AN ERDŐS-KO-RADO THEOREM FOR THE DERANGEMENT
GRAPH OF PGL₃(𝑞) ACTING ON THE PROJECTIVE PLANE

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Abstract. In this paper we prove an Erdős-Ko-Rado-type theorem for intersecting sets of permutations. We show that an intersecting set of maximal size in the projective general linear group PGL₃(𝑞), in its natural action on the points of the projective line, is either a coset of the stabilizer of a point or a coset of the stabilizer of a line. This gives the first evidence to the veracity of Conjecture 2 from K. Meagher, P. Spiga, An Erdős-Ko-Rado theorem for the derangement graph of PGL(2, 𝑞) acting on the projective line, J. Comb. Theory Series A 118 (2011), 532–544.

1. General results

The Erdős-Ko-Rado theorem [11] determines the cardinality and describes the structure of a set of maximal size of intersecting 𝑘-subsets from {1, . . . , 𝑛}. Specifically, the theorem says that provided that 𝑛 > 2𝑘, a set of maximal size of intersecting 𝑘-subsets from {1, . . . , 𝑛} has cardinality \(\binom{𝑛−1}{k−1}\) and is the set of all 𝑘-subsets that contain a common fixed element. Analogous results hold for many other objects other than sets, and in this paper we are concerned with an extension of the Erdős-Ko-Rado theorem to permutation groups.

Let \(G\) be a permutation group on \(Ω\). A subset \(S\) of \(G\) is said to be intersecting if for every \(g, h ∈ S\) the permutation \(gh^{-1}\) fixes some point of \(Ω\) (note that this implies that \(α^g = α^h\), for some \(α ∈ Ω\)). As with the Erdős-Ko-Rado theorem, we are interested in finding the cardinality of an intersecting set of maximal size in \(G\) and classifying the sets that attain this bound. This problem can be formulated in graph-theoretic terminology. We denote by \(Γ_G\) the derangement graph of \(G\); the vertices of this graph are the elements of \(G\) and the edges are the pairs \(\{g, h\}\) such that \(gh^{-1}\) is a derangement, that is, \(gh^{-1}\) fixes no point. An intersecting set of \(G\) is simply an independent set or a coclique of \(Γ_G\).

The natural extension of the Erdős-Ko-Rado theorem for the symmetric group Sym(𝑛) was proved in [8] and [15]. These papers, using different methods, showed that every independent set of \(Γ_{\text{Sym}(𝑛)}\) has size at most \((𝑛 − 1)!\). They both further showed that the only independent sets meeting this bound are the cosets of the stabilizer of a point. The same result was also proved in [12] using the character theory of Sym(𝑛). The approach in [12] is similar to the approach used by Wilson in [21]; in this paper Wilson gives an excellent algebraic proof of the exact bound in the Erdős-Ko-Rado theorem. This approach is also used by Newman [17] to prove that a version of the Erdős-Ko-Rado theorem holds for vector spaces over a finite field.

1991 Mathematics Subject Classification. Primary 05C35; Secondary 05C69, 20B05.
Key words and phrases. derangement graph, independent sets, Erdős-Ko-Rado theorem.
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Recently there have been many papers proving that the natural extension of the Erdős-Ko-Rado theorem holds for specific permutation groups $G$ (see [2, 14, 16, 20]) and there are also two papers, [3] and [4], that consider when the natural extension of the Erdős-Ko-Rado theorem holds for transitive and two-transitive groups. This means asking if the largest independent sets in the derangement graph $\Gamma_G$ are the cosets in $G$ of the stabilizer of a point. Typically, for a general permutation group, the derangement graph may have independent sets of size larger than the size of the stabilizer of a point, let alone hope that every such independent set is the coset of the stabilizer of a point. However, a behavior very similar to $\text{Sym}(n)$ is offered by $\text{PGL}_2(q)$ in its natural action on the projective line [16, Theorem 1]; the independent sets of maximal size in the derangement graph of $\text{PGL}_2(q)$ are exactly the cosets of the stabilizer of a point. It is not hard to see that for a generic projective linear group there are maximum independent sets in the derangement graph that are not the stabilizer of a point. This leads to the following conjecture:

