CUBICAL BLAKERS-MASSEY THEOREM FOR CDGA

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Abstract

We state and prove an algebraic version of the Blakers-Massey Theorem.

The Blakers-Massey Theorem is a classical result in homotopy theory that measures the obstruction to homotopy excision. It also measures how far a homotopy pushout square of topological spaces is from being a homotopy pullback. This theorem can be generalized to higher-dimensional cubical diagrams of topological spaces, where it measures how far a cubical homotopy colimit is from being a homotopy limit.

This work is inspired by Quillen’s rational homotopy theory, where commutative differential graded algebras (or CDGAs for short) over the field \( \mathbb{Q} \) of rational numbers are algebraic models. We construct the Blakers-Massey Theorem for \( n \)-cubes of (simply connected) CDGAs, measuring how far an \( n \)-cube of CDGAs which is a homotopy limit is from being a homotopy colimit.
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Chapter 1

Introduction

This introductory chapter is aimed at presenting the main result (Theorem 6.1.1) of the thesis, at the same time explaining the historical background that motivates this study (Section 1.1). Also explained here is the relation of our result to some existing work (Section 1.2), as well as the organization of the thesis (Section 1.3).

1.1 Background and Motivation

We begin by observing that homotopy groups have a feature that differs from that of (co)homology groups. Recall that the construction of homotopy groups is rather direct: for a non-negative integer $n$, the \textit{n-th homotopy group} $\pi_n X$ of a based topological space $X$ is simply the set $[S^n, X]$ of homotopy classes of based continuous maps from the $n$-sphere $S^n$ to $X$. (When $n = 1$ this is a group, and when $n \geq 2$ this is an abelian group.) If $A$ is a subspace of $X$ having the same base point, then the $n$-th homotopy
group \(\pi_n(X, A)\) of the pair \((X, A)\) is defined to be the set \([(D^n, S^{n-1}), (X, A)]\) of homotopy classes of maps \(f : (D^n, S^{n-1}) \to (X, A)\). (Here \(f : D^n \to X, f(S^{n-1}) \subset A,\) and \(f\) sends the common base point of \(D^n\) and \(S^{n-1}\) to the common base point of \(X\) and \(A\).) On the other hand, however, the computations of homotopy groups turn out to be much harder. This is in striking contrast to (co)homology groups: while the definitions often require some discussion, the computation turns out to be a much easier task.

We recall that a (covariant) functor \(H_* : \text{Top}_* \to \text{GrAbgrp}\) from the category of based topological spaces to the category of \(\mathbb{Z}\)-graded abelian groups is called a *homology theory* if it satisfies the Eilenberg-Steenrod axioms. Among these axioms, the long exact sequence of a pair, and the excision property, are two main reasons that explain the computability of homology groups. Recall that an *excisive triad* refers to a triple \((X; A, B)\) of based spaces where \(A, B\) are two subspaces of \(X\) whose interiors cover \(X\), such that \(C := A \cap B\) is non-empty, and that the common base point \(* \in C\). The excision property says that the inclusion map \(i : (A, C) \to (X, B)\) induces isomorphisms on all relative homology groups. That is, \(i_* : H_n(A, C) \xrightarrow{\cong} H_n(X, B),\) \(\forall n \in \mathbb{Z}\).

An important consequence of the excision property is the existence of the Mayer-Vietoris sequence. Let \((X; A, B)\) be an excisive triad, and let \(C = A \cap B\). We have inclusion maps \(C \xrightarrow{i_A} A, C \xrightarrow{i_B} B, A \xrightarrow{j_A} X, B \xrightarrow{j_B} X\). There are boundary
homomorphisms $\partial_n : H_n(X) \to H_{n-1}(C)$ making the following a long exact sequence:

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial_n} H_n(C) \xrightarrow{(i^A_\ast i^B_\ast)} H_n(A) \oplus H_n(B) \xrightarrow{j^A_\ast - j^B_\ast} H_n(X) \xrightarrow{\partial_{n-1}} H_{n-1}(C) \to \cdots.$$  

(See [21, Section 14.5].) The importance of the Mayer-Vietoris sequence is that it makes possible obtaining the homological information of a space out of the given homological information of its local pieces.

The homotopy functor $\pi_\ast$ fails to be a homology theory: although the relative homotopy groups $\pi_\ast(X,A)$ do fit into the following long exact sequence:

$$\cdots \to \pi_n(A) \xrightarrow{i_\ast} \pi_n(X) \xrightarrow{j_\ast} \pi_n(X,A) \xrightarrow{\partial_n} \pi_{n-1}(A) \xrightarrow{i_\ast} \cdots,$$

(where $A \hookrightarrow X, (X,*) \xrightarrow{j} (X,A)$ are based inclusions), we no longer have the excision property. That is, if $(X;A,B)$ is an excisive triad then the inclusion map $i : (A,A \cap B) \to (X,B)$ no longer induces isomorphisms $\pi_n(A,A \cap B) \to \pi_n(X,B)$ for all $n$.  
(See [18, Chapter 4].)

The classical Blakers-Massey theorem in topology.

This is, however, not a complete failure. It turns out that homotopy excision does hold for an excisive triad $(X;A,B)$, but only within a range of dimensions, which is controlled by the connectivity (see Section 4.1) of the inclusion maps $A \cap B \to A$ and $A \cap B \to B$. The Blakers-Massey theorem is a classical result in homotopy theory that investigates this phenomenon, which measures the obstruction to homotopy excision. Below is a precise statement.
Theorem 1.1.1. (The Blakers-Massey Theorem, [22, Theorem 4.1.1]) Let \((X; A, B)\) be an excisive triad with path-connected intersection \(C := A \cap B\). If \(C \to A\) is \(r\)-connected, and if \(C \to B\) is \(s\)-connected, \(r, s \geq 1\), then the map \(\pi_n(A, C) \to \pi_n(X, B)\) induced by the inclusion \((A, C) \to (X, B)\) is an isomorphism when \(1 \leq n < r + s\), and a surjection when \(n = r + s\).

The above theorem is also called the homotopy excision theorem. (Notice that this form of the homotopy excision theorem is not exactly the same as the one that appeared in the original paper [4] of Blakers-Massey. See Section 1.2 for more detailed historical sketch.) There is a more precise statement. In fact, one could define, for \(n \geq 2\), the homotopy groups \(\pi_n(X; A, B)\) of a triad such that there is a long exact sequence:

\[
\cdots \to \pi_n(X; A, B) \to \pi_n(A, C) \to \pi_n(X, B) \to \pi_{n-1}(X; A, B) \to \cdots .
\]

(See [21, Section 11.3] for details.) The Blakers-Massey theorem could then be thought of giving a range of dimensions within which the triad homotopy groups vanish. Namely, \(\pi_n(X; A, B) = 0\) for \(2 \leq n \leq r + s\). (Under the same conditions as in Theorem 1.1.1.)

The Blakers-Massey theorem has other interpretations. Despite the point of view that it measures the failure to homotopy excision, one could also regard it as a result measuring how far a homotopy pullback diagram of based spaces is from being a homotopy pushout. (See Section 3.2 for the definition of homotopy pullbacks and
Theorem 1.1.2. ([22, Theorem 4.2.1]) Let the following be a homotopy pushout diagram in $\text{Top}_*$.

$$
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
\downarrow{g} & & \downarrow{}
\end{array}
\begin{array}{ccc}
B & \longrightarrow & X
\end{array}
$$

Let $A \times_X B$ denote the homotopy pullback of the diagram $A \to X \leftarrow B$. If $f$ is is $r$-connected, and if $g$ is $s$-connected, then the canonical map $C \to A \times_X B$ is $(r + s - 1)$-connected.

We refer the reader to [22, Section 4.1] for the explanation why Theorem 1.1.2 implies Theorem 1.1.1.

The last version of the Blakers-Massey theorem immediately inspires generalizations to higher cubical diagrams. For $n \geq 1$, let $\mathbf{n}$ be the category whose objects are subsets of \{1, \cdots, n\} and whose morphisms are inclusions of subsets. An $n$-cube of based topological spaces is understood as a contravariant functor $\mathcal{A} : \mathbf{n} \to \text{Top}_*$. We write $A_T$ for the space that $\mathcal{A}$ assigns to $T \subset \mathbf{n}$. (See Section 5.1 for details.) We say that an $n$-cube is homotopy (co)cartesian if it “looks like a homotopy (co)limit”. (Definition 5.1.1.) A natural question to ask is how far a homotopy cocartesian cube in $\text{Top}_*$ is from being homotopy cartesian. This is answered by the following result of Goodwillie [17].

Theorem 1.1.3. ([22, Theorem 6.2.1]) Let $\mathcal{A}$ be a homotopy cocartesian diagram in
Topology. If the map \( A_n \to A_{n-\{i\}} \) is \( r_i \)-connected, \( i = 1, \ldots, n \), then \( A \) is \((\sum_{i=1}^n r_i) - 1\)-cartesian.

We call this theorem the classical \( n \)-dimensional Blakers-Massey theorem for topological spaces.

The algebraic side.

This thesis is devoted to the study of an algebraic version of Theorem 1.1.3, namely the Blakers-Massey theorem for \( n \)-cubes of CDGAs. (Section 6.1.)

This is inspired by rational homotopy theory, which, roughly speaking, is the study of rational homotopy types of spaces. To be precise, there is a rationalization construction assigning to each (simply-connected) space \( X \) a map \( f : X \to X_\mathbb{Q} \) from \( X \) to a rational space \( X_\mathbb{Q} \), which induces isomorphisms \( \pi_n(X) \otimes \mathbb{Q} \to \pi_n(X_\mathbb{Q}) \), where \( \mathbb{Q} \) is the field of rational numbers. A rational space refers to a space \( Y \) such that \( \pi_\ast(Y) \) is a graded \( \mathbb{Q} \)-vector space, and by the rational homotopy type of \( X \) we mean the weak homotopy type of its rationalization \( X_\mathbb{Q} \). (Two spaces are said to have the same weak homotopy type if they have isomorphic homotopy groups in all dimensions.) We refer to the reader Quillen’s original paper [23] for a full exposition of rational homotopy theory. [15] is a standard textbook. [19] is also an excellent introductory paper to this subject.

Here by a CDGA we mean a commutative differential graded algebra, over some
commutative ring. (See Section 2.2.) CDGAs, as purely algebraic objects, are of fundamental importance in rational homotopy theory since one learns from the theory of Sullivan [26] that CDGAs over \( \mathbb{Q} \) provide algebraic models for rational homotopy types. There is a Sullivan functor \( A_{PL} : \text{Top} \to \text{CDGA} \) from the category of topological spaces to that of CDGAs over \( \mathbb{Q} \), such that for any simply connected space \( X \), the rational homotopy type of \( X \) is completely determined by any CDGA quasi-isomorphic (Section 2.2) to \( A_{PL}(X) \). This suggests that the study of rational homotopy types of spaces can be regarded as the study of “homotopy types” of CDGAs. More precisely, the functor \( A_{PL} \) induces an equivalence between the homotopy category of simply connected rational space of finite type and the homotopy category of 1-connected CDGAs over \( \mathbb{Q} \) of finite type.

Just like the category of topological spaces, the category \( \text{CDGA} \) has the structure of a model category (Section 2.1) so that we can “do homotopy theory”. One could make sense of homotopy limits and colimits (Definition 3.2.8.) in the category. Again we say that a cubical diagram is homotopy (co)cartesian if it “behaves like a homotopy (co)limit”. Now one is in the position to ask a similar (but dual, since the Sullivan functor is contravariant) question: given a homotopy cartesian \( n \)-cube of CDGAs, how far it is from being homotopy cocartesian?

Our study answers this question by giving a precise measurement (Theorem 6.1.1),
again via the important notion of connectivity (Section 4.1). We also proved a generalized version of the Blakers-Massey theorem (Theorem 6.1.2).

1.2 Literature Review

The investigation of the problem of homotopy excision was initiated by Blakers and Massey in their 1951 paper [4]. They defined the homotopy groups of $\pi_*(X; A, B)$ of a triad $(X; A, B)$ [4, Section 2.1], and then deduced the long exact sequence [4, Theorem 3.5.4], which could be regarded as a precise obstruction to homotopy excision.

A year later (1952) they had another paper [5] where the homotopy excision theorem was first stated and proved. Their statement [5, Theorem I] of this theorem, compared with Theorem 1.1.1, assumed an additional condition on the triad $(X; A, B)$. In 1953 Blakers and Massey gave a computational result on the triad homotopy groups [6, Theorem I], namely that the map $\pi_r(A, C) \otimes \pi_s(B, C) \to \pi_{r+s-1}(X; A, B)$ given by the generalized Whitehead product is an isomorphism. (Here we assume the same conditions as in Theorem 1.1.1.) This would immediately guarantee the desired range of dimensions within which the triad homotopy groups vanish.

Since then the homotopy excision theorem, or the classical 2-dimensional Blakers-Massey theorem for topological spaces, was reviewed, improved, and re-proved by many other authors. Here we mention, among others, the results [9, Theorem 1.B] of Chachólski, [12, Theorem 6.4.1] of Dieck, [18, Theorem 4.23] of Hatcher, and [21,
11.3] of May. A recent work is [10, Corollary 2.2]. A more detailed historical sketch can be found in [22, Section 4.1].

The generalization of the Blakers-Massey theorem to dimension $n$ was first seen in the 1956 paper [2] of Barratt and Whitehead. In 1987 Ellis and Steiner gave another proof [13]. Also important is Goodwillie’s proof in his 1991 paper [17]. A more recent exposition is the book [22] of Munson and Volić in 2015. Munson-Volić provide another proof of Theorem 1.1.3 in [22, Chapter 6]. We would also like to mention here that it is their induction argument [22, Section 6.4] that inspires one part of the proof of our algebraic results (Theorems 6.1.1 and 6.1.2).

Also the Blakers-Massey theorem was brought into various other contexts.

In a 2005 lecture note [25] of C. Rezk on “homotopy topos theory”, he presented a result [25, Proposition 8.16] measuring how far a homotopy pushout square in a model topos is from being a homotopy pullback.

In 2013 Lumsdaine, Finster, and Licata announced work on the Blakers-Massey theorem in homotopy type theory, with the paper yet to appear. Later in 2016, Hou (Favonia), together with the above three authors presented a mechanized proof of the Blakers-Massey theorem [14].

In 2014 Ching and Harper established a sequence of results on the Blakers-Massey theorem on structured ring spectra [11]. They proved higher dimensional homotopy excision and higher dimensional Blakers–Massey theorems [11, Theorems 1.4 through
1.11], for algebras and left modules over operads in the category of modules over a commutative ring spectrum.

In a recent paper [1] launched in March 2017, Anel, Biedermann, Finster, and Joyal gave a version of the Blakers-Massey theorem valid in an arbitrary higher topos and with respect to an arbitrary modality [1, Theorem 4.2].

1.3 Plan of the Thesis

This thesis is divided into six chapters.

After this introductory Chapter we would like to provide some background material in Chapters 2 and 3.

The purpose of Chapter 2 is to introduce the language of model categories. Before we formally give the definition of a model category in Section 2.1, we will first have a quick recap on classical homotopy theory, reviewing the notions of cofibrations, fibrations, and weak equivalences in the category of topological spaces. In Section 2.2 we will discuss three algebraic categories, namely the category of (non-negative) chain complexes, that of CDGAs, and that of differential graded modules over a fixed CDGA. We will mention how forgetful functors play with these model structures.

In Chapter 3 we will make precise the notion of homotopy limits and colimits, via derived functors. Section 3.1 discusses the projective and injective model structure on a category of diagrams, and Section 3.2 presents the construction of derived functors.
The last three chapters are devoted to the statement and the proof of our main result (Theorem 6.1.1), which is the algebraic analogue of Theorem 1.1.3. We will use an induction argument. However, it turned out that we need to apply induction to a slightly more generalized version (Theorem 6.1.2).

Chapter 4 is the proof of the ground case, namely the Blakers-Massey theorem for squares of CDGAs (Theorem 4.2.1). This is the exact algebraic analogue of Theorem 1.1.2. We will measure, using the notion of the connectivity of a map (Section 4.1), how far a homotopy pullback square of CDGAs is from being a homotopy pushout. Also stated and proved is a generalized version (Theorem 4.3.4). A subsidiary aim of this chapter is to get the reader familiar with the game we play with diagrams (Lemmas 4.3.1 through 4.3.3). In later chapters our notations and proofs will become significantly more complicated, but the spirit will remain the same.

In Chapter 5 we will work on the technical tools required for the proof of the main theorem. First in Section 5.1 we will introduce our notation and terminology of cubical diagrams. Then in Section 5.2 we study in detail the $n$-cubes of CDGAs. We will see how to learn the cocartesian property of a cube (i.e., how far a cube is from being a homotopy colimit) out of the cocartesian property of its associated cube of homotopy cofibers, which is a cube of dimension one less. It is this result (Proposition 5.2.3) that enables the induction. Also indispensable is Theorem 5.3.1 which tells us the cartesian property of the associated cube of homotopy cofibers. It
is this algebraic step that makes the induction applicable to the generalized version of Blakers-Massey. The key idea here is the use of the bar construction (Section 5.3).

In the last Chapter we will finish the proof of the main theorems. In Section 6.1 we will state our main result and give an overview of the proof. We will first redo the ground case $n = 2$, now using the bar construction. We then show that if both Theorem 6.1.1 and Theorem 6.1.2 hold for dimensions $\leq n - 1$ then Theorem 6.1.1 holds for dimension $n$. This will be a quick proof, thanks to the algebraic step (Proposition 5.3.1). Next we show that if Theorem 6.1.1 holds for dimensions $\leq n$ and if Theorem 6.1.2 holds for dimension $\leq n - 1$ then Theorem 6.1.2 will hold for dimension $n$. The proof of this step is completely formal, but rather lengthy. (This proof is completely dual to the argument in the topological setting in [22, Chapter 6].) Therefore we would like to first give the proof, assuming two subsidiary lemmas. In Section 6.2 we will prove these two lemmas, which will finish the proof.
Chapter 2

Homotopy Theory and Model Categories

In this chapter we review the basics of classical homotopy theory, and give the definition of a model category. We also introduce several algebraic objects to be studied in later chapters, and fix the model structures on the corresponding categories.

2.1 From Classical Homotopy Theory to Model Categories

Modern homotopy theory began with the concepts of cofibrations, fibrations, and weak homotopy equivalences. Here we recall their definitions.

Cofibrations.

Let $f : X \to Y$ be a map of topological spaces. Say that $f$ is a cofibration, if it satisfies the homotopy extension property. That is, given any other space $Z$ and maps $g : Y \to Z$, $h : X \times I \to Z$ such that $g \circ f = h \circ i_0$ (where $i_0$ is the map
there exists a map $\tilde{h} : Y \times I \to Z$ such that $\tilde{h} \circ i_0 = g$, $\tilde{h} \circ (f \times id) = h$.

The following diagram summarizes this definition.

$$
\begin{array}{ccc}
X & \xrightarrow{i_0} & X \times I \\
\downarrow{f} & & \downarrow{f \times id} \\
Y & \xrightarrow{i_0} & Y \times I \\
\downarrow{g} & & \downarrow{h} \\
\end{array}
$$

A particular situation is when $i : A \to X$ is an inclusion of a subspace. Then $i$ is a cofibration if given any map $g$ on the whole space $X$ and any homotopy on the subspace $A$ that agrees with $g$ at the 0-th level, this homotopy can be extend to $X$ with $g$ as its 0-th level. A fundamental example of a cofibration is the inclusion $S^n \hookrightarrow D^{n+1}$ of the $n$-sphere as the boundary of the $(n+1)$-disk ($n \geq 0$).

So intuitively cofibrations could be understood as “nice inclusions”. Below is an example that exhibits this nice feature. Consider the pushout of the diagram $* \leftarrow S^n \rightarrow *$. This simply identifies two points and the result is therefore a single point. However, if we replace the map $S^n \to *$ by the cofibration $S^n \to D^{n+1}$ (note that $D^{n+1}$ is homotopy equivalent to a point) and consider, instead, the pushout of the new diagram $* \leftarrow S^n \rightarrow D^{n+1}$, then the result is no longer so trivial as a point. Indeed, this identifies the boundary of the $(n+1)$-disk with a point, which gives the $(n+1)$-sphere $S^{n+1}$. This example will re-appear in Section 3.2 as we discuss homotopy pushouts.
In the category of topological spaces one could always replace a map by a cofibration. To be precise, given any map \( f : X \to Y \), there is a space \( Y' \) together with maps \( f' : X \to Y' \), \( q : Y' \to Y \) such that \( f' \) is a cofibration, that \( q \) is a homotopy equivalence, and that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & Y' \\
\downarrow{f} & \searrow{q} & \downarrow{=} \\
Y &   & Y
\end{array}
\]

(We use the hooked arrow to denote a cofibration.)

Indeed, we can take \( Y' \) to be the mapping cylinder \( Mf \) of \( f \). Here the mapping cylinder \( Mf := \left( X \times I \right) \coprod Y / \sim \) is defined as a quotient of the disjoint union of \( X \times I \) with \( Y \), where the equivalence relation \( \sim \) is generated by \((x,0) \sim f(x)\). In other words \( Mf \) is formed by the following pushout:

\[
\begin{array}{ccc}
X \times I & \xrightarrow{f} & Y \\
\downarrow{j_0} & & \downarrow{j} \\
X \times I & \xrightarrow{i} & Mf
\end{array}
\]

We construct the map \( q : Mf \to Y \) by letting \( q(y) = y \) for \( y \in Y \), and \( q(x,t) = f(x) \) for \((x,t) \in X \times I\). Note that this is a well-defined continuous map. We then let \( f' : X \to Mf \) be such that \( x \mapsto (x,1) \). Now it is easy to check that \( f' \) is a cofibration and that \( q \circ f' = f \).

**Fibrations.**

A fibration is the “dual” notion to a cofibration. The precise definition goes as follows.
A map $f : X \to Y$ of topological spaces is called a fibration, if it is surjective, and if it satisfies the covering homotopy property. That is, given any other space $Z$ and maps $g : Z \to X$, $h : Z \to Y^I$ that satisfy $f \circ g = p_0 \circ h$, there exists a map $\tilde{h} : Z \to X^I$ such that $p_0 \circ \tilde{h} = g$, and that $f^I \circ \tilde{h} = h$. (Here $X^I$ is the space of maps $I = [0,1] \to X$, with compact-open topology. The map $p_0 : \sigma \mapsto \sigma(0)$ takes a map to its starting point, and the map $f^I$ takes a map $\sigma$ in $X$ to the map $f \circ \sigma$ in $Y$.) This definition can be summarized by the following diagram.

