time during pseudo-steady-state flow regime, which however is only suitable for depleted reservoirs. Chen (2012) investigated the mechanical interaction of multiple fractures and analyzed induced maximum and minimum
horizontal stresses by propped hydraulic fractures, whereas the analysis is limited to plane strain elastic media without considering the depletion-induced stress of multiple fractures due to poroelasticity.

To meet the challenges of EOR planning and fracture closure problems, this chapter proposes a boundary integral approach to model induced stress evolution by the depletion of MFHW in a heterogeneous reservoir. Analytic fundamental solutions form the backbone of the boundary integral approach so that the modeling results have near analytical accuracy. Moreover, heterogeneities (i.e. flow barriers, sweet spots, and swarms of natural fractures) with various permeability and porosity around the well are explicitly considered in the model, and the induced stress components are to be correlated with fluid flow behavior affected by multiple fractures and reservoir heterogeneities.

4.2 Methodology

4.2.1 Problem definition and assumptions

As shown in Figure 4.1, a heterogeneous system is consisting of five locally homogeneous regions with average fluid and reservoir properties. Within each region, rock and fluid properties are assumed to be uniform with effective average values. Outflow outer boundary of the reservoir is denoted as $I_{\text{outer}}$. Various flow conditions can be prescribed including Dirichlet, Neuman, and mixed boundary conditions. For simplicity, it is assumed that the pressure is initially uniform throughout the reservoir and the outer boundary condition is prescribed as a no-flow condition. A horizontal well with multiple transverse fractures is located in one of the regions with a stimulated reservoir volume (SRV). Sizes and geometries of the hydraulic fractures are arbitrary.
Figure 4.1 A heterogeneous system with five heterogeneous geo-bodies (not to scale). Each region has uniform average porosity and permeability. A MFHW with arbitrarily-shaped fractures is located in region III, where a SRV colored in grey develops around the horizontal well. Permeability and porosity of the five regions and SRV are dependent on interpretation of geology.
A stimulated reservoir volume (SRV) region, as shown in grey color, develops around the stimulated horizontal well. Shear-slip natural fractures with high conductivity may also exist to contribute to matrix-fracture flow. Respecting to real geology, physical boundaries of all different scales of heterogeneous geo-bodies can be arbitrary and irregular.

The objective of this study is to model transient fluid flow and depletion-induced stress evolution in a heterogeneous reservoir with MFHW. The coupled fluid flow and solid deformation problem is highly non-linear in the theory of poroelasticity. A lot of pioneer works have been done to couple geomechanical effects with fluid flow models to account for stress-sensitive permeability and porosity, in which permeability change is usually correlated to pore pressure and effective stress (Bachman et al. 2011; Lin et al. 2015). It is important to investigate the change of stress-dependent matrix properties and fracture closure by comprehensive experimental and theoretical studies (Clarkson et al., 2013). However, in this study, I focus on studying the overall transient behavior of depletion-induced stress changes through theoretical derivation. Studies on how stress changes can affect matrix properties and how dynamic stress domain is going to vary during production process are two folds of this technical domain that need to be examined. This chapter is focusing on the dynamic stress part. Detournay and Cheng (1988) investigated induced stress field by fluid flow around a wellbore and introduced anisotropy of stress field to the coupled solutions. Displacement discontinuity method (DDM) used to solve fully coupled problems for infinite homogeneous systems have been investigated and found to require more computation effort than uncoupled approaches (Detournay and Cheng, 1987; Carvalho and Curran, 1998).
Inspired by the analytical solutions of diffusivity equations in well testing, the proposed work assumes plane strain conditions to derive the coupled diffusivity-deformation equation to have similar forms usually solved in well testing. Accordingly, layered problems and deformation in vertical directions are beyond the scope in current stage of this study. Nonetheless, hydraulic heterogeneity and fracture morphologies are explicitly considered to model transient fluid flow behavior and spatio-temporal stress evolution induced by both fluid flow and mechanical loadings. Some major assumptions of the defined problem are:

1. The reservoir is a static poroelastic media with a single phase fluid flow.
2. The formation has constant thickness and hydraulic fractures are under plane strain condition, which indicates no solid deformation in the vertical direction.
3. The mechanical properties of the formation such as shear modulus, Biot coefficient, and Poisson’s ratio are uniform and isotropic. The contrasts between the reservoir and surrounding rock are not considered in the proposed model.
4. The hydraulic properties of the formation such as permeability and porosity are heterogeneous, however, independent of stress change.
5. Geometries of heterogeneous geo-bodies such as swarms of natural fractures, sweet spots, barriers, and geological facies are arbitrary.

4.2.2 Linear poroelasticity

Biot (1941) introduced the theory of static poroelasticity which assumes a linear and isotropic fluid-saturated porous media. The poroelasticity theory was later improved by many researchers and applied to various areas including the reservoir engineering in oil and gas industry. Poroelasticity physically describes mechanical responses of a porous
rock saturated with moving fluid. The linear poroelastic processes incorporated with the Terzaghi effective stress principle, can be described as an elasticity equation with a fluid coupling term, and a diffusion equation with a solid coupling term (Biot and Willis, 1957; Rice and Cleary, 1976; Rudnicki, 1987; Detournay and Cheng, 1993). Under the plane strain condition, constitutive equations of the diffusion-deformation process become:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \frac{2Gv}{1-2v}\varepsilon - \alpha\delta_{ij}p \quad , \quad i=x, y; j=x, y \quad [4.1]$$

$$\zeta = \frac{\alpha}{3K_b}\sigma_{kk} + S_{\sigma}p \quad [4.2]$$

where, $K_b$ is the drained bulk modulus. $\zeta$ is the fluid variation content. $i, j, k$ are indices representing directions in Cartesian coordinates system, where a comma followed by subscripts means spatial differentiation, and repeated indices in the same form denote summation over the range of indices (as for the plane strain condition, the indices are $x$ and $y$ directions).

$S_{\sigma}$, in terms of porosity and compressibility, is similar to the product of porosity times total compressibility used in well testing (Carroll, 1980). The Biot coefficient is also related to solid and bulk compressibility according to Biot and Willis (1957).

$$S_{\sigma} = \phi C_{\text{fluid}} - (1+\phi)C_{\text{solid}} + C_b$$

$$\alpha = 1 - \frac{C_{\text{solid}}}{C_b} \quad [4.3]$$

where, $C_b$ is the bulk compressibility, $C_{\text{fluid}}$ is the fluid compressibility, and $C_{\text{solid}}$ is the solid compressibility. All these parameters can be obtained by experimental studies (Zimmerman et al., 1986; Hart and Wang, 1995).

The poroelastic media is static so that the balance laws can be described as
\[ \sigma_{ij,j} = -F_i \]  \[4.4\]

where, \( F \) is the body force of the poroelastic media.

The continuity equation of compressible fluid flow derived from mass conservation is

\[ \frac{\partial \zeta}{\partial t} + q_{i,i} = \gamma \]  \[4.5\]

where \( q_{i,i} \) is the flux derivative in the \( i \) direction. \( \gamma \) is a source/sink term.

Fluid flow in the poroelastic media obeys the Darcy’s law:

\[ q_i = -\frac{k}{\mu} (p_{,i} - f_i) \]  \[4.6\]

where, \( \mu \) and \( k \) are viscosity and permeability respectively. \( p_{,i} \) is the pressure gradient in the \( i \) direction and \( f \) is the fluid body force.

Substituting Eq. [4.2] and Eq. [4.6] into Eq. [4.5], the diffusion equation with a solid coupling term is obtained (Detournay and Cheng, 1993):

\[ \frac{\alpha}{3K_b} \frac{\partial \sigma_{kk}}{\partial t} + S_\sigma \frac{\partial p}{\partial t} = \nabla \cdot \left( \frac{k}{\mu} \nabla p \right) + \gamma - \frac{k}{\mu} f_{i,i} \]  \[4.7\]

In a homogeneous system with constant permeability and fluid viscosity, ignoring the body forces, Eq. [4.7] reduces to

\[ \frac{\alpha}{3K_b} \frac{\partial \sigma_{kk}}{\partial t} + S_\sigma \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla^2 p + \gamma \]  \[4.8\]

Combining the constitutive relation Eq. [4.1] and compatibility equation Eq. [4.2], the field equation in terms of displacement with a fluid coupling term was derived as (Detournay and Cheng, 1993):
The following compatibility equation describes relations between strain and displacement in the poroelastic media:

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \]  \[\text{[4.10]}\]

where, \( u_i \) and \( \varepsilon_i \) denote the displacement and strain respectively.

Eq. [4.9] can be expressed in two-dimensional as

\[ G\nabla^2 u_i + \frac{G}{1-2v}u_{k,ki} = \alpha p_{,i} - F_i \]  \[\text{[4.9]}\]

Accordingly, the diffusivity equation for pressure and the Navier-type equation for displacement are general governing equations of linear poroelasticity. More detailed derivation processes are referred to pioneer works of Rice and Cleary (1976) and Detournay and Cheng (1993).

4.2.3 **General diffusivity equation and its boundary integral form**

According to the linear poroelasticity, Shahri and Miska (2015) gave the following relation between the principal stresses and pore pressure under the plane strain condition:

\[ \sigma_{kk} = 2\alpha \frac{(2v-1)}{(1-v)} p \]  \[\text{[4.12]}\]
Substituting Eq. [4.12] to Eq. [4.8], the general diffusion equation considering a coupling solid deformation term becomes:

$$ S_{ps} \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla^2 p + \gamma $$  \hspace{1cm} [4.13]$$

where,

$$ S_{ps} = \frac{2}{3} \alpha \left( \frac{1-2v}{1-v} \right) \left( C_{solid} - C_b \right) + S_\sigma $$ \hspace{1cm} [4.14]$$

$S_{ps}$ is the storage capacity for homogeneous poroelastic media under plane strain condition, which is different from the generally used storativity defined as a product of porosity times total compressibility. With well-imposed hydraulic boundary conditions, Eq. [4.13] considering solid deformation has a form similar to the uncoupled diffusivity equation in well-testing and can be solved to obtain pore pressure change in the poroelastic reservoir.

Boundary integral formulations are advantageous because they explicitly consider arbitrary shapes of boundaries and sources. Once boundary unknowns are obtained by solving the linear equations, solutions at any point will be calculated without subdividing the domain. In the framework of boundary element method (BEM) according to Kikani and Horne (1992, 1993), unsteady state fluid flow described in the general diffusion equation in Eq. [4.13] has the following singular integral solutions without considering the source/sink term:

$$ \theta p(x,t) = \int_t \left[ \int_{\Gamma} \left( G_{fsg}(\chi - x, t - \tau) q_i(\chi, \tau) n_i(\chi) - \frac{\partial G_{fsg}(\chi - x, t - \tau)}{\partial n_i(\chi)} p(\chi, \tau) \right) d\Gamma \right] d\tau $$ \hspace{1cm} [4.15]$$
where, $\theta$ is the angle between two adjacent boundary elements, if the evaluation point is in the domain, $\theta = 2\pi$. $G_{\mu g}$ is the free-space Green’s function of a point source. $q$ and $p$ are boundary values of flux and pressure respectively.

4.2.4 A hybrid boundary integral approach

Boundary integral forms of poroelastic field equations were derived by Cheng and Detournay (1988) based on the Betti reciprocal theorem. The general expression for linear poroelasticity is

$$\sigma_{ij}^1 \varepsilon_{ij}^2 + p^1 \zeta_{ij}^2 = \sigma_{ij}^2 \varepsilon_{ij}^1 + p^2 \zeta_{ij}^1$$  \[4.16\]

where, superscripts 1 and 2 represents two independent systems of stresses and strains.

By assigning physical quantities for the first system and using free-space Green’s functions for the second system, singular integral equations for stresses without considering body forces were derived by Cheng and Detournay (1998):

$$\theta \sigma_{kl}(x,t) = \int \left\{ \int \left[ u_{ijkl}^* (\chi - x, t - \tau) \sigma_{ij} (\chi, \tau) n_j (\chi) \right] dS \right\} d\tau$$

$$- \int \left\{ \int \left[ q_{ijkl}^* (\chi - x, t - \tau) n_j (\chi) p (\chi, \tau) \right] dS \right\} d\tau$$  \[4.17\]

where $\theta = 2\pi \rightarrow if (x,y) \in \Omega$. $k,l,i,j = x,y$. $u_{ijkl}^*$, $\sigma_{ijkl}^*$, $q_{ijkl}^*$, and $p_{kl}^*$ are fundamental solutions of an infinite homogeneous poroelastic media derived by Cheng and Detournay (1998). $q$ and $p$ are boundary values of flux and pressure, respectively. $\sigma_{ij}$ and $u_i$ are boundary values of stresses and displacements, respectively.
The singular integral solutions described in Eq. [4.15] can be solved with well-imposed boundary conditions of flux and pressure. Accordingly, the unknown values of \( p \) and \( q \), can be obtained independent of stress and displacement conditions on the boundary. The stress changes due to poroelasticity in Eq. [4.17] are thus linearized as superposition of induced stresses by two mechanisms shown in Eq. [4.18]. The first integral in Eq. [4.17] represents the induced stresses by mechanical loadings such as far-field stresses and local forces. The second integral represents the induced stresses by fluid flow and pore pressure change. Eq. [4.19] and Eq. [4.20] express the two mechanisms respectively. Once boundary conditions of fluid flow and pressure are known, Eq. [4.15] will be solved in the frame of boundary integral approaches. Accordingly, stress changes induced by fluid at any point in the domain can be obtained by evaluating Eq. [4.20] with appropriate fundamental solutions of stresses induced by fluid flow (Xiao et al., 2017).

\[
\sigma_{kl}(x,t) = \sigma_{kl}^{\text{mechanical}}(x,t) + \sigma_{kl}^{\text{fluid}}(x,t)
\]  

\[
\theta \sigma_{kl}^{\text{mechanical}}(x,t) = \int_\Gamma \left\{ \int_\Gamma \left[ u_{ikl}^*(\chi, x, t - \tau)\sigma_{ij}(\chi, \tau)n_j(\chi) - \sigma_{ijkl}^*(\chi, x, t - \tau)n_j(\chi)u_i(\chi, \tau) \right] dS \right\} d\tau
\]  

\[
\theta \sigma_{kl}^{\text{fluid}}(x,t) = \int_\Gamma \left\{ \int_\Gamma \left[ p_{kl}^*(\chi, x, t - \tau)q_i(\chi, \tau)n_i(\chi) - q_{ijkl}^*(\chi, x, t - \tau)n_i(\chi)p(\chi, \tau) \right] dS \right\} d\tau
\]

Similarly, induced stresses by mechanical loadings can be obtained by solving Eq. [4.19] with well-imposed boundary conditions of displacement and tractions. However, in the development of unconventional reservoirs, existence of hydraulic and natural fractures makes the domain discontinuous so that it is cumbersome to use the continuum-
based integral formulations to solve Eq. [4.19]. Cheng and Detournay (1998) related the indirect and direct methods of boundary integral equations through mathematical derivations. Displacement discontinuity method (DDM) was derived from an indirect form of boundary integral approach in their work.