**Conjecture 1.1** ([16, Conjecture 2]). *The independent sets of maximal size in the derangement graph of $\text{PGL}_{n+1}(q)$ acting on the points of the projective space $\text{PG}_{n}(q)$ are exactly the cosets of the stabilizer of a point and the cosets of the stabilizer of a hyperplane.*

This conjecture is appealing in that, like the other versions of the Erdős-Ko-Rado theorem, it characterizes the independent sets of maximal size in the derangement graph of a group. However, in this case there are two distinct families of independent sets of maximal size – this is similar to the situation for $k$-subsets from an $n$-set in which any two subsets have at least $t$ elements in common (see [1] for more details).

The main result of this paper gives the first important contribution towards a proof of this conjecture.

**Theorem 1.2.** *The independent sets of maximal size in the derangement graph of $\text{PGL}_{3}(q)$ acting on the points of the projective plane $\text{PG}_{2}(q)$ are exactly the cosets of the stabilizer of a point and the cosets of the stabilizer of a line.*

In particular, Theorem 1.2 settles Conjecture 2 of [16] for $n = 2$. (The case $n = 1$ was already settled in [16, Theorem 1].) Our proof uses the method developed in [12] and hence we make use of some information on the character theory of $\text{PGL}_{3}(q)$. Some, but not all, of our arguments in the proof of Theorem 1.2 work for any $n \geq 2$. Therefore, we have decided to present all of our results uniformly with $n = 2$.

There are two critical junctures where we either need some geometric properties of $\text{PG}_2(q)$ or some algebraic properties of the irreducible characters of $\text{PGL}_{3}(q)$; these are preventing us from using this method for all $\text{PGL}_{n+1}(q)$. First, we are unsure of how to prove that the eigenvalue of the derangement graph for $\text{PGL}_{n+1}(q)$ that we conjecture to be the smallest, is indeed the smallest (for $n = 2$, this is Lemma 4.3). Second, Proposition 3.3 describes a basis for the right kernel of a particular matrix $M$ related to $\text{PGL}_{3}(q)$. We show that this is indeed a basis by determining a basis for all the eigenspaces of $M$ (this is done in Section 5); it is not clear how to generalize this work for larger values of $n$. For example, at several points in our proofs we use the fact that any two lines intersect in exactly one point. We suspect that the natural generalization of these results will require using hyperplanes, but we are unsure of how a basis of these eigenspaces will look like when $n \geq 3$. Proposition 3.3 and the description of the eigenspaces of $M$ seem to
be fundamental to our work; indeed, they are what allows us to use the methods from [12] and [16] with the group PGL₃(q).

To some extent, the proof of Theorem 1.2 is more involved than the proof of the Erdös-Ko-Rado theorem for PGL₂(q) [16] and for Sym(n) [12]. In our opinion, this increase of complexity is natural since the independent sets of maximal size are not all of the same type.

2. Notation

We let \( q \) be a power of a prime and we denote by \( \mathbb{GF}(q) \) a field of size \( q \), and by \( \mathbb{GF}(q)^3 \) the 3-dimensional vector space over \( \mathbb{GF}(q) \) of row vectors with basis \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \). The (Desarguesian) projective plane \( \text{PG}_2(q) \) is the pair \( (\mathcal{P}, \mathcal{L}) \), where the elements of \( \mathcal{P} \), respectively \( \mathcal{L} \), are the 1-dimensional, respectively 2-dimensional, subspaces of \( \mathbb{GF}(q)^3 \). The elements of \( \mathcal{P} \) are called points and the elements of \( \mathcal{L} \) are called lines of the plane \( \text{PG}_2(q) \). For denoting the points we will use Greek letters, and for the lines Roman letters.