![Diagram showing the definition of a fibration](image)

**Remark.** The definition we give above is that of Hurewicz fibrations. We also have the notion of a Serre fibration, whose definition is exactly the same as above, except that we loosen the condition by only requiring that $Z$ be any CW-complex.

An important example of a fibration is the path space fibration $p : X^I \to X$, $\sigma \mapsto \sigma(1)$. When $X$ is a based space, the path space fibration refers to the map $p : P(X) \to X$, where $P(X)$ is the space of based maps $I \to X$, i.e., an element in $P(X)$ is a map $\sigma : I \to X$ such that $\sigma(0)$ is the base point of $X$. Note that $P(X)$ itself has a natural base point, namely the constant map to the base point of $X$; and that $P(X)$ is contractible, i.e., homotopy equivalent to a point.
Intuitively, fibrations could be understood as “nice surjections.” Here is an example that exhibits this nice feature. Consider the pullback of the diagram \( * \to S^n \leftarrow * \), where on \( S^n \) we have specified a base point and the two maps \( * \to S^n \) both send the point \(*\) to that base point of \( S^n \). With these assumptions we know that the pullback is nothing but a single point. However, if we replace one of the maps \( * \to S^n \) by the fibration \( P(S^n) \to S^n \) and consider, instead, the pullback of the new diagram \( * \to S^n \leftarrow P(S^n) \), then what we get is the loop space \( \Omega(X) \) of \( X \). (Recall that this is the space of maps \( \sigma : I \to X \) such that \( \sigma(0) = \sigma(1) = \) the base point of \( X \).) This example will also re-appear in Section 3.2 as we discuss homotopy pullbacks.

We can always replace a map by a fibration. That is, given any map \( f : X \to Y \) of topological spaces there exists a space \( X' \), together with maps \( j : X \to X' \), \( f' : X' \to Y \) such that \( f' \) is a fibration, that \( j \) is a homotopy equivalence, and that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{j} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xRightarrow{\simeq} & \\
\end{array}
\]

(Here we use the double-headed arrow to denote a fibration.)

Indeed we could do that. We take \( X' \) to be the mapping path space \( Pf \) of \( f \), which is the subspace of the cartesian product \( X \times Y^I \) containing pairs \((x, \sigma)\) such
that \( f(x) = \sigma(0) \), i.e., the following pullback:

\[
\begin{array}{ccc}
Pf & \rightarrow & X \\
\downarrow & & \downarrow f \\
Y & \rightarrow & Y \\
\end{array}
\]

We define \( j : X \rightarrow Pf \) by \( x \mapsto \sigma_{f(x)} \), where \( \sigma_{f(x)} : t \mapsto f(x) \) is the constant path at \( f(x) \in Y \). Also we let \( f' : Pf \rightarrow Y \) be such that \( (x, \sigma) \mapsto \sigma(1) \). It is easy to check that \( f' \) is a fibration, and that \( f = f' \circ j \).

**Weak homotopy equivalences.**

Finally we give the definition of weak homotopy equivalences. A map \( f : X \rightarrow Y \) is called a weak homotopy equivalence, if \( f \) induces isomorphisms on all homotopy groups, i.e., \( f_* : \pi_i(X) \xrightarrow{\cong} \pi_i(Y), \forall i \geq 0 \).

A homotopy equivalence is always a weak homotopy equivalence. The converse is not true in general. In the category of CW-complexes, however, Whitehead’s theorem tells us that a weak homotopy equivalence is also a homotopy equivalence.

More generally, we say that a map \( f : X \rightarrow Y \) is \( n \)-connected, if \( f_* : \pi_i(X) \xrightarrow{\cong} \pi_i(Y) \) is an isomorphism for \( i < n \) and a surjection when \( i = n \).

**Model categories.**

The notions of cofibrations, fibrations, and weak homotopy equivalences in the category of topological spaces inspire an abstraction of the classical homotopy theory.
Just as in any other aspect of mathematics, a good abstraction often clarifies ideas and makes some otherwise cumbersome arguments formal. On the other hand, there are many other circumstances in mathematics where people would expect to “do homotopy theory”. People are in want of a general context under which homotopy-theoretic methods could make sense. The invention of the concept of a model category, due to Quillen’s work [24] in the 1960’s, provides the perfect machinery for these purposes.

Roughly speaking, a model category is a category $\mathcal{M}$ equipped with three classes of maps, namely cofibrations, fibrations, and weak equivalences, satisfying several axioms. We will give the precise definition below. This follows [16, Section 3]. In this paper we will focus on the model structures on several algebraic categories, especially on the category $CDGA_k$ of commutative differential graded algebras over a commutative ring $k$. (See Section 2.2.) Also important is the model structures on the category $\mathcal{M}^D$ of diagrams in a model category $\mathcal{M}$ of shape $D$, which will be treated in Section 3.1.

Before we define a model category (Definition 2.1.1), we need to recall a few concepts. First is the notion of retract. Let $\mathcal{C}$ be a general category, and let $f : X \to X'$, $g : Y \to Y'$ be two morphisms in $\mathcal{C}$. We say that $f$ is a retract of $g$, if there exists
a commutative diagram of the following form:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{i'} & Y'
\end{array}
\]

\[X \xrightarrow{r} X\]
\[r \circ i = \text{id}_X, \quad r' \circ i' = \text{id}_{X'}\]

So the third axiom above says that if \(f\) is a retract of \(g\) and if \(f\) belongs to any of the three classes \(W, C,\) or \(F,\) then so does \(g.\) Secondly, by an acyclic fibration we mean a map that is both a fibration and a weak equivalence, and by an acyclic cofibration we mean a map that is both a cofibration and a weak equivalence. Finally, we explain the lifting properties. Let \(f : X \to X',\) \(g : Y \to Y'\) be two maps in a general category \(C.\) We say that \(f\) has the left lifting property with respect to \(g,\) or that \(g\) has the right lifting property with respect to \(f,\) if for any commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g} \\
X' & \xrightarrow{g} & Y'
\end{array}
\]

in \(C,\) there exists the dashed arrow \(X' \to Y\) making the two triangles commute. Let \(L\) be a collection of morphisms in \(C.\) We say that a map \(f\) has the left (right) lifting property with respect to \(L,\) if \(f\) has the left (right) lifting property with respect to any \(g \text{ in } L.\)

**Definition 2.1.1.** A model category is a category \(\mathcal{M}\) that has all small limits and colimits, and is equipped with three classes \(W, C, F\) of morphisms (called the class of weak equivalences, cofibrations, and fibrations, respectively), satisfying the following axioms:
(i). \( W, C, \) and \( F \) are all closed under composition.

(ii). The class \( W \) of weak equivalences satisfies the “2-out-of-3” property: if \( f, g \) are two maps in \( M \) such that \( g \) can be composed with \( f \), then any two of the three maps \( f, g, g \circ f \) being a weak equivalence can imply the third also being a weak equivalence.

(iii). \( W, C, \) and \( F \) are all closed under retracts.

(iv). Cofibrations satisfy the left lifting property with respect to acyclic fibrations, and fibrations satisfy the right lifting property with respect to acyclic cofibrations.

(v). Any map \( f \) in \( M \) can be factored as a cofibration followed by a acyclic fibration, and can be factored as a acyclic cofibration followed by a fibration.

We say that \((W, C, F)\) is a model structure on \( M \). We will use \( \hookrightarrow \) to denote a cofibration, \( \twoheadrightarrow \) to denote a fibration, and \( \simeq \) to denote a weak equivalence.

We remark that a model category admits both an initial and a terminal object. (See [16, Remark 3.4].) In a model category \( M \) we say that an object \( X \) is cofibrant, if the unique map \( \emptyset \rightarrow X \) from the initial object in \( M \) to \( X \) is a cofibration; and we say that \( X \) is fibrant, if the unique map \( X \rightarrow * \) from \( X \) to the terminal object in \( M \) is a fibration. By a (co)fibrant replacement of an object \( X \) we mean a (co)fibrant object \( X' \) weak equivalent to \( X \). (Co)Fibrant replacements exist in a model category (with initial and terminal objects), due to the factorization axiom (v).

We point out that the axioms above are in a sense redundant. In fact, in a model category \( M \) the cofibrations are precisely those that have the left lifting property
with respect to the class of acyclic fibrations; and conversely, a map is an acyclic fibration if and only if it has the right lifting property with respect to all cofibrations. Similarly, fibrations are precisely those having the right lifting property with respect to all acyclic cofibrations; and conversely, a map is an acyclic cofibration if and only if it has the left lifting property with respect to all fibrations. (See [16, Proposition 3.13] for a proof of this.) A consequence is that to define a model structure on a category one only needs to specify the classes \( W, C \) of weak equivalences and cofibrations, and describe the fibrations via the lifting property; or one could just specify the weak equivalences \( W \) and the fibrations \( F \), and define cofibrations via the lifting property.

We would like to mention the following property of (co)fibrations: any pushout of a cofibration is again a cofibration, and any pullback of a fibration is again a fibration. This can be checked easily by verifying the lifting properties.

The category \( \text{Top} \) of topological spaces admits many model structures. In fact, if we let the weak equivalences be the weak homotopy equivalences, and the fibrations be the Hurewicz fibrations, then this makes \( \text{Top} \) a model category. Here the cofibrations are precisely those cofibrations in classical homotopy theory defined above (i.e., via homotopy extension property). Another common model structure on \( \text{Top} \) is the following: the weak equivalences are weak homotopy equivalences, and the fibrations are Serre fibrations. Under this model structure the cofibrations are retracts of relative cell complexes. (We refer the reader to [16, Section 8] for the verification.)
The homotopy category.

The homotopy category of a model category is an important concept in this paper. We will give the definition via the localization construction.

Let $C$ be a general category, and let $L$ be a collection of morphisms in $C$. The localization of $C$ with respect to $L$ is a pair $(F, D)$ where $D$ is a category and $F$ is a functor $C \to D$, such that $F(f)$ is an isomorphism in $D$ whenever $f \in L$, and that $(F, D)$ is universal with respect to this property. That is, if $(G, D')$ is another pair such that $G(f)$ is an isomorphism for all $f \in L$, then there is a unique functor $H : D \to D'$ such that $G = H \circ F$. We will write $L^{-1}C$ for the localization of $C$ with respect to $L$.

Now let $\mathcal{M}$ be a model category, and let $W$ be its class of weak equivalences. Then the localization $W^{-1}\mathcal{M}$ exists. (See [16, Sections 5 and 6].) We call it the homotopy category of $\mathcal{M}$, and denote by $Ho(\mathcal{M})$.

2.2 Algebraic Categories and Their Model Structures

The purpose of this section is to introduce a sequence of algebraic notions, especially that of a CDGA, which is the main object of study in this thesis. In this section we let $k$ be a commutative ring with multiplicative identity, unless otherwise stated.

Graded modules.
A graded module over \( k \) is a \( \mathbb{Z} \)-indexed family \( V := \{ V^i \}_{i \in \mathbb{Z}} \) of \( k \)-modules. We write the degrees as superscripts. The degree of an element \( v \in V \) is denoted by \( |v| \), or \( \deg v \). Write \( V^{\geq l} \) for the subspace of \( V \) containing elements of degree at least \( l \).

A map \( f : V \to W \) of graded \( k \)-modules is a sequence \( f = \{ f^i : V^i \to W^i \}_{i \in \mathbb{Z}} \) of \( k \)-linear maps. A graded module \( V \) is said to be non-negative, if \( V^i = 0 \) for all \( i < 0 \).

The category of graded \( k \)-modules is denoted by \( \text{GrMod}_k \). When \( k \) is a field then we use \( \text{GrVect}_k \) to denote the category of graded \( k \)-vector spaces.

Let \( V = \{ V^i \}_{i \in \mathbb{Z}} \) and \( W = \{ W^j \}_{j \in \mathbb{Z}} \) be two graded \( k \)-modules. The tensor product of \( V \) and \( W \) is a new graded \( k \)-module \( U = \{ U^n \}_{n \in \mathbb{Z}} \) with \( U^n = \bigoplus_{i+j=n} V^i \otimes W^j \). We will frequently use \( V \otimes W \) to denote the tensor product of \( V \) and \( W \). The suspension of a graded module \( V \) is a new graded module \( sV \) defined by \( (sV)^i = V^{i+1} \). Write \( sv \) for the element in \( sV \) corresponding to an element \( v \) in \( V \). Note that if \( |v| = n \) then \( |sv| = n - 1 \).

**Chain complexes.**

A chain complex over \( k \) is a graded \( k \)-module \( V := \{ V^i \}_{i \in \mathbb{Z}} \) that is equipped with a \((+1)\)-differential, i.e., a collection \( d = \{ d^i : V^i \to V^{i+1} \}_{i \in \mathbb{Z}} \) of \( k \)-linear maps with the property that \( d^{i+1} \circ d^i = 0 \), \( \forall i \in \mathbb{Z} \). (We remark that what we call a chain complex here is usually called a cochain complex, but we prefer to use the word chain complex in order to save terminology.) A map \( f : (V,d_V) \to (W,d_W) \) of chain complexes is a chain map, i.e., a collection \( f = \{ f^i : V^i \to W^i \}_{i \in \mathbb{Z}} \) of
$k$-linear maps such that $d_W^i \circ f^i = f^{i+1} \circ d_V^i$, $\forall i$. The $i$-th cohomology of a chain complex $V$ is defined to be the quotient $\text{Ker}(d^i)/\text{Im}(d^{i-1})$, and is denoted by $H^i(V)$.

A chain map $f : V \to W$ is called a quasi-isomorphism if it induces isomorphisms on cohomologies: $H^i(f) : H^i(V) \xrightarrow{\cong} H^i(W)$, $\forall i \in \mathbb{Z}$. We use the symbol “$\simeq$” to denote a quasi-isomorphism. A chain complex $(V, d)$ is non-negative if $V^i = 0$ for all $i < 0$.

We denote by $Ch(k)$ the category of chain complexes over $k$, and by $Ch_{\geq 0}(k)$ the category of non-negative chain complexes over $k$.

**Remark.** A graded module is nothing but a chain complex with zero differential. The commutative ring $k$ itself could be regarded as a chain complex concentrated in degree zero, with zero differential. We will write $(k, 0)$, or sometimes just $k$, for this structure.

One could define tensor products in the category of chain complexes. If $(V, d_V)$, $(W, d_W)$ are two objects in $Ch(k)$ then their tensor product $(V \otimes W, d)$ is the object in $Ch(k)$ whose underlying graded module is the tensor product $V \otimes W$ of the underlying graded modules, and whose differential $d$ is defined by $d(v \otimes w) = (d_V v) \otimes w + (-1)^{|v|} v \otimes d_W w$.

**Graded algebras.**

A graded $k$-module $V$ becomes a (unital, associative) graded $k$-algebra, if there
exist chain maps $\epsilon : k \to V$ and $\mu : V \otimes V \to V$ satisfying the property that

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id) : V \otimes V \otimes V \to V,$$

and that

$$\mu \circ (\epsilon \otimes id_V) = id_V = \mu \circ (id_V \otimes \epsilon) : V \to V,$$

which could be summarized into the following commutative diagrams:

$$\begin{align*}
V \otimes V \otimes V & \xrightarrow{\mu \otimes id_V} V \otimes V \\
\downarrow^{id_V \otimes \mu} & \downarrow^{\mu} \\
V \otimes V & \xrightarrow{\mu} V \\
\\
V \otimes k & \cong V \cong k \otimes V \xrightarrow{\epsilon \otimes id_V} V \\
\downarrow^{id_V \otimes \epsilon} & \downarrow^{id_V} \\
V & \xrightarrow{\mu} V.
\end{align*}$$

In a more fancier language, a graded algebra is nothing but a monoid object in the category $GrMod_k$ of graded $k$-modules.

Note that since $\epsilon$ and $\mu$ are chain maps, we have $\epsilon(k) \subset V^0$, and $\mu(V^i \otimes V^j) \subset V^{i+j}$.

The element $\epsilon(1_k) \in V^0$ is called the multiplicative identity of $V$, and is dentoted by $1_V$, or simply 1. Note also that the axioms do not exclude the case where $\epsilon$ is the zero map. In this case $V$ is necessarily zero and is called the zero algebra.

We will write $ab$ for the product $a \otimes b$ of two elements $a$ and $b$. A graded algebra $V$ is commutative if its multiplication is commutative in the graded sense: $ab = (-1)^{|a||b|}ba$, $\forall a, b \in V$. A map $f : V \to W$ of graded $k$-algebras is a map of graded $k$-modules such that $f(ab) = f(a)f(b)$, $\forall a, b \in V$, and that $f(1_V) = 1_W$. Denote by $GrAlg_k$ the category of graded $k$-algebras, and by $CGA_k$ that of commutative graded $k$-algebras.
(Commutative) differential graded algebras.

A differential graded algebra, or a DGA for short, over \( k \), is a non-negative chain complex \((A,d)\) which is also a graded algebra, and the two structures are compatible in the sense that both the unit map \( \epsilon \) and the multiplication map \( \mu \) are chain maps, and that the graded Leibniz rule is satisfied: \( d(ab) = (da)b + (-1)^{|a|}adb, \forall a, b \in A \).

Say that a DGA is commutative if its underlying graded algebra is commutative. A commutative DGA is called a CDGA. A map \( f \) of (C)DGAs is a map of graded algebras which is also a chain map. We say that \( f \) is a quasi-isomorphism if it is a quasi-isomorphism as a chain map.

We will simply write \( A \) for a (C)DGA instead of \((A,d)\), whenever the differential is clear from the context. A connected (C)DGA is a (C)DGA \( A \) such that \( A^0 = k \), and that the unit map \( k \to A \) is the identity map \( k \to k = A^0 \). Note that the graded Leibniz rule implies \( d(1) = 0 \). Therefore a connected (C)DGA \( A \) will always have the property that \( A^0 = k = H^0(A) \).

Almost all (C)DGAs we deal with in later chapters will be connected. Although it appears convenient to work within the category of connected CDGAs, in order to ensure the existence of a model structure one needs to introduce the category of augmented (C)DGAs. Let \((A,d)\) be a (C)DGA over \( k \). An augmentation of \((A,d)\) is a (C)DGA map \((A,d) \to (k,0)\). A co-augmentation of \((A,d)\) is a (C)DGA map \((k,0) \to (A,d)\). A (co-)augmented (C)DGA is one with a chosen (co-)augmentation.
Note that any connected (C)DGA admits a unique augmentation and a unique co-augmentation.

The augmented CDGAs over k form a category, which is denoted by $CDGA_k$. The objects in the category are pairs $(A, \eta)$ where $A$ is a CDGA and $\eta : A \rightarrow k$ is an augmentation of $A$. A morphism between two augmented CDGAs $(A, \eta)$ and $(A', \eta')$ is a CDGA map $f : A \rightarrow A'$ such that $\eta = \eta' \circ f$. Similarly there is a category of augmented DGAs over $k$ which will be denoted by $DGA_k$. Finally we remark that both these categories admit a zero object, namely the CDGA $(k, 0)$ with id$_k$ as augmentation.

(Relative) Sullivan algebras.

Sullivan algebras $(\Lambda V, d)$ form an important class of examples of CDGAs. Temporarily we let $k$ be a field of characteristic zero, and let $V = \{V^j\}_{j \geq 1}$ be a graded vector space over $k$ that is concentrated in the positive dimensions. Here $\Lambda V$ denotes the free graded commutative $k$-algebra on $V$. The differential $d$ is required to have a nilpotence condition. To be precise, we ask that there is an increasing sequence of subspaces $0 = V(-1) \subset V(0) \subset V(1) \subset \cdots \subset V(n) \subset \cdots$ such that $V = \bigcup_{i=0}^\infty V(i)$ and that $d(V(i)) \subset \Lambda V(i-1)$, $\forall i \geq 0$. Let $A$ be a CDGA. By a Sullivan model of $A$ we mean a Sullivan algebra $\Lambda V$ together with a CDGA map $\Lambda V \xrightarrow{\sim} A$ which is a quasi-isomorphism. Any CDGA $A$ satisfying $H^0(A) = k$ admits a Sullivan model [15, Proposition 12.1]. A Sullivan algebra $(\Lambda V, d)$ is minimal, if for any element $x \in \Lambda V$,
\(d(x)\) has no linear part.

We also have the important notion of relative Sullivan algebras. A *relative Sullivan algebra* is a CDGA of the form \((A \otimes \Lambda V, d)\), where \(A\) is a CDGA with \(H^0(A) = k\) and \(V = \{V^i\}_{i \geq 1}\) is positively-indexed, which satisfies that \(A \cong A \otimes 1 \subset A \otimes \Lambda V\) is a sub-CDGA, and that the differential \(d\) has the nilpotence condition. Namely, there is an increasing sequence of subspaces \(V(0) \subset V(1) \subset \cdots \subset V(n) \subset \cdots\) such that \(V = \bigcup_{i=0}^{\infty} V(i)\), that \(d(V(i)) \subset A \otimes \Lambda V(i-1), \forall i \geq 1\), and that \(d(V(0)) \subset A\). Often the inclusion map \(A \overset{f}{\rightarrow} A \otimes \Lambda V\) is also called a relative Sullivan algebra.

In a relative sullivan algebra \((A \otimes \Lambda V, d)\) we always have \(d(A) \subset A\), since by definition \(A \cong A \otimes 1\) is a sub-algebra. However it is not true that \(d(\Lambda V) \subset \Lambda V\). In fact the differential on \(V\) often involves with elements in \(A\). We define the Sullivan fiber [15, Section 14] of the Sullivan algebra \((A \otimes \Lambda V, d)\) to be the Sullivan algebra \((\Lambda V, \bar{d}) := (A \otimes \Lambda V, d) \otimes_A k\). Here the \(A\)-module structure on \(k\) is induced by augmentation.

The Sullivan model of a CDGA map \(A \overset{f}{\rightarrow} B\) is a CDGA map \(A \otimes \Lambda V \overset{q}{\rightarrow} B\) where \(A \otimes \Lambda V\) is a relative Sullivan algebra, and \(q\) is a quasi-isomorphism extending \(f\). That is, the following diagram commutes:

\[
\begin{array}{ccc}
A \otimes \Lambda V & \overset{\sim}{\rightarrow} & \Lambda V \\
\downarrow{\approx} & & \downarrow{q} \\
A & \overset{f}{\rightarrow} & B
\end{array}
\]

The Sullivan model exists for any CDGA map \(A \overset{f}{\rightarrow} B\) with the properties that \(H^0(A) = k = H^0(B)\) and that \(H^1(f)\) is surjective [15, Proposition 14.3].
Differential graded modules.