A complementary region is introduced to share a common surface of the problem domain (Ω is bounded by Γ) as shown in Figure 4.2. The complementary region is denoted as Ω'. Normal vector n and n' are associated with Ω and Ω' respectively, where n = −n'. Eq. [4.19] can be written for the complementary region Ω' as

\[
0 = \int \int_{\Gamma} \left[ \begin{array}{c}
u^*_ijkl(\chi - x, t - \tau)\sigma'_{ij}(\chi, \tau)n'_{j}(\chi) \\
- \sigma^*_{ijk}(\chi - x, t - \tau)n'_{j}(\chi)u'_i(\chi, \tau) \\
\end{array} \right] dS d\tau \tag{4.21}
\]

Summing Eq. [4.19] and [4.21] produces

\[
\theta \sigma_{kl}^{\text{mechanical}}(x, t) = \int \int_{\Gamma} \left[ \begin{array}{c}
u^*_ijkl(\chi - x, t - \tau)(\sigma_{ij}(\chi, \tau) - \sigma'_{ij}(\chi, \tau))n_{j}(\chi) \\
- \sigma^*_{ijkl}(\chi - x, t - \tau)n_{j}(\chi)(u_i(\chi, \tau) - u'_i(\chi, \tau)) \\
\end{array} \right] dS d\tau \tag{4.22}
\]

If the boundary tractions along the common boundary Γ between the two domains are equal, then Eq. [4.22] reduces to (Cheng and Detournay, 1998):

\[
\theta \sigma_{kl}^{\text{mechanical}}(x, t) = \int \int_{\Gamma} \left[ \begin{array}{c}\sigma_{ijkl}(\chi - x, t - \tau)D_{ij}(\chi, \tau) \\
\end{array} \right] dS d\tau \tag{4.23}
\]

\[
D_{ij} = D_{ij} n_{j}(u'_i - u_i) \tag{4.24}
\]

where \(D_{ij}\) is the displacement discontinuity tensor. In fractured reservoirs, the normal displacement discontinuity usually represents the widths of opening cracks. \(\sigma^*_{ijkl}\) is the free-space Green’s function for a point displacement discontinuity in a linear poroelastic media.
Figure 4.2 Complementary regions bounded by a common surface (Cheng and Detournay, 1998).
For a homogeneous system with displacement discontinuities (i.e. fractures and faults), pore pressure change and fluid flow will be obtained by solving Eq. [4.15]; induced stresses by fluid flow are consequently calculated by Eq. [4.20] with known boundary values of $p$ and $q_i$; induced stresses by mechanical loadings will be computed by the DDM using Eq. [4.23] with well-imposed tractions along discontinuities and the far-field stress conditions. Eventually, stress changes due to fluid flow and mechanical loadings in the linear poroelastic media will be obtained by superposition shown in Eq. [4.18]. Therefore, transient fluid flow and stress changes will be effectively calculated in a manner of hybrid boundary integral approach, which includes the DDM for stress solution and BIA for transient fluid flow. Appendix C summarizes all the fundamental solutions and free-space Green’s functions used in the boundary integral approaches.

### 4.2.5 Definition of dimensionless variables

Dimensionless variables not only simplify equations but also provide universal solutions. Therefore, the following dimensionless parameters are introduced to convert field governing equations and constitutive relations of stress and strain into dimensionless forms (Xiao et al., 2017).

\[
D = \frac{k}{\mu S_{ps} l^2} t
\]  

\[
x_D = \frac{x}{l}; \quad y_D = \frac{y}{l}; \quad D_{id} = \frac{D_{ij} n_j (u_i - u_i)}{l}
\]  

\[
Q_D = \frac{q}{q_B}; \quad p_D = 2\pi \frac{-kh}{q_B \mu} (p_i - p)
\]
\[
\begin{align*}
\mathbf{u}_D &= \frac{\mathbf{u}_i}{l} ; \quad \mathbf{e}_{ijD} = \frac{1}{2}(\mathbf{u}_{iD,j} + \mathbf{u}_{jD,i}) ; \quad \mathbf{e}_D = \frac{\partial \mathbf{u}_{xD}}{\partial x_D} + \frac{\partial \mathbf{u}_{yD}}{\partial y_D} \quad \quad [4.28] \\
\sigma_{ijD} &= 2\pi(\sigma_{ij}^{\text{initial}} - \sigma_{ij})/(q_B k h), \quad i=x, y; \quad j=x, y \quad \quad [4.29]
\end{align*}
\]

4.2.6 BIA for transient fluid flow and induced stresses in heterogeneous media

For the heterogeneous problems defined in Figure 4.1, Chapter 3 has documented a boundary integral approach to subdivide the domain along local homogeneous regions with average permeability and porosity. To physically relate different regions, the dimensionless transmissibility ratio \((C_{Tn})\), storativity ratio \((C_{Sn})\), and diffusivity ratio \((C_{Rn})\) are introduced for each region.

\[
C_{Tn} = \left(\frac{k_h}{\mu_n}\right) / \left(\frac{k_h}{\mu_l}\right) ; \quad C_{Sn} = \left(\frac{S_{ps\ n}}{S_{ps\ l}}\right) ; \quad C_{Rn} = \frac{C_{Tn}}{C_{Sn}} , \quad n=1, N_R \quad \quad [4.30]
\]

Theoretically, any region can be the reference to calculate \(C_{Tn}, C_{Sn}, C_{Rn}\). Accordingly, each homogeneous region with average rock and fluid parameters can be normalized in dimensionless forms. The integral solution for each homogeneous region described in Eq. [4.15] becomes in dimensionless form:

\[
\theta_n p_{Dn}(x_D, y_D, C_{Rn}, t_D) = \int_{C_{Rn}D} \left\{ \int_{\Gamma_n} \left[ \frac{1}{C_{Tn}} G_{n} q_{Dn} - p_{Dn} \frac{\partial G_{n}}{\partial n} \right] dS \right\} d\tau \quad \quad [4.31]
\]

\[
\begin{align*}
\theta_n &= 2\pi \rightarrow \text{if} \,(x_D, y_D) \in \Omega_n , \\
\theta_n &= \theta_n \rightarrow \text{if} \,(x_D, y_D) \in \Gamma_n , \quad \quad n=1, N_R \quad \quad [4.32]
\end{align*}
\]
where $\theta_n$ are the internal angles between two adjacent elements on the boundaries. $N_R$ is the number of local homogeneous regions. $q_{Dn}$ and $p_{Dn}$ are boundary values of flux and pressure for the $n^{th}$ region respectively.

$G_n$ is the free space Green’s function for the $n^{th}$ region, which physically represents the pressure responses due to an instantaneous point source of unit strength at local coordinates $(\xi_D, \zeta_D)$ removed at time $\tau_D$ (Carslaw and Jaeger, 1959). The free space Green’s function in dimensionless form is given:

$$G_n(x_D, y_D, C_{Re}t_D, \xi_D, \zeta_D, \tau_D) = \frac{H(C_{Re}(t_D - \tau_D))}{4\pi C_{Re}(t_D - \tau_D)} \exp[-\left(\frac{x_D - \xi_D}{2\sqrt{C_{Re}(t_D - \tau_D)}}\right)^2] \exp[-\left(\frac{y_D - \zeta_D}{2\sqrt{C_{Re}(t_D - \tau_D)}}\right)^2]$$  \[4.33\]

Kikani and Horne (1993) proposed a BEM approach for a two-zone composite reservoir. Their work showed promise of using BEM to solve multi-scale and irregularly shaped heterogeneous systems. Pecher and Stanislav (1996) solved a compartmentalized problem with three consecutive zones. According to the partition and coupling strategy proposed in Chapter 3, linear equations with the same number of unknown values along boundaries will be obtained in a complicated heterogeneous reservoir. Due to the continuity of pressure and flux at the interface boundaries of adjacent regions, taking the interface boundary between region II and III for example:

$$C_{T2}p_{D2}((x_D, y_D) \in \Gamma_{23}, C_{R2}t_D) = C_{T3}p_{D3}((x_D, y_D) \in \Gamma_{23}, C_{R3}t_D)$$  \[4.34\]

By solving the systematic linear equations in the frame of BIA, I have obtained all boundary values of flux and pressure. Then, pressure drop at any point in the heterogeneous reservoir is evaluated by using Eq. [4.31] again. Once all boundary values
of flux and pressure of the heterogeneous problems are known, Eq. [4.36] will be used to obtain induced stresses by transient fluid flow in the \( n^{th} \) region of the heterogeneous poroelastic media.

\[
\theta_n \sigma_{\text{fluid}}^{kDn}(x_D, y_D, C_R, t_D) = \int_{C_{nD}} \left\{ \int_{\Gamma_n} \frac{1}{C_{\tau n}} p^*_{kDn} q_{Dn} - d^*_{kDn} P_{Dn} \right\} dS \right\} d\tau \tag{4.36}
\]

\[
\theta_n = 2\pi \quad \text{if} \quad (x_D, y_D) \in \Omega_n,
\]

\[
\theta_n = \theta_n \quad \text{if} \quad (x_D, y_D) \in \Gamma_n \quad n=1, N_R \tag{4.37}
\]

**4.2.7 Fracture-matrix interaction**

Each hydraulic fracture of the MFHW is treated as a line source in the matrix, which is the integration of the free space Green’s function over a line source. As shown in **Figure 4.3**, I divide the outer boundary \( \Gamma \) of the region with MFHW and hydraulic fractures into \( N_j \) boundary elements and \( N_w \) fracture elements respectively. Pressure drop evaluated at the center of each element is obtained by superposition of pressure drops caused by all fracture and boundary elements. The local coordinates system is used to account for arbitrary morphologies of fractures. Relatively more elements will be used for curvature boundaries and fractures than straight ones.

If the region with MFHW is set as the reference region, then \( C_{\tau n} \) for this region becomes one. Subsequently, Eq. [4.31] for the region with MFHW is written in a discrete form at the evaluating point \((\xi_{Dl}, \zeta_{Dl})\) in local coordinates as:
Figure 4.3 Discretization of the boundary and hydraulic fractures into different elements.

The boundary $\Gamma$ is divided into $N_j$ boundary elements. Fractures are divided into $N_w$ fracture elements.
\[ \theta_P G_{Dl}(\xi, \zeta, t_D) = \int \left\{ \sum_{l=1}^{N_l} \int_{\xi_{ll}^{L}}^{\xi_{ll}^{H}} \left[ G_n \frac{\partial P_{Di}(\xi, \zeta, t)}{\partial n} - P_{Di}(\xi, \zeta, t) \frac{\partial G_n}{\partial n} \right] d\xi + \sum_{w=1}^{N_w} Q_{Dw} G_{Lw} \right\} d\tau \] \quad [4.38]

\[ \theta_l = 1 \rightarrow \text{if} (\xi_{Di}, \zeta_{Di}) \in \Omega, \quad l=1, N_l; \quad w=1, N_w \] \quad [4.39]

where, \( a_{Di} \) is the dimensionless half-length of each boundary element around the region with MFHW.