Given two distinct points \( \alpha \) and \( \alpha' \), we denote by \( \alpha \lor \alpha' \) the line spanned by \( \alpha \) and \( \alpha' \), that is, the line of \( \text{PG}_2(q) \) containing both \( \alpha \) and \( \alpha' \). Moreover, given two distinct lines \( \ell \) and \( \ell' \), we denote by \( \ell \land \ell' \) the point of the intersection of \( \ell \) and \( \ell' \).

We denote by \( G \) the permutation group \( \text{PGL}_3(q) \) in its action on \( \mathcal{P} \), and by \( \mathcal{D} \) the set of derangements of \( G \). As is standard, the subgroup of \( G \) that fixes the point \( \alpha \) is denoted by \( G_{\alpha} \), and, similarly, the subgroup that fixes the line \( \ell \) is denoted by \( G_{\ell} \).

As usual, \( \mathbb{C}[G] \) is the group algebra of \( G \) over the complex numbers \( \mathbb{C} \). We only need the vector space structure on \( \mathbb{C}[G] \), and a basis for \( \mathbb{C}[G] \) is indexed by the group elements \( g \in G \). Given a subset \( S \) of \( G \), we denote by \( \chi_S \in \mathbb{C}[G] \) the characteristic vector of \( S \), that is, \( (\chi_S)_g = 1 \) if \( g \in S \), and \( (\chi_S)_g = 0 \) otherwise. The all ones vector is denoted by \( 1 \) (the length of the vector will be clear by context).

There exists a natural duality between the points and the lines of \( \text{PG}_2(q) \) (see [7] for details) and this duality is preserved by \( G = \text{PGL}_3(q) \). Hence, for each \( g \in G \), the number of elements of \( \mathcal{P} \) fixed by \( g \) coincides with the number of elements of \( \mathcal{L} \) fixed by \( g \). In particular, we have the equality

\[
\sum_{\alpha \in \mathcal{P}} \chi_{G_{\alpha}} = \sum_{\ell \in \mathcal{L}} \chi_{G_{\ell}}.
\]

In our arguments we assume that the reader is familiar with the basic properties of \( \text{PGL}_3(q) \) and also with the work of Steinberg [19] on the character theory of \( \text{PGL}_3(q) \). For the benefit of some readers, we have included some important facts about \( \text{PGL}_3(q) \) and details on its complex irreducible characters in the Appendix (Section 6).

3. Proof of Theorem 1.2

To prove Theorem 1.2, we will show that the characteristic vector for every independent set of maximal size in the derangement graph of \( G \) is a linear combination of a specific set of vectors. Then we will show that the only such possible linear combinations are the characteristic vectors for either the cosets of the stabilizer of a point or the cosets of the stabilizer of a line.

Let \( A \) be the \( \{0,1\} \)-matrix where the rows are indexed by the elements of \( G \), the columns are indexed by the ordered pairs of points from \( \mathcal{P} \) and \( A_{g, (\alpha, \beta)} = 1 \)
if and only if $\alpha^g = \beta$. In particular, $A$ has $|G| = q^4(q^3 - 1)(q^2 - 1)$ rows and $|P|^2 = (q^2 + q + 1)^2$ columns.

We fix a particular ordering of the rows of $A$ so that the first rows are labeled by the derangements $D$ of $G$, and the remaining rows are labeled by the elements of $G \setminus D$. With this ordering, we get that $A$ is the following block matrix

$$A = \begin{pmatrix} M \\ B \end{pmatrix}.$$ 

In particular, the rows of the submatrix $M$ are labeled by elements of $D$ and the columns of $M$ are labeled by pairs of elements of $P$.

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Since the columns of $A$ have coordinates indexed by the elements of $G$, we can view each column of $A$ as an element of $C[G]$. Further, if we define $V$ to be the $C$-vector space whose basis consists of all $e_{\alpha\beta}$, where $(\alpha, \beta)$ is an ordered pair of elements of $P$, then the rows of $A$ can be viewed as characteristic vectors in $V$.