Let \((R, d_R)\) be a CDGA. A differential graded module over \(R\) (also called an \((R, d_R)\)-module, or an \(R\)-dg module), is a graded \(R\)-module \(M\) equipped with a graded module map \(R \otimes M \rightarrow M, r \otimes m \mapsto r \cdot m\), together with a \((+1)\)-differential \(d\) that satisfies the graded Leibniz rule: \(d(r \cdot m) = d_R(r) \cdot m + (-1)^{|r|} r \cdot d(m)\). A map of \(R\)-dg modules is a map of graded \(R\)-modules which is also a chain map. We write \(DGM_R\) for the category of differential graded modules over \(R\).

An \(R\)-dg module \(M\) is semi-free if \(M\) admits an increasing sequence \(0 = M(-1) \subset M(0) \subset \cdots \subset M(n) \subset \cdots\) of sub \(R\)-dg modules such that \(M = \bigcup_{i=0}^{\infty} M(i)\), and that \(M(i)/M(i-1)\) is \(R\)-free on a basis of cocycles, \(\forall i \geq 0\). (See [15, Chapter 6] for a detailed explanation.) Let \(M\) be an \(R\)-dg module. By a semi-free resolution of \(M\) we mean a semi-free \(R\)-dg module \(P\) together with a \(R\)-dg module map \(P \xrightarrow{\cong} M\) which is a quasi-isomorphism. Any \(R\)-dg module admits a semi-free resolution [15, Proposition 6.6].

Let \(M\) be an \(R\)-dg module. By a semi-free extension of \(M\) we mean an inclusion \(i: M \rightarrow N\) of \(R\)-dg modules such that there is an increasing sequence \(N(-1) \subset N(0) \subset N(1) \subset \cdots \subset N(n) \subset \cdots\) of sub \(R\)-dg modules of \(N\) satisfying that \(N = \bigcup_{i=0}^{\infty} N(i)\), that \(i(M) = N(-1)\), and that \(N(i)/N(i-1)\) is \(R\)-free on a basis of cocycles, \(\forall i \geq 0\).

Here we recall the following properties of differential graded modules.

Lemma 2.2.1. ([15, Lemma 14.1]) Let \(f: A \rightarrow A \otimes \Lambda V\) be a relative Sullivan algebra.
Then $f$ is a semi-free extension.

**Lemma 2.2.2.** ([15, Proposition 6.7 (ii)]) Let $f : M \to M'$ be a map of $R$-dg modules, and let $g : N \to N'$ be a map of semi-free $R$-dg modules. If both $f$ and $g$ are quasi-isomorphisms then the canonical map $M \otimes_R N \overset{f \otimes g}{\longrightarrow} M' \otimes_R N'$ is a quasi-isomorphism.

**Model structures on algebraic categories.**

The categories $CDGA_k$, $DGM_R$, and $Ch(k)$ all admit model structures. The following describes a model structure on all the three categories. We call it the *projective model structure*, following tradition.

- The weak equivalences are quasi-isomorphisms.
- The fibrations are degree-wise surjections.

The projective model structure on $CDGA_k$ is justified in [7, Theorem 4.3]. In $CDGA_k$ relative Sullivan algebras are examples of cofibrations, and the Sullivan algebras are examples of cofibrant objects. (See [19, Section 1.2]). Any object is fibrant.

We remark that not all cofibrations are relative Sullivan algebras, as indicated in [7, 4.4]. However, if $A \overset{f}{\to} B$ is a CDGA map such that $H^0(A) = k = H^0(B)$ and that $H^1(f)$ is surjective, then $f$ can be replaced by a relative Sullivan algebra [15, Proposition 14.3]. Similarly, any CDGA $A$ such that $H^0(A) = k$ admits a Sullivan algebra as its cofibrant replacement [15, Proposition 12.1].
The justification for the projective model structure on $Ch(k)$ can be found, for example, in [3], where it is more generally proved that the model structure holds in the case where $k$ is a commutative ring [3, Theorem 1.4, Proposition 1.5]. The cofibrations are degree-wise injections with degree-wise projective cokernel [3, Proposition 1.7, Proposition 1.9].

The justification for the projective model structure on $DGM_R$ can be found in [3, Theorem 3.3]. The cofibrations are retracts of semi-free extensions and the cofibrant objects are semi-free $(R,d)$-modules.

**Behavior of forgetful functors.**

Let $(A,d) \xrightarrow{f} (B,d)$ be a map of CDGAs. We note that there is a sequence of forgetful functors:

$$CDGA_k \xrightarrow{F_1} DGM_A \xrightarrow{F_2} Ch(k) \xrightarrow{F_3} GrVect_k.$$  

Here the last forgetful functor takes $(A,d)$ to $(A,0)$, i.e., forgets the differential.

Consider the four maps $f$, $F_1(f)$, $F_2 \circ F_1(f)$, and $F_3 \circ F_2 \circ F_1(f)$. A direct consequence of the projective model structures is that if any of the four maps above is a fibration, then so are the other three.

Also if any of the three maps $f$, $F_1(f)$, $F_2 \circ F_1(f)$ is a weak equivalence, then so are the other two. Note that if $F_3 \circ F_2 \circ F_1(f)$ is a weak equivalence, i.e., if $(A,0) \xrightarrow{} (B,0)$ is an isomorphism, then $f : (A,d) \rightarrow (B,d)$ is necessarily a quasi-isomorphism.
Lastly, note that if $f$ is a relative Sullivan algebra then by Lemma 2.1 $F_1(f)$ is a cofibration of $A$-dg modules.
Chapter 3

Homotopy Limits and Colimits

This chapter provides background knowledge of homotopy limits and colimits.

3.1 Diagrams in Model Categories

We fix a model category $\mathcal{M}$. Let $\mathcal{D}$ be a small category (i.e., both the objects and the morphisms form sets). Denote by $\mathcal{M}^\mathcal{D}$ the category whose objects are functors $X : \mathcal{D} \to \mathcal{M}$, and whose morphisms are natural transformations of these functors. This is called the diagram category of shape $\mathcal{D}$. An object in the category is called a $\mathcal{D}$-shaped diagram in $\mathcal{M}$. When $\mathcal{D}$ is simple enough we often depict it with dots and arrows. For example $\mathcal{D} = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ is the category with three objects and two non-identity morphisms as shown.

We can lift the given model structure on $\mathcal{M}$ to model structures on the diagram category $\mathcal{M}^\mathcal{D}$. The following two model structures are the most commonly used. The
first is used to define homotopy limits and the second is used to define homotopy colimits.

- The **projective model structure**. Both weak equivalences and fibrations are defined object-wise. That is, a map $X \to Y$ in $\mathcal{M}^D$ is a weak equivalence (resp., fibration), if for any object $d$ in $D$, $X(d) \to Y(d)$ is a weak equivalence (resp., fibration) in $\mathcal{M}$. Cofibrations are defined via the left lifting property.

- The **injective model structure**. Both weak equivalences and cofibrations are defined object-wise. That is, a map $X \to Y$ in $\mathcal{M}^D$ is a weak equivalence (resp., cofibration), if for any object $d$ in $D$, $X(d) \to Y(d)$ is a weak equivalence (resp., cofibration) in $\mathcal{M}$. Fibrations are defined via the right lifting property.

We remark that these model structures do not exist for every $\mathcal{M}$ and $D$. However, if $D$ is a direct category ([20, Definition 5.1.1]), then we can guarantee the existence of the projective model structure on $\mathcal{M}^D$. Also if $D$ is an inverse category ([20, Definition 5.1.1]), then the injective model structure on $\mathcal{M}^D$ exists. (See [20, Theorem 5.1.3] for a proof.) In particular, note that the category of $n$-cubes (see Section 5.1) admits both the projective and the injective model structure, since its indexing category $n$ (see Section 5.1) is both direct and inverse in the sense of [20, Definition 5.1.1].
3.2 Derived Functors, Homotopy Limits and Colimits

Let’s begin by observing that pushouts and pullbacks are not well-behaved from the homotopy-theoretic point of view. Here let’s revisit the example mentioned in the beginning of Section 2.1.

**Example 3.2.1.** Let $\mathcal{X}$ be the following commutative diagram of topological spaces:

$$
\begin{array}{ccc}
* & \leftarrow & S^n \\
\downarrow & = & \downarrow \\
* & \leftarrow & S^n \\
\end{array}
$$

Here $n$ is any non-negative integer. The map $S^n \hookrightarrow D^{n+1}$ includes the $n$-sphere $S^n$ as the boundary of the $(n + 1)$-disk $D^{n+1}$. Note that the vertical maps in $\mathcal{X}$ are all homotopy equivalences. So the top line and the bottom line in $\mathcal{X}$ are weakly equivalent (3.1), as objects in $\text{Top}^\mathcal{D}$, where the diagram category $\mathcal{D}$ is of the shape $\{\bullet \leftarrow \bullet \rightarrow \bullet\}$.

However one observes that the top line and the bottom line do not produce homotopy equivalent pushouts. In fact the universal property of pushouts gives a canonical map $S^{n+1} \to *$, which is not a homotopy equivalence.

$$
\begin{array}{ccc}
S^n & \hookrightarrow & D^{n+1} \\
\downarrow & & \downarrow \\
* & \rightarrow & S^{n+1} \\
\end{array} \quad \begin{array}{ccc}
S^n & \rightarrow & * \\
\downarrow & & \downarrow \\
* & \rightarrow & * \\
\end{array}
$$

$S^{n+1}$ is **NOT** homotopic to $*$. 

Similar problems can occur when doing pullbacks.
Example 3.2.2. Consider the following diagrams of based topological spaces:

\[
\begin{array}{ccc}
* & \longrightarrow & S^n \\
\downarrow & & \downarrow \\
* & \longrightarrow & S^n
\end{array}
\]

Again the top line and the bottom line are weak equivalent, as objects in the category \( \text{Top}_D \), where \( D \) is the category \( \{ \bullet \to \bullet \leftarrow \bullet \} \). However, it is again not true that the pullbacks are equivalent.

\[
\begin{array}{ccc}
\ast & \longrightarrow & \ast \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & S^n
\end{array}
\]

\[
\begin{array}{ccc}
\Omega S^n & \longrightarrow & PS^n \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & S^n
\end{array}
\]

\( \ast \) is NOT homotopic to \( \Omega S^n \).

There are certain lessons one could learn from the examples. First of all, one observes that equivalent diagrams may not be able to produce equivalent pushouts (pullbacks). The pushout operation, in a category \( C \), is nothing but a functor \( C^D \longrightarrow C \), or further, a functor \( C^D \longrightarrow C \longrightarrow \text{Ho}(C) \), where \( D \) is the category \( \{ \bullet \leftarrow \bullet \to \bullet \} \).

In other words, the pushout functor does not factor through the homotopy category \( \text{Ho}(C^D) \):

\[
\begin{array}{ccc}
C^D & \longrightarrow & C \\
\downarrow & & \downarrow \\
\text{Ho}(C^D) & \longrightarrow & \text{Ho}(C)
\end{array}
\]

So now the best thing we could do is to approximate. This is the exact idea of constructing homotopy pushouts and pullbacks, which are understood as derived functors (Definitions 3.2.3 and 3.2.4). The examples also suggest how to actually take
the homotopy pushouts and pullbacks. To take the homotopy pushout (pullback) of a diagram one should first replace one of the maps by a “nice” one, namely a cofibration (fibration), and then take the actual pushout (pullback) of the replaced diagram. (Propositions 3.2.5 and 3.2.6.)

Now we begin the definition of derived functors. We often find ourselves in the situation where we have a functor between two model categories and would like to extend it to the homotopy categories:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow & & \downarrow \\
\text{Ho}(\mathcal{C}) & \longrightarrow & \text{Ho}(\mathcal{D})
\end{array}
\]

The extension, however, is not always possible, as illustrated by the Examples 3.2.1 and 3.2.2. Can we invent a construction that, in a sense, could be understood as the best possible approximation to such an extension? By answering we are lead to the definition of derived functors. Consider a functor \( F : \mathcal{C} \to \mathcal{C}' \) from a model category to a general category. Let \( q \) be the canonical functor \( \mathcal{C} \to \text{Ho}(\mathcal{C}) \).

**Definition 3.2.3.** A left derived functor of \( F \) is a pair \((LF, \alpha)\), where \( LF \) is a functor \( \text{Ho}(\mathcal{C}) \to \mathcal{C}' \) and \( \alpha \) is a natural transformation \( LF \circ q \to F \), satisfying the following universal property: given any other pair \((G, \beta)\), where \( G : \text{Ho}(\mathcal{C}) \to \mathcal{C}' \) is a functor and \( \beta : G \circ q \to F \) a natural transformation, there exists a unique natural transformation \( \theta : G \to LF \) such that \( \beta = \alpha \circ (\theta \circ q) \).
The universal property of left derived functors.

From now on we shall denote the left derived functor of $F$ by simply $LF$, without explicitly mentioning the natural transformation. Note that it is completely justifiable to speak of "the" left derived functor, since the universal property dictates that any two derived functors of $F$ are naturally equivalent.

In a similar way we can define right derived functors. Again let $F : C \to C'$ be a functor from a model category to a general category.

**Definition 3.2.4.** A right derived functor of $F$ is a pair $(RF, \alpha)$, where $RF$ is a functor $Ho(C) \to C'$ and $\alpha$ is a natural transformation $F \to LF \circ q$, satisfying the following universal property: given any other pair $(G, \beta)$, where $G : Ho(C) \to C'$ is a functor and $\beta : F \to G \circ q$ a natural transformation, there exists a unique natural transformation $\eta : RF \to G$ such that $\beta = (\eta \circ q) \circ \alpha$.

The universal property of right derived functors.
Again the universal property guarantees that right derived functors of $F$ are unique up to natural equivalences. We shall denote the right derived functor of $F$ simply by $RF$.

A first question to ask after the definition is whether or not derived functors exist. We hope to set up conditions under which derived functors can be taken. Here is a naive case. Suppose that $F : C \to C'$ has the property that $F(f)$ is an isomorphism in $C'$ whenever $f$ is a weak equivalence in $C$. Then $F$ itself factors through $Ho(C)$. (In fact, let $\bar{F}$ be the functor from $Ho(C)$ to $C'$ such that $\bar{F}(X) = F(X)$ for each object $X$, and that for each element $\sigma$ in $[X,Y]$ we pick an arbitrary $f \in Hom_C(X,Y)$ that represents it and set $\bar{F}(\sigma) = F(f)$. This is well-defined.) Now we have $F = \bar{F} \circ q$, and therefore $\bar{F}$ (together with the identity natural transformation) is the left derived functor of $F$.

The following proposition provides a necessary condition that is less trivial.

**Proposition 3.2.5.** ([16, Proposition 9.3]) Let $F : C \to C'$ be a functor from a model category to a general category. If $F$ takes weak equivalences between cofibrant objects in $C$ to isomorphisms in $C'$, then the left derived functor $LF$ of $F$ exists. Moreover, for any cofibrant object $X$ in $C$, the map $LF(X) \to F(X)$ is an isomorphism.

Below is the parallel result for right derived functors.

**Proposition 3.2.6.** Let $F : C \to C'$ be a functor from a model category to a general category. If $F$ takes weak equivalences between fibrant objects in $C$ to isomorphisms
in \( C' \), then the right derived functor \( RF \) of \( F \) exists. Moreover, for any fibrant object \( X \) in \( C \), the map \( F(X) \to RF(X) \) is an isomorphism.

Another aspect of derived functors one would be interested in knowing concerns their behavior with respect to adjoint functors. More specifically, let’s consider two model categories \( C, C' \). Suppose that \( F : C \to C' \) is a functor that is left adjoint to \( F' : C' \to C \). Consider the left derived functor \( L(q \circ F) \) of \( C \to C' \to \text{Ho}(C') \), and the right derived functor \( R(q \circ F') \) of \( C' \to C \to \text{Ho}(C) \). One expects that \( L(q \circ F) : \text{Ho}(C) \to \text{Ho}(C') \) remains to be left adjoint to \( R(q \circ F') : \text{Ho}(C') \to \text{Ho}(C) \).

This is indeed true, provided that a condition is satisfied.

**Proposition 3.2.7.** (\([16, \text{Theorem 9.7}]\)) If \( F \) preserves cofibrations and \( F' \) preserves fibrations then the derived functors \( L(q \circ F) \), \( R(q \circ F') \) exist, and they form an adjoint pair.

**Remark.** The condition in the above proposition is equivalent to \( F \) preserving cofibrations and acyclic cofibrations, or that \( F' \) preserves fibrations and acyclic fibrations.

Now we are in the position to define homotopy limits and colimits, as derived functors.

**Definition 3.2.8.** (i) The homotopy limit functor \( \text{holim} : \text{Ho}(\mathcal{M}^D) \to \text{Ho}(\mathcal{D}) \) is the right derived functor of the limit functor \( \text{lim} : \mathcal{M}^D \to \mathcal{D} \).

(ii) The homotopy colimit functor \( \text{hocolim} : \text{Ho}(\mathcal{M}^D) \to \text{Ho}(\mathcal{D}) \) is the left derived functor of the colimit functor \( \text{colim} : \mathcal{M}^D \to \mathcal{D} \).
Homotopy limits and colimits exist, as a consequence of Proposition 3.2.7. Indeed, since there are adjoint functors:

$$\text{colim} : \mathcal{M}^D \rightleftarrows \mathcal{M} : \Delta, \text{ and } \Delta : \mathcal{M}^D \rightleftarrows \mathcal{M} : \text{lim}.$$  

Here the functor $\Delta$ is the “constant diagram functor”. The conditions in Proposition 3.2.7 are satisfied. (See [16, Section 10].)

**Remark.** Notice that Propositions 3.2.5 and 3.2.6 tell us that $\text{holim}(X)$ is weakly equivalent to $\text{lim}(X')$ for any fibrant object $X'$ weakly equivalent to $X$, and that $\text{hocolim}(X)$ is weakly equivalent to $\text{colim}(X'')$ for any cofibrant object $X''$ weakly equivalent to $X$. Therefore there is a standard way of taking homotopy (co)limits. To take the homotopy limit of a diagram $X$, we first replace $X$ by a fibrant diagram $X'$ (using the injective model structure), and then take the actual limit of $X'$. To take the homotopy colimit of $X$, we first replace $X$ by a cofibrant diagram $X''$ (using the projective model structure), and then take the actual colimit of $X''$.

Homotopy pushouts (pullbacks) are particular examples of homotopy colimits (limits). The homotopy pushout functor is the left derived functor of the pushout functor $\mathcal{M}^D \to \mathcal{D}$, where $\mathcal{D}$ is of the form $\{\bullet \leftarrow \bullet \to \bullet\}$. The homotopy pullback functor is the right derived functor of the pullback functor $\mathcal{M}^D \to \mathcal{D}$, where $\mathcal{D}$ is of the form $\{\bullet \to \bullet \leftarrow \bullet\}$. Furthermore we have the following definition.

**Definition 3.2.9.** *(Homotopy pushouts and homotopy pullbacks.)* Let the following
be a commutative diagram in a model category $\mathcal{M}$.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow f \\
C & \rightarrow & D \\
\end{array}
\]

Denote by $h_0$ the canonical map $\text{hocolim}(C \leftarrow A \rightarrow B) \rightarrow D$, and by $h_1$ the map $A \rightarrow \text{holim}(C \leftarrow D \rightarrow B)$. We say that the diagram is

- a homotopy pushout, or homotopy cocartesian, if $h_0$ is a weak equivalence;
- a homotopy pullback, or homotopy cartesian, if $h_1$ is a weak equivalence.
- $r$-cocartesian, if $h_0$ is $r$-connected;
- $r$-cartesian, if $h_1$ is $r$-connected.

**Definition 3.2.10. (Homotopy (co)fiber.)** Let $\mathcal{M}$ be a model category. Denote by $\emptyset$ the initial object of $\mathcal{M}$, and by $*$ the terminal object of $\mathcal{M}$. Let $f : A \rightarrow B$ be a map in $\mathcal{M}$. The homotopy fiber of $f$ is the homotopy pullback along the unique map $\emptyset \rightarrow B$, i.e., $\text{holim}(\emptyset \rightarrow B \leftarrow A)$. The homotopy cofiber of $f$ is the homotopy pushout along the unique map $A \rightarrow *$, i.e., $\text{hocolim}(\star \rightarrow A \rightarrow B)$. 

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Chapter 4

The Blakers-Massey Theorem for Squares

In this chapter we state and prove the Blakers-Massey theorem for squares of CDGAs (Theorem 4.2.1), and a generalized version (Theorem 4.3.4). Although this is a special case of our main result, we decide to present the proofs here for the following reasons. Firstly, we use a straightforward argument to prove Theorem 4.2.1, which is different from the bar construction approach in the last chapter. Secondly, we hope that the sequence of lemmas we use to prove Theorem 4.3.4 will help the reader understand the spirit of the formal argument in the last Chapter.

In this chapter we will work over a field of characteristic zero.

4.1 The Notion of Connectivity

Definition 4.1.1. Let \( r \) be a positive integer. We say that a CDGA \( A \) is \( r \)-connected, if \( H^i(A) = 0 \) for \( 0 < i \leq r \).
So an \( r \)-connected CDGA is one that is cohomologically trivial up to degree \( r \). By definition if \( 0 < s < r \) then an \( r \)-connected CDGA is also \( s \)-connected. A CDGA is \( \infty \)-connected if it is \( r \)-connected for every positive integer \( r \). Such a CDGA is also said to be acyclic. We say that a CDGA \( A \) is simply connected, if \( A^0 = k \) and \( A^1 = 0 \).

**Definition 4.1.2.** Let \( r \) be a positive integer. A map \( f : A \to B \) of CDGAs is \( r \)-connected, if its homotopy cofiber \( \text{hocofib}(f) \) is \( (r - 1) \)-connected.

We say that \( f \) is \( \infty \)-connected if it is \( r \)-connected for every \( r \), or equivalently, if its homotopy cofiber is acyclic.

If a CDGA \( A \) is \( r \)-connected, then we write that \( \text{con}(A) = r \), and say that “\( A \) has connectivity \( r \)”;
also if a CDGA map \( f \) is \( r \)-connected, then we write that \( \text{con}(f) = r \), and say that “\( f \) has connectivity \( r \)”.

The following relates the connectivity of a map to its cohomological behavior.

**Lemma 4.1.3.** Let \( f : A \to B \) be a map of CDGAs. Then \( f \) is \( r \)-connected if and only if \( H^i(f) : H^i(A) \to H^i(B) \) is an isomorphism when \( 0 \leq i < r \) and an injection when \( i = r \).