The mathematical expression for a line source element with a half-length \( L_{Dw} \) is described as

\[ G_{Lw}(x, y, t, \xi, \zeta, \tau) = \int_{\xi_{ll}^{L}}^{\xi_{ll}^{H}} \frac{H(t_D - \tau)}{4\pi(t_D - \tau)} \exp\left[ -\left( \frac{x - \xi_D}{2\sqrt{C_{Rw}(t_D - \tau)}} \right)^2 \right] \exp\left[ -\left( \frac{y - \zeta_D}{2\sqrt{C_{Rw}(t_D - \tau)}} \right)^2 \right] d\xi \] \quad [4.40]

### 4.2.8 Laplace transformation

The Laplace transformation shown in Eq. [4.41] is applied to convert the convolution in the time domain to algebraic multiplication in the Laplace domain, which is more convenient to solve.

\[ L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \tilde{f}(s) \] \quad [4.41]

The aforementioned singular integral solutions of pressure and stresses shown in Eq. [4.15], [4.18], [4.19] and [4.20] become in Laplace domain as:

\[ \partial \tilde{p}(x, s) = \int_{\Gamma} \tilde{G}_{fog}(\chi - x, s) \tilde{q}_i(\chi, s) n_i(\chi) - \frac{\partial \tilde{G}_{fog}(\chi - x, s)}{\partial n_i(\chi)} \tilde{p}(\chi, s) \right] dS \] \quad [4.42]

\[ \partial \tilde{\sigma}_{kl}^{\text{fluid}}(x, s) = \int_{\Gamma} \left[ \tilde{p}_{kl}(\chi - x, s) \tilde{q}_i(\chi, s) n_i(\chi) - \tilde{q}_{kl}(\chi - x, s) n_i(\chi) \tilde{p}(\chi, s) \right] dS \] \quad [4.43]
\[
\bar{\Theta} \bar{\sigma}_{kl}^{\text{mechanical}}(x,s) = \int_{\Gamma} \left[ \tilde{\sigma}_{ijkl}^*(\chi - x, s) \tilde{D}_j(\chi, s) \right] dS \tag{4.44}
\]

\[
\tilde{\sigma}_{kl}(x, s) = \tilde{\sigma}_{kl}^{\text{mechanical}}(x, s) + \tilde{\sigma}_{kl}^{\text{fluid}}(x, s) \tag{4.45}
\]

where, the superscript “~” denotes the Laplace transformation and \( s \) is the Laplace variable, \( i, j, k, \) and \( l \) are Einstein indices for the plane strain problems.

Using the Stehfest inverse Laplace algorithm (Stehfest, 1970), dimensionless pore pressure and induced stress components of a MFHW in the time domain will be obtained.

### 4.3 Model Benchmarking

To verify the solution domain of the proposed modeling technique, I need to compare the modeled mechanical-induced stress, depletion-induced stress, and pore pressure change with their corresponding analytical solutions. Moreover, limiting homogeneous cases are used to benchmark the results of the heterogeneous problems. Induced stresses and pore pressure are all in their dimensionless forms.

#### 4.3.1 Mechanical-induced stress

For a static hydraulic fracture in a homogeneous media, assuming the net pressure inside the fracture is constant and uniform, the analytical solution of induced stress due to elasticity is given by Sneddon and Elliot (1946). Since the proposed modeling technique considers a poroelastic media, the early time solution corresponding to the undrained condition is used to compare with the analytical solutions. Figure 4.4 shows excellent agreement between the solutions in this work and Sneddon’s analytical solutions.
**Figure 4.4** Induced stress components of a hydraulic fracture with constant net pressure along the direction perpendicular to the fracture plane.
The mechanical-induced stress components are normalized by the net pressure; and distances along the direction perpendicular to fracture are normalized by the fracture half length. In the work of Sneddon and Elliot (1946), the positive sign means increase of compressive stresses. It clearly shows that the opening of the hydraulic fracture increases the normal component of total compressive stress ($\sigma_{yy\_mechanic}$) in the direction perpendicular to the fracture, which decays further away from the fracture. On the other hand, the normal component of total compressive stress ($\sigma_{xx\_mechanic}$) in the direction parallel to the fracture is increased near the center of the fracture, however, it is decreased further from the fracture due to tensile stresses induced around the fracture tip.

4.3.2 Depletion-induced stress

Homogeneous Cases For the depletion-induced stresses, Detornay and Cheng (1988) described them as mode II loading and derived analytical solutions of the tangential and radial stresses induced by production from a borehole. By reducing the fracture length to be infinitely small, the fracture solutions in this work will be a close approximation to the point source solutions in the work of Detornay and Cheng (1988). Production from a well reduces the pore pressure and total stresses. Figure 4.5 compares the induced tangential stress variation with radius at different time calculated by the proposed model and the analytical solution. In the work of Detornay and Cheng (1988), the negative sign means reduction of total compressive stresses. The induced tangential stress is tensile around the wellbore; however, it becomes compressive farther away from the wellbore. Figure 4.6 also shows good agreement between the analytical solution and the modeling result of the induced radial stress variation with radius.
Figure 4.5 Induced tangential stress variation with radius at different time calculated by the proposed model and the analytical solution. Following the work of Detournay and Cheng (1988), the negative sign means reduction of total compressive stresses.
Figure 4.6 Induced radial stress variation with radius at different time calculated by the proposed model and the analytical solution. Following the work of Detornay and Cheng (1988), the negative sign means reduction of total compressive stresses.
**Heterogeneous Cases** As shown in Figure 4.7, a rectangular heterogeneous reservoir consists of 25 regions which can have various values of permeability and porosity. Regions are numbered in progression starting with 1 in the NW corner of the reservoir and ending with 25 in the SE corner. For the sake of simplicity, each rectangular region is identical and has a dimensionless size of four by four, whereas the fracture half-length is one. If all regions have the same porosity and permeability, the heterogeneous case (Scenario 1) becomes identical to a homogeneous case with a size of 20 by 20 (Scenario 2). Moreover, if I assign all regions except for the most inner region (#13) extremely low permeability and porosity, the heterogeneous case (Scenario 3) should be very close to a homogeneous case with a reduced reservoir size of four by four (Scenario 4) because the outer regions all act like a no-flow boundary. Table 4.1 summarizes the four scenarios in more detail. Figure 4.8 compares the induced normal stress components in the $x$ direction ($\sigma_{xx}$) and their corresponding derivative responses ($d\sigma_{xx}/d\log t$). To be consistent with the dimensionless pressure drop at the wellbore in well testing, Eq. [4.29] is used to calculate dimensionless depletion-induced stress components and thus positive sign means reduction of compressive stresses. Scenarios 1 and 2 have the same depletion-induced stresses and the great agreement between their derivative responses proves the accuracy of the proposed model. Scenario 3 has almost the same response as Scenario 4 for quite a long time because the extremely poor outer region acts like a no-flow boundary. However, the late time response of Scenario 3 is reaching to the one of Scenario 2 because eventually the diffusive transients will reach the real reservoir boundary.
Figure 4.7 A rectangular heterogeneous reservoir consisting of 25 regions can have various permeability and porosity. A MFHW or fractured vertical well is located in the region #13.
<table>
<thead>
<tr>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
<th>Scenario 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>The reservoir has 25 four by four regions. All regions have the same porosity and permeability ($C_R=1$). The reservoir size is $X=Y=20$.</td>
<td>The homogeneous case with uniform porosity and permeability ($C_R=1$). The reservoir size is $X=Y=20$.</td>
<td>The reservoir has 25 four by four regions. All regions have the extremely low porosity and permeability ($C_R=1e-4$) except for region 13 which has $C_R=1$. The reservoir size is $X=Y=20$.</td>
<td>The homogeneous case with uniform porosity and permeability ($C_R=1$) and the reservoir size is reduced to $X=Y=4$.</td>
</tr>
</tbody>
</table>
Figure 4.8 The depletion-induced normal stress components in the x direction ($\sigma_{xx}$) and their corresponding derivative responses ($d\sigma_{xx}/d \log t$).
4.3.3 Transient pore pressure change

The proposed boundary integral approach in this work explicitly considers transient fluid flow in the reservoir. Transient pore pressure change commonly analyzed in the well testing is simultaneously obtained in this work. Analytical solutions of fractured wells are widely available in the commercial well testing software. A fractured vertical well model available in the Ecrin v4.30 is used to compare with solutions obtained by the proposed boundary integral approach (KAPPA, 2013). The obtained pore pressure change and its derivatives have excellent agreement with the analytical solutions as shown in Figure 4.9. The perfect match of the two solutions has validated the capacity of the proposed model to deal with transient fluid flow in fractured wells.

4.4 Results and Discussion

In this section, various factors affecting induced stresses will be investigated based on modeling results. The transient behavior of depletion-induced stress change around a fractured well is comprehensively presented and is found to be very consistent with commonly monitored pressure and pressure derivative change in well testing.

4.4.1 Effects of net pressure and fracture widths on mechanical-induced stresses

If hydraulic fractures are stimulated with constant and uniform net pressure, the fracture opening can be predicted with the DDM method (Tao et al., 2010). Widths of hydraulic fractures increase with larger net pressure and remain constant with perfectly placed proppant. Relatively larger compressive stresses will be induced by propped fractures with higher normalized net pressure as shown in Figure 4.10.
Figure 4.9 Dimensionless pore pressure change and its derivative of a fractured vertical well producing under constant production rate.
Figure 4.10 Mechanical-induced compressive stresses by propped fractures.
4.4.2 Transient behavior of depletion-induced stresses of a fractured vertical well

In well tests and interference tests, pore pressure change at the wellbore could be monitored by using down-hole pressure gauges, which record continuous time-lapse pressure change. However, it is extremely difficult to monitor the in-situ stress state for a continuous period. The mini-fracture test is one of the very few practical ways to measure the minimum principal stress at the single point time of measurement. Therefore, it is of great importance to develop theoretical models to predict spatio-temporal stress evolution at the wellbore. The modeling results in this work have provided patterned correlations for stress evolution (Xiao et al., 2017).

Figure 4.11 shows a fractured vertical well and a MFHW in a bounded reservoir with no flow boundary. Hydraulic fractures are assumed to be parallel with the maximum horizontal stress (the $x$ direction). Fluid flow due to depletion will induce two normal stress components in the $x$ and $y$ directions, namely $\sigma_{xx}$ and $\sigma_{yy}$ respectively. The pore pressure change and induced stress components at the well are shown in Figure 4.12. The log-log plots of pressure derivatives are informative in transient pressure analysis to reveal transient fluid flow regimes. Accordingly, Figure 4.13 shows the log-log plots of derivatives of pore pressure and induced stress components of the fractured vertical well. The stress evolution is found to be consistent with pore pressure change and also shows characteristic trends as a function of time. The pressure derivative curve shows four distinctive flow regimes commonly identified in well testing. From dimensionless time $1\times10^{-5}$ to $1\times10^{-3}$, the fluid flow is in linear flow regime with a linear trend of $\frac{1}{2}$ slope. Later, a transitional flow regime occurs when the slope gradually evolves to a zero slope line, which indicates the occurrence of the pseudo-radial flow regime.
Figure 4.11 Schematic of (a) a fractured vertical well and (b) a MFHW with hydraulic fractures parallel to the maximum horizontal stress in the $x$ direction.
Figure 4.12 The pore pressure change and induced stress components in the $x$ and $y$ directions, namely $\sigma_{xx}$ and $\sigma_{yy}$ respectively.
Figure 4.13 The log-log plots of derivatives of pore pressure and induced stress components of the fractured vertical well.
When the pressure transients diffuse to the no-flow reservoir boundary, a boundary dominated flow regime occurs with a unit slope. Correspondingly, the two induced stress components on the fracture show four distinctive trends. The two stress components have different characteristics during the linear and transitional flow periods. The $\sigma_{xxD}$ has a $\frac{1}{2}$ slope line during the linear flow regime while the $\sigma_{yyD}$ has a unit slope trend. However, after the transitional flow period, the two induced stress components become identical with a near zero slope line during the pseudo-radial flow regime and a constant unit slope when the boundary dominated flow regime occurs. Because of the consistency between the pore pressure change and the depletion-induced stress components, it is possible to correlate pore pressure data recorded at the well with the induced stress components on the hydraulic fractures.

The ratios of the two induced stress components and pore pressure change are also calculated. Figure 4.14 indicates that the ratios of the two induced stress components and pore pressure change have distinctive characteristics at different time. During the early time period, the $\sigma_{xxD}/p_D$ has a constant value of 1.34 and gradually changes with different slopes as producing time increases. On the other hand, the $\sigma_{yyD}/p_D$ has a relatively unit slope and gradually changes during the transitional fluid flow period. Eventually, when the producing time is longer enough, the two ratios $\sigma_{xxD}/p_D$ and $\sigma_{yyD}/p_D$ become the same with a constant value of 0.34 in this case which is consistent with the late time stress-pressure relations.
Figure 4.14 The ratios of the two induced stress components and pore pressure change of the fractured vertical well.
The modeling results reveal the fact that the depletion-induced stress at the fractured well is time dependent and non-linear with pore pressure change. Therefore the typical stress path monitored at the well may change with time and become non-linear with pore pressure change. Meanwhile, the characteristic trends of $\sigma_{xxD} / p_D$ and $\sigma_{yyD} / p_D$ changing with time may provide predictive tools to evaluate the depletion-induced stress path change of the fractured vertical well.

### 4.4.3 Transient behavior of depletion-induced stresses of a MFHW

The depletion-induced stress and pore pressure relation is non-linear and time dependent as revealed by the aforementioned responses of a fractured vertical well. It is also important to investigate the spatio-temporal stress evolution of a MFHW with multiple fractures (Xiao et al., 2017). Figure 4.15 shows the induced stress components on the center fracture varying with different numbers of fractures. Generally, more fractures induced smaller pore pressure drop at the wellbore and reduce relatively less total stresses. As shown in Figure 4.16, their derivative responses clearly reveal the transient behavior of depletion-induced stresses during different flow regimes. Interference between multiple fractures causes a concave feature on the $\sigma_{xxD}$ derivative responses during the transitional flow period. More fracture stages will have a deeper concave on the derivative responses. After the pseudo-radial flow period, all derivatives merge to one stem because of the boundary dominated flow. Interestingly, interference of multiple fractures does not show much influence on the induced $\sigma_{yyD}$ derivatives. More fractures will induce relatively smaller values of $\sigma_{yyD}$ which evolve consistently with pore pressure change.
Figure 4.15 The induced stress components on the center fracture varying with different numbers of fractures of the MFHW.
Figure 4.16 The derivative responses of the induced stress components on the center fracture varying with different numbers of fracture stages of the MFHW.
Figure 4.17 shows that the induced $\sigma_{xx_D}/p_D$ and $\sigma_{yy_D}/p_D$ vary with different numbers of stages. The ratio between the induced $\sigma_{xx_D}$ and pore pressure during the early time flow period is not affected by fracture stages and has a constant value of 1.34. However, during the transitional flow period, the ratio becomes variable and more fracture stages will result in a lower value of $\sigma_{xx_D}/p_D$. On the other hand, the induced $\sigma_{yy_D}/p_D$ have a linear trend of unit slope during the early time flow period and become smaller with more fracture stages. During the late producing time, the induced $\sigma_{xx_D}/p_D$ and $\sigma_{yy_D}/p_D$ both become constant and are independent of fracture stages. Figure 4.18 also shows the induced $\sigma_{xx_D}/p_D$ and $\sigma_{yy_D}/p_D$ on different fracture stages. Generally, the centre fractures have much severer fracture interference than the edge fractures so that their values of $\sigma_{xx_D}/p_D$ and $\sigma_{yy_D}/p_D$ are different during the transitional flow period.