Propositions 3.2 and 3.3 will give the right kernels of $A$ and $M$, respectively. To describe these kernels, we define three families of vectors from $V$. Two of these families are indexed by the points $\alpha \in P$ and the third by the lines $\ell \in L$:

$$e^1_\alpha = \sum_{\beta \in P} e_{\alpha\beta}, \quad e^2_\alpha = \sum_{\beta \in P} e_{\beta\alpha}, \quad e_\ell = \sum_{\beta, \beta' \in \ell} e_{\beta\beta'}.$$ 

(2)

We now state three pivotal properties of the matrices $A$ and $M$; we postpone the technical proofs of all three properties to Sections 4 and 5.

**Proposition 3.1.** If $S$ is an independent set of maximal size of $\Gamma_G$, then $\chi_S$ is a linear combination of the columns of $A$.

**Proposition 3.2.** The matrix $A$ has rank $(|P| - 1)^2 + 1$. Moreover, given $\bar{\alpha} \in P$, the subspace

$$\langle e^1_{\alpha} - e^1_{\bar{\alpha}}, e^2_{\alpha} - e^2_{\bar{\alpha}} \mid \alpha \in P \rangle$$

of $V$ is the right kernel of $A$.

**Proposition 3.3.** Given $\bar{\alpha} \in P$ and $\bar{\ell} \in L$, the subspace

$$\langle e_{\alpha \bar{\alpha}}, e^1_{\bar{\alpha}} - e^1_{\alpha}, e^2_{\bar{\alpha}} - e^2_{\alpha}, e_\bar{\ell} - e_\ell \mid \alpha \in P, \ell \in L \rangle$$

of $V$ is the right kernel of $M$.

Before proving Theorem 1.2 (using these three yet unproven propositions), we need to show that if the columns of the matrix $B$ are arranged so that the first $q^2 + q + 1$ columns correspond to the ordered pairs of the form $(\alpha, \alpha)$, then there is an arrangement of the rows such that the top left corner of $B$ forms a $(q^2 + q + 1) \times (q^2 + q + 1)$ identity matrix. To prove this, it is enough to show that for every $\alpha \in P$ there is a permutation in $G$ that has $\alpha$ as its only fixed point. (To simplify some of the computations in the proof of Theorem 1.2, we actually prove something slightly stronger.)

**Lemma 3.4.** For every $\alpha \in P$ and for every $\ell \in L$, there exists $g \in G$ with $g$ fixing only the element $\alpha$ of $P$ and only the element $\ell$ of $L$.

**Proof.** Observe that $G$ acts transitively on the sets $\{ (\alpha, \ell) \mid \alpha \in P, \ell \in L, \alpha \in \ell \}$ (this is the set of flags of $\text{PG}_2(q)$) and $\{ (\alpha, \ell) \mid \alpha \in P, \ell \in L, \alpha \notin \ell \}$ (this is the set of anti-flags of $\text{PG}_2(q)$). In particular, replacing $\alpha$ and $\ell$ by $\alpha^h$ and $\ell^h$ for some
\[ h \in G, \text{ we may assume that either } \alpha = \langle \varepsilon_1 \rangle \text{ and } \ell = \langle \varepsilon_1, \varepsilon_2 \rangle, \text{ or } \alpha = \langle \varepsilon_1 \rangle \text{ and } \ell = \langle \varepsilon_2, \varepsilon_3 \rangle. \]  

In the first case,  
\[ g = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in G \]
fixes only the element \( \alpha \) of \( P \) and only the element \( \ell \) of \( L \). In the second case, if  
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
is a \( 2 \times 2 \)-matrix with coefficients in \( \text{GF}(q) \) and with irreducible characteristic polynomial, then  
\[ g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \in G \]
fixes only the element \( \alpha \) of \( P \) and only the element \( \ell \) of \( L \).

Now we have all the tools needed to prove the main result of this paper.