**Proof.** First replace \( f \) by a cofibration.

\[
\begin{array}{ccc}
A \otimes \Lambda V & \xrightarrow{=} & B \\
\downarrow & & \\
A & \longrightarrow & B
\end{array}
\]

Denote by the same symbol \( f \) the relative Sullivan algebra \( A \to A \otimes \Lambda V \).
Now the homotopy cofiber of $f$ is the pushout:

\[
\begin{align*}
(A, d_A) &\longrightarrow (A \otimes \Lambda V, d) \\
\downarrow &\downarrow \\
(k, 0) &\longrightarrow (\Lambda V, \bar{d})
\end{align*}
\]

Our goal is to see that $H^i(\Lambda V, \bar{d}) = 0$ for $0 < i \leq r - 1$ if and only if $H^i(f)$ is an isomorphism for $i < r - 1$ and an injection when $i = r$.

We use a spectral sequence argument to prove this. Consider the decreasing filtration $F$ on $A \otimes \Lambda V$ defined by $F^p(A \otimes \Lambda V) = A^{\geq p} \otimes \Lambda V$. We have that:

\[
E_2^{p,q} = H^p(A, d_A) \otimes H^q(\Lambda V, \bar{d}) \Rightarrow H^*(A \otimes \Lambda V, d).
\]

So the terms on the final page are:

\[
E_\infty^{p,q} = \frac{F^p H^{p+q}(A \otimes \Lambda V)}{F^{p+1} H^{p+q}(A \otimes \Lambda V)} = \frac{Im(H^{p+q}(F^p(A \otimes \Lambda V)) \to H^{p+q}(A \otimes \Lambda V, d))}{Im(H^{p+q}(F^{p+1}(A \otimes \Lambda V)) \to H^{p+q}(A \otimes \Lambda V, d))}.
\]

In particular we have:

\[
E_\infty^{p,0} = Im(H^p(A, d_A) \to H^p(A \otimes \Lambda V, d)).
\]

Now suppose that $(\Lambda V, \bar{d})$ is $(r - 1)$-connected. Then $E_2^{p,q} = 0$ for $1 \leq q \leq r - 1$. We therefore see that for $0 \leq p \leq r$ entries $E_2^{p,0} = H^p(A, d_A)$ all survive to the final page. That is,

\[
H^p(A, d_A) \xrightarrow{\approx} Im(H^p(A, d_A) \to H^p(A \otimes \Lambda V, d)), 0 \leq p \leq r.
\]

This is to say that $H^i(f)$ are injections, $0 \leq i \leq r$. Furthermore, note that in the final page $E_\infty^{*,*}$, elements along the diagonal line $p + q = n$ all vanish except possibly
for $E^n_{\infty,0}$, whenever $1 \leq n \leq r - 1$. This implies that

$$H^n(A \otimes \Lambda V, d) \cong \text{Im}(H^n(A, d_A) \to H^n(A \otimes \Lambda V, d)), 0 \leq n \leq r - 1.$$ 

To sum up, $H^i(f)$ are isomorphisms for $0 \leq i \leq r - 1$ and an injection when $i = r$.

Conversely, suppose that $H^i(f)$ is an isomorphism for $i < r - 1$ and an injection when $i = r$. So for $0 \leq p \leq r$ we have

$$E^{p,0}_2 = H^p(A, d_A) \xrightarrow{\cong} \text{Im}(H^p(A, d_A) \to H^p(A \otimes \Lambda V, d)) = E^{p,0}_\infty.$$ 

Let $n$ be the smallest positive integer such that $H^n(A \otimes \Lambda V, d) \neq 0$. If $n \leq r - 1$ then in the final page we have entries $E^{1,n-1}_\infty = E^{2,n-2}_\infty = \cdots = E^{n-1,1}_\infty = 0$ along the diagonal line $p + q = n$. So we have the short exact sequence:

$$0 \to E^{n,0}_\infty \to H^n(A \otimes \Lambda V, d) \to E^{0,n}_\infty \to 0.$$ 

Note that $E^{0,n}_2$ will survive to the final page unless the differential $d_{n+1} : E^{0,n}_n \to E^{n+1,0}_n$ is nontrivial. However this cannot happen since otherwise $E^{n+1,0}_2$ won’t survive to the final page, i.e., $H^{n+1}(A, d_A) \xrightarrow{\cong} \text{Im}(H^{n+1}(A, d_A) \to H^{n+1}(A \otimes \Lambda V, d))$ won’t be an isomorphism. This violates our hypothesis. (Note that $n + 1 \leq r$.)

**Corollary 4.1.4.** A map of CDGAs is $\infty$-connected if and only if it is a quasi-isomorphism.

**Proof.** Direct from Lemma 4.1.3. 

Proof.
Lemma 4.1.5. Let the following be a commutative diagram of CDGAs.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{} & C
\end{array}
\]

(i) If \( f \) is \( r \)-connected and if \( g \) is \( s \)-connected then \( h \) is \( \min\{r, s\} \)-connected.
(ii) If \( g \) is \( r \)-connected and if \( h \) is \( s \)-connected then \( f \) is \( \min\{r, s\} \)-connected.
(iii) If \( f \) is \( r \)-connected and if \( h \) is \( s \)-connected, with \( r \geq s \), then \( g \) is \( s \)-connected.
(iv) If \( f \) is \( r \)-connected and if \( h \) is \( s \)-connected, with \( r < s \), then \( g \) is \( (r - 1) \)-connected.

Proof. These all follow from the definition and dimension-counting. \( \square \)

The notion of connectivity can be extended to other algebraic categories.

Definition 4.1.6. Let \( \mathcal{M} \) be any of the categories: \( DGM_R, Ch_{\geq 0}(k) \), or \( GrMod_k \).

Let \( r \) be a positive integer.

(i) We say that an object \( A \) in \( \mathcal{M} \) is \( r \)-connected, if \( H^i(A) = 0 \) for \( 0 \leq i \leq r \). Say that \( A \) is \( \infty \)-connected (or acyclic), if \( A \) is \( r \)-connected for every \( r \).

(ii) We say that a map \( f : A \to B \) in \( \mathcal{M} \) is \( r \)-connected, if and only if \( H^i(f) : H^i(A) \to H^i(B) \) is an isomorphism when \( 0 \leq i < r \) and an injection when \( i = r \). Say that \( f \) is \( \infty \)-connected, if \( f \) is \( r \)-connected for every \( r \) (i.e., if \( f \) is a quasi-isomorphism).

Remark. Again let’s recall the sequence of forgetful functors:

\[
CDGA_k \xrightarrow{F_1} DGM_A \xrightarrow{F_2} Ch(k) \xrightarrow{F_3} GrVect_k.
\]
Let \( f : (A, d_A) \to (B, d_B) \) be a map of CDGAs. Consider the three maps \( f, F_1(f), \) and \( F_2 \circ F_1(f) \). It follows from the definition that if any of the three maps is \( r \)-connected then so are the other two. If \( F_3 \circ F_2 \circ F_1(f) : (A, 0) \to (B, 0) \) is \( r \)-connected, then so are \( f, F_1(f), \) and \( F_2 \circ F_1(f) \). The converse is not true in general. Also note that a map \( (A, 0) \to (B, 0) \) is \( r \)-connected if and only if its kernel is \( r \)-connected.

### 4.2 The Blakers-Massey Theorem for Squares

Recall the concept of an \( r \)-cocartesian square from Definition 3.2.9.

**Theorem 4.2.1. (The Blakers-Massey theorem for squares of CDGAs)**

Let the following be a homotopy pullback diagram of CDGAs:

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow f \\
C & \longrightarrow & D \\
\end{array}
\]

Suppose that \( f \) is \( r_1 \)-connected and that \( g \) is \( r_2 \)-connected. Then the diagram is \((r_1 + r_2 - 1)\)-cocartesian.
Proof. First let’s take the homotopy pushout of the diagram $C \leftarrow A \to B$.

This amounts to first replacing the maps $A \to B$, $A \to C$ by cofibrations and then taking the actual pushout. Let $h$ denote the canonical map $A \otimes \Lambda V \otimes \Lambda W \to D$.

Since the original diagram is a homotopy pullback we know that $\text{con}(A \to C) = \text{con}(B \to D) = r_1$. Also since $A \otimes \Lambda W \to C$ is a quasi-isomorphism we have $\text{con}(A \hookrightarrow A \otimes \Lambda W) = \text{con}(A \to C)$. Therefore $\text{con}(A \hookrightarrow A \otimes \Lambda W) = r_1$ and similarly $\text{con}(A \hookrightarrow A \otimes \Lambda V) = r_2$. By definition this means that $(\Lambda W, \bar{d})$ is $(r_1 - 1)$-connected, and that $(\Lambda V, \bar{d})$ is $(r_2 - 1)$-connected.

We want to learn the connectivity of the map $h$. Let $F$ be the forgetful functor $\text{CDGA} \to \text{DGM}$. Let’s apply $F$ to the above diagram, and take the pushout of $A \otimes \Lambda W \leftarrow A \to A \otimes \Lambda V$ in the category $\text{DGM}$. Then we have the following
Let’s focus on the commutative triangle:

\[
\begin{array}{ccc}
A \otimes \Lambda W & \rightarrow & A \otimes \Lambda V \\
\downarrow & & \downarrow f \\
A \otimes \Lambda W \oplus_A A \otimes \Lambda V & \rightarrow & A \otimes \Lambda V \otimes \Lambda W \\
\end{array}
\]

Since a cofibration in \(CDGA\) is also a cofibration in \(DGM\), we know that the pushout

\[
\begin{array}{ccc}
A & \rightarrow & A \otimes \Lambda V \\
\downarrow & & \downarrow f \\
A \otimes \Lambda W & \rightarrow & A \otimes \Lambda W \oplus_A A \otimes \Lambda V \\
\end{array}
\]

is a homotopy pushout in \(DGM\). Also since in the category \(DGM\), a square diagram is a homotopy pushout if and only if it is a homotopy pullback, we know that the map \(A \otimes \Lambda W \oplus_A A \otimes \Lambda V \rightarrow D\) is a weak equivalence. So if we can show that \(A \otimes \Lambda W \oplus_A A \otimes \Lambda V \rightarrow A \otimes \Lambda V \otimes \Lambda W\) is \((r_1 + r_2)\)-connected then it follows by Lemma 4.1.5 that \(A \otimes \Lambda V \otimes \Lambda W \rightarrow D\) is \((r_1 + r_2 - 1)\)-connected.

To see that \(A \otimes \Lambda W \oplus_A A \otimes \Lambda V \rightarrow A \otimes \Lambda V \otimes \Lambda W\) is \((r_1 + r_2)\)-connected it suffices to verify that its homotopy cofiber, \(A \otimes \Lambda V \otimes \Lambda W\), is \((r_1 + r_2 - 1)\)-connected. Note
that in $A \otimes \Lambda V \otimes \Lambda W$ we have $d(A) \subset A$, $d(\Lambda V) \subset \Lambda V$, and $d(\Lambda W) \subset \Lambda W$. Since we are working over a field, the Künneth formula implies that

$$H^*(A \otimes \Lambda V \otimes \Lambda W) \cong H^*(A) \otimes H^*(\Lambda V) \otimes H^*(\Lambda W).$$

Since $(\Lambda W, \bar{d})$ is $(r_1 - 1)$-connected, we know that if $w$ is an element of $\Lambda W$ such that it represents a non-zero cohomology class in $H^*(\Lambda W)$ of the least positive degree, then we must have $\text{deg}(w) \geq r_1$. Similarly if $v$ is an element of $\Lambda V$ such that it represents a non-zero cohomology class in $H^*(\Lambda V)$ of the least positive degree, then $\text{deg}(v) \geq r_2$. Now by the above isomorphism $H^0(A) = k$ we learn that $1 \otimes v \otimes w$, an element in $A \otimes \Lambda V \otimes \Lambda W$ of degree $r_1 + r_2$, represents a non-zero cohomology class and no element of lower degree can represent a non-zero cohomology class. It follows that $A \otimes \Lambda V \otimes \Lambda W$ is $(r_1 + r_2 - 1)$-connected, as desired.

\[\square\]

### 4.3 The Generalized Blakers-Massey Theorem for Squares

**Lemma 4.3.1.** Let the following be a commutative diagram of CDGAs:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

The diagram is $r$-cocartesian if and only if

$$\text{hocofib}(A \rightarrow C) \longrightarrow \text{hocofib}(B \rightarrow D)$$

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is $r$-connected.

**Proof.** Assume that the diagram is cofibrant, i.e., is equivalent to the following:

\[
\begin{array}{ccc}
A & \rightarrow & A \otimes \Lambda V \\
\downarrow & & \downarrow \\
A \otimes \Lambda W & \rightarrow & A \otimes \Lambda V \otimes \Lambda W \otimes \Lambda U
\end{array}
\]

where any map is a cofibration and the canonical map

\[ h : A \otimes \Lambda V \otimes \Lambda W \rightarrow A \otimes \Lambda V \otimes \Lambda W \otimes \Lambda U \]

is also a cofibration.

By definition the original diagram is $r$-cocartesian if and only if $h$ is $r$-connected, that is, if and only if $\text{hocofib}(h) = (\Lambda U, \bar{d})$ is $(r - 1)$-connected.

On the other hand, note that

\[ \text{hocofib}(B \rightarrow D) \simeq \text{cofib}(A \otimes \Lambda V \rightarrow A \otimes \Lambda V \otimes \Lambda W \otimes \Lambda U) \simeq (\Lambda W \otimes \Lambda U, \bar{d}). \]

One concludes that the map between homotopy cofibers

\[ (\Lambda W, \bar{d}) \rightarrow (\Lambda W \otimes \Lambda U, \bar{d}) \]

is $r$-connected if and only if its homotopy cofiber, $(\Lambda U, \bar{d})$, is $(r - 1)$-connected.

This finishes the proof. \qed
Lemma 4.3.2. Let the following be a commutative diagram of CDGAs:

\[
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
\]

If the left diagram is \(r\)-cocartesian and if the right diagram is \(s\)-cocartesian then the big diagram is \(\min\{r, s\}\)-cocartesian.

Proof. Without loss of generality assume that both the left and the right squares are cofibrant (hence the big diagram is also cofibrant). Let \(CF := \text{hocofib}(A \to B)\), \(CF' := \text{hocolim}(A' \to B')\), and \(CF'' := \text{hocolim}(A' \to B')\). Then we have a commutative diagram:

\[
\begin{array}{ccc}
CF & \longrightarrow & CF'' \\
& \searrow & \\
& CF' & 
\end{array}
\]

By assumption and Lemma 4.3.1 we know that the map \(CF \to CF'\) is \(r\)-connected, and that the map \(CF' \to CF''\) is \(s\)-connected. By Lemma 4.1.5 their composition \(CF \to CF''\) is \(\min\{r, s\}\)-connected. Again by Lemma 4.3.1 the big diagram is \(\min\{r, s\}\)-cocartesian. \(\square\)

Lemma 4.3.3. Let the following be a commutative diagram of CDGAs:

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

If \(A \to C\) is \(r\)-connected and if \(B \to D\) is \(s\)-connected then the diagram is \(\min\{r-1, s-1\}\)- cocartesian.
Proof. Let $CF := \text{hcofib}(A \to C)$ and let $CF' := \text{hcofib}(B \to D)$. The condition says that $CF$ is $(r - 1)$-connected and that $CF'$ is $(s - 1)$-connected. So for $0 < i \leq \min\{r - 1, s - 1\}$ we have $H^i(CF) = H^i(CF') = 0$. This forces that maps $H^i(CF) \to H^i(CF')$ be isomorphisms whenever $i$ is within the range. So we learn that $CF \to CF'$ is $\min\{r - 1, s - 1\}$-connected and hence, by Lemma 4.3.1, the original diagram is $\min\{r - 1, s - 1\}$-cocartesian.

Now we are ready to prove a generalized version of Theorem 4.2.1.

**Theorem 4.3.4. (Generalized Blakers-Massey theorem for squares of CDGAs)**

Let the following be an $r$-cartesian diagram of CDGAs:

\[
\begin{array}{ccc}
A & \xrightarrow{g'} & B \\
\downarrow{f'} & & \downarrow{f} \\
C & \xrightarrow{g} & D
\end{array}
\]

Suppose that $f$ is $r_1$-connected and that $g$ is $r_2$-connected then the diagram is $\min\{r_1 + r_2 - 1, r - 1\}$-cocartesian.

**Proof.** Assume that the diagram is fibrant.

Let $PB$ be the pullback of $C \to D \leftarrow B$. Then both $f$ and $g$ are fibrations, and $PB$ is the homotopy pullback. By the condition the map $h : A \to PB$ is $r$-connected.
Note that we have the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & PB & \xrightarrow{f''} & C \\
\downarrow{g'} & & \downarrow{g''} & & \downarrow{g} \\
B & \xrightarrow{=} & B & \xrightarrow{f} & D
\end{array}
\]

where right square is a homotopy pullback. Then the Blakers-Massey theorem 4.2.1 implies that the right square is \((r_1 + r_2 - 1)\)-cocartesian.

On the other hand, we note that in the left square the map \(h\) is \(r\)-connected, and its parallel map \(B \to B\) is an isomorphism. It then follows form Lemma 4.3.3 that the left square is \((r - 1)\)-cocartesian.

Now by Lemma 4.3.2 we know that the whole diagram, i.e., our original diagram, is \(\text{min}\{r_1 + r_2 - 1, r - 1\}\)-cocartesian, as desired.

\(\square\)
Chapter 5

Technical Tools for the Proof

This is a preparatory chapter dealing with the technical tools needed for the proof of the main theorem. In this chapter we will always work over a field of characteristic zero, unless otherwise stated.

5.1 Cubical Diagrams in CDGAs

We need the following concepts to proceed our discussion. Some of our notations here follow from [22, Chapter 6].

Cubical diagrams.

Let $n$ be a positive integer. Denote by $n$ the category whose objects are subsets
of \{1, \cdots, n\}, and for two objects \(T\) and \(S\),

\[
\text{Hom}_{\mathbf{n}}(T, S) = \begin{cases} 
\text{the unique inclusion } T \subset S, & \text{if } T \text{ is a subset of } S; \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

We will also denote by \(\mathbf{n}\) the set \(\{1, \cdots, n\}\) itself, in situations where no possible confusion can occur. Write \(P(n)\) for the power set of \(\{1, \cdots, n\}\), i.e., the set of all subsets of \(\{1, \cdots, n\}\).

Let \(\mathcal{M}\) be a model category. A \textit{cubical diagram of dimension }n\textit{ in }\mathcal{M}\textit{, or simply an }n\textit{-cube in }\mathcal{M}\textit{, is a contravariant functor }A : \mathbf{n} \to \mathcal{M}\text{. A map of }n\text{-cubes is a natural transformation of these functors. Let }A, B\text{ be two }n\text{-cubes. A map from }A\text{ to }B\text{ is written as }A \to B\text{.}

We remark that a map of two \(n\)-cubes can be regarded as an \((n+1)\)-cube. Indeed, if \(\phi : A \to B\) is a map of \(n\)-cubes then we can associate to it an \((n + 1)\) cube \(C(\phi)\) as follows. Let \(T\) be a subset of \(\mathbf{n+1}\). We set:

\[
C(\phi)(T) = \begin{dcases} 
B(T), & \text{if } T \subset \mathbf{n}; \\
A(T - \{n + 1\}), & \text{otherwise.}
\end{dcases}
\]

We write that \(C = (A \to B)\). Conversely an \(n\)-cube can be realized, in many possible ways, as a map of two \((n - 1)\)-cubes.

Let \(A : \mathbf{n} \to \mathcal{M}\) be an \(n\)-cube. Let \(T\) be a subset of \(\{1, \cdots, n\}\). The object \(A(T)\) of \(\mathcal{M}\) that \(A\) associates to \(T\) will sometimes be denoted by \(A_T\) in later sections. We
say that $A_T$ is the *corner* of $\mathcal{A}$ at $T$. When $T$ is specific, say $T = \{1, 2, 4\}$, we will simply write $A_{124}$ instead of $A_{\{1,2,4\}}$. Also we write $A_0$ instead of $A_{\emptyset}$.

An $n$-cube need not be indexed by $\mathbf{n}$. Consider a finite set $T$ of $n$-elements. (We say that $T$ has *cardinality* $n$ and write $|T| = n$.) Let $\mathbf{T}$ be the category whose objects are subsets of $T$, and whose morphisms are subset inclusions. Then a functor $\mathbf{T} \to \mathcal{M}$ is also called an $n$-cube in $\mathcal{M}$.

**Sub-diagrams of cubes.**

Let $\mathcal{A}$ be an $n$-cube, and let $\mathbf{S}$ be a subset of $P(n)$. Note that $\mathbf{S}$ could be regarded as a subcategory of $\mathbf{n}$. Therefore $\mathbf{S}$ determines a functor $\mathbf{S} \to \mathbf{n} \xrightarrow{A} \mathcal{M}$. This functor is called the *sub-diagram* of $\mathcal{A}$ determined by $\mathbf{S}$, and is denoted by $\mathcal{A}(\mathbf{S})$. If $\mathbf{S}$ is a subset of $P(n)$, and if $T$ is an element of $P(n)$, then we write $\mathbf{S} \cap T$ for the subset $\{T' \cap T : T' \in \mathbf{S}\} \subset P(n)$. Similarly, write $\mathbf{S} \cup T$ for the subset $\{T' \cup T : T' \in \mathbf{S}\} \subset P(n)$. Say that a subset $\mathbf{S} \subset P(n)$ is *convex*, if it has the following property: if $T' \subset T \subset \mathbf{n}$ and if $T \in \mathbf{S}$ then $T' \in \mathbf{S}$.

Let $T \subset \mathbf{n}$. Denote by $P(T)$ the set of all subsets of $T$, by $P_1(T)$ the set of all proper subsets of $T$, and by $P_0(T)$ the set of all non empty subsets of $T$. The sub-diagrams $\mathcal{A}(P_1(T))$ and $\mathcal{A}(P_0(T))$ are called the punctured cubes determined by $T$. If $|T| = m$ then they are called punctured $m$-cubes.