Fracture spacing not only affects transient fluid flow but also alters the stress state around a MFHW. Figure 4.19 shows the derivative responses of depletion-induced stress components on the center fracture varying with fracture spacing. The MFHW has seven hydraulic fractures with equal lengths and spacing. Adjacent fractures start affecting the induced stresses after the linear flow period. Smaller fracture spacing will have an earlier interference characterized with a concave feature. When fracture spacing becomes larger, the concave period becomes longer and shifts more to the later producing time. When the flow transients diffuse to the reservoir boundary, the boundary dominated flow regime occurs and all derivative curves merge to one stem with a unit slope.
Figure 4.17 The induced $\sigma_{x_D}/p_D$ and $\sigma_{y_D}/p_D$ vary with different numbers of fractures of the MFHW.
Figure 4.18 The induced $\sigma_{xxD}/p_D$ and $\sigma_{yyD}/p_D$ on different fractures of the MFHW.
Figure 4.19 The derivatives responses of depletion-induced stress components change on the center fracture of a MFHW with various fracture spacing.
Similarly, Figure 4.20 and Figure 4.21 show that fracture spacing drastically affects the ratios of the two induced stress components and pore pressure change. The $\sigma_{yy}/p_D$ has a unit slope during the early time and the slope gradually changes during the transitional flow period. During the transitional flow regime, smaller fracture spacing will result in an earlier change of slopes and reaches to a smaller value of $\sigma_{yy}/p_D$; the induced $\sigma_{xx}/p_D$ are also affected by fracture spacing significantly.

4.4.4 Transient behavior of depletion-induced stresses in a heterogeneous reservoir

Transient behavior of depletion-induced stresses in heterogeneous reservoirs is comprehensively analyzed using the proposed boundary integral approach. Characterization of reservoir heterogeneity has been significantly improved by high resolution geophysical data and 3-D geo-modeling techniques. This work proposes an approach that can integrate seamlessly with heterogeneous geological models to calculate fluid flow and spatio-temporal stress evolution. Though the reservoir heterogeneity presented in this study is arbitrary without detailed geological study, the purpose is to showcase how the depletion-induced stresses and pore pressure change will evolve with different forms of reservoir heterogeneity.

Table 4.2 shows three heterogeneous cases with arbitrary distributions of permeability and porosity. The proposed approach calculated depletion-induced stresses on the hydraulic fractures of a MFHW producing under constant rate. The pressure derivative responses monitored at the well show different flow regimes affected by reservoir heterogeneity.
Figure 4.20 The ratios of the induced stress components in the y direction and pore pressure change, $\sigma_{yD}/p_D$, on the center fracture of a MFHW with various fracture spacing.
Figure 4.21 The ratios of the induced stress components in the x direction and pore pressure change, $\sigma_{xxD}/p_D$ on the center fracture of a MFHW with various fracture spacing.
Table 4.2 Three heterogeneous cases with arbitrary distributions of permeability and porosity

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
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<tbody>
<tr>
<td>The MFHW is located in region 13.</td>
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</tr>
<tr>
<td>The yellow color denotes the dimensionless diffusivity ratio as one and the dark blue color denotes $C_R=1\times10^{-3}$</td>
<td>The yellow color denotes the dimensionless diffusivity ratio as one and the dark blue color denotes $C_R=1\times10^{-3}$</td>
<td>The yellow color denotes the dimensionless diffusivity ratio as one and the dark blue color denotes $C_R=1\times10^{-3}$. The region colored in red has relatively 100 times higher diffusivity than the region with MFHW.</td>
</tr>
<tr>
<td><img src="image1.png" alt="Case 1 Diagram" /></td>
<td><img src="image2.png" alt="Case 2 Diagram" /></td>
<td><img src="image3.png" alt="Case 3 Diagram" /></td>
</tr>
</tbody>
</table>

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As shown in Figure 4.22, the pressure and pressure derivatives vary with each other due to the reservoir heterogeneity. It is worth noticing that the derivatives of depletion-induced stress components also show consistent transient behavior with the pressure responses as shown in Figure 4.23. When an extremely low-permeable region exists in the vicinity of the well, a close to unit slope line occurs on the log-log plots of derivatives of the induced stress responses.

The severe interference between multiple fractures shows a concave feature on the stress derivative responses instead of a close to unit slope line that occurs on the pressure derivatives. The depletion-induced stresses are sensitive to reservoir heterogeneity and consistent with pore pressure change. As a matter of fact that bottom-hole pressure monitoring is very important to predict depletion-induced stress evolution on hydraulic fractures.

Effects of heterogeneity on the induced \( \sigma_{xxD} / p_D \) and \( \sigma_{yyD} / p_D \) are further investigated as shown in Figure 4.24 and Figure 4.25. All cases have the same values during the early time and late time flow periods because the fracture network and reservoir properties around the MFHW are the same. When the local heterogeneities start influencing the fluid flow during the transitional flow period, ratios of induced stress components and pore pressure change become various because of different permeability and porosity in the outer regions. The local heterogeneity with better permeability will result in larger values of \( \sigma_{xxD} / p_D \) and \( \sigma_{yyD} / p_D \) during the transitional flow period.
Figure 4.22 Dimensionless pressure and pressure derivative responses of the three heterogeneous cases.
Figure 4.23 The derivative responses of depletion-induced stress components in the $x$ direction of the three heterogeneous cases.
Figure 4.24 The induced $\sigma_{xxD}/P_D$ of the three heterogeneous cases
Figure 4.25 The induced $\sigma_{yyD}/p_D$ of the three heterogeneous cases.
4.4.5 Time-dependent stress path $\Delta \sigma_h / \Delta p$ evolution

Under the plane strain condition, the maximum and minimum horizontal stresses are defined as (Ugural and Fenster, 2011):

$$\sigma_H = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \left(\sigma_{xy}\right)^2}$$  \[4.46\]

$$\sigma_h = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \left(\sigma_{xy}\right)^2}$$  \[4.47\]

For a fractured vertical well shown in Figure 4.11, the initial maximum stress prior to production is assumed to be parallel to the hydraulic fracture (Zoback et al., 2003); and a constant stress deviator ($\sigma_{\text{deviator}} = \sigma_H - \sigma_h$) exists between the initial maximum and minimum horizontal stresses. Once the well starts producing, the fluid flow around the fractured well will induce spatio-temporal stress evolution. Accordingly, the induced maximum and minimum horizontal stresses on the fracture can be derived in terms of depletion-induced stress components. Assuming the shear stress components on the fractures are negligible, the induced maximum and minimum stresses can be obtained from the modeling results of $\sigma_{xxD}$ and $\sigma_{yyD}$:

$$\Delta \sigma_h = \sigma_{yyD}, \Delta \sigma_H = \sigma_{xxD}, \text{ if } \sigma_{xxD} \leq \sigma_{\text{deviator}} + \sigma_{yyD}$$  \[4.48\]

$$\Delta \sigma_h = \sigma_{xxD} - \sigma_{\text{deviator}}, \Delta \sigma_H = \sigma_{yyD} + \sigma_{\text{deviator}}, \text{ if } \sigma_{xxD} > \sigma_{\text{deviator}} + \sigma_{yyD}$$  \[4.49\]

When $\sigma_{xxD} > \sigma_{\text{deviator}} + \sigma_{yyD}$, the stress reversal happens because the maximum and minimum horizontal stresses rotate $90^\circ$ accordingly. In such cases, the induced stress path $\Delta \sigma_h / \Delta p$ is mainly affected by the change of $\sigma_{xxD}$ since $\sigma_{\text{deviator}}$ is constant. For a
fractured vertical well with a symmetric hydraulic fracture parallel to the direction of initial maximum stress, the depletion-induced $\sigma_{xxD}/p_D$ and $\sigma_{yyD}/p_D$ were illustrated previously in Figure 4.14. Because of symmetry, the induced shear stress component on the fracture is negligible. Therefore, Eq. [4.48] and [4.49] can be used to calculate the induced stress path $\Delta\sigma_h/\Delta p$. When the stress deviator is zero ($\sigma_{\text{deviator}} = 0$), the induced $\sigma_{xxD}$ of the fractured vertical well is always larger than the induced $\sigma_{yyD}$. According to Eq. [4.49], when the stress reversal happens the induced stress path evolution equals to the change of $\sigma_{xxD}/p_D$. On the other hand, if the stress deviator is large enough (i.e. $\sigma_{\text{deviator}} = 1.0$ in this case), no stress reversal will happen and the induced stress path evolution will have the same feature with $\sigma_{yyD}/p_D$. As shown in Figure 4.26, if stress reversal happens $\Delta\sigma_h/\Delta p$ behaves similarly to the $\sigma_{xxD}/p_D$, whereas the induced minimum horizontal stress has a linear relation with pore pressure change during the early time of production. However, if there is no stress reversal $\Delta\sigma_h/\Delta p$ is mainly influenced by the $\sigma_{yyD}/p_D$, whereas the $\Delta\sigma_h/\Delta p$ during the early time period will evolve proportionally with a unit slope. Moreover, stress reversal is time dependent and significantly affected by the value of the stress deviator. Figure 4.27 shows the effects of stress deviator on the induced stress path $\Delta\sigma_h/\Delta p$. When the stress reversal does not happen, the induced stress path $\Delta\sigma_h/\Delta p$ equals to the $\sigma_{yyD}/p_D$. Once the stress reversal occurs, the induced stress path will deviate from the $\sigma_{yyD}/p_D$. Also, the deviation occurs earlier with smaller stress deviator.
Figure 4.26 The induced stress path evolution with and without stress reversal.
Figure 4.27 The effects of stress deviator on the induced stress path $\Delta \sigma_h / \Delta p$. 
For a MFHW with multiple stages of hydraulic fractures, the induced stress components evolve differently with the fractured vertical well due to interference from adjacent fractures. As previously shown in Figure 4.17, for a MFHW with 11 fractures, the induced $\sigma_{xxD}$ becomes smaller than the induced $\sigma_{yyD}$ after dimensionless time 0.1. Therefore, no matter how small the stress deviator is, it is impossible for the stress reversal to occur after dimensionless time 0.1. Figure 4.28 shows the effects of stress deviator $\sigma_{\text{deviator}}$ on the induced stress path evolution of a MFHW with 11 fractures. For this particular case, the induced stress path on the centre fracture always equals to the induced $\sigma_{yyD} / p_D$ after dimensionless time 0.1.

### 4.5 Summary

The boundary integral approach proposed in the chapter explicitly considers non-planar hydraulic fractures and reservoir heterogeneity. It solves the coupled fluid flow and solid deformation problem in a heterogeneous reservoir with near analytical accuracy. It is the first time to correlate the depletion-induced stress evolution with transient flow regimes occurring in a life time of a producing MFHW. The transient behavior of stress evolution on the hydraulic fractures is found to be consistent with flow regimes identified in transient pressure analysis. During the early linear flow period, the derivative of induced stress component parallel with the fracture plane has a 1/2 slope on the log-log plots with time. When the transitional flow regime occurs, the slope gradually changes and is significantly affected by the interference between multiple fractures. More fracture stages will result in a larger concave feature on the derivative responses of induced stress components. When the flow transients reach the local heterogeneities (i.e. flow barriers
with low permeability and sweet spots with high permeability), derivatives of pore pressure and induced stress components will have similar characteristics of slope change. The ratios of induced stress components and pore pressure are found to be within a range of constant values, which are affected by Poisson’s ratio and Biot coefficient. Both $\sigma_{xx}/p_D$ and $\sigma_{yy}/p_D$ are time dependent and their values change with distinctive trends during different flow regimes. The induced spatio-temporal stress path $\Delta\sigma_h/\Delta p$ is anisotropic and predictive during various flow regimes by the proposed model. The informative trends of $\Delta\sigma_h/\Delta p$ versus time calculated by the proposed approach provides possible ways to predict the stress path evolution once the pore pressure change and stress deviator are known.
Figure 4.28 The effects of stress deviator $\sigma_{\text{deviator}}$ on the induced stress path evolution of a MFHW with 11 stages.
Nomenclature

Notations

\( a_{pl} \) = dimensionless half-length of each reservoir boundary element

\( C_b \) = bulk compressibility, \( 1/\text{Pa} \)

\( C_{\text{fluid}} \) = fluid compressibility, \( 1/\text{Pa} \)

\( C_{Rj} \) = diffusivity ratio between the \( j \)-th and the reference subsystem

\( C_{Sj} \) = storativity ratio between the \( j \)-th and the reference subsystem

\( C_{Tj} \) = transmissibility ratio between the \( j \)-th and the reference subsystem

\( C_{\text{solid}} \) = solid compressibility, \( 1/\text{Pa} \)