**Proof of Theorem 1.2.** Let \( S \) be an independent set of maximal size of \( \Gamma G \). We aim to prove that \( S \) is a coset of the stabilizer of a point or a line of \( \text{PG}_2(q) \). Up to multiplication of \( S \) by a suitable element of \( G \), we may assume that the identity of \( G \) is in \( S \). In particular, we have to prove that \( S \) is the stabilizer of a point or a line of \( \text{PG}_2(q) \).

From Proposition 3.1, the characteristic vector \( \chi_S \) of \( S \) is a linear combination of the columns of \( A \), and hence, for some vector \( x \), we have  
\[ \chi_S = Ax = \begin{pmatrix} M \\ B \end{pmatrix} x = \begin{pmatrix} Mx \\ Bx \end{pmatrix}. \]

As the identity of \( G \) is in \( S \), there are no derangements in \( S \). Hence, by our choice of the ordering of the rows of \( A \), we get  
\[ \chi_S = \begin{pmatrix} 0 \\ t \end{pmatrix} \]
and thus \( Mx = 0 \) and \( Bx = t \). Hence \( x \) lies in the right kernel of \( M \). Therefore, by Proposition 3.3, given a fixed point \( \bar{\alpha} \) and a fixed line \( \bar{\ell} \), we have  
\[ x = \sum_{\alpha \in P} c_\alpha e_{\alpha \alpha} + \sum_{\alpha \in P \setminus \{ \bar{\alpha} \}} c_\alpha^1 (e_{\alpha}^1 - e_{\alpha}^1) + \sum_{\alpha \in P \setminus \{ \bar{\alpha} \}} c_\alpha^2 (e_{\alpha}^2 - e_{\alpha}^2) + \sum_{\ell \in L \setminus \{ \bar{\ell} \}} c_\ell (e_\ell - e_\ell), \]
for some scalars \( c_\alpha, c_\alpha^1, c_\alpha^2, c_\ell \in \mathbb{C} \).

From Proposition 3.2, the vectors \( e_{\alpha}^1 - e_{\alpha}^1 \) and \( e_{\alpha}^2 - e_{\alpha}^2 \) are in the right kernel of \( A \), and hence \( B(e_{\alpha}^1 - e_{\alpha}^1) = B(e_{\alpha}^2 - e_{\alpha}^2) = 0 \). In particular,  
\[ t = Bx = B \left( \sum_{\alpha \in P} c_\alpha e_{\alpha \alpha} + \sum_{\ell \in L \setminus \{ \bar{\ell} \}} c_\ell (e_\ell - e_\ell) \right); \]
and hence we may assume that \( c_\alpha^1 = c_\alpha^2 = 0 \), for every \( \alpha \in P \).
For $\alpha \in \mathcal{P}$, we have $Be_{\alpha \alpha} = \chi_{G_\alpha}$, where $\chi_{G_\alpha}$ is the characteristic vector of the stabilizer $G_\alpha$ of $\alpha$. Moreover, for $\ell \in \mathcal{L}$ and $g$ an element of $G$ that is not a derangement, we have

$$(3) \quad (Be_\ell)_g = \sum_{\alpha, \beta \in \ell} B_{g, (\alpha, \beta)} = \sum_{\alpha \in \ell, \alpha \neq \ell} 1 = |\{\alpha \in \ell \mid \alpha^g \in \ell\}| = \begin{cases} q + 1 & \text{if } g \in G_\ell, \\ 1 & \text{if } g \notin G_\ell. \end{cases}$$

(For the last equality observe that $\ell \land \ell^g$ is a point of $\text{PG}_2(q)$ when $\ell \neq \ell^g$.) Thus $B(e_\ell) = q\chi_{G_\ell} + 1$, and it follows from (3) that $B(e_\ell - e_\ell) = q(\chi_{G_\ell} - \chi_{G_\ell})$. Putting these two facts together, we get that

$$(4) \quad t = Bx = \sum_{\alpha \in \mathcal{P}} c_\alpha \chi_{G_\alpha} + q \sum_{\ell \in \mathcal{L} \setminus \{\ell\}} c_\ell (\chi_{G_\ell} - \chi_{G_\ell}).$$