Faces form an important collection of sub-diagrams, and will be used intensively throughout this paper. Most of our notations here follow from those used in [22].
Let $A$ be an $n$-cube. Suppose that $S \subset T \subset \mathbb{n}$ are subsets with $|T| = l$, $|S| = m$, $l > m$. Then the pair $(S, T)$ determines an $(l - m)$-cube $\partial^T_S A$ as follows. If $R$ is a subset of $T - S$ then $\partial^T_S A(R) = A_{R \cup S}$. The sub-diagram $\partial^T_S A$ of $A$ is called the face determined by $(S, T)$. In short, a face of a cube is a sub-diagram which is itself a cube.

In particular we have the face $\partial^T_\emptyset A$ determined by the pair $(\emptyset, T)$, where $T$ is some subset of $\mathbb{n}$. This is called the fundamental face of $A$ determined by $T$, and will also be denoted by $A^T$. Note that an $n$-cube $A$ can be written as a map $\partial^{\mathbb{n}}_{\{i\}} A \to \partial^{-\{i\}}_\emptyset A$, $\forall i \in \mathbb{n}$.

**Cartesian and cocartesian cubes.**

The cartesian and cocartesian properties of cubical diagrams in algebraic categories are the main objects of study in this paper. Here by an algebraic category we mean the category $Ch(k)$, $DGM_R$, or $CDGA_k$, where $k$ is a field of characteristic zero.

Let $A$ be an $n$-cube in some algebraic category. If $S$ is a subset of $P(n)$, i.e., a collection of subsets of $\mathbb{n}$, then the (homotopy) limit of the sub-diagram $A(S)$ will be denoted by $(ho)\lim A(S)$, or by $(ho)\lim_{T \in S} A_T$. Similarly write $(ho)\colim A(S)$, or $(ho)\colim_{T \in S} A_T$, for the (homotopy) colimit of the diagram $A(S)$.

In particular we can take $S = P_1(n)$, and consider the homotopy limit of the diagram $A(P_1(n))$. There is a canonical map $h_1 : A_{\mathbb{n}} \to h\lim A(P_1(n))$. Similarly
when \( S = P_0(n) \) we can consider the homotopy colimit of the diagram \( \mathcal{A}(P_0(n)) \).

There is a canonical map \( h_0 : \text{hocolim}\mathcal{A}(P_0(n)) \to A_0 \).

**Definition 5.1.1.** We say that \( \mathcal{A} \) is:

- \( r \)-cartesian, if \( h_1 \) is \( r \)-connected;
- homotopy cartesian, if \( h_1 \) is a weak equivalence;
- \( r \)-cocartesian, if \( h_0 \) is \( r \)-connected;
- homotopy cocartesian, if \( h_0 \) is a weak equivalence.

Also we say that \( \mathcal{A} \) is:

- strongly homotopy cartesian, if any square face of \( \mathcal{A} \) is homotopy cartesian;
- strongly homotopy cocartesian, if any square face of \( \mathcal{A} \) is homotopy cocartesian.

**Remark.** From now on, an \( n \)-cube \( \mathcal{A} \) of CDGAs will be assumed to have the property that the connectivity of \( A_i \to A_0 \) is at least 2, \( i = 1, \ldots, n \), unless otherwise stated.

**Remark.** Again we would like to discuss the behavior of forgetful functors with respect to these notions. Recall our forgetful functors \( \text{CDGA}_k \overset{F_1}{\to} \text{Ch}(k) \overset{F_2}{\to} \text{GrVect}_k \). If \( \mathcal{A} \) is an \( n \)-cube in \( \text{CDGA}_k \), then we let \( F_1(\mathcal{A}) \) be the \( n \)-cube in \( \text{Ch}(k) \) defined by \( T \mapsto F_1(A_T), T \subset n \).

Also recall from Section 2.2 that quasi-isomorphisms and fibrations are defined in the same way in \( \text{CDGA}_k \) and in \( \text{Ch}(k) \). It then follows from the definition that a
cube $A$ is $r$-cartesian in $\text{CDGA}_k$ if and only if $F_1(A)$ is $r$-cartesian in $\text{Ch}(k)$. On the other hand, if $F_2 \circ F_1(A)$ is $r$-cartesian in $\text{GrVect}_k$ then $A$ is itself $r$-cartesian in $\text{CDGA}_k$.

We note also that, using the above language, cofibrant and fibrant $n$-cubes have the following description. A cofibrant $n$-cube $A$ is an $n$-cube such that for any subset $T \subset \underline{n}$, the fundamental face $A^T$ determined by $T$ is homotopy cocartesian; and a fibrant cube is one such that for any subset $T \subset \underline{n}$, the fundamental face $A^T$ is homotopy cartesian. (This follows form the discussion in [16, Section 10.13].)

**Replace cubes by nice ones.**

In later sections, we will often start a proof by assuming that a cubical diagram of CDGAs satisfies some desired properties. This will often reduce the length of a proof without losing generality.

First of all, it is standard in model category theory that any $n$-cube $A$ of CDGAs admits a fibrant replacement and a cofibrant replacement. That is, there exist a fibrant $n$-cube $A'$ that is weak equivalent to $A$, and a cofibrant $n$-cube $A''$ weak equivalent to $A$. Furthermore, we have the following result.

**Lemma 5.1.2.** Let $A$ be a strongly cartesian $n$-cube of simply connected CDGAs. There exists an $n$-cube $A'$ weakly equivalent to $A$, that satisfies the following property: the connectivity of $(A'_i, d_{A'_i}) \to (A'_0, d_{A'_0})$ is equal to the connectivity of $(A'_i, 0) \to (A'_0, 0)$, $i = 1, \ldots, n$. 62
This can be reduced to the following lemma.

**Lemma 5.1.3.** Let \( f : A \to B \) be an \( r \)-connected map of simply connected CDGAs. Here \( r > 1 \), and \( B = \Lambda V \) is a minimal Sullivan algebra. Then there exists a CDGA \( A' \), together with CDGA maps \( f' : A' \to B \), \( g : A' \to A \), such that \( f' = f \circ g \), that \( g \) is a quasi-isomorphism, and that \( f^i : (A')^i \to B^i \) is an isomorphism for \( i \leq r - 1 \) and an injection for \( i = r \).

\[
\begin{array}{ccc}
A' & \xrightarrow{g} & A \\
\downarrow{f'} & & \downarrow{f} \\
B = \Lambda V & & 
\end{array}
\]

**Proof of Lemma 5.1.2, assuming Lemma 5.1.3.**

Let \( f_i \) denote the map \( A_i \to A_0 \) in \( A \), \( i = 1, \ldots, n \). First of all we claim that \( A \) is equivalent to a cube \( A'' \) where \( A''_i = \Lambda V_i \) is a minimal Sullivan algebra, \( i = 0, \ldots, n \). Indeed, recall that any simply connected CDGA admits a Sullivan model. (See [15, Proposition 12.2].) So for each \( i, i = 0, \ldots, n \), there exists a Sullivan algebra \( \Lambda V_i \) and a map \( \varphi_i : \Lambda V_i \to A_i \) that is a quasi-isomorphism. Also recall that \( f_i \circ \varphi_i \) admits a lift \( \hat{f}_i : \Lambda V_i \to \Lambda V_0 \) such that \( f_i \circ \varphi_i = \varphi_0 \circ \hat{f}_i, i = 1, \ldots, n \). (See [15, Lemma 12.4].)

We let \( f''_i := \hat{f}_i, i = 1, \ldots, n \).

\[
\begin{array}{ccc}
\Lambda V_i & \xrightarrow{f''_i = f_i} & \Lambda V_0 \\
\varphi_i & \simeq & \varphi_0 \\
\downarrow & & \downarrow \\
A_i & \xrightarrow{f_i} & A_0 \\
\end{array}
\]
Now for $T = \{n_1, n_2\} \subset \mathbf{n}$ is of cardinality 2 we let $A''_T := \text{holim}(A''_{n_1} \leftarrow A''_0 \rightarrow A''_{n_2})$.

Inductively, if $A''_T$ is defined for any $T \subset \mathbf{n}$ of cardinality less than $m$, then for a subset $S \subset \mathbf{n}$ of cardinality $m$ we define $A''_S := \text{holim} A''(P_i S)$. This completes the construction of $\mathcal{A}''$ and we have an equivalence $\mathcal{A}'' \simeq \mathcal{A}$.

We then start to construct $\mathcal{A}'$. We set $A'_0 = \Lambda V_0$. For each $i = 1, \ldots, n$, by Lemma 5.1.3 there exists a triple $(A'_i, f'_i, g_i)$, where $g_i : A'_i \rightarrow \Lambda V_i$ is a quasi-isomorphism, $f'_i : A'_i \rightarrow \Lambda V_0$ is an isomorphism up to degree $r - 1$, and an injection on degree $r$, which makes the following diagram commute:

$$
\begin{array}{c}
A'_i \\
\downarrow \sim \\
\Lambda V_i \\
\downarrow f'_i \\
\Lambda V_0
\end{array}
$$

By definition $(A'_i, 0) \rightarrow (A'_0, 0)$ has connectivity $r$. Also since $g_i$ is a quasi-isomorphism we know that $\text{con}(f'_i) = \text{con}(f''_i) = \text{con}(f_i) = r$. This finishes the definition of $A'_i$, $i = 0, \ldots, n$. Now if $T = \{n_1, n_2\} \subset \mathbf{n}$ is of cardinality 2, then we set $A'_T := \text{holim}(A'_{n_1} \leftarrow A'_0 \rightarrow A'_{n_2})$. Inductively, if $A'_T$ is defined for any $T \subset \mathbf{n}$ of cardinality less than $m$, then for a subset $S \subset \mathbf{n}$ of cardinality $m$ we define $A'_S := \text{holim} A'(P_i S)$. This completes the construction of $\mathcal{A}'$. It is easy to see that there is an equivalence $\mathcal{A}' \simeq \mathcal{A}''$, and hence an equivalence $\mathcal{A}' \simeq \mathcal{A}$.

Now let’s prove Lemma 5.1.3.

Proof. We proceed through the following steps.

Step 1. Solution of a lifting problem.
We claim that the following lifting problem can be solved. Here \( \lambda : \Lambda(V^{\leq r-1}) \to B \) is the inclusion map.

\[
\begin{array}{ccc}
\Lambda(V^{\leq r-1}) & \xrightarrow{\lambda} & B = \Lambda V \\
\downarrow f & & \\
A & \xrightarrow{\hat{\lambda}} & & \end{array}
\]

Indeed, the minimality and the simply-connectedness of \( B = \Lambda V \) guarantee that \( \Lambda(V^{\leq r-1}) \) is again a Sullivan algebra. This allows us to construct the lifting \( \tilde{\lambda} \) inductively. Suppose that \( V^{\leq r-1} \) is the union of \( V^{\leq r-1}(0) \subset \cdots \subset V^{\leq r-1}(k) \subset \cdots \). There exists a lift for \( \Lambda(V^{\leq r-1}(0)) \) since \( d(V^{\leq r-1}(0)) = 0 \). Suppose that \( \tilde{\lambda} \) has been defined on \( V^{\leq r-1}(k) \). Write \( V^{\leq r-1}(k+1) = V^{\leq r-1}(k) \oplus V_k \). For \( v \in V_k \), note that \( \tilde{\lambda}(dv) \) is defined already since \( dv \in V^{\leq r-1}(k) \), and that \( \tilde{\lambda}(dv) \) is a cycle: \( d(\tilde{\lambda}(dv)) = \tilde{\lambda}(d^2v) = 0 \).

Now that \( H^r(f)([\tilde{\lambda}(dv)]) = [f(\tilde{\lambda}(dv))] = [\lambda(dv)] = [d\lambda(v)] = 0 \), and that \( H^r(f) \) is injective, we know that \( \tilde{\lambda}(dv) \) must be a boundary. Say \( \tilde{\lambda}(dv) = da \) for some \( a \in A \).

We then define \( \check{\lambda}(v) = a \).

**Step 2. Overview of the construction.**

We are going to construct a triple \((A', j, g)\), where \( A' \) is a CDGA of the form \( A' = (\Lambda(V^{\leq r-1}) \otimes \Lambda W, d) \), \( j : \Lambda(V^{\leq r-1}) \to A' \) is the inclusion, \( g : A' \to A \) is a quasi-isomorphism that extends \( \check{\lambda} \) (i.e., \( \check{\lambda} = g \circ j \)), such that the composition \( f' := f \circ g \) has the desired property, namely that \((f')^i : (A')^i \to B^i \) is an isomorphism for \( i \leq r - 1 \) and an injection for \( i = r \). This is done in an inductive way. We will define a triple \((A'(n), j(n), g(n))\) for each \( n \geq r \), such that \( A'(n+1) = A'(n) \otimes \Lambda W(n+1) \), \( j(n) \) is
an inclusion, \( g(n) \) is \((n+1)\)-connected, and \( f'(n) := f \circ g(n) \) has the property that \((f'(n))^i\) is an isomorphism for \( i \leq r - 1 \) and an injection for \( i = r \). Then \((A', j, g)\) is obtained by taking colimits.

\[
\begin{array}{ccc}
A'(n) & \xrightarrow{\text{\text{(n+1)-connected}}} & A \\
\downarrow j(n) & & \downarrow f \\
\Lambda(V^{\leq r-1}) & \xrightarrow{\lambda} & B
\end{array}
\quad
\begin{array}{ccc}
A' & \xrightarrow{\sim} & A \\
\downarrow j & & \downarrow f \\
\Lambda(V^{\leq r-1}) & \xrightarrow{\lambda} & B
\end{array}
\]

**Step 3. Construction of \((A'(r), j(r), g(r))\).**

Let's fix a basis \( \alpha_1 = [a_1], \ldots, \alpha_s = [a_s] \) for \( \text{Ker} H^{r+1}(\tilde{\lambda}) \), and fix a basis \( \beta_1 = [b_1], \ldots, \beta_t = [b_t] \) for \( \text{Coker} H^r(\tilde{\lambda}) \). Further, we pick elements \( a'_1, \ldots, a'_s \in A^r \) such that \( da'_i = \tilde{\lambda}(a_i), i = 1, \ldots, n \). Then we let \( W(r) = k \langle v_1, \ldots, v_s; w_1, \ldots, w_t \rangle \) be the graded \( k \)-vector space on the basis \( v_1, \ldots, v_s, w_1, \ldots, w_t \), each of degree \( r \). We define

\[
A'(r) = (\Lambda(V^{\leq r-1}) \otimes \Lambda W(r), d),
\]

where the differential \( d \) extends the differential on \( \Lambda(V^{\leq r-1}) \), and on \( W(r) \) it is given by \( d(v_i) = a_i, i = 1, \ldots, s \), and \( d(w_i) = 0, i = 1, \ldots, t \). We define \( j(r) \) to be the inclusion \( \Lambda(V^{\leq r-1}) \to \Lambda(V^{\leq r-1}) \otimes \Lambda W(r) \). We define \( g(r) \) as follows. On \( \Lambda(V^{\leq r-1}) \) it is equal to \( \tilde{\lambda} \). On \( W(r) \) it is given by \( g(r)(v_i) = a'_i, i = 1, \ldots, s \), and \( g(r)(w_i) = b_i, i = 1, \ldots, t \). Finally, we denote by \( f'(r) \) the composition \( f \circ g(r) \).

**Step 4. Show that \((f'(r))^i\) is an isomorphism for \( i \leq r - 1 \) and an injection for \( i = r \).**

It follows directly from the construction that \((f'(r))^i\) is an isomorphism for \( i \leq r - 1 \). We show that \( \text{Ker}(f'(r))^r = 0 \). Let \( x \in (A'(r))^r = (\Lambda(V^{\leq r-1}) \otimes \Lambda W(r))^r \) be such
that \( f'(r)(x) = 0 \). Our goal is to prove that \( x = 0 \). We discuss the following cases.

**Case (i).** \( x = c_1v_1 + \cdots + c_sv_s \) is a linear combination of \( v_1, \ldots, v_s \). We have that

\[
0 = f'(r)(x) = f(g(r)(x)) = f(g(r)(c_1v_1 + \cdots + c_sv_s)) = f(c_1g(r)(v_1) + \cdots + c_sg(r)(v_s)) = f(c_1a'_1 + \cdots + c_s a'_s).
\]

It then follows that

\[
0 = df(c_1a'_1 + \cdots + c_s a'_s) = c_1f(da'_1) + \cdots + c_sf(da'_s) = c_1f(\tilde{\lambda}(a_1)) + \cdots + c_s f(\tilde{\lambda}(a_s)) = c_1\lambda(a_1) + \cdots + c_s \lambda(a_s) = c_1a_1 + \cdots + c_s a_s.
\]

But this implies that \( c_1[a_1] + \cdots + c_s[a_s] = 0 \) in \( KerH^{r+1}(\tilde{\lambda}) \). Since \([a_1], \ldots, [a_s]\) is a basis we know that \( c_1 = \cdots = c_s = 0 \), so \( x = 0 \).
Case (ii). $x = c_1 w_1 + \cdots + c_t w_t$ is a linear combination of $w_1, \ldots, w_t$. We have

$$0 = f'(r)(x) = f(g(r)(x)) = f(g(r)(c_1 w_1 + \cdots + c_t w_t)) = f(c_1 g(r)(w_1) + \cdots + c_t g(r)(w_t)) = f(c_1 b_1 + \cdots + c_t b_t).$$

This is to say that $c_1[b_1] + \cdots + c_t[b_t] \in \text{Ker} H^r(f)$. By condition $f$ is $r$-connected, so $H^r(f)$ is injective. It then follows that $c_1[b_1] + \cdots + c_t[b_t] = 0$. Since $[b_1], \ldots, [b_t]$ is a basis for $\text{Coker} H^r(\lambda)$ we know that $c_1 = \cdots = c_t = 0$, so $x = 0$.

Case (iii). $x$ is a general element of $W(r)$. Then we write $x = x_1 + x_2$ where $x_1 = \sum_{i=1}^s c_i v_i$, $x_2 = \sum_{i=1}^t c'_i w_i$. We have

$$0 = f'(r)(x) = f(g(r)(x)) = f(\sum_{i=1}^s c_i a'_i + \sum_{i=1}^t c'_i b_i).$$

It then follows that

$$0 = df(\sum_{i=1}^s c_i a'_i + \sum_{i=1}^t c'_i b_i) = fd(\sum_{i=1}^s c_i a'_i + \sum_{i=1}^t c'_i b_i) = c_1 a_1 + \cdots + c_s a_s.$$  

We know that $c_1 = \cdots = c_s = 0$, so $x_1 = 0$. It then follows from Case (ii) that $x_2 = 0$.

Case (iv). $x$ is a general element of $A'(r)$. Recall that $A'(r) = \Lambda(V^{\leq r-1}) \otimes AW(r)$. Since $x$ is of degree $r$, $x$ must be of the form $x = x_1 + x_2$, where $x_1 \in \Lambda(V^{\leq r-1})$, and

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$x_2 \in W(r)$. Now we have

$$0 = f'(r)(x) = f(g(r)(x_1)) + f(g(r)(x_2))$$

$$= f(\tilde{\lambda}(x_1)) + f(g(r)(\sum_{i=1}^s c_i v_i + \sum_{i=1}^t c_i' w_i))$$

$$= \lambda(x_1) + f(\sum_{i=1}^s c_ia_i' + \sum_{i=1}^t c_i'b_i)$$

$$= x_1 + \sum_{i=1}^s c_if(a_i') + \sum_{i=1}^t c_i'f(b_i).$$

It then follows that

$$0 = d(x_1 + \sum_{i=1}^s c_if(a_i') + \sum_{i=1}^t c_i'f(b_i))$$

$$= dx_1 + \sum_{i=1}^s c_idf(a_i') + \sum_{i=1}^t c_i'df(b_i)$$

$$= dx_1 + \sum_{i=1}^s c_if(da_i') + \sum_{i=1}^t c_i'f(db_i)$$

$$= dx_1 + \sum_{i=1}^s c_i\tilde{\lambda}(a_i))$$

$$= dx_1 + \sum_{i=1}^s c_i\lambda(a_i)$$

$$= dx_1 + \sum_{i=1}^s c_ia_i.$$ 

We learn that $\sum_{i=1}^s c_i[a_i] = [d(-x_1)] = 0$. Since $[a_1], \cdots, [a_s]$ is a basis for $\ker H^{r+1}(\tilde{\lambda})$ we know that $c_1 = \cdots = c_s = 0$. Now we know that $x_1$ is a cycle and that $x_1 + \sum_{i=1}^t c_i'f(b_i) = 0$. Note that $x_1 = \tilde{\lambda}(x_1)$. It follows that $f(\tilde{\lambda}(x_1) + \sum_{i=1}^t c_i' b_i) = 0$. But this is to say that $[\tilde{\lambda}(x_1)] + \sum_{i=1}^t c_i' [b_i] \in \ker H^r(f)$. It follows from the injectivity of $H^r(f)$ that $[\tilde{\lambda}(x_1)] + \sum_{i=1}^t c_i' [b_i] = 0$. Then $\sum_{i=1}^t c_i' [b_i] = H^r(\tilde{\lambda})([-x_1]) \in \text{Im} H^r(\tilde{\lambda}).$
Since also $\sum_{i=1}^{t} c'_i[b_i] \in \text{Coker} H^r(\tilde{\lambda})$, we must have $\sum_{i=1}^{t} c'_i[b_i] = 0$. But then $c_1 = \cdots = c_t = 0$ since $[b_1], \ldots, [b_t]$ is a basis. We have proved that $x_2 = 0$. It then follows that $x_1 = 0$, and $x = 0$.

\textit{Step 5. Show that $g(r)$ is $(r+1)$-connected.}

It is clear from the construction that $H^i(g(r))$ is an isomorphism for $i \leq r$. We show that $H^{r+1}(g(r))$ is injective. Since $B$ is simply connected we know that there is no element of degree one in $V^{\leq r-1}$, and hence in $\Lambda(V^{\leq r-1})$. Also note that the degree of any homogeneous element in $\Lambda W(r)$ must be an integer multiple of $r$, where $r > 1$.

So due to this degree reason a cohomology class $\alpha = [a] \in H^{r+1}(A'(r))$ must belong to $H^{r+1}(\Lambda(V^{\leq r-1}))$. Note that $g(r)|_{\Lambda(V^{\leq r-1})} = \tilde{\lambda}$, by construction. So if $H^{r+1}(g(r))(\alpha) = 0$ then $0 = [g(r)(a)] = [\tilde{\lambda}(a)]$, i.e., $\alpha = [a] \in \text{Ker} H^{r+1}(\lambda)$. Then $\alpha = c_1\alpha_1 + \cdots + c_s\alpha_s$ is a linear combination of the basis elements. It follows that $a = c_1a_1 + \cdots + c_s a_s + da'$ for some $a' \in \Lambda(V^{\leq r-1})$. Now $a = c_1dv_1 + \cdots + c_s dv_s + da' = d(c_1v_1 + \cdots + c_s v_s + a')$, and therefore $\alpha = [a] = 0$.