\( D \) = displacement discontinuity tensor

\( F \) = body force of the poroelastic media, \( \text{N} \)

\( f \) = body force of the fluid, \( \text{N} \)

\( G \) = shear modulus, \( \text{Pa} \)

\( h \) = formation thickness, \( \text{m} \)

\( K_b \) = drained bulk modulus, \( \text{Pa} \)

\( k \) = reservoir permeability, \( \text{m}^2 \)

\( L_{Dw} \) = dimensionless half-length of a fracture element

\( N_j \) = number of boundary elements for each region

\( N_w \) = number of fracture elements

\( p \) = pressure at any location, \( \text{Pa} \)

\( p_i \) = initial pressure, \( \text{Pa} \)

\( p \) = pressure at any location, \( \text{Pa} \)
\( Q \) = production rate from the source, m³/s
\( q \) = flux rate, m³/s
\( q_b \) = base production rate, m³/s
\( s \) = Laplace variable
\( S_\sigma \) = storativity in terms of porosity and compressibility, 1/Pa
\( S_{ps} \) = storage capacity under plane strain condition defined in Eq. [4.14], 1/Pa
\( t \) = time, s
\( u \) = displacement, m
\( x \) = horizontal coordinate in Cartesian coordinates system, m
\( y \) = vertical coordinate in Cartesian coordinates system, m

**Greek letters**

\((\xi, \zeta)\) = local coordinates, m
\( \mu \) = fluid viscosity, Pa·s
\( \theta \) = internal angles between two adjacent elements
\( \tau \) = time when sources are removed, s
\( \phi \) = porosity
\( \Omega \) = solution domain
\( \Omega' \) = solution domain of the complementary region described in Figure 4.2
\( \sigma \) = stress component, Pa
\( \sigma_{\text{initial}} \) = initial state of stress component before depletion, Pa
\( \sigma_{\text{mechanical}} \) = induced stress component by mechanical loading, Pa
\( \sigma_{\text{fluid}} \) = induced stress component by fluid flow, Pa
\( \sigma_{xx_{\text{mechanic}}} \) = normalized mechanical-induced stress component in the x direction

\( \sigma_{yy_{\text{mechanic}}} \) = normalized mechanical-induced stress component in the y direction

\( \chi \) = location of the removal of instantaneous source along the x direction, m

\( \zeta \) = fluid variation content

\( \alpha \) = Biot coefficient

\( \gamma \) = a source/sink in the porous media, \( m^3/s \)

\( \nu \) = Poisson’s ratio

\( \varepsilon \) = strain component

\( \Gamma^{\text{outer}} \) = outer boundary of the reservoir

\( \Gamma \) = boundary of the subdivided domain

**Functions**

\( \text{erf} \) = error function

\( \exp \) = exponential function

\( H \) = Heaviside step function

\( G_{\text{fg}} \) = general free space Green’s function of a point source

\( G_{k} \) = free space Green’s function of a line source

\( G_{n} \) = free space Green’s function for the \( n^{\text{th}} \) region

\( L_{\{\ldots\}} \) = Laplace transformation

\( \delta \) = Dirac delta function

**Subscript and superscript**

\( D \) = dimensionless term

\( n \) = the \( n^{\text{th}} \) region of the heterogeneous system

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\( l \) = the \( l^{th} \) discretized boundary element

\( w \) = the \( w^{th} \) discretized fracture element

\( ijk \) = indices representing directions in a 2-D Cartesian coordinates system, \( i, j, k=x,y; \)

“\( \sim \)” = Laplace transformation of a function/variable

“\( \ast \)” = fundamental solutions of related quantities

**SI Metric Conversion Factors**

\[
\begin{align*}
\text{bbl} \times 1.589874 \quad & E-01 = \text{m}^3 \\
\text{cp} \times 1.0* \quad & E-03 = \text{Pa} \cdot \text{s} \\
\text{ft} \times 3.048* \quad & E-01 = \text{m} \\
\text{mD} \times 9.869233 \quad & E-04 = \mu\text{m}^2 \\
\text{lbm} \times 4.535924 \quad & E-01 = \text{kg} \\
\text{psi} \times 6.894757 \quad & E+00 = \text{kPa}
\end{align*}
\]

*Conversion factor is exact*
CHAPTER 5  EVALUATION OF REFRUCTURING UPSIDE IN A
SOUTHERN SASKATCHEWAN BAKKEN POOL

Tight formations and shale gas reservoirs have been successfully unlocked since the application of hydraulic fracturing. After the boom of massive multistage fractured horizontal wells (MFHW), tight oil and shale gas production decline sharply because of rapid reservoir depletion and low drilling rig activities in such a low commodity price environment. Rather than acquiring more land and spending capital on drilling, operators tend to look at alternative technologies to increase production rate of tight and shale reservoirs as alternatives to meet production budget with lower capital costs. Refracturing (also called restimulation by some operators) is to recomplete an existing well to increase production rate by adding new fractures to contact virgin reservoir and stimulating existing perforations to improve original fracture conductivity. It has gained popularity among tight reservoir operators due to the minimum usage of land and surface infrastructures and quick response of production enhancement. In this chapter, the boundary integral approaches proposed in Chapter 3 and Chapter 4 are applied to model depletion-induced stress reorientation around fractures and production uplift of MFHW by adding new fractures. Moreover, the theoretical model is tuned and updated with an actual refractured MFHW in a Southern Saskatchewan Bakken pool. With the tuned model, evaluation of refracturing upside in a Southern Saskatchewan Bakken pool is completed, and the field evaluation study is also discussed with operational considerations.
5.1 Introduction

A recent literature review study has shown over 140 published papers on the topic of refracturing design and application (Vincent, 2011). Successful field applications have been documented in a wide variety of reservoirs including Canadian Bakken fields in Manitoba and Saskatchewan, Foothills Cardium formations, Viking formations, shallow-gas wells in Southern Alberta, and Eagle Ford fields (Oruganti et al., 2015). The success of refracturing has been credited to restoration of original fracture’s conductivity, contact of new reservoir area, optimization of proppant type and schedule, and fracture reorientation due to local stress alteration. Restimulation treatment was reported as early as in 1950s (Sallee and Rugg, 1953). As a result of quick declining rate of Canadian Bakken horizontal wells, refracturing activities have been reported in recent years to improve the ultimate oil recovery. Vincent (2011) summarized a detailed review of lessons learned from the refractured Bakken MFHW. The paper has provided comprehensive analysis of refracturing opportunities and challenges existing in unconventional reservoirs with both public data and operators’ experiences. It also revealed the complexity and technical issues faced in the practice of refracturing in unconventional reservoirs. The success of refracturing jobs varies on a well-by-well basis due to consideration of different completion designs, selection of proppant, and local geology and reservoir characterization.

Lantz et al. (2007) reported that refractured production uplift after refracturing in the Elm Coulee field of Montana could exceed the original peak production of the initial fracture treatment. Ceramic proppant is more likely to be used while increasing the number of perforation clusters to enhance fracture conductivity and reduce fluid velocity.
and proppant flowback (Besler et al., 2007). Reeves et al. (1999) investigated refracturing potential for tight gas reservoirs and found that most incremental is likely to be confined to limited wells. Moore and Ramakrishnan (2006) pointed out that candidate selection criteria of restimulated wells should not be universal and must be customized to fit particular situations. Higher success rate is related to the learning process of existing experience in the field and thorough understanding of governing parameters of production rate return. Majority of the identified refractured wells in Eagle Ford and Bakken reservoirs have shown an average of 69% EUR increase in a recently published data mining study (Oruganti et al., 2015). Case studies in the Barnett shale and Woodford shale were also reviewed to identify the production incremental of refractured MFHW (Craig et al., 2012; French et al., 2014). Indras and Blankenship (2015) conducted a net present value (NPV) analysis of refracturing upside in three mature plays – the Bakken, Barnett and Haynesville. They showed that refractured MFHW in Bakken and Haynesville would generate higher NPV than infill well drilling in a low commodity pricing environment; however, it would be a negative investment in some Barnett wells due to reduced recoveries and value erosion. Integrated studies of petrophysics, geomechanics, hydraulic fracturing, reservoir modeling workflow, and field observations are identified as a necessary process to select refracturing candidate and optimize restimulation strategies (Malpani et al., 2015). Reservoir pressure, stress alteration, and geological heterogeneity are found to be critical parameters in the modeling processes.

5.2 Theoretical Modeling of Stress Reorientation and Production Uplift

According to the boundary integral approach of modeling fluid flow and solid deformation problems in Chapter 4, the depletion-induced stress changes due to
poroelastic effects will cause stress reorientation. It is also considered as the first criterion indicative of success in a refracturing treatment by Roussel and Sharma (2013). Tiltmeter measurements in the field are usually effective to detect the orientation of original and restimulated fractures (Wolhart et al., 2007). Because of stress alteration, restimulated fractures (also called refracs) intend to propagate along the direction of altered maximum horizontal stress, which usually has curvature within an elliptical region. The universal boundary integral approaches proposed in this study not only consider the effect of poroelasticity on induced stress changes but also become capable of modeling production uplift of refractured MFHW with both curved and planar refracs.

5.2.1 Depletion-induced stress reorientation

Warpinski and Branagan (1989) pointed out that mechanical and poroelastic effects are two major contributors to stress reorientation. The stress deviator between initial maximum and minimum horizontal stresses also determines the timing and extent of stress reversal region, where the direction of maximum horizontal stress is switched 90° degrees. In Chapter 4, the time-dependent stress path evolution has shown that directions of maximum and minimum stresses may be altered due to depletion of the reservoir. Marongiu-Porcu et al. (2015) used a finite element model to calculate the change in stress magnitudes and orientations. Their modeling results suggest that the stresses reorient within the elliptically depleted region around the MFHW when horizontal stress anisotropy increases. In Chapter 4, Figure 4.27 shows the effects of stress deviator on the induced stress path $\Delta \sigma_h / \Delta \rho$. Once the stress reversal occurs with certain range of stress deviator, the induced stress path deviates from the $\sigma_{3D} / p_D$. Figure 5.1 shows the areal
distribution of maximum horizontal stress around a fractured well at dimensionless time 0.1 with a dimensionless stress deviator of 0.1. The initial direction of maximum horizontal stress is along the fracture direction (the x axis). As a result of production, the maximum stress direction reorients 90° degrees within certain distance perpendicular to the fracture and gradually aligns along the initial maximum stress direction. An elliptical region of stress reorientation exists around the initial fracture with the longer axis along the fracture direction. The modeled reoriented stress distribution agrees with the observation of depletion-induced stress reorientation by Roussel and Sharma (2013).

Several papers have addressed the mechanisms of complex fracture propagation during hydraulic fracturing (Wu and Olson, 2016; Weng et al., 2011). The dynamic fracture propagation is beyond the scope of the current study. It is assumed that the newly added refracs will propagate along the direction of altered maximum horizontal stress (Wolhart et al., 2007; Roussel and Sharma, 2013). Therefore, Figure 5.2 shows a possible realization of refracs initiated from the same perforations with the initial fractures due to stress reorientation. The refracs propagate perpendicularly to the maximum stress reversal point and gradually become parallel with the initial fracture. In occurrence of stress reorientation, refracs have curved morphologies and are no longer planar fractures because of the altered maximum stress direction with curvature.
Figure 5.1 The areal distribution of maximum horizontal stress around the initial fracture at dimensionless time 0.1 with a dimensionless stress deviator of 0.1. The direction of initial maximum horizontal stress is along the fracture direction (the x axis).
**Figure 5.2** A possible realization of refracs initiated from the same perforations with the initial fractures due to stress reorientation (no propagation modeling has been completed). The direction of initial maximum horizontal stress is along the initial fracture direction (the $x$ axis). The refracs have a curved morphology along the altered maximum horizontal stress.
5.2.2 Production uplift of a refractured MFHW

The boundary integral approach proposed in Chapter 4 has been extended to model transient production rate response of a refractured MFHW, whereas the newly added refracs do not exist until time \( t_{rf} \). The refracs have no effect on the reservoir flow until the refracturing treatment is completed and the well is put on production. After the refracturing time \( t_{rf} \), the dimensionless free space Green’s function of fractures, as shown in Eq. [4.38], should be modified to account for the refracs.