As the identity of $G$ is in $S$, the coordinate of $t$ corresponding to the identity of $G$ is 1 and hence, using the formula in (4) for $t$, we get

$$(5) \quad \sum_{\alpha \in \mathcal{P}} c_\alpha = 1.$$ 

Applying Lemma 3.4 to $\alpha \in \mathcal{P}$ and to the line $\ell$, we get $g \in G$ that fixes only the point $\alpha$ and only the line $\ell$. As $t$ is a $\{0, 1\}$-vector, the coordinate of $t$ corresponding to $g$ is either 0 or 1; by taking the coordinate corresponding to $g$ on the right-hand side of (4), we get

$$(6) \quad c_\alpha + q \sum_{\ell \in \mathcal{L} \setminus \{\ell\}} c_\ell \in \{0, 1\},$$

for every $\alpha \in \mathcal{P}$. Applying this argument to $\alpha \in \mathcal{P}$ and to $\ell \in \mathcal{L} \setminus \{\ell\}$, we obtain

$$(7) \quad c_\alpha - qc_\ell \in \{0, 1\},$$

for every $\alpha \in \mathcal{P}$ and every $\ell \in \mathcal{L} \setminus \{\ell\}$.

Write $c = \sum_{\ell \in \mathcal{L} \setminus \{\ell\}} c_\ell$. From (6), we have $c_\alpha = -qc$ or $c_\alpha = -qc + 1$, for every $\alpha \in \mathcal{P}$. Define the sets

$$\mathcal{P}_{-qc} = \{\alpha \in \mathcal{P} \mid c_\alpha = -qc\}, \quad \mathcal{P}_{-qc+1} = \{\alpha \in \mathcal{P} \mid -qc + 1\}.$$

We will first show that if these sets are both non-empty then $S$ is the stabilizer of a point. Next we will show if the first set is empty then $S$ is the stabilizer of $\ell$, and if the second is empty then $S$ is the stabilizer of some line in $\mathcal{L} \setminus \{\ell\}$.

Suppose that both $\mathcal{P}_{-qc}$ and $\mathcal{P}_{-qc+1}$ are non-empty. Let $\alpha \in \mathcal{P}_{-qc+1}$ and let $\beta \in \mathcal{P}_{-qc}$. Applying (7) to $\alpha$ and $\beta$, we get

$$-qc + 1 -qc_\ell \in \{0, 1\}, \quad -qc -qc_\ell \in \{0, 1\},$$

for each $\ell \in \mathcal{L} \setminus \{\ell\}$. If $-qc + 1 -qc_\ell = 0$, for some $\ell \in \mathcal{L} \setminus \{\ell\}$, then $-qc -qc_\ell = -1$, but this is a contradiction. Thus $-qc + 1 -qc_\ell = 1$ (and hence $c_\ell = -c$), for every $\ell \in \mathcal{L} \setminus \{\ell\}$. Since $c = \sum_{\ell \in \mathcal{L} \setminus \{\ell\}} c_\ell$, we obtain

$$c = \sum_{\ell \in \mathcal{L} \setminus \{\ell\}} (-c) = -c(q^2 + q).$$

This implies that $c = 0$, and hence $c_\ell = 0$, for each $\ell \in \mathcal{L} \setminus \{\ell\}$. This shows that $t = \sum_{\alpha \in \mathcal{P}} c_\alpha \chi_{G_\alpha}$. Now, (6) gives $c_\alpha \in \{0, 1\}$, for every $\alpha \in \mathcal{P}$, and hence (5) implies that there exists a unique $\alpha' \in \mathcal{P}$ with $c_{\alpha'} = 1$ and all other scalars are zero. Thus $t = \chi_{G_{\alpha'}}$ and $S$ is the stabilizer of the point $\alpha'$. 