\textit{Step 6. Completion of the proof.}

Our next job is to define $(A'(r+1), j(r+1), g(r+1))$. We are to choose an appropriate graded vector space $W(r+1)$ and set $A'(r+1) = A'(r) \otimes \Lambda(W(r+1))$, define $j(r+1)$ as the inclusion map, and define an $(r+2)$-connected map $g(r+1) : A'(r+1) \rightarrow A'$. Note that for this purpose there is no need to introduce elements of degree $r$ in $W(r+1)$,
since $H^{r+1}(g(r))$ is already injective. Then the composition $f'(r + 1) := f \circ g(r + 1)$ will again have the property that $(f'(r + 1))^i$ is an isomorphism for $i \leq r - 1$ and an injection for $i = r$. Continue the construction inductively.

Finally we define $A' = \text{colim}_{n \geq r} A'(n) = \Lambda(V^{\leq r-1}) \otimes (\bigotimes_{n \geq r} \Lambda W(n))$. We let $j = \text{colim}_{n \geq r} j(n)$, which is the inclusion; and let $g = \text{colim}_{n \geq r} g(n)$. For each $n \geq r$ the map $f'(n) := f \circ g(n)$ satisfies that $(f'(n))^i$ is an isomorphism for $i \leq r - 1$ and an injection for $i = r$. We define $f' := \text{colim}_{n \geq r} f'(n) = f \circ g$. Now the triple $(A', f', g)$ has the desired property. □

**Factorizations of cubes.**

Let $\mathcal{X}, \mathcal{Y}$ be $n$-cubes. Say that $\mathcal{X}$ can be composed by $\mathcal{Y}$, if there exist $(n - 1)$-cubes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that $\mathcal{X} = (\mathcal{B} \to \mathcal{C})$ and that $\mathcal{Y} = (\mathcal{A} \to \mathcal{B})$. If this is the case then we define the composition of $\mathcal{X}$ with $\mathcal{Y}$ to be the $n$-cube $Z := (\mathcal{A} \to \mathcal{B} \to \mathcal{C})$, written $Z = \mathcal{X} \mathcal{Y}$.

Let $Z, Z_1, Z_2, \ldots, Z_m$ be $n$-cubes. Say that $Z_1 Z_2 \ldots Z_m$ is a factorization of $Z$, if $Z_i$ can be composed by $Z_{i+1}$, $1 \leq i \leq m - 1$, and $Z = Z_1 Z_2 \ldots Z_m$ is the composition.

**The cube of homotopy limits induced by a convex subset.**

Let $\mathcal{A}$ be an $n$-cube and let $S$ be a convex subset of $P(n)$. Here we construct a new $n$-cube, denoted by $\mathcal{A}_S$, as follows. For any $T \subset \{1, \cdots, n\}$ we let $\mathcal{A}_S : T \mapsto$
holim\(A(S \cap P(T))\). This diagram \(A_S\) is called the cube of homotopy limits induced by \(S\).

So according to the definition, to learn the corner of \(A_S\) at \(T\) one should first find all subsets of \(T\) that appear in the collection \(S\), and then take the homotopy limit of the corresponding diagram.

**The associated cube of homotopy (co)fibers.**

Let \(A\) be an \(n\)-cube. Define an \((n-1)\)-cube \(C\) by setting

\[
C(T) := \text{ho(co)fib}(A_{T \cup \{n\}} \to A_T), \quad \forall T \subset \{1, \cdots, n-1\}.
\]

Call \(C\) the diagram of homotopy (co)fibers associated to \(A\).

### 5.2 Diagram Analysis

This section is a further investigation on cubical diagrams of CDGAs. The properties here will be used in the proof of the main theorems.

Our first result is Proposition 5.2.3, which inspires an induction argument in the proof of the main theorem. (Note that it is the generalization of Lemma 4.3.1 to higher dimensions.) To prove it we need two preparatory lemmas, which are important themselves.
Lemma 5.2.1. Let $\mathcal{A}$ be an $n$-cube of CDGAs. The following is a homotopy pushout:

$$
\begin{array}{ccc}
\text{hocolim} \mathcal{A}(P_0(n-1) \cup \{n\}) & \longrightarrow & A_n \\
\downarrow & & \downarrow \\
\text{hocolim} \mathcal{A}(P_0(n-1)) & \longrightarrow & \text{hocolim} \mathcal{A}(P_0(n))
\end{array}
$$

Proof. Assume that $\mathcal{A}$ is cofibrant, so that there exists a Sullivan algebra $A$, together with a family $\{\Lambda V_S\}_{S \in P(n)}$ of Sullivan algebras, with $\Lambda V_0 = (k, 0)$, such that the corner of $\mathcal{A}$ at $T$ is $A_T = A \otimes (\bigotimes_{S \subseteq n-T} \Lambda V_S)$, $\forall T \subset n$. The diagram to consider now becomes:

$$
\begin{array}{ccc}
A \otimes \bigotimes_{S \in P_0(n-1)} \Lambda V_S & \longrightarrow & A \otimes \bigotimes_{S \in P(n-1)} \Lambda V_S \\
\downarrow & & \downarrow \\
A \otimes \bigotimes_{S \in (P_0(n-1) \cup (n-1))} \Lambda V_S & \longrightarrow & A \otimes \bigotimes_{S \in P_0(n)} \Lambda V_S
\end{array}
$$

It is clear that this is a homotopy pushout. 

\[\square\]

Lemma 5.2.2. Let $\mathcal{A}$ be an $n$-cube of CDGAs, and let $\mathcal{C}$ be its associated $(n-1)$-cube of homotopy cofibers. Then we have an isomorphism of CDGAs:

$$
\text{hocolim} \mathcal{C}(P_0(n-1)) \cong \text{hcofib}(\text{hocolim} \mathcal{A}(P_0(n-1) \cup \{n\}) \rightarrow \text{hocolim} \mathcal{A}(P_0(n-1))).
$$

Proof. Again assume that $\mathcal{A}$ is of the form in the above proof. Now we have

$$
C_T = \text{hcofib}(A_{T \cup \{n\}} \rightarrow A_T) = \bigotimes_{S \in (P(n-1-T) \cup \{n\})} \Lambda V_S, \forall T \subset n-1
$$

and therefore

$$
\text{hocolim} \mathcal{C}(P_0(n-1)) = \bigotimes_{S \in (P_0(n-1) \cup \{n\})} \Lambda V_S.
$$
On the other hand, note that

\[ \text{hocofib}(\text{hocolim}\mathcal{A}(P_0(n - 1) \cup \{n\}) \to \text{hocolim}\mathcal{A}(P_0(n - 1))) = \text{hocofib}(A \otimes \bigotimes_{S \in P_0(n - 1)} \Lambda V_S \to A \otimes \bigotimes_{S \in (P_0(n - 1) \cup \{n\})} \Lambda V_S). \]

This proves the result.

\[ \square \]

**Proposition 5.2.3.** Let \( \mathcal{A} \) be an \( n \)-cube of CDGAs, and let \( \mathcal{C} \) be its associated \( (n-1) \)-cube of homotopy cofibers. Then \( \mathcal{A} \) is \( r \)-cocartesian if and only if \( \mathcal{C} \) is \( r \)-cocartesian.

**Proof.** We point out that it is enough to prove this for \( n = 2 \). Suppose that \( \mathcal{A} \) is an \( n \)-cube. By definition \( \mathcal{A} \) is \( r \)-cocartesian if and only if \( \text{hocolim}\mathcal{A}(P_0(n)) \to A_0 \) is \( r \)-connected. By Lemma 5.2.1 we know that this is true if and only if

\[ \begin{array}{ccc}
\text{hocolim}\mathcal{A}(P_0(n - 1) \cup \{n\}) & \to & A_n \\
\downarrow & & \downarrow \\
\text{hocolim}\mathcal{A}(P_0(n - 1)) & \to & A_0
\end{array} \]

is \( r \)-cocartesian. Let \( CP \) be the homotopy cofiber of \( \text{hocolim}\mathcal{A}(P_0(n - 1) \cup \{n\}) \to \text{hocolim}\mathcal{A}(P_0(n - 1)). \) If the case \( n = 2 \) is settled then the above diagram is \( r \)-cocartesian if and only if the map \( CP \to C_0 \) is \( r \)-connected. But by Lemma 5.2.2 we have \( CP = \text{hocolim}\mathcal{C}(P_0(n - 1)). \) So \( CP \to C_0 \) is \( r \)-connected if and only if \( \mathcal{C} \) is \( r \)-connected. This finishes the proof.

Now let’s prove it for \( n = 2 \). Assume \( \mathcal{A} \) to be cofibrant, so that it is of the
following form:

\[
\begin{array}{c}
A \\
\downarrow \\
A \otimes \Lambda U \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \end{array}
\begin{array}{c}
A \otimes \Lambda V \\
\downarrow \\
A \otimes \Lambda U \otimes \Lambda V \otimes \Lambda W \\
\end{array}
\]

Note that \(\text{hocofib}(A \to A \otimes \Lambda U) = \Lambda U\), that \(\text{hocofib}(A \otimes \Lambda V \to A \otimes \Lambda U \otimes \Lambda V \otimes \Lambda W) = \Lambda U \otimes \Lambda W\), and that \(\text{hocolim}(A \otimes \Lambda U \leftarrow A \to A \otimes \Lambda V) = A \otimes \Lambda U \otimes \Lambda V\). We are to prove that the connectivity of \(A \otimes \Lambda U \otimes \Lambda V \to A \otimes \Lambda U \otimes \Lambda V \otimes \Lambda W\) is equal to that of \(\Lambda U \to \Lambda U \otimes \Lambda W\). This follows from the fact that both the homotopy cofibers are \(\Lambda W\).

\[\square\]

The following two lemmas give examples of the situation where one could deduce the cocartesian property of a cube from the cocartesian properties from existing cubes. Note that they are generalizations of Lemma 4.3.2 and Lemma 4.3.3, respectively.

**Lemma 5.2.4.** Suppose that \(Z\) is an \(n\)-cube of CDGAs and that there is a factorization \(Z = \mathcal{X}\mathcal{Y}\). If \(\mathcal{X}\) is \(r\)-cocartesian and \(\mathcal{Y}\) is \(s\)-cocartesian then \(Z\) is \(\min\{r, s\}\)-cocartesian.

*Proof.* Let \(C_Z, C_{\mathcal{X}}, C_{\mathcal{Y}}\) be the \((n-1)\)-cubes of homotopy cofibers associated to \(Z\), \(\mathcal{X}\), \(\mathcal{Y}\), respectively. By definition the factorization \(C_Z = C_{\mathcal{X}}C_{\mathcal{Y}}\) holds. This allows us to reduce to the case \(n = 1\). Now let \(f : A \to B\) and \(g : B \to C\) be CDGA maps such that \(\text{con}(f) = r\) and \(\text{con}(g) = s\), then \(g \circ f\) is \(\min\{r, s\}\)-connected, which follows directly from the definition of connectivity.

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Lemma 5.2.5. Let \( \mathcal{Z} \) be an \( n \)-cube of CDGAs which is regarded as a map \( A \rightarrow B \) of two \( (n - 1) \)-cubes. If \( A \) is \( r \)-cocartesian and \( B \) is \( s \)-cocartesian then \( \mathcal{Z} \) is \( \text{min} \{ r - 1, s - 1 \} \)-cocartesian.

Proof. We first reduce to the case \( n = 2 \). Assume that \( A = \partial_{(n)} \mathcal{Z}, B = \partial_{\emptyset}^{n-1} \mathcal{Z} \). By definition \( A \) is \( r \)-cocartesian if and only if the map \( \text{hocolim} \mathcal{Z}((P_0(n - 1)) \cup \{n\}) \rightarrow Z_n \) is \( r \)-connected, and \( B \) is \( s \)-cocartesian if and only if the map \( \text{hocolim} \mathcal{Z}(P_0(n - 1)) \rightarrow Z_0 \) is \( s \)-connected.

If the case \( n = 2 \) is settled, then it will imply that the diagram

\[
\begin{array}{ccc}
\text{hocolim} \mathcal{Z}(P_0(n - 1)) \cup \{n\} & \longrightarrow & Z_n \\
\downarrow & & \downarrow \\
\text{hocolim} \mathcal{Z}(P_0(n - 1)) & \longrightarrow & Z_0
\end{array}
\]

is \( \text{min} \{ r - 1, s - 1 \} \)-cocartesian. By Proposition 5.2.3 this means that the map

\[
\text{hocofib}(\text{hocolim} \mathcal{Z}(P_0(n - 1)) \cup \{n\}) \rightarrow \text{hocolim} \mathcal{Z}(P_0(n - 1))) \rightarrow \text{hocofib}(Z_n \rightarrow Z_0)
\]

is \( \text{min} \{ r - 1, s - 1 \} \)-connected. But this is true if and only if the diagram of homotopy cofibers associated to \( \mathcal{Z} \) is \( \text{min} \{ r - 1, s - 1 \} \)-cocartesian, and hence, by Proposition 5.2.3, if and only if \( \mathcal{Z} \) is itself \( \text{min} \{ r - 1, s - 1 \} \)-cocartesian.

Now we give the proof of the case \( n = 2 \). Assume that \( \mathcal{Z} \) is cofibrant. Then it is of the form:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \otimes \Lambda V \\
\downarrow & & \downarrow \\
A \otimes \Lambda U & \xrightarrow{g} & A \otimes \Lambda U \otimes \Lambda V \otimes \Lambda W
\end{array}
\]
We are to show that if \( f \) is \( r \)-connected and if \( g \) is \( s \)-connected then the diagram is \( \text{min} \{r - 1, s - 1\} \)-cocartesian. By Proposition 5.2.3 it is enough to show that \( \text{hocofib}(f) \rightarrow \text{hocofib}(g) \), i.e., \( \Lambda V \rightarrow \Lambda V \otimes \Lambda W \), is \( \text{min} \{r - 1, s - 1\} \)-connected. By the definition of connectivity \( \Lambda V \) is \( (r - 1) \)-connected, and \( \Lambda V \otimes \Lambda W \) is \( (s - 1) \)-connected. This forces \( H^i(\Lambda V) \rightarrow H^i(\Lambda V \otimes \Lambda W) \) to be a (trivial) isomorphism whenever \( i \leq \text{min} \{r - 1, s - 1\} \). So \( \Lambda V \rightarrow \Lambda V \otimes \Lambda W \) is \( \text{min} \{r - 1, s - 1\} \)-connected.

Remark. We remark that there is a parallel result about the cartesian property. Let \( Z \) be an an \( n \)-cube of CDGAs which is a map \( A \rightarrow B \) of two \( (n - 1) \)-cubes. If \( A \) is \( r \)-cartesian and if \( B \) is \( s \)-cartesian then \( Z \) is \( \text{min} \{r - 1, s - 1\} \)-cartesian. We prefer to omit the proof since it is similar to that of 5.2.5.

The following concerns homotopy limits and plays a vital role in the proof of the main theorem.

Lemma 5.2.6. Let \( A \) be an \( n \)-cube of CDGAs, and let \( S_1, S_2 \) be two convex subsets of \( P(n) \). Then the following diagram is a homotopy pullback:

\[
\begin{array}{ccc}
\text{holim} A(S_1 \cup S_2) & \longrightarrow & \text{holim} A(S_1) \\
\downarrow & & \downarrow \\
\text{holim} A(S_2) & \longrightarrow & \text{holim} A(S_1 \cap S_2)
\end{array}
\]

Proof. Assume that \( A \) is fibrant. Then our goal is to see that \( \text{lim} A(S_1 \cup S_2) \) and \( \text{lim}(\text{lim} A(S_1) \rightarrow \text{lim} A(S_1 \cap S_2) \leftarrow \text{lim} A(S_2)) \) are the same.

We begin by pointing out that if \( S \) is a convex subset of \( n \) and if \( S_1, \ldots, S_k \) are all the maximal elements of \( S \) then \( \text{lim} A(S) \) is a subset of \( \oplus_{i=1}^k A_{S_i} \). An element \((a_i)\)
is in the subset if and only if for any $S \in S$ and any $S_j, S_l$ (1 ≤ j, l ≤ k) containing S as a subset, we have $f^{S_j}_S(a_j) = f^{S_l}_S(a_l) \in S$. (Here for $T' \subset T \subset \mathbb{N}$ we denote by $f_T$ the map $A_T \rightarrow A_{T'}$ in the cube $A$.)

Now let $M = \{M_1, \ldots, M_j\}$ be the collection of all maximal elements in $S_1$, and let $N = \{N_1, \ldots, N_l\}$ be the collection of all maximal elements in $S_2$.

On one hand, by definition $\lim A(S_1 \cup S_2)$ is a subset of $\bigoplus_{T \in M \cup N} A_T$ such that an element $(a_T)$ is in the subset if and only if for any $S \in S_1 \cup S_2$ and any $T, T' \in M \cup N$ containing $S$, we have $f^{T'}_S(a_T) = f^{T'}_S(a_T)$.

On the other hand, $\lim(\lim A(S_1) \rightarrow \lim A(S_1 \cap S_2) \leftarrow \lim A(S_2))$ is a subset of $\bigoplus_{T \in M \cup N} A_T$. First of all, note that if $M_i = N_k$ for some $1 \leq i \leq j$, $1 \leq k \leq l$ then we have $a_{M_i} = f^{M_i}_{M_i}(a_{M_i}) = f^{N_k}_{N_k}(a_{N_k}) = a_{N_k}$. So we learn that $\lim(\lim A(S_1) \rightarrow \lim A(S_1 \cap S_2) \leftarrow \lim A(S_2))$ is actually a subset of $\bigoplus_{T \in M \cup N} A_T$. Secondly, if an element $(a_T)$ is in the subset, then the condition it has to satisfy is exactly the same as the condition for an element in $\lim A(S_1 \cup S_2)$ has to satisfy.

This finishes the proof.

In fact, the above lemma can be generalized. We will also need this generalized version for the proof of the main theorem.

**Lemma 5.2.7.** Let $A$ be an n-cube, and let $S_1, \ldots, S_m$ be a finite collection of convex subsets of $P(n)$. Define an m-cube $L$ by setting $L_T = \lim A(\bigcap_{i \in m-T} S_i)$ for $T \subseteq m$. 

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and \( L_m = A(\bigcup_{i=1}^m S_i) \). Then \( \mathcal{L} \) is homotopy cartesian.

\( \square \)

**Proof.** By induction on \( m \). The ground case \( m = 2 \) is done in Lemma 5.2.6.

**Lemma 5.2.8.** Let \( A \) be an \( n \)-cube of CDGAs. Suppose that for each subset \( T \subset \{1, \cdots, n\} \), the map \( A_T \rightarrow \text{holim} A(P_1(T)) \) is \( r_T \)-connected. Now let \( S \) be any convex subset of \( P(n) \) and let \( S \) be a maximal element of \( S \). Then the map \( \text{holim} A(S) \rightarrow \text{holim} A(S - S) \) is \( r_S \)-connected.

\( \square \)

**Proof.** This result is an application of Lemma 5.2.6. In fact, let’s take two convex subsets \( S_1 = S - S \) and \( S_2 = P(S) \). Note that \( S_1 \cup S_2 = S \), and that \( S_1 \cap S_2 = P_1(S) \).

By Lemma 5.2.6 the following diagram is a homotopy pullback:

\[
\begin{array}{ccc}
\text{holim} A(S) & \longrightarrow & \text{holim} A(P(S)) \\
\downarrow & & \downarrow \\
\text{holim} A(S - S) & \longrightarrow & \text{holim} A(P_1(S))
\end{array}
\]

The right map \( \text{holim} A(P(S)) \rightarrow \text{holim} A(P_1(S)) = A_S \) is \( r_S \)-connected, by definition.

Therefore the left map \( \text{holim} A(S) \rightarrow \text{holim} A(S - S) \) is also \( r_S \)-connected.

\( \square \)

**5.3 The Algebraic Step**

This section is devoted to the proof of Proposition 5.3.1. From now on, by an \( n \)-cube of CDGAs we will always mean an \( n \)-cube of simply connected CDGAs, unless otherwise stated.
Proposition 5.3.1. Let $\mathcal{A}$ be a strongly homotopy cartesian $n$-cube of CDGAs, and let $\mathcal{C}$ be its associated diagram of homotopy cofibers. Suppose that for each $i \in \{1, \cdots, n\}$, the map $A_i \to A_0$ is $r_i$-connected. Then $\mathcal{C}$ is $((\sum_{i=1}^{n} r_i) - 1)$-cartesian.

The proof uses the bar construction. Here we present a brief summary of the knowledge needed. For a more detailed exposition see [15, Chapter 19].

The bar construction.

The bar construction, in its essence, is a functor from the category of augmented differential graded algebras over $k$ to that of co-augmented differential graded coalgebras over $k$.

Let $V$ be a graded $k$-module. We associate to $V$ a new graded $k$-module $TV$, as follows. We let $TV := \bigoplus_{i=0}^{\infty} T^i(V)$ where $T^0(V) = k$, $T^i(V) = \underbrace{V \otimes \cdots \otimes V}_{i \text{-times}}$, $i \geq 1$. Here the tensor products are over $k$. There is a canonical algebra structure and a coalgebra structure on $TV$. Here it is the coalgebra structure that we care about. To give such a structure we need to give the diagonal map $\Delta : TV \to TV \otimes TV$, and it is enough to specify it on basic tensors. The convention is to write a basic tensor in the form $[v_1|\cdots|v_m]$, instead of $v_1 \otimes \cdots v_m$. (This follows [Chapter 19, Félix-Halperin-Thomas]. However in Chapter 6 of this paper we use the second notation, as no possible confusion can occur there.) We let:

$$\Delta : [v_1|\cdots|v_m] \mapsto [v_1|\cdots|v_m] \otimes 1 + \sum_{j=1}^{m-1} [v_1|\cdots|v_j] \otimes [v_{j+1}|\cdots|v_m] + 1 \otimes [v_1|\cdots|v_m].$$
Note that $TV$ is co-augmented. A co-augmentation can be given by including $k$ as $T^0(V)$ in $TV$.