Consequently, Eq. [4.31] for the region with refractured MFHW is subsequently written in a discrete form at the evaluating point \((ξ_D, ζ_D)\) in local coordinates as

\[
θ_l P_{DL}(ξ_D, ζ_D, t_D) = \int_{t_D} \left\{ \sum_{l=1}^{N_j} \int_{ξ_{li}}^{ξ_{li+1}} \left[ \frac{∂P_{DL}(ξ_D, ζ_D, t_D, τ)}{∂n} \right] dξ + \sum_{w=1}^{N_w} Q_{Dw} G_{Lw} \right\} dτ \tag{5.1}
\]

\[
G_{Lw}(x_D, y_D, t_D, ξ, ζ, τ) = \int_{ξ_{li}}^{ξ_{li+1}} \frac{H(t_D - τ)}{4π(t_D - τ)} \exp\left[-\frac{(x_D - ξ)^2}{2C_{Re}(t_D - τ)}\right] \exp\left[-\frac{(y_D - ζ)^2}{2C_{Re}(t_D - τ)}\right] dξ \tag{5.2}
\]

where,

\[
θ_l = 1 \rightarrow if(ξ_D, ζ_D) ∈ Ω, \quad l=1, N_j ; w=1, N_w, \quad t_D < t_{rf} \tag{5.3}
\]

\[
θ_l = 0.5 \rightarrow if(ξ_D, ζ_D) ∈ Γ, \quad l=1, N_j ; w=1, N_wrf, \quad t_D >= t_{rf} \tag{5.4}
\]

\[
θ_l P_{DL}(ξ_D, ζ_D, t_D) = \int_{t_D} \left\{ \sum_{l=1}^{N_j} \int_{ξ_{li}}^{ξ_{li+1}} \left[ \frac{∂P_{DL}(ξ_D, ζ_D, t_D, τ)}{∂n} \right] dξ + \sum_{w=1}^{N_w} Q_{Dw} G_{Lw} + \sum_{wrf=1}^{N_wrf} Q_{Dwrf} G_{Lwrf} \right\} dτ \tag{5.5}
\]

\[
G_{Lwrf}(x_D, y_D, t_D, ξ, ζ, τ) = \int_{ξ_{li}}^{ξ_{li+1}} \frac{H(t_D - τ)}{4π(t_D - τ)} \exp\left[-\frac{(x_D - ξ)^2}{2C_{Re}(t_D - τ)}\right] \exp\left[-\frac{(y_D - ζ)^2}{2C_{Re}(t_D - τ)}\right] dξ \tag{5.6}
\]

where,

\[
θ_l = 1 \rightarrow if(ξ_D, ζ_D) ∈ Ω, \quad l=1, N_j ; w=1, N_wrf, \quad t_D >= t_{rf} \tag{5.7}
\]
\( a_{pl} \) is the dimensionless half-length of each boundary element around the region with MFHW; \( L_{Dw} \) and \( L_{Dwrf} \) are the dimensionless half-length of each fracture element of the initial fractures and refracs respectively.

The local coordinate system is used to account for the curvature of refracs. As shown in Figure 5.3, each refrac element has its own local coordinates which can be calculated by Eq. [5.7].

\[
\begin{align*}
\xi &= x \cos \beta + y \sin \beta \\
\zeta &= -x \sin \beta + y \cos \beta 
\end{align*}
\]  

[5.7]

where, \((\xi, \zeta)\) is the local coordinates, \((x, y)\) is the global coordinates, and \(\beta\) is the angle between the \(\zeta\) and \(x\) plane in different coordinates systems.

With the modified matrix equations, the solution process for the transient production rate of refractured MFHW is the same to the process addressed in Chapter 4.

### 5.3 Operational Significance of Identifying Fracture Reorientation

By isolating the depleted areas and existing initial fractures, the key of successful refracturing treatment is to generate fractures contacting the virgin reservoir (French et al., 2014; Diakhate et al., 2015). Microseismic monitoring and production logs suggest that restimulated lateral coverage could be as low as 25% due to the challenges of poor settling of proppant. Two main methods of multistage fractured horizontal wells used in the Bakken formation are cemented liner plug-and-perf and openhole ball drop fracturing system.
**Figure 5.3** The relationship between global and local coordinates systems. Fracture elements are assigned along both initial fracture and refracs. Local coordinates are calculated according to the locations of different fracture elements.
The cemented liner plug-and-perf technology uses the bridge plugs as the isolation between stages and pumps down coiled tubing to perforate and fracture the reservoir matrix as shown in Figure 5.4 a (Rivenbark and Appleton, 2013). On the other hand, the openhole ball drop fracturing system sets packers and size-specific actuation balls to isolate stages and stimulates through frac sleeves to create hydraulic fractures (Figure 5.4 b).

When it comes to refracturing the existing MFHW, both systems have operational challenges in either treating existing initial fractures or adding new fractures at intact intervals. Mechanical isolations are better methods to refracture a cemented liner system at untreated zones. On the other hand, the openhole system has more operational challenges in isolating existing fracture stages and adding infill fractures (Strother et al., 2013). New technologies are available such as recloseable frac sleeves and retrievable fracturing sleeve system to allow wellbores to be ready for refracturing treatment (Paneitz et al., 2012; Wellhoefer and Simmons, 2016). However, they might not be cost effective comparing to mature technologies in the current low commodity price environment. Operationally, it is important to decide whether the refracturing treatment should be at the existing perforations or at the untreated zones by using advanced isolation technologies, especially for initial fractures with openhole ball drop fracturing system. The existence of stress reorientation allows refracs to propagate perpendicularly from the same perforations of initial fractures stages towards intact reservoir regions without advanced isolation technologies.
Figure 5.4 Schematic of (a) cemented liner multistage, plug-and-perf completion; and (b) openhole, multistage system completion (Appleton and Rivenbark, 2013).
For a MFHW with openhole ball drop fracturing system, the identification of stress reorientation will have significant operation advantages by refracturing at existing frac sleeves. Figure 5.5 shows an example of the modeled dimensionless pressure distribution of a MFHW with five initial fractures. The blue regions represent intact reservoirs while the red ones are more likely depleted. Figure 5.6 a shows the reoriented refracs developed from the existing perforations due to stress reorientation; and Figure 5.6 b shows infilling refracs developed from untreated zones between initial stages.

The pressure distribution profiles of the two refracturing designs are different due to various morphologies of refracs. Nonetheless, both reoriented refracs and infilling refracs have reached to the intact reservoir regions. Figure 5.7 shows pressure profiles at the late producing time when the flow transients have reached the boundaries. Figure 5.8 compares the production performances of the two refracturing strategies. It shows that refracturing at existing perforations with reoriented refracs have similar production uplift with the infilling refracs developed from untreated zones. It has operational significance to identify stress reorientation so that optimized refracturing designs could be conducted to reduce operational costs.
Figure 5.5 The modeled dimensionless pressure distribution of a MFHW with five initial fractures; the blue regions represent intact reservoirs while the red ones are more likely depleted.
Figure 5.6 Schematic of (a) the reoriented refracs developed from the existing perforations due to stress reorientation; and (b) The infilling refracs developed from untreated zones between initial stages without stress reorientation.
Figure 5.7 Schematic of (a) late time pressure distribution of the MFHW with reoriented refracs; and (b) late time pressure distribution of the MFHW with infilling refracs.
Figure 5.8 Dimensionless production rates of the two refracturing designs with respect to dimensionless producing time. The refracturing treatment occurs at about dimensionless time 1000.
5.4 Model Tuning for an Actual Refractured Bakken Well

From the public database (i.e. the AccuMap by IHS and Canadian Discovery Frac database), an actual refractured well from the Southern Saskatchewan Bakken formation was identified and confirmed from information provided by operators. According to the report from Saskatchewan Ministry of Energy and Resources (2010), the Bakken tight oil play with about 25 m in thickness consists of a lower organic-rich shale member, a middle siltstone and sandstone member, and an upper organic-rich shale member. The middle member of the Canadian Bakken is the main target for hydraulic fracturing; it has porosity from 15 to 25 percentage and permeability ranging from several milidarcies to hundreds of milidarcies. The oil in the Bakken formation is very light with an API of 40°. Table 5.1 summarizes the effective reservoir properties and completion information of the actual refractured well.

Reflecturing treatment of the well was conducted by isolating initial fracture stages and stimulating untreated zones. Additional 14 infilling refracs were added between the 16 initial fracture stages, reducing fracture spacing from about 90 m to 45 m. Unfortunately, tiltmeter and geomechanics data were not found in the public database. Accordingly, stress reorientation around the refractured well was not considered in the current study and all initial fractures and refracs were assumed to be planar fractures. By plotting the actual production rate versus material balance time on a log-log scale, the characteristic flow regimes before and after the refracturing treatment were identified and successfully matched with the modeled type curves.
Table 5.1 Summary of the completion information and effective reservoir properties of the actual refractured Bakken well

<table>
<thead>
<tr>
<th></th>
<th>Initial Fracture</th>
<th>Refracs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well – Completed Zone</td>
<td>Bakken</td>
<td>Bakken</td>
</tr>
<tr>
<td>Well – Completion Date</td>
<td>2/11/2010</td>
<td>9/5/2013</td>
</tr>
<tr>
<td>Well - Open or Cased</td>
<td>Cased</td>
<td>Cased</td>
</tr>
<tr>
<td>Well – Stages Actual (#)</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>Well – Estimated Completed Length (m)</td>
<td>1359.33</td>
<td>1359.33</td>
</tr>
<tr>
<td>Well – TVD (m)</td>
<td>1635.1</td>
<td>1635.1</td>
</tr>
<tr>
<td>Reservoir - Porosity</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Reservoir – Formation Thickness (m)</td>
<td>1.8</td>
<td>1.8</td>
</tr>
<tr>
<td>Oil – Viscosity (cp)</td>
<td>1.4</td>
<td>1.4</td>
</tr>
</tbody>
</table>
As shown in Figure 5.9, a linear flow regime followed by a boundary dominated flow regime was identified before the refracturing treatment. The interpreted initial fracture half-length was about 75 m and the effective matrix permeability was around 1.0 mD. A new linear flow regime with shorter period occurred after the refracturing treatment with higher production rates. It is because that the refractures established new accesses to intact reservoirs and shorter linear flow occurred from the reservoir to refractures. A new boundary dominated flow regime was also identified when the flow transients of the whole system with refractures reached to the no-flow reservoir boundaries. It indicates that flow regimes alteration contributes to the increase of production rate after the refracturing treatment. Figure 5.10 shows the matched results of production rate with respect to the normalized producing time. The production uplift occurred with about 45% of the initial peak rate after the refracturing treatment on the normalized 47th month. The oil production increased about four folds to 45 bbl/day and gradually declined for the next 30 months. The refractured Bakken MFHW shows the success of refracturing and also provides a field base case to update the theoretical model to evaluate a Southern Saskatchewan Bakken pool.

5.5 Refracturing Upside in a Southern Saskatchewan Bakken Pool

It is critically important but challenging to optimize candidate selection for refracturing treatment in mature plays (Indras and Blankenship, 2015). Refractured MFHW in the US Bakken formations have exhibited an incremental of 120% compared to the initial peak production rate according to the work of Indras and Blankenship (2015).
Figure 5.9 Type curve matching of transient flow regimes of an actual refractured MFHW in the Southern Saskatchewan Bakken Formation. Production rate is plotted with respect to the material balance time on a log-log scale.
Figure 5.10 Type curve matching of monthly production rate and cumulative production of an actual refractured MFHW in the Southern Saskatchewan Bakken Formation.
Based on the identified 22 refractured Bakken wells on the US side, the average peak production rate after refracturing was about 92% of initial peak rate and the refracturing treatment in Bakken formation seems to be more successful than the Eagle Ford refracturing campaigns (Oruganti et al., 2015). Most of the refracturing candidates screening studies are based on statistical analysis of initial production, play thickness, production rate of the first three months, declining rate and so forth. Malpani et al. (2015) pointed out that the refracturing candidate selection is the key for a successful refracturing project. They integrated petrophysical, geomechanical, hydraulic fracturing, and reservoir modeling workflows to properly design refracturing jobs. The integrated study could effectively reduce the uncertainty of estimated production uplift after refracturing. However, detailed simulation work on a well-by-well basis is too cumbersome to be applicable in the preliminary screening process of mature plays.

The theoretical model developed in this work considers the basic physics of MFHW such as well lateral lengths, fracture stages, fracture half lengths, reservoir properties, and morphologies of refractures. Moreover, the modeling results of transient flow regimes and production rate type curves provide quicker analytical approaches than numerical simulation studies to evaluate potential candidate wells for refracturing. The type curve matching process also considers more physics than statistical evaluation approaches without suffering computation speed. A Southern Saskatchewan Bakken pool, with about 50 wells drilled and completed by the MFHW technology since 2010, was selected to use the model-based type curve matching to evaluate the refracturing upside.
As shown in **Figure 5.11**, 18 wells highlighted in blue were selected with consideration of the data quality and well locations. Using the actual refractured Southern Saskatchewan Bakken well as the benchmark and assuming similar refracturing strategies across the studied pool, evaluation of refracturing upside of the 18 MFHW has been completed by the proposed model-based type curve matching technique. Since the actual refractured Southern Saskatchewan Bakken well was recompleted with infilling refracs, I assume that similar refracturing treatment will be applied to the 18 wells by adding infilling refractures between initial fracture stages.

By type curve matching the actual monthly production data, the modeling result for each well was successfully validated and used to estimate production uplift after the refracturing treatment. **Figure 5.12** and **Figure 5.13** show the type curve matching results of transient flow regimes and production rates of a selected well, respectively. After matching on the historical production data, production uplift of each well after the refracturing treatment was then estimated based on the modeled type curves. **Figure 5.14** illustrates the estimated production uplift and incremental cumulative production after refracturing. It is worthy to noticing that the production uplift will vary significantly with different refracturing designs (i.e. refracs lengths, heights, and conductivities). For the sake of simplicity and consistency, the properties of refracs of each well were assumed to be the same with their initial fractures. **Figure 5.15** shows the refracturing upside of the evaluated 18 wells from the Southern Saskatchewan Bakken pool at the 5 bbl/day cutoff rate. An average of 38,000 bbls/well of refracturing incremental production is estimated for the Southern Saskatchewan Bakken pool.
Figure 5.11 The location of the studied Southern Saskatchewan Bakken Pool, where the evaluated wells are highlighted in blue.
Figure 5.12 Type curve matching of transient flow regimes of one of the evaluated MFHW. Production rate is plotted with respect to the material balance time on a log-log scale.
Figure 5.13 Type curve matching of monthly production rate and cumulative production of one of the evaluated MFHW.
Figure 5.14 The estimated production uplift and cumulative production rate by refracturing of one of the evaluated MFHW.
Figure 5.15 The refracturing upside of the evaluated 18 wells from the Southern Saskatchewan Bakken pool at the 5 bbl/day cutoff rate.
Generally, good wells make better candidates for refracturing. Initial production rate, reservoir pressure depletion, effective permeability and porosity, and fracture properties are major factors affecting the refracturing upside. Significant variance of refractured wells from the same pool suggests the critical importance of refracturing candidate selection. In addition to the statistical methods, the proposed approach for refracturing evaluation is advantageous to quickly identify good candidates based on production history, completion data, and local reservoir properties and depletion history. The success of a refracturing campaign is highly contingent on candidate well selection. A poorly selected refracturing candidate could result in overestimation of production uplift and negative impact on offset wells.