Moreover, (7) gives for each \( \ell \) all other lines \( b \) have 
\[ S(\ell) = \ell \] for each \( b \) and let \( P \). We have the equations
\[ cq = 0, \]
for each \( \ell \in L \setminus \{ \ell \} \). Let \( a \) be the number of lines \( \ell \in L \setminus \{ \ell \} \) with \( c_\ell = -c \) and let \( b \) be the number of lines \( \ell \in L \setminus \{ \ell \} \) with \( c_\ell = -c - 1/q \). As \( |L \setminus \{ \ell \}| = q^2 + q \), we have \( a + b = q^2 + q \). Moreover, by the definition of the parameter \( c \), we have
\[ a(-c) + b(-c - 1/q) = c. \]
Putting these together we have \(-c(q^2 + q) - b/q = c\) which implies \( b = -cq(q^2 + q + 1) = 1 \). In particular, there exists a unique \( \ell' \in L \setminus \{ \ell \} \) with \( c_{\ell'} = -c - 1/q \) and all other lines \( \ell \in L \setminus \{ \ell \} \) have \( c_\ell = -c \). Therefore, using (1), we get
\[ t = -cq \sum_{a \in P} \chi_{G_a} + q(-c) \sum_{\ell \in L \setminus \{ \ell \}} (\chi_{G_\ell} - \chi_{G_{\ell'}}) + q \left( \frac{-1}{q} \right) (\chi_{G_\ell} - \chi_{G_{\ell'}}) \]
\[ = -cq \left( \sum_{a \in P} \chi_{G_a} - \sum_{\ell \in L} \chi_{G_\ell} \right) - qc \sum_{\ell \in L} \chi_{G_\ell} - (\chi_{G_\ell} - \chi_{G_{\ell'}}) \]
\[ = -cq(q^2 + q + 1) \chi_{G_\ell} - (\chi_{G_\ell} - \chi_{G_{\ell'}}) = \chi_{G_\ell} - (\chi_{G_\ell} - \chi_{G_{\ell'}}) = \chi_{G_{\ell'}}. \]
In this case \( S \) is the stabilizer of the line \( \ell' \).

Finally, suppose that \( P_{-qc+1} = \emptyset \). Thus \( P = P_{-qc} \) and \( c_\alpha = -qc \), for each \( \alpha \in P \). In particular, (5) gives \((-qc+1)(q^2 + q + 1) = 1\), that is, \( c = (q+1)/(q^2 + q + 1) \). Furthermore, (7) gives \(-qc - qc_\ell \in \{0,1\}\), that is,
\[ c_\ell \in \{-c, -c + 1/q\}, \]
for each \( \ell \in L \setminus \{ \ell \} \). Again, let \( a \) be the number of lines \( \ell \in L \setminus \{ \ell \} \) with \( c_\ell = -c + 1/q \) and let \( b \) be the number of lines \( \ell \in L \setminus \{ \ell \} \) with \( c_\ell = -c \). As in the previous case, we have the equations
\[ a + b = q^2 + q, \quad a(-c + 1/q) + b(-c) = c, \]
and hence \(-c(q^2 + q) + a/q = c\). Thus \( a = cq(q^2 + q + 1) = q^2 + q \) and \( b = 0 \). Therefore, using (1), we get
\[ t = (-cq + 1) \sum_{a \in P} \chi_{G_a} + q(-c + 1/q) \sum_{\ell \in L \setminus \{ \ell \}} (\chi_{G_\ell} - \chi_{G_{\ell'}}) \]
\[ = (-cq + 1) \left( \sum_{a \in P} \chi_{G_a} - \sum_{\ell \in L} \chi_{G_\ell} \right) + (-cq + 1) \sum_{\ell \in L} \chi_{G_\ell} \]
\[ = (-cq + 1)(q^2 + q + 1) \chi_{G_\ell} = \chi_{G_{\ell'}}. \]
In this case \( S \) is the stabilizer of the line \( \ell' \).