The bar construction of $(A, d_A)$ is the co-augmented differential graded coalgebra $(BA, d)$, where $BA = T(sA)$ as co-augmented graded coalgebra, and the differential $d$ is the sum $d_0 + d_1$ of two coderivations:

$$d_0 : [sa_1 | \cdots | sa_m] \mapsto \sum_{j=1}^{m} (-1)^{n_j+1} [sa_1 | \cdots | sd_A(a_j) | \cdots | sa_m];$$

and

$$d_1 : [sa] \mapsto 0, \ [sa_1 | \cdots | sa_m] \mapsto \sum_{j=2}^{m} (-1)^{n_j} [sa_1 | \cdots | sa_{j-1}a_j | \cdots | sa_m].$$

Here the number $n_j$ is defined to be $\sum_{i<j} \deg(sa_i)$. We write $B^i A := T^i(sA)$ for elements in $BA$ of bar length $i$, and $B^{\geq i} A$ for those of bar length at least $i$.

Note that any map $f : A \to A'$ between CDGAs induces a map $B(f) : BA \to BA'$ on their bar constructions, in such a way that $B(f) : [sa_1 | \cdots | sa_m] \mapsto [sf(a_1) | \cdots | sf(a_m)].$

This finishes the definition of the bar construction functor.

Next we define the bar construction of a CDGA $(A, d_A)$ with coefficients in some $A$-dg module $(M, d_M)$. As a chain complex, this is $B(A; M) := (BA \otimes M, d)$, whose differential $d = d_0 + d_1$ is again the sum of two parts, where:

$$d_0 : [sa_1 | \cdots | sa_m] \otimes x \mapsto \sum_{j=1}^{m} (-1)^{n_j+1} [sa_1 | \cdots | sd_A(a_j) | \cdots | sa_m] \otimes x$$

$$+ (-1)^{n_{m+1}+1} [sa_1 | \cdots | sa_m] \otimes d_M x,$$
and

\[ d_1 : [sa_1] \cdots [sa_m] \otimes x \mapsto \sum_{j=2}^{m} (-1)^{n_j} [sa_1] \cdots [sa_{j-1}a_j] \cdots [sa_m] \otimes x \]

\[ + (-1)^{n_{m+1}} [sa_1] \cdots [sa_{m-1}] \otimes a_m \cdot x. \]

(Also \(d_0 x = d_M x, d_1 x = 0\), and \(d_1([sa] \otimes x) = (-1)^{|a|+1} a \cdot m.\)) Note that there is a natural \(A\)-multiplication on the right, making \(B(A; M)\) a right \(A\)-dg module.

We finish our introduction of the bar construction by pointing out the following important property of \(B(A; A)\). Here temporarily we require that \(k\) is a field.

**Lemma 5.3.2.** The map \(\epsilon \otimes \epsilon : B(A; A) = BA \otimes A \rightarrow (k, 0)\) induced by the augmentations is a quasi-isomorphism. Furthermore \(B(A; A)\) is a semi-free \(A\)-dg module.

**Proof.** [15, Proposition 19.2].

The importance of the above lemma is that it provides a way to replace the map \((A, d_A) \rightarrow (k, 0)\) by a cofibration, in the category \(DGM_A\).

We also need some other preparatory lemmas for the proof of Proposition 5.3.1.

**Lemma 5.3.3.** Let \(V, W\) be graded vector spaces over \(k\), and let \(V_1 \subset V, W_1 \subset W\) be subspaces. Then we have a canonical isomorphism:

\[ V \otimes W/(V_1 \otimes W + V \otimes W_1) \cong V/V_1 \otimes W/W_1. \]

**Proof.** We construct a map \(f : V/V_1 \otimes W/W_1 \rightarrow V \otimes W/(V_1 \otimes W + V \otimes W_1)\) as follows. On a basic tensor \((x + V_1) \otimes (y + W_1)\) we let \(f((x + V_1) \otimes (y + W_1)) = [x \otimes y].\)
(Here \([x \otimes y]\) denotes the class of \(x \otimes y\) in the quotient.) Note that this is well defined. Extend bilinearly.

Also define \(g : V \otimes W/(V_1 \otimes W + V \otimes W_1) \to V/V_1 \otimes W/W_1\) as follows. On the class \([x \otimes y]\) represented by a basic tensor \(x \otimes y \in V \otimes W\), we set \(g([x \otimes y]) = (x + V_1) \otimes (y + W_1)\). Note that this is well-defined. Extend bilinearly.

It is easy to check that \(g \circ f = f \circ g = id\).

\[\Box\]

**Corollary 5.3.4.** Let \(V\) be a graded vector space, and \(V_1 \subset V\) a subspace. Then:

\[
\ker(T^n(V) \to T^n(V/V_1)) \cong \bigoplus_{j=0}^{n-1} T^j(V) \otimes V_1 \otimes T^{n-1-j}(V).
\]

**Proof.** By Lemma 5.3.3 and induction on \(n\).

\[\Box\]

**Lemma 5.3.5.** Let \(A\) be a fibrant \(n\)-cube in \(Ch_{\geq 0}(k)\). Then we have:

\[
\ker(A_n \to \lim A(P_1(n))) = \bigcap_{i=1}^n \ker(A_n \to A_{n-(i)}).
\]

**Proof.** Denote by \(g_i\) the map \(A_n \to A_{n-(i)}\), and by \(g\) the map \(A_n \to \lim A(P_1(n))\). Note that \(\lim A(P_1(n))\) is a subset of \(\bigoplus_{i=1}^n A_{n-(i)}\) and that \(g = (g_1, \ldots, g_n)\). We see that \(g(x) = 0\) for an element \(x \in A_n\) precisely when \(g_i(x) = 0\) for all \(i\).

\[\Box\]

**Lemma 5.3.6.** Let \(A\) be a strongly cartesian \(n\)-cube in \(CDGA_k\). We associate to \(A\)
a new $n$-cube $BA$, as follows:

$$T \mapsto BA_T, \ T \subset n.$$ 

Here $BA_T$ is the bar construction of $A_T$. Suppose that $A_i \xrightarrow{f_i} A_0$ is $r_i$-connected, $i = 1, \ldots, n$. Then $BA$ is $((\sum_{i=1}^n r_i) - 1)$-cartesian.

Proof. Assume that $A$ is fibrant. The goal is to show that the map $BA_n \xrightarrow{p} \lim BA(P_1(n))$ is $((\sum_{i=1}^n r_i) - 1)$-connected. That is, in Ker($p$) the smallest possible degree of a nonzero element is $\sum_{i=1}^n r_i$. For $i = 1, \ldots, n$, we denote by $p_{-i}$ be the projection map:

$$A_1 \oplus A_0 \cdots \oplus A_0 A_n \rightarrow A_1 \oplus A_0 \cdots \oplus A_0 \overset{\sim}{A_i} \cdots \oplus A_0 A_n.$$

Here the notation $\overset{\sim}{A_i}$ means that $A_i$ does not appear as a summand. We write $Bp_{-i}$ for the induced map:

$$B(A_1 \oplus A_0 \cdots \oplus A_0 A_n) \rightarrow B(A_1 \oplus A_0 \cdots \oplus A_0 \overset{\sim}{A_i} \cdots \oplus A_0 A_n).$$

Also we write $B^k p_{-i}$ for the induced map:

$$B^k(A_1 \oplus A_0 \cdots \oplus A_0 A_n) \rightarrow B^k(A_1 \oplus A_0 \cdots \oplus A_0 \overset{\sim}{A_i} \cdots \oplus A_0 A_n).$$

(Recall that $B^k A = T^k(sA)$ denotes the elements in $BA$ of bar length $k$.)

By Lemma 5.3.5 we know that $\text{Ker}(p) = \bigcap_{i=1}^n \text{Ker}(Bp_{-i})$. Furthermore, we note that $\text{Ker}(Bp_{-i}) = \bigoplus_{k \geq 1} \text{Ker}(B^k p_{-i})$, since $Bp_{-i}$ preserves bar length. We let $V := s(A_1 \oplus A_0 \cdots \oplus A_0 A_n)$, and let $V_i := s(A_1 \oplus A_0 \cdots \oplus A_0 \overset{\sim}{A_i} \cdots \oplus A_0 A_n)$, for $i = 1, \ldots, n$. Then $B^k p_{-i}$ is the map $T^k(V) \rightarrow T^k(V_i)$ induced by $p_{-i}$. 
Let’s investigate $\text{Ker}(B^k p_{-i})$. First observe that $\text{Ker}(p_{-i}) \cong \text{Ker}(A_i \to A_0)$. Indeed, recall that $A_1 \oplus A_0 \cdots \oplus A_0 A_n$ is the subset of $A_1 \oplus \cdots \oplus A_n$ containing elements $(a_1, \ldots, a_n)$ such that $f_1(a_1) = \cdots = f_n(a_n) \in A_0$. Since $p_{-i}(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1} \cdots, a_n)$ we know that $(a_1, \ldots, a_n) \in \text{Ker}(p_{-i})$ if and only if $a_1 = \cdots = a_{i-1} = a_{i+1} \cdots = a_n = 0$, and $f_i(a_i) = 0$. Hence $\text{Ker}(p_{-i}) \cong \text{Ker}(A_i \to A_0)$.

For simplicity we write $K_i := \text{Ker}(p_{-i})$, $i = 1, \ldots, n$. It then follows that $\text{Ker}(V \to V_i) = sK_i$. Now by Corollary 5.3.4, we have

$$\text{Ker}(B^k p_{-i}) \cong \bigoplus_{j=0}^{k-1} T^j(V) \otimes sK_i \otimes T^{k-1-j}(V).$$

To sum up, we have obtained that

$$\text{Ker}(p) = \bigcap_{i=1}^{n} \text{Ker}(Bp_{-i}) = \bigcap_{i=1}^{n} \bigoplus_{k \geq 1} \text{Ker}(B^k p_{-i}) = \bigoplus_{k \geq 1} \bigcap_{i=1}^{n} \text{Ker}(B^k p_{-i})$$

$$= \bigoplus_{k \geq 1} \bigcap_{i=1}^{n} \sum_{j=0}^{k-1} T^j(V) \otimes sK_i \otimes T^{k-1-j}(V).$$

We claim that $\bigcap_{i=1}^{n} \sum_{j=0}^{k-1} T^j(V) \otimes sK_i \otimes T^{k-1-j}(V) = 0$, when $k < n$; and when $k = n$, it is equal to

$$\sum_{(t_1, \ldots, t_n) \text{ is a permutation of } (1, \ldots, n)} sK_{t_1} \otimes \cdots \otimes sK_{t_n}.$$  

To verify this, let’s first observe that $sK_i \cap \sum_{1 \leq j \leq n, j \neq i} sK_j = 0$ for $i = 1, \ldots, n$. Indeed, by definition if an element $(sa_1, \ldots, sa_n) \in V$ is in $sK_i$ then only $sa_i$ can be nonzero. But any element in the sum $\sum_{1 \leq j \leq n, j \neq i} sK_j$ must have zero as its $i$-th coordinate. It then follows that the sum $\sum_{i=1}^{n} sK_i$ is a direct sum. Hence there exists a subspace
Let $B := \{ v^{(i)}_a \}_{a \in I_i}$ be a basis for $sK_i$, $i = 1, \ldots, n$, and let $B^{(0)} = \{ v^{(0)}_\beta \}_{\beta \in I_0}$ be a basis for $V'$. Then $B = B^{(0)} \cup B^{(1)} \cup \cdots \cup B^{(n)}$ is a basis for $V$. We then see that

$$\{ v^{(t_1)}_{\alpha_1} \otimes \cdots \otimes v^{(t_k)}_{\alpha_k} : 0 \leq t_1, \ldots, t_k \leq n, (\alpha_1, \ldots, \alpha_k) \in I_{t_1} \times \cdots \times I_{t_k} \}$$

is a basis for $T^k(V)$. Furthermore, note that $\mathcal{V} := \{ v^{(t_1)}_{\alpha_1} \otimes \cdots \otimes v^{(t_k)}_{\alpha_k} : 0 \leq t_1, \ldots, t_k \leq n, (\alpha_1, \ldots, \alpha_k) \in I_{t_1} \times \cdots \times I_{t_k}, \exists l \text{ such that } \alpha_l \in I_i \}$ is a basis for $\bigcup_{j=1}^{k-1} T^j(V) \otimes sK_i \otimes T^{k-1-j}(V)$, $i = 1, \ldots, n$. It follows that $\mathcal{V} := \bigcap_{i=1}^n \mathcal{V}^{(i)}$ gives a basis for $\bigcap_{i=1}^n \bigcup_{j=1}^{k-1} T^j(V) \otimes sK_i \otimes T^{k-1-j}(V)$. Note that an element $v^{(t_1)}_{\alpha_1} \otimes \cdots \otimes v^{(t_k)}_{\alpha_k}$ is in $\mathcal{V}$ should satisfy the following property - at least one of the $\alpha_l$’s must be in $I_i$, $\forall i : 1 \leq i \leq n$. Now if $k < n$ then by the pigeonhole principle there is at least one $l$, $1 \leq l \leq k$, such that there exist $i, i' : 1 \leq i \neq i' \leq n$ with $\alpha_l \in I_i \cap I_{i'}$. But since $sK_i \cap sK_{i'} = 0$ as $i \neq i'$, no element $v^{(t_1)}_{\alpha_1} \otimes \cdots \otimes v^{(t_k)}_{\alpha_k}$ can have that property. So $\mathcal{V} = \emptyset$ and hence $\bigcap_{i=1}^n \bigcup_{j=1}^{k-1} T^j(V) \otimes sK_i \otimes T^{k-1-j}(V) = 0$. However when $k = n$, $v^{(t_1)}_{\alpha_1} \otimes \cdots \otimes v^{(t_k)}_{\alpha_k} \in \mathcal{V}$ if and only if $(t_1, \ldots, t_n)$ is a permutation of $(1, \ldots, n)$. The result follows.

Now that the claim is verified, we see that an element of the least positive degree in $sK_1 \otimes \cdots \otimes sK_n$ is also of the least positive degree in $\text{Ker}(p)$.

Let $x_i$ be a nonzero element of $K_i$ having the least degree. Since $K_i \cong \text{Ker}(A_i \rightarrow A_0)$ and $\text{con}(A_i \rightarrow A_0) = r_i$ we know that $\deg(x_i) = r_i + 1$, and hence $\deg(sx_i) = r_i$. The element $sx_1 \otimes \cdots \otimes sx_n$ is an element of the least positive degree in $sK_1 \otimes \cdots \otimes sK_n$, \[86\]
and hence in $Ker(p)$. We see that $deg(sx_1 \otimes \cdots \otimes sx_n) = \sum_{i=1}^n deg(sx_i) = \sum_{i=1}^n r_i$, as desired. \hfill \Box

Now we are ready to prove Proposition 5.3.1.

**Proof of Proposition 5.3.1.**

*Proof.* Assume that $\mathcal{A}$ is fibrant. Also by Lemma 5.1.2 one could assume that $\mathcal{A}$ has the property that the connectivity of $(A_i', d_{\mathcal{A}}') \to (A_0', d_{\mathcal{A}}')$ is equal to the connectivity of $(A_i', 0) \to (A_0', 0)$, $i = 1, \ldots, n$.

We now describe the cube $C$ of homotopy cofibers associated to $\mathcal{A}$. For $T \subset \mathfrak{n-1}$, we have

$$C_T = hocolim(A_T \cup \{n\} \to A_T)$$

$$= colim(A' \leftarrow A_T \cup \{n\} \to A_T)$$

$$\cong A' \otimes_{A_T \cup \{n\}} A_T.$$  

Here $A_T \cup \{n\} \to A'$ is some cofibration that replaces $A_T \cup \{n\} \to (k, 0)$. Recall that the bar construction provides a way to resolve $(k, 0)$ as a semi-free $(A, d)$-module (Lemma 5.3.2). Then by Lemma 2.2.2 we further have

$$C_T \cong A' \otimes_{A_T \cup \{n\}} A_T \cong B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T.$$  

as $A_T \cup \{n\}$-dg modules, and hence, as chain complexes. Therefore in the category $Ch(k)$, $C$ is equivalent to the cube $T \mapsto B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T$. We denote
this new cube by $C'$. Recall (the remark following Theorem 5.1.1) that a cube is $r$-cartesian in $CDGA_k$ if and only if it is $r$-cartesian in $Ch(k)$. It is therefore enough to show that $C'$ is $((\sum_{i=1}^n r_i) - 1)$-cartesian.

Consider a new $(n-1)$ cube $C''$: $T \mapsto B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T$. There is a natural map $C'' \to C'$ of $(n-1)$-cubes, given by $B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T \to B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T$. We regard this map $C'' \to C'$ as an $n$-cube (in the category $Ch(k)$). Let $F$ denote the associated diagram of homotopy fibers of this $n$-cube. That is,

$$F_T = hofib(C''_T \to C'_T)$$

$$= holim(C'_T \leftarrow C''_T \to 0)$$

$$\cong Ker(B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T \to B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} A_T)$$

$$= B(A_T \cup \{n\}; A_T \cup \{n\}) \otimes_{A_T \cup \{n\}} Ker(A_T \to A_T).$$

Here the last equality holds because we are working over a field.

Note that $C''$ is acyclic at each corner. It is obvious from the definition that such a cube must be homotopy cartesian. Then it is enough to show that $F$ is $((\sum_{i=1}^n r_i) - 2)$-cartesian. (Recall the remark following Lemma 5.2.5.) Furthermore, it is enough to show that $F$ is $((\sum_{i=1}^n r_i) - 2)$-cartesian in the category of graded vector spaces.
(Again, recall the remark following Theorem 5.1.1.) In that category, we have

\[ F_T = B(A_{T \cup \{n\}}; A_{T \cup \{n\}}) \otimes_{A_{T \cup \{n\}}} \text{Ker}(A_{T \cup \{n\}} \to A_T) \]

\[ \cong (B A_{T \cup \{n\}} \otimes A_{T \cup \{n\}}) \otimes_{A_{T \cup \{n\}}} \text{Ker}(A_{T \cup \{n\}} \to A_T) \]

\[ \cong B A_{T \cup \{n\}} \otimes \text{Ker}(A_{T \cup \{n\}} \to A_T). \]

By our assumption, \( \text{Ker}(A_{T \cup \{n\}} \to A_T) \cong \text{Ker}(A_{T' \cup \{n\}} \to A_{T'}) \), for any \( T, T' \subset n-1 \).

Now we see that it is enough to show that the cube \( T \mapsto B A_{T \cup \{n\}} \) is \( (\sum_{i=1}^{n-1} r_i - 2) \)-cartesian. But this is guaranteed by Lemma 5.3.6. \( \square \)
Chapter 6

Main Results

This chapter is devoted to the proof of the main results of the thesis. In this chapter we will continue to work over a field $k$ of characteristic zero, and continue to assume that an $n$-cube of CDGAs is an $n$-cube of simply connected CDGAs.

6.1 The Statement and the Outline of Proof

The following theorems are the main results of the thesis. Here $n \in \mathbb{Z}_{>0}$.

**Theorem 6.1.1.** *(The Blakers-Massey Theorem for $n$-cubes of CDGAs)* Let $\mathcal{A}$ be a strongly homotopy cartesian $n$-cube of CDGAs.

Suppose that for each $i \in \{1, \cdots, n\}$, the map $A_i \to A_0$ is $r_i$-connected. Then $\mathcal{A}$ is $((\sum_{i=1}^n r_i) - (n - 1))$-cocartesian.

**Theorem 6.1.2.** *(Generalized Blakers-Massey Theorem for $n$-cubes of CDGAs)* Let $\mathcal{A}$ be an $n$-cube of CDGAs.
Suppose that for any nonempty subset $T \subset \{1, \ldots, n\}$, the fundamental face $A^T$ determined by $T$ is $r_T$-cartesian, and that $r_{T'} \leq r_T$ whenever $T' \subset T$. Let

$$r = \min \left\{ \sum_j r_{T_j} : \{T_j\} \text{ is a partition of } \{1, \ldots, n\} \right\}.$$ 

Then $A$ is $(r - (n - 1))$-cocartesian.

Outline of the proof.

We would like to first give a summary of the proof.

We will use induction. When $n = 1$ both theorems are trivially true. Indeed, an 1-cube is nothing but a map $f : A_1 \to A_0$. Without loss of generality assume that $A_1$ is cofibrant. Then by definition $\text{holim}(f) = A_0$ and $\text{hocolim}(f) = A_1$, and therefore both the canonical maps $A_1 \to \text{holim}(f)$ and $\text{hocolim}(f) \to A_0$ are equal to $f$ itself.

Now both theorems can be verified easily.

Also note that the case $n = 2$ has already been settled in Chapter 4. In fact, Theorem 4.2.1 is a special case of Theorem 6.1.1 and Theorem 4.3.4 is a special case of Theorem 6.1.2. Below is a different proof of Theorem 6.1.1 when $n = 2$, using the bar construction.

**Proof of Theorem 6.1.1 for $n = 2$.**

Proof. We are to show that if the following square $\mathcal{A}$:

\[
\begin{array}{ccc}
A_{12} & \xrightarrow{f_{12}^2} & A_2 \\
\downarrow{f_{12}^1} & & \downarrow{f_2} \\
A_1 & \xrightarrow{f_1} & A_0
\end{array}
\]
is a homotopy pullback and if \( f_i \) is \( r_i \)-connected, \( i = 1, 2 \), then \( \mathcal{A} \) is \( (r_1 + r_2 - 1) \)-cocartesian.

By Proposition 5.2.3 it is enough to show that the associated diagram \( \mathcal{C} \) of homotopy cofibers is \( (r_1 + r_2 - 1) \)-cocartesian. That is, the map \( hcofib(f_1^{12}) \to hcofib(f_2) \) is \( (r_1 + r_2 - 1) \)-connected. But it follows from Proposition 5.3.1 that \( \mathcal{C} \) is \( (r_1 + r_2 - 1) \)-cartesian, which means exactly that above map is \( (r_1 + r_2 - 1) \)-connected. \qed

The above argument, though very simple, demonstrates well the philosophy of our proof. To learn the cocartesian property of an \( n \)-cube, one investigates the cocartesian property of its associated \( (n - 1) \)-cube of homotopy cofibers. For this purpose one should first study the cartesian property of this \( (n - 1) \)-cube and then apply Theorem 6.1.2. Finally one finds that the desired cartesian property is provided by Proposition 5.3.1.

So we proceed as follows. We show that if both theorems are true for cubes of dimensions \( \leq n - 1 \), then Theorem 6.1.1 holds for \( n \)-cubes \( (n \geq 2) \). Next we show that if Theorem 6.1.1 holds for cubes of dimension \( \leq n \) and if Theorem 6.1.2 holds for cubes of dimension \( \leq n - 1 \), then Theorem 6.1.2 holds for dimension \( n \) \( (n \geq 2) \).

We remark that this formal argument is inspired by and is parallel to that in [22, Section 6.3].