5.6 Summary

Stress reorientation of a MFHW due to production is investigated by the proposed theoretical models in this study. In the occurrence of stress reorientation, refractures could have curved morphologies and develop into intact regions from initial fracture stages. If stress reorientation is identified, restimulation at the existing perforations will have similar production uplift with the infilling refractures stimulated from untreated zones. It is of operational significance to identify stress reorientation so that optimized refracturing designs could be conducted to reduce operational costs.

The proposed model-based type curve matching process considers the basic physics of MFHW such as well lateral lengths, fracture stages, fracture half lengths, reservoir properties, and morphologies of refractures. Moreover, the modeling results of transient flow regimes and production rate type curves provide quicker analytical approaches than
numerical simulation studies to evaluate potential candidate wells for refracturing. The type curve matching process also considers more physics than statistical evaluation approaches without sacrificing computation speed.

An actual refractured Southern Saskatchewan Bakken well has been used to update the theoretical model to the field condition. Using the actual field case as the benchmark, a full field evaluation of refracturing upside in a Southern Saskatchewan Bakken pool has been completed. An average incremental production of 38,000 bbls/well refracturing upside has been estimated.

**Nomenclature**

**Notations**

\( a_{Di} \) = dimensionless half-length of each reservoir boundary element

\( C_{Rj} \) = diffusivity ratio between the \( j \)-th and the reference subsystem

\( C_{Sj} \) = storativity ratio between the \( j \)-th and the reference subsystem

\( C_{Tj} \) = transmissibility ratio between the \( j \)-th and the reference subsystem

\( L_{DW} \) = dimensionless half-length of a fracture element

\( N_j \) = number of boundary elements for each region

\( N_w \) = number of fracture elements

\( p \) = pressure at any location, Pa

\( Q \) = production rate from the source, \( \text{m}^3/\text{s} \)

\( s \) = Laplace variable

\( t \) = time, \( \text{s} \)

\( x \) = horizontal coordinate in Cartesian coordinates system, \( \text{m} \)
$y$ = vertical coordinate in Cartesian coordinates system, m

**Greek letters**

$(\xi, \zeta)$ = local coordinates, m

$\theta$ = internal angles between two adjacent elements

$\tau$ = time when sources are removed, s

$\Omega$ = solution domain

**Functions**

$erf$ = error function

$exp$ = exponential function

$H$ = Heaviside step function

$G_L$ = free space Green’s function of a line source

$G_n$ = free space Green’s function for the $n^{th}$ region

$L\{\ldots\}$ = Laplace transformation

$\delta$ = Dirac delta function

**Subscript and superscript**

$D$ = dimensionless term

$n$ = the $n^{th}$ region of the heterogeneous system

$w$ = the $w^{th}$ discretized fracture element

“~” = Laplace transformation of a function/variable

“*” = fundamental solutions of related quantities

**SI Metric Conversion Factors**

$\text{bbl} \times 1.589874 = \text{m}^3$
\[ \text{cp} \times 1.0^* \times 10^{-3} = \text{Pa} \cdot \text{s} \]

\[ \text{ft} \times 3.048^* \times 10^{-1} = \text{m} \]

\[ \text{mD} \times 9.869233 \times 10^{-4} = \mu\text{m}^2 \]

\[ \text{lb}_m \times 4.535924 \times 10^{-1} = \text{kg} \]

\[ \text{psi} \times 6.894757 \times 10^0 = \text{kPa} \]

*Conversion factor is exact*
CHAPTER 6  CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

During the course of the PhD study, universal boundary integral approaches (BIA) have been proposed to solve the research problems stated in Chapter 1.

Firstly, the BIA was used to model fluid flow in CHOPS reservoirs with complex networks of wormholes. Arbitrary geometries of wormholes and reservoir boundaries were effectively modeled by using local coordinate systems and integrating along discretized wormhole and boundary elements. The modeling results were successfully applied to estimate effective wormhole coverage through analysis of rate-time behavior.

Secondly, dimensionless variables were introduced to describe local heterogeneity in heterogeneous reservoirs in terms of diffusivity and storability ratios. As an extension of boundary element method (BEM), a BIA with systematic partition and coupling strategy was proposed to solve for transient fluid flow in heterogeneous reservoirs. Transient pressure and production responses of multiple wells producing in the fluvial system were obtained simultaneously. The proposed BIA, exemplified with a fluvial system, is flexible of discretizing boundaries of partitioned subsystems and highly compatible with realistic geology. The modeling results are independent of gridding size and time steps because of the analytical nature of BIA and Laplace transformation.

Thirdly, an integrated study of reservoir engineering and geomechanics was conducted to investigate the stress changes due to fluid flow and mechanical loading. A hybrid boundary integral approach was proposed to account for depletion-induced stress changes and mechanical-induced stress change of propped fractures by using the
principle of superposition. The proposed BIA explicitly considers non-planar fractures and reservoir heterogeneity and solves the diffusion-deformation problem with near analytical accuracy. It is the first time to correlate depletion-induced stress evolution with different transient flow regimes occurring in a producing multistage fractured horizontal well (MFHW).

Key conclusions drawn from this PhD research include:

- Effective wormhole coverage and overall wormhole intensity dominantly affect transient pressure and rate-time behavior regardless of the detailed wormhole morphologies.
- The slope of the transitional flow regime shown on log-log plots of rate versus material balance time is close to negative one due to strong interference between wormholes.
- Log-log plots of production rate versus material balance time are more informative than linear scale plots of rate versus real time. They show straight decline trends with negative half, near negative unit, and negative unit slopes.
- Monthly production data of CHOPS wells are useful to characterize wormholes in terms of effective wormhole coverage. Sand production data are also important to reduce the uncertainties.
- The proposed systematic partition and coupling strategy is the key to model heterogeneous systems by using boundary integral approaches, which are compatible with both object and pixel based geologic modeling techniques.
- The transient behavior of stress evolution on the hydraulic fractures is found to be consistent with flow regimes identified in transient pressure analysis.
• Transient fluid flow influences stress changes significantly in the development of unconventional reservoirs with MFHW. More fracture stages will result in a larger concave feature on the derivative responses of induced stress components. Such concave feature is different from the close to unit slope line shown on the pressure derivatives.

• When the flow transients reach the local heterogeneities (i.e. flow barriers with low permeability and sweet spots with high permeability), derivatives of pore pressure and induced stress components will have similar characteristics of slope change.

• Both $\sigma_{xxD} / p_D$ and $\sigma_{yyD} / p_D$ are time dependent and their values change with distinctive trends related to different flow regimes. The induced spatio-temporal stress path $\Delta \sigma_h / \Delta p$ is anisotropic and has different characteristics related to transient flow regimes.

• The informative trends of $\Delta \sigma_h / \Delta p$ versus time calculated by the proposed approach provides possible ways to predict the stress path evolution once the pore pressure change and stress deviator are known.

• In the refracturing candidate selection process, the model-based type curve matching process provides quicker analytical approaches than numerical simulation studies to evaluate potential candidate wells for refracturing. It also considers more physics than statistical evaluation approaches without sacrificing computation speed.
• An average incremental production of 38,000 bbls/well of the studied Southern Saskatchewan Bakken pool has been estimated by using the model-based refracturing type-cruve matching technology.

6.2 Recommendations

Future Research The universal boundary integral approaches open a door to future research in the areas of effective and accurate semi-analytical modeling strategies:

1) Moving source and moving boundary problems in heterogeneous systems.
2) Coupled thermal and fluid flow problems with reservoir heterogeneity.
3) Multi-disciplinary studies such as hydraulic fracturing propagation with shear-slip natural fractures, finite fracture conductivity variation, fracture closure due to depletion, and refracturing candidate selection.
4) Boundary integral approaches based simulators.

Recommendation For Operations To reduce the uncertainties of field application of theoretical studies, the data acquisition and analysis process needs the following considerations:

1) Daily production rate of each well is usually prorated and largely affected by daily operations. Monthly production data are more representative of overall productivity and provide more useful characteristics of production trends.
2) Sand production data for CHOPS wells need to be properly recorded by operators because they are important to calibrate wormhole coverage, which is very crucial in future development of the unconsolidated sand formation.
3) Bottom hole pressure data is important to be tested and measured for unconventional wells. Values obtained from various approaches need to be cross checked to reduce the uncertainties of reservoir pressure.

4) Tiltmeter measurement is useful to detect stress field alteration. Geomechanical properties of the studied formation need to be acquired before and during the development of unconventional reservoirs.

5) Besides of the widely used statistical studies and numerical simulations, user-friendly semi-analytical models are desired to allow operators to efficiently analyze mega-size data sets and quickly provide useful information in the decision process.
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APPENDIX A: DERIVATION OF THE FOAMY OIL FLOW IN CHOPS RESERVOIRS

Based on mass conservation and Darcy’s law, the general diffusivity equation describing fluid flow in porous media can be described as:

$$\nabla \left[ \rho(p) \cdot \frac{k(p)}{\mu(p)} \cdot \nabla p \right] = \frac{\partial [\rho(p) \cdot \phi(p)]}{\partial t}$$  \hspace{1cm} [A.1]

According to the compressibility definition of foamy oil by Smith (1988), foamy oil density can be approximated by:

$$\rho(p) = \rho_b \left( \frac{p}{p_b} \right)^\beta \times \exp\left[ c_o (p - p_b) \right]$$  \hspace{1cm} [A.2]

where, $\rho_b$ is a base density at base pressure $p_b$. $\beta$ is the foamy oil compressibility constant. $c_o$ is the oil compressibility.

Assuming rock compressibility is constant, substitute the density function into the right hand side of Eq. [A.1] and obtain:

$$\frac{\partial [\rho(p) \cdot \phi(p)]}{\partial p} \frac{\partial p}{\partial t} = \rho_b \left( \frac{p}{p_b} \right)^\beta \exp\left[ c_o (p - p_b) \right] \frac{(\beta \rho + c_o + c_f) \cdot \phi(p)}{\partial \phi(p)}$$  \hspace{1cm} [A.3]

where, $c_f = \frac{1}{\phi(p)} \frac{d\phi(p)}{dp}$ is the constant rock compressibility.

Substituted the approximated density function, the left hand side of Eq. [A.1] becomes:

$$\nabla \left[ \rho(p) \cdot \frac{k(p)}{\mu(p)} \cdot \nabla p \right] = \rho_b \left( \frac{p}{p_b} \right)^\beta \times \exp\left[ c_o (p - p_b) \right] \left[ \frac{k(p)}{\mu(p)} \cdot \nabla p \right]$$  \hspace{1cm} [A.4]
Consequently, the diffusivity equation becomes:

\[
\nabla \left[ k(p) \frac{\mu(p)}{\mu(p)} \nabla p \right] = \phi(p) \cdot C_i(p) \cdot \frac{\partial p}{\partial t}
\]

where (Smith, 1988),

\[
C_i(p) = \frac{\beta}{p} + c_a + c_f
\]

Pseudo variables of pressure and time are defined as:

\[
M(p) = \int_{p_s}^{p} \frac{k(p)}{\mu(p)} dp
\]

\[
t_a = \int_{0}^{t} \frac{k(p)}{\phi(p) \mu(p) c_i(p)} dt
\]

According to the chain rule:

\[
\frac{\partial M(p)}{\partial r} = \frac{\partial M(p)}{\partial p} \cdot \frac{\partial p}{\partial r} = \frac{k(p)}{\mu(p)} \cdot \frac{\partial p}{\partial r}
\]

\[
\frac{\partial M(p)}{\partial t} = \frac{\partial M(p)}{\partial p} \cdot \frac{\partial p}{\partial t} = \frac{k(p)}{\mu(p)} \cdot \frac{\partial p}{\partial t}
\]

\[
\frac{\partial M(p)}{\partial t} = \frac{\partial M(p)}{\partial t_a} \cdot \frac{\partial t_a}{\partial t} = \frac{k(p)}{\phi(p) \mu(p) c_i(p)} \cdot \frac{\partial M(p)}{\partial t_a}
\]

Substitute the above equations to Eq. [A.5], then the diffusivity equation becomes in terms of pseudo variables of pressure and time:

\[
\nabla^2 M(p) = \frac{\partial M(p)}{\partial t_a}
\]
APPENDIX B: DETAILS OF THE SOLUTION PROCESS OF A
RADIAL COMPOSITE SYSTEM WITH IRREGULAR
SUBSYSTEMS (OR GRIDS)

Because of the symmetry in the subsystems (or girds) shown in Figure 3.5 (a), there are essentially four different subsystems with various scales and shapes. Figure B1 shows that each subsystem is assigned with boundary elements along the boundaries. To account for the arbitrary shape of each subsystem, the local coordinates system is used when moving fictitious sources along all boundaries. As shown in Figure B2, each boundary element has its own local coordinates system and the local coordinates can be calculated by Eq. [B.1]. Boundary elements between adjacent subsystems are numbered in opposite directions. If the subsystem 4 is numbered clockwise, then the adjacent subsystems 1 and 5 have to be numbered counterclockwise.

\[
\begin{align*}
\xi &= x \cos \beta + y \sin \beta \\
\zeta &= -x \sin \beta + y \cos \beta 
\end{align*}
\]  
[B.1]

where, \((\xi, \zeta)\) is the local coordinates, \((x, y)\) is the global coordinates, and \(\beta\) is the angles between the \(\xi\) and \(x\) plane in different coordinates systems.