4. Proof of Propositions 3.1 and 3.2

For a moment, we leave some of the notation that we have set so far and consider general groups. Let \( G \) be a permutation group on \( \Omega \) and let \( \Gamma_G \) be its derangement graph. Since the right regular representation of \( G \) is a subgroup of the automorphism group of \( \Gamma_G \), we see that \( \Gamma_G \) is a Cayley graph. Namely, if \( D \) is the set of derangements of \( G \), then \( \Gamma_G \) is the Cayley graph on \( G \) with connection set \( D \), i.e.
Γ₆ = Cay(G, D). Clearly, D is a union of G-conjugacy classes, so Γ₆ is a normal Cayley graph.

As usual, we simply say that the complex number ξ is an eigenvalue of the graph Γ if ξ is an eigenvalue of the adjacency matrix of Γ. We use Irr(G) to denote the irreducible complex characters of the group G and given χ ∈ Irr(G) and a subset S of G we write χ(S) for ∑ₖ∈S χ(k). In the following lemma we recall that the eigenvalues of a normal Cayley graph on G are determined by the irreducible complex characters of G, see [5].

**Lemma 4.1.** Let G be a permutation group on Ω and let D be the set of derangements of G. The spectrum of the graph Γ₆ is {χ(D)/χ(1) | χ ∈ Irr(G)}. Also, if τ is an eigenvalue of Γ₆ and χ₁,...,χₘ are the irreducible characters of G such that τ = χᵢ(D)/χᵢ(1), then the dimension of the τ-eigenspace of Γ₆ is ∑ᵢ=1 χᵢ(1)².

The next result is the well-known ratio-bound for independent sets in a graph; for a proof see (for example) [16, Lemma 3].

**Lemma 4.2.** Let G be a permutation group, let τ be the minimum eigenvalue of Γ₆, let d be the valency of Γ₆ and let S be an independent set of Γ₆. Then

|S| ≤ |G|/(1 − d/τ).

If the equality is met, then χ₆S = |S|/|G|χ₆ is an eigenvector of Γ₆ with eigenvalue τ.

We are now ready to return to only considering the group G = PGL₃(q). Let π be the permutation character of G. As G is 2-transitive, we have π = 1 + χ₀, where 1 is the principal character of G and χ₀ is an irreducible character of degree q² + q.

**Lemma 4.3.** Let Γ₆ be the derangement graph of G = PGL₃(q). Then the following hold:

1. The largest eigenvalue of Γ₆ is |D| = (q² − 1)q³/3.
2. The minimum eigenvalue of Γ₆ is τ = −(q − 1)(q² − 1)q³/3 and, provided that q > 2, the eigenvalue τ has multiplicity (gcd(3, q − 1) − 1) + (q² + q)².
3. For χ ∈ Irr(G),

χ(D)/χ(1) = τ = −(q − 1)(q² − 1)q³/3

if and only if χ = χ₀, or χ = ξ where ξ is one of the (gcd(3, q − 1) − 1) non-principal irreducible linear characters of G.

**Proof.** The character table and information on the conjugacy classes of PGL₃(q) are given in [19, Section 3]. The eigenvalues of the derangement graph of G can be found from a direct inspection of these tables; these are given in Table 3 in the appendix. (Details for the calculation of the eigenvalues of the derangement graph are also included in the appendix, Section 6.) The lemma follows directly from Table 3.

Applying Lemma 4.2 with the values for τ and d as in Lemma 4.3 we obtain the maximal size of an independent set of Γ₆.

**Corollary 4.4.** The maximal size of an independent set of Γ₆ is q³(q² − 1)(q − 1).

**Proof.** Using the value of |D| and of the minimum eigenvalue τ of Γ₆ obtained in Lemma 4.3, Lemma 4.2 shows that an independent set of Γ₆ has size no more than |G|/(1 − |D|/τ) = q³(q² − 1)(q − 1). The stabilizer of a point is an independent set of Γ₆ of this size. □
The same argument can be applied