**Theorem 6.1.2 for dimension \( n-1 \) \( \Rightarrow \) Theorem 6.1.1 for dimension \( n \).**
Proof. Let \( C \) be the diagram of homotopy cofibers associated to \( A \). By Proposition 5.2.3 it suffices to show that \( C \) is \(((\sum_{i=1}^{n} r_i) - (n - 1))\)-cocartesian.

By Proposition 5.3.1 we know that \( C \) is \(((\sum_{i=1}^{n} r_i) - 1)\)-cartesian. Consider an arbitrary subset \( T \subset \{1, \cdots, n - 1\} \). Note that the fundamental face \( A^T \) determined by \( T \) is also strongly homotopy cartesian, and that its diagram of homotopy cofibers is nothing but the fundamental face \( C^T \) of \( C \). Now we can apply Proposition 5.3.1 to \( A^T \). This gives that \( C^T \) is \( r_T := ((\sum_{i \in T} r_i) + r_n - 1)\)-cartesian.

Note that we have \( r_T' \leq r_T \) whenever \( T' \subset T \subset \{1, \cdots, n - 1\} \), since the \( r_i \)'s are always positive. Now we can apply Theorem 6.1.2 to \( C \). This gives that \( C \) is \((r - (n - 2))\)-cocartesian, where

\[
r = \min \left\{ \sum_{j} r_{T_j} : \{T_j\} \text{ is a partition of } \{1, \cdots, n - 1\} \right\}.
\]

Note that if \( \{T_j\}_{j \in J} \) is a partition of \( \{1, \cdots, n - 1\} \) such that the cardinality of \( J \) is \( s \), then

\[
\sum_{j \in J} r_{T_j} = \left( \sum_{i=1}^{n-1} r_i \right) + sr_n - s.
\]

Since \( r_n \geq 1 \) we have

\[
\left( \sum_{i=1}^{n-1} r_i \right) + sr_n - s \geq \left( \sum_{i=1}^{n} r_i \right) - 1.
\]

So we have \( r = (\sum_{i=1}^{n} r_i) - 1 \). It then follows that \( C \) is \(((\sum_{i=1}^{n} r_i) - 1 - (n - 2))\)-cocartesian, i.e., \(((\sum_{i=1}^{n} r_i) - (n - 1))\)-cocartesian, as desired.

\[\Box\]
Theorem 6.1.1 for dimension $n \Rightarrow$ Theorem 6.1.2 for dimension $n$.

Proof. Assume that $\mathcal{A}$ is fibrant. Then it follows from the conditions that for each $T \in P(n)$, the map $A_T \to \lim A(P_1(T))$ is $r_T$-connected.

The idea is to factorize $\mathcal{A}$ into a composition of $n$-cubes such that for each factor we know how cocartesian it is.

In fact, let $S_1 \subset S_2 \subset \ldots \subset S_m$ be an increasing sequence of convex subsets of $P(n)$ such that:

- $S_1 = \{\emptyset, \{1\}, \ldots, \{n\}\}$;
- $S_m = P(n)$;
- $S_i - S_{i-1} = \{S_i\}$, a single subset of $P(n)$;
- The cardinality of $S_i$ is non-decreasing in $i$.

We claim that there exist $n$-cubes $Z_i$, $2 \leq i \leq m$, such that $A_{S_i} = A_{S_{i-1}} Z_i$, so that we have the factorization $\mathcal{A} = A_{S_1} Z_2 \cdots Z_m$.

The cubes $Z_i$ are constructed as follows. Suppose that $S_i = \{a_1, \ldots, a_k\}$, with $a_1 < \cdots < a_k$. Let

$$Z_i(T) = \begin{cases} A_{S_{i-1}}(T \cup \{a_1\}), & \text{if } a_1 \notin T; \\ A_{S_i}(T), & \text{if } a_1 \in T. \end{cases}$$

Then the factorization $A_{S_i} = A_{S_{i-1}} Z_i$ follows. Indeed, note that these cubes can be written as maps:
\( \mathcal{A}_s = (\partial^n_{\{a_1\}} \mathcal{A}_s \to \partial^{n-\{a_1\}}_0 \mathcal{A}_s), \)

\( \mathcal{A}_{s_{i-1}} = (\partial^n_{\{a_1\}} \mathcal{A}_{s_{i-1}} \to \partial^{n-\{a_1\}}_0 \mathcal{A}_{s_{i-1}}), \)

\( Z_i = (\partial^n_{\{a_1\}} Z_i \to \partial^{n-\{a_1\}}_0 Z_i). \)

By definition, we have

\[ \partial^n_{\{a_1\}} Z_i = \partial^n_{\{a_1\}} \mathcal{A}_s, \]

\[ \partial^{n-\{a_1\}}_0 Z_i = \partial^{n-\{a_1\}}_0 \mathcal{A}_{s_{i-1}}. \]

Also note that for any \( T \subset n - \{a_1\}, \)

\[ \mathcal{A}_s(T) = \lim A(S_i \cap P(T)) = \lim A(S_{i-1} \cap P(T)) = \mathcal{A}_{s_{i-1}}(T) \]

so we have \( \partial^{n-\{a_1\}}_0 \mathcal{A}_s = \partial^{n-\{a_1\}}_0 \mathcal{A}_{s_{i-1}}. \) We therefore see that \( \mathcal{A}_{s_{i-1}} \) and \( Z_i \) are composable, with composition \( \mathcal{A}_s. \)

By definition the diagram \( \mathcal{A}_s \) is homotopy cartesian. It follows from Theorem 6.1.1 that this diagram is \((\sum^n_{i=1} r_i) - (n - 1)\)-cocartesian.

**Claim.** Let \( R_i = \min \left\{ \sum_j r_{T_j} \right\} \) where the minimum is taken over all partitions \( \{T_j\} \) of \( \{1, \cdots, n\} \) which contain \( S_i \) as an element, and any other element in the partition belongs to \( S_{i-1}. \) Then the diagram \( Z_i \) is \( R_i - (n-1)\)-cocartesian, \( 2 \leq i \leq m. \)

We are done once the claim is verified. Indeed, it will then follow from Lemma 5.2.4 that \( A \) is \( \min \left\{ (\sum^n_{i=1} r_i) - (n - 1), R_1 - (n - 1), \ldots, R_m - (n - 1) \right\} \)-cocartesian,
and this number is obviously no less than the number \((r - (n - 1))\), where
\[
   r = \min \left\{ \sum_j r_{T_j} : \{T_j\} \text{ is a partition of } \{1, \cdots, n\} \right\}.
\]
as in the statement of Theorem 6.1.2.

The above claim will be an immediate corollary of two facts, whose proofs are rather lengthy (although straightforward). Therefore we would like to state these facts as lemmas and prove them in the next section.

\[\square\]

6.2 The Last Lemmas

**Lemma 6.2.1.** Again write \(S_i = \{a_1, \ldots, a_k\}\), with \(a_1 < \cdots < a_k\). The cube \(Z_i\) is \(s\)-cocartesian, provided that \(\partial_{S_i - \{a_1\}}^n Z_i\) is \((s + k - 1)\)-cocartesian.

**Lemma 6.2.2.** The cube \(\partial_{S_i - \{a_1\}}^n Z_i\) is \((R_i + k - n)\)-cocartesian. Here \(R_i\) is the same as in the above proof.

**Verification of the last claim.**

**Proof.** Lemma 6.2.2 guarantees that \(\partial_{S_i - \{a_1\}}^n Z_i\) is \((R_i + k - n)\)-cocartesian. Now take \(s = R_i - (n - 1)\), and apply Lemma 6.2.1.

\[\square\]

**Proof of Lemma 6.2.1.**

**Proof.** The cube \(Z_i\) can be written as the following map of two \((n - 1)\)-cubes:
\[ Z_i = (\partial_{\{a_k\}}^n Z_i \to \partial_{\emptyset}^n \setminus \{a_k\} Z_i). \]

Again, the cube \( \partial_{\{a_k\}}^n Z_i \) is a map of \((n - 2)\)-cubes:

\[ \partial_{\{a_k\}}^n Z_i = (\partial_{\{a_{k-1}, a_k\}}^n Z_i \to \partial_{\{a_k\}}^{n-\{a_{k-1}\}} Z_i). \]

Continuing in a similar fashion, we obtain that for any \( j, 2 \leq j \leq k, \)

\[ \partial_{\{a_{j+1}, \ldots, a_k\}}^n Z_i = (\partial_{\{a_{j}, \ldots, a_k\}}^n Z_i \to \partial_{\{a_{j+1}, \ldots, a_k\}}^{n-\{a_j\}} Z_i). \]

Note that all the cubes \( \partial_{\{a_{j+1}, \ldots, a_k\}}^{n-\{a_j\}} Z_i \) are homotopy cocartesian. This is because

\[ \partial_{\{a_{j+1}, \ldots, a_k, a_1\}}^{n-\{a_j\}} Z_i = (\partial_{\{a_{j+1}, \ldots, a_k, a_1\}}^{n-\{a_j, a_1\}} Z_i) \]

is a map of identical cubes. To see that the two cubes \( \partial_{\{a_{j+1}, \ldots, a_k, a_1\}}^{n-\{a_j\}} Z_i \) and \( \partial_{\{a_{j+1}, \ldots, a_k\}}^{n-\{a_j, a_1\}} Z_i \) are indeed identical, note that for any \( T \subset n - \{a_j, a_{j+1}, \ldots, a_k, a_1\}, \)

\[ \partial_{\{a_{j+1}, \ldots, a_k, a_1\}}^{n-\{a_j\}} Z_i(T) = Z_i(T \cup \{a_{j+1}, \ldots, a_k, a_1\}) \]

\[ = \mathcal{A}_S(T \cup \{a_{j+1}, \ldots, a_k, a_1\}) \]

\[ = \lim \mathcal{A}(S \cap P(T \cup \{a_{j+1}, \ldots, a_k, a_1\})) \]

is equal to

\[ \partial_{\{a_{j+1}, \ldots, a_k\}}^{n-\{a_j, a_1\}} Z_i(T) = Z_i(T \cup \{a_{j+1}, \ldots, a_k\}) \]

\[ = \mathcal{A}_{S_{n-1}}(T \cup \{a_{j+1}, \ldots, a_k\} \cup \{a_1\}) \]

\[ = \lim \mathcal{A}(S_{n-1} \cap P(T \cup \{a_{j+1}, \ldots, a_k, a_1\})). \]
Indeed, recall that $S_i$ and $S_{i-1}$ differ by a single set $S_i = \{a_1, \ldots, a_k\}$.
Therefore $S_i \cap P(T \cup \{a_{j+1}, \ldots, a_k, a_1\}) = S_{i-1} \cap P(T \cup \{a_{j+1}, \ldots, a_k, a_1\})$ since $S_i \not\subset (T \cup \{a_{j+1}, \ldots, a_k, a_1\})$.

Now that $\partial_{n}^{-\{a_1\}}Z_i$ is homotopy cocartesian, and we have $\partial_{\{a_{j+1}, \ldots, a_k\}}^{n}Z_i = (\partial_{\{a_{j}, \ldots, a_k\}}^{n-\{a_1\}}Z_i \to \partial_{\{a_{j+1}, \ldots, a_k\}}^{n-\{a_1\}}Z_i)$, it follows from Lemma 5.2.5 that if $\partial_{\{a_{j}, \ldots, a_k\}}^{n}Z_i$ is $s$-cocartesian then $\partial_{\{a_{j+1}, \ldots, a_k\}}^{n}Z_i$ is $(s-1)$-cocartesian.

Therefore if $\partial_{S_i-\{a_1\}}^{n}Z_i$ is $(s + k - 1)$-cocartesian, i.e., $\partial_{\{a_{j}, \ldots, a_k\}}^{n}Z_i$ is $(s + k - 1)$-cocartesian, then repeating the above argument as $j$ runs from 2 to $k$ gives that $Z_i$ is $s$-cocartesian.

**Proof of Lemma 6.2.2.**

Proof. Our goal is to see how cocartesian the cubes $\partial_{S_i-\{a_1\}}^{n}Z_i$ are. The idea is to apply Theorem 6.1.2 for lower dimensions. To do this we need to learn how cartesian the fundamental faces are. Note that the fundamental faces of $\partial_{S_i-\{a_1\}}^{n}Z_i$ are $\partial^{T}_{S_i-\{a_1\}}Z_i$, where $S_i - \{a_1\} \subset T \subset n$.

We divide the discussion into two cases.

First, consider the case where $|T - (S_i - \{a_1\})| = 1$. Now the face $\partial^{T}_{S_i-\{a_1\}}Z_i$ is nothing but a map $Z_i(T) \to Z_i(S_i - \{a_1\})$.

If $a_1 \in T$, then by definition the map is $A_{S_i}(T) \to A_{S_{i-1}}(S_i)$, i.e., $\lim A(S_i \cap P(T)) \to \lim A(S_{i-1} \cap P(S_i))$. We have $S_i \cap P(T) = P(S_i)$, because now $T = S_i$. On the other hand, recall the condition that $S_i - S_{i-1} = \{S_i\}$, and that the
$S_i$'s are convex. It follows that $S_{i-1} \cap P(S_i) = P(S_i) - S_i = P_1(S_i)$. So our map becomes $\lim A(P(S_i)) = A_{S_i} \rightarrow \lim A(P_1(S_i))$. The condition says that this map is $r_{S_i}$-connected.

If $a_1 \notin T$, then by definition the map is $A_{S_{i-1}}(T \cup \{a_1\}) \rightarrow A_{S_{i-1}}(S_i)$, i.e., $\lim A(S_{i-1} \cap P(T \cup \{a_1\})) \rightarrow \lim A(S_{i-1} \cap P(S_i))$. Now $T$ is of the form $(S_i - \{a_1\}) \cup \{a\}$, for some $a \in n$. $a \notin S_i$. Our map becomes $\lim A(S_{i-1} \cap P(S_i \cup \{a\})) \rightarrow \lim A(P_1(S_i))$.

We claim that this map is $r_{\{a\}}$-connected. Indeed, let's choose a sequence of convex subsets of $P(n)$: $T_1 \subset \cdots \subset T_q$ such that $T_1 = P_1(S_i)$, $T_q = S_{i-1} \cap P(S_i \cup \{a\})$, and that for $2 \leq j \leq q$, $T_j - T_{j-1} = \{T_j\}$, a single set between $P_1(S_i)$ and $S_{i-1} \cap P(S_i \cup \{a\})$. Lemma 5.2.8 implies that the map $\lim A(T_j) \rightarrow \lim A(T_{j-1})$ is $r_{T_j}$-connected, $2 \leq j \leq q$. It then follows from Lemma 5.2.4 that the map $\lim A(T_q) \rightarrow \lim A(T_1)$, i.e., the map $\lim A(S_{i-1} \cap P(S_i \cup \{a\})) \rightarrow \lim A(P_1(S_i))$, is $\min \{r_{T_j} : 2 \leq j \leq q\}$-connected. However, we have $\min \{r_{T_j} : 2 \leq j \leq q\} = r_{\{a\}}$.

To see this, note that any set between $P_1(S_i)$ and $S_{i-1} \cap P(S_i \cup \{a\})$ must contain $a$ as an element, and one of the $T_j$'s must be $\{a\}$. Also recall that we have condition $r_T \leq r_{T'}$ whenever $T' \subset T$. It then follow that $r_{\{a\}}$ is indeed the minimum.

Now let's consider the second case, where $|T - (S_i - \{a_1\})| \geq 2$. In this case, to investigate how cartesian the face $\partial_{S_i - \{a_1\}}^T Z_i$ is, one is to determine the connectivity of the map $Z_i(T) \rightarrow \lim Z_i(P_1(T - (S_i - \{a_1\})) \cup (S_i - \{a_1\}))$. (Let’s recall our notation: this limit is taken over all $Z'_T$ with $S_i - \{a_1\} \subset T' \subset T$.) Again there are
two situations.

If \( a_1 \in T \), then we show that the above map is \( \infty \)-connected. First let’s observe that the following diagram is a (homotopy) pullback:

\[
\begin{array}{ccc}
\lim Z_i(P_1(T - (S_i - \{a_1\})) \cup (S_i - \{a_1\})) & \rightarrow & \lim Z_i(P_1(T - S_i) \cup S_i) \\
\downarrow & & \downarrow \\
\lim Z_i(P(T - \{a_1\})) & \rightarrow & \lim Z_i(P_1(T - S_i) \cup (S_i - \{a_1\}))
\end{array}
\]

This follows from Lemma 5.2.6. Indeed, note that

\[
(P_1(T - S_i) \cup S_i) \cap P(T - \{a_1\}) = P_1(T - S_i) \cup (S_i - \{a_1\}),
\]

and that

\[
(P_1(T - S_i) \cup S_i) \cup P(T - \{a_1\}) = P_1(T - (S_i - \{a_1\})) \cup (S_i - \{a_1\}).
\]

So Lemma 5.2.4 can be applied.

Now to see that \( Z_i(T) \rightarrow \lim Z_i(P_1(T - (S_i - \{a_1\})) \cup (S_i - \{a_1\})) \) is a weak equivalence, it is enough to show that the diagram

\[
\begin{array}{ccc}
Z_i(T) & \rightarrow & \lim Z_i(P_1(T - S_i) \cup S_i) \\
\downarrow & & \downarrow \\
Z_i(T - \{a_1\}) & \rightarrow & \lim Z_i(P_1(T - S_i) \cup (S_i - \{a_1\}))
\end{array}
\]

is a homotopy pullback. (Note that \( \lim Z_i(P(T - \{a_1\})) = Z_i(T - \{a_1\}) \).) Since \( a_1 \in T \), by the construction of the \( Z_i \)'s, the above diagram is:

\[
\begin{array}{ccc}
A_{S_i}(T) & \rightarrow & \lim A_{S_i}(P_1(T - S_i) \cup S_i) \\
\downarrow & & \downarrow \\
A_{S_{i-1}}(T) & \rightarrow & \lim A_{S_{i-1}}(P_1(T - S_i) \cup S_i)
\end{array}
\]
Now by the construction of the $\mathcal{A}_{S_i}$’s, the above diagram is:

$$
\begin{array}{ccc}
\lim A(S_i \cap P(T)) & \longrightarrow & \lim A(S_i \cap (P_1(T - S_i) \cup S_i)) \\
\downarrow & & \downarrow \\
\lim A(S_{i-1} \cap P(T)) & \longrightarrow & \lim A(S_{i-1} \cap (P_1(T - S_i) \cup S_i))
\end{array}
$$

Note that

$$(S_{i-1} \cap P(T)) \cap (S_i \cap (P_1(T - S_i) \cup S_i)) = S_{i-1} \cap (P_1(T - S_i) \cup S_i),$$

and that

$$(S_{i-1} \cap P(T)) \cup (S_i \cap (P_1(T - S_i) \cup S_i)) = S_i \cap P(T).$$

So again as a consequence of Lemma 5.2.6, the above diagram is a homotopy pullback.

If we are in the opposite situation $a_1 \notin T$, then the map to consider:

$$Z_i(T) \to \lim Z_i (P_1(T - (S_i - \{a_1\})) \cup (S_i - \{a_1\}))$$

becomes, by definition, the following:

$$\mathcal{A}_{S_{i-1}}(T) \to \lim \mathcal{A}_{S_{i-1}} (P_1(T - (S_i - \{a_1\})) \cup S_i).$$

Again by definition, this is:

$$\lim A(S_{i-1} \cap P(T)) \to \lim A(S_{i-1} \cap (P_1(T - (S_i - \{a_1\})) \cup S_i)).$$

If $T - (S_i - \{a_1\}) \notin S_{i-1}$, then Lemma 5.2.7 implies that this is a weak equivalence. If $T - (S_i - \{a_1\}) \in S_{i-1}$, then we could use the trick used previously.

Choose a sequence $T_1 \subset \cdots \subset T_q$ of convex subsets of $P(n)$ such that $T_1 = S_{i-1} \cap \cdots \cap \cdots \cap$.
\((P_1(T - (S_i - \{a_1\})) \cup S_i), T_q = S_{i-1} \cap P(T),\) and that for \(2 \leq j \leq q, T_j - T_{j-1} = \{T_j\},\) a single set between \(S_{i-1} \cap (P_1(T - (S_i - \{a_1\})) \cup S_i)\) and \(S_{i-1} \cap P(T).\) By Lemma 5.2.8 we know that the map \(\lim A(T_j) \rightarrow \lim A(T_{j-1})\) is \(r_{T_j}\)-connected, \(2 \leq j \leq q.\) It then follows from Lemma 5.2.4 that the map \(\lim A(T_q) \rightarrow \lim A(T_1),\) i.e., the map \(Z_i(T) \rightarrow \lim Z_i(P_1(T - (S_i - \{a_1\})) \cup (S_i - \{a_1\})),\) is \(\min \{r_{T_j} : 2 \leq j \leq q\}\)-connected. This minimum is equal to \(r_{T - (S_i - \{a_1\})}.\) Indeed, any set between \(S_{i-1} \cap (P_1(T - (S_i - \{a_1\})) \cup S_i)\) and \(S_{i-1} \cap P(T)\) must contain \(T - (S_i - \{a_1\}),\) and one of the \(T_j\)'s must be equal to \(T - (S_i - \{a_1\}).\)

To sum up, we have proved that the diagram \(\partial_{S_i - \{a_1\}} T Z_i\) is:

- \(r_{S_i}\)-connected, if \(|T - (S_i - \{a_1\})| = 1, a_1 \in T;\)
- \(r_a\)-connected, if \(T = (S_i - \{a_1\}) \cup \{a\},\) for some \(a \notin S_i;\)
- \(\infty\)-cartesian, if \(|T - (S_i - \{a_1\})| \geq 2, a_1 \in T;\)
- \(\infty\)-cartesian, if \(|T - (S_i - \{a_1\})| \geq 2, a_1 \notin T, T - (S_i - \{a_1\}) \notin S_{i-1};\)
- \(r_{T - (S_i - \{a_1\})}\)-cartesian, if \(|T - (S_i - \{a_1\})| \geq 2, a_1 \notin T, T - (S_i - \{a_1\}) \in S_{i-1}.\)

We now have complete knowledge of the cartesian properties of the various faces \(\partial_{S_i - \{a_1\}} T Z_i.\) Applying Theorem 6.1.2 to the cube \(\partial_{S_{i-1} - \{a_1\}} T Z_i\) of dimension \((n - k + 1)\) gives the desired cocartesian property. (Note that since \(k \geq 2,\) the dimension \((n - k + 1)\) is strictly less than \(n.)\) \qed
Bibliography