The irregular boundaries of subsystems are represented by certain numbers of boundary elements as shown in Figure B1. Between adjacent subsystems, the same number of boundary elements and properties are assigned along the inter-boundary. For example, there are four common boundary elements between subsystems 1 and 2 along their inter-boundary, while seven common boundary elements exist between subsystems
2 and 5. By using the Eq. [3.14], Eq. [3.16] and Eq. [3.17], a linear equations system will be formed by moving evaluating points along all boundary elements. Linear solver such as the Gaussian elimination can be used to solve the matrix system for the unknown value on each boundary element.
**Figure B1** Boundary elements are assigned along the boundaries of four different subsystems with various scales and shapes to solve for the complex heterogeneous problems shown in **Figure 3.4 a**. Boundary elements between adjacent subsystems are numbered in opposite directions.
Figure B2 The relationship between global and local coordinates systems. Boundary elements are assigned along the boundary of subsystem 5. Local coordinates are calculated according to the locations of different boundary elements.
**APPENDIX C: FUNDAMENTAL SOLUTIONS USED FOR THE**

**BOUNDARY INTEGRAL APPROACHES**

**Fundamental solutions of induced stresses by fluid flow**

Revised from the work of Detournay and Cheng (1987), the dimensionless free space Green’s functions of induced stresses by fluid flow with a point source are obtained as:

\[
\begin{align*}
    r_D &= \sqrt{(\chi - x_D)^2 + y_D^2}, \quad \xi_D^2 = \frac{r_D^2}{4(t_D - \tau)} \quad \text{[C.1]} \\
    p_{Diij}^*(\chi - x_D, t_D - \tau) &= \eta \frac{1}{r_D^2} \left[ (\delta_{ij} - 2r_{D,i}r_{D,j})(1 - e^{-\xi_D^2}) - 2(\delta_{ij} - r_{D,i}r_{D,j})\xi_D^2 e^{-\xi_D^2} \right] \quad \text{[C.2]} \\
    q_{Diij}^*(\chi - x_D, t_D - \tau) &= \frac{\partial p_{Diij}^*(\chi - x_D, t_D - \tau)}{\partial n_y} = \frac{\partial p_{Diij}^*(\chi - x_D, t_D - \tau)}{\partial r_D} \frac{y_D}{r_D} \quad \text{[C.3]}
\end{align*}
\]

Applying the Laplace transformation to Eq. [C.2] and [C.3] to obtain the fundamental solutions in the Laplace domain as

\[
\begin{align*}
    \tilde{p}_{Diij}^*(\chi - x_D, s) &= \eta \left\{ \frac{1}{s} - \frac{2(\chi - x_D)^2}{r_D^4} - \frac{(1 - (\chi - x_D)^2)}{r_D^2}K_0(rDs) \right\} \\
    \tilde{q}_{Diij}^*(\chi - x_D, s) &= \eta \left\{ \frac{1}{s} - \frac{2(\chi - x_D)^2}{r_D^4} - \frac{(1 - (\chi - x_D)^2)}{r_D^2}K_0(rDs) \right\} \\
    \tilde{p}_{Dii}^*(\chi - x_D, s) &= \eta \left\{ \frac{1}{s} - \frac{2y_D^2}{r_D^4} - \frac{(1 - y_D^2)}{r_D^2}K_0(rDs) \right\} \\
    \tilde{q}_{Dii}^*(\chi - x_D, s) &= \eta \left\{ \frac{1}{s} - \frac{2y_D^2}{r_D^4} - \frac{(1 - y_D^2)}{r_D^2}K_0(rDs) \right\}
\end{align*}
\]

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\[
\tilde{p}_D^*(\chi - x_D, s) = \eta \left\{ -\frac{2(\chi - x_D) y_D}{r_D^3} + \frac{(\chi - x_D) y_D K_0(r_D s)}{r_D^3} \right\} + \frac{2(\chi - x_D) y_D}{r_D^3 \sqrt{s}} K_1(r_D s) \]  \hspace{4cm} [C.6]

\[
\tilde{q}_{Dxx}^*(\chi - x_D, s) = \eta \left\{ -\frac{2 y_D}{r_D^4 s} + \frac{8(\chi - x_D)^2 y_D}{r_D^6 s} + \left[ \frac{y_D \sqrt{s}}{r_D} - \frac{(\chi - x_D)^2 y_D \sqrt{s}}{r_D^3} \right] K_0(r_D \sqrt{s}) + \left[ \frac{3(\chi - x_D)^2 y_D}{r_D^4} \right] K_1(r_D \sqrt{s}) \right\} \hspace{4cm} [C.7]

\[
\tilde{q}_{Dyy}^*(\chi - x_D, s) = \eta \left\{ -\frac{2 y_D}{r_D^4 s} + \frac{8 y_D^3}{r_D^6 s} + \left[ \frac{y_D \sqrt{s}}{r_D} - \frac{y_D \sqrt{s}}{r_D^3} + \frac{y_D \sqrt{s}}{r_D^5} - \frac{6 y_D^3}{r_D^7} \right] K_0(r_D \sqrt{s}) \right\} \hspace{4cm} [C.8]

\[
\tilde{q}_{Dyy}^*(\chi - x_D, s) = \eta \left\{ \frac{8(\chi - x_D) y_D^2}{r_D^6 s} - \left[ \frac{(\chi - x_D) y_D^2 \sqrt{s}}{r_D^3} + \frac{6(\chi - x_D) y_D^2}{r_D^5 \sqrt{s}} \right] K_0(r_D \sqrt{s}) \right\} \hspace{4cm} [C.9]

Fundamental solutions of induced pore pressure by fluid flow

Dimensionless free-space Green’s function of the general diffusion equation with an instantaneous point source is given (Kikani and Horne, 1992):

\[
G_D(\chi - x_D, t_D - \tau) = \frac{1}{4(t_D - \tau)} e^{-\frac{x_D^2}{4(t_D - \tau)}} = \frac{1}{4(t_D - \tau)} e^{-\frac{(x_D - \chi)^2}{4(t_D - \tau)}} e^{-\frac{y_D^2}{4(t_D - \tau)}} \] \hspace{4cm} [C.10]

For a straight line source segment with constant length of 2a_D, the fundamental solutions of a line source can be obtained by integrating the point source solution along the straight line.
Special functions in Eq. [C.11] and Eq. [C.12] are defined according to the work of Zhao and Thompson (2002):

\[ \hat{\Theta}_L(z_1, z_2, t) = \frac{1}{2} \left[ \text{erf} \left( \frac{z_2}{\sqrt{t}} \right) - \text{erf} \left( \frac{z_1}{\sqrt{t}} \right) \right], \text{where } \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt \]  

\[ \Theta_L(z, t) = \frac{1}{\sqrt{\pi t}} \exp \left( -\frac{z^2}{\sqrt{t}} \right), \text{where } \exp \text{ denotes the exponential function} \]  

Fundamental solutions of induced stresses by displacement discontinuity

Detournay and Cheng (1987) derived fundamental solutions of induced stresses by point displacement discontinuity. Carvalho (1990) integrated point fundamental solutions of DDM along a straight fracture segment with a length of 2a_D. Based on their work, the defined dimensionless variables are used to obtain the dimensionless fundamental solutions of induced stresses by a straight displacement discontinuity element with constant length of 2a_D.
Induced stresses by dimensionless continuous normal displacement discontinuity

\( \text{(DnD)} \):

\[
\sigma_{Lxx}^{\text{Dn}}(x_D, t_D) = \int_{t_D} \left[ \int \sigma_{xx}^{\text{Dn}}(\chi - x_D, t_D - \tau) dS \right] d\tau = \int_{t_D - a_D} \left[ \sigma_{xx}^{\text{Dn}}(\chi - x_D, t_D - \tau) \right] d\chi d\tau
\]

\[
= -\frac{\beta}{(1 - v_a)} \left\{ \frac{(x_D - \chi)^3 - (x_D - \chi) y_D^2}{r_D^4} \right\} \bigg|_{t_D - a_D}^{a_0} + \frac{-\beta(v_u - v)}{(1-v)(1-v_a)} \left\{ \frac{3(x_D - \chi) y_D^2}{r_D^4} \right\} \bigg|_{-a_D}^{a_0} \tag{C.15}
\]

\[
\sigma_{Lyy}^{\text{Dn}}(x_D, t_D) = \int_{t_D} \left[ \int \sigma_{yy}^{\text{Dn}}(\chi - x_D, t_D - \tau) dS \right] d\tau = \int_{t_D - a_D} \left[ \sigma_{yy}^{\text{Dn}}(\chi - x_D, t_D - \tau) \right] d\chi d\tau
\]

\[
= -\frac{\beta}{(1 - v_a)} \left\{ \frac{(x_D - \chi)^3 + 3(x_D - \chi) y_D^2}{r_D^4} \right\} \bigg|_{-a_D}^{a_0} + \frac{-\beta(v_u - v)}{(1-v)(1-v_a)} \left\{ \frac{(x_D - \chi)^3 - 3(x_D - \chi) y_D^2}{r_D^4} \right\} \bigg|_{-a_D}^{a_0} \tag{C.16}
\]

\[
\sigma_{Lxy}^{\text{Dn}}(x_D, t_D) = \int_{t_D} \left[ \int \sigma_{xy}^{\text{Dn}}(\chi - x_D, t_D - \tau) dS \right] d\tau = \int_{t_D - a_D} \left[ \sigma_{xy}^{\text{Dn}}(\chi - x_D, t_D - \tau) \right] d\chi d\tau
\]

\[
= -\frac{\beta}{(1 - v_a)} \left\{ \frac{(x_D - \chi)^2 y_D - y_D^3}{r_D^4} \right\} \bigg|_{-a_D}^{a_0} + \frac{-\beta(v_u - v)}{(1-v)(1-v_a)} \left\{ \frac{y_D^3 - 3(x_D - \chi)^2 y_D^2}{r_D^4} \right\} \bigg|_{-a_D}^{a_0} \tag{C.17}
\]

Induced stresses by dimensionless continuous shear displacement discontinuity

\( \text{(DsD)} \):
\[
\sigma_{kxx}(x_D, t_D) = \int_{t_0}^{t} \int_{t_0}^{t} \left[ \sigma_{kx}^{Dx}(\chi - x_D, t_D - \tau) \right] dS d\tau = \int_{t_0}^{t_0 + a_D} \int_{t_0}^{t_0 + a_D} \left[ \sigma_{kx}^{Dx}(\chi - x_D, t_D - \tau) \right] d\tau
\]

\[
= \frac{\beta}{(1 - \nu)} \left\{ \frac{3(x_D - \chi)^2 y_D + y_D^3}{r_D^4} \right\}_{a_D}^{a_D} + \\
- \beta(\nu - \nu) \left\{ \frac{-y_D^3 + 3(x_D - \chi)^2 y_D (1 - e^{-\frac{\xi_0}{\xi_D}})}{r_D^4} \right\}_{a_D}^{a_D} \frac{\xi_2^2}{\xi_D^2} + \frac{2y_D^3 e^{-\frac{\xi_0}{\xi_D}}}{r_D^4} \right\}_{a_D}^{a_D}
\]

[C.18]

\[
\sigma_{kyy}(x_D, t_D) = \int_{t_0}^{t} \int_{t_0}^{t} \left[ \sigma_{ky}^{Dy}(\chi - x_D, t_D - \tau) \right] dS d\tau = \int_{t_0}^{t_0 + a_D} \int_{t_0}^{t_0 + a_D} \left[ \sigma_{ky}^{Dy}(\chi - x_D, t_D - \tau) \right] d\tau
\]

\[
= -\frac{\beta}{(1 - \nu)} \left\{ \frac{(x_D - \chi)^2 y_D - y_D^3}{r_D^4} \right\}_{a_D}^{a_D} + \\
- \beta(\nu - \nu) \left\{ \frac{y_D^3 - 3(x_D - \chi)^2 y_D (1 - e^{-\frac{\xi_0}{\xi_D}})}{r_D^4} \right\}_{a_D}^{a_D} \frac{\xi_2^2}{\xi_D^2} + \frac{2(x_D - \chi)^2 y_D e^{-\frac{\xi_0}{\xi_D}}}{r_D^4} \right\}_{a_D}^{a_D}
\]

[C.19]

\[
\sigma_{kxy}(x_D, t_D) = \int_{t_0}^{t} \int_{t_0}^{t} \left[ \sigma_{ky}^{Dy}(\chi - x_D, t_D - \tau) \right] dS d\tau = \int_{t_0}^{t_0 + a_D} \int_{t_0}^{t_0 + a_D} \left[ \sigma_{ky}^{Dy}(\chi - x_D, t_D - \tau) \right] d\tau
\]

\[
= -\frac{\beta}{(1 - \nu)} \left\{ \frac{(x_D - \chi)^3 - (x_D - \chi) y_D^2}{r_D^4} \right\}_{a_D}^{a_D} + \\
- \beta(\nu - \nu) \left\{ \frac{3(x_D - \chi) y_D^2 - (x_D - \chi)^3 (1 - e^{-\frac{\xi_0}{\xi_D}})}{r_D^4} \right\}_{a_D}^{a_D} \frac{\xi_2^2}{\xi_D^2} + \frac{2(x_D - \chi) y_D^2 e^{-\frac{\xi_0}{\xi_D}}}{r_D^4} \right\}_{a_D}^{a_D}
\]

[C.20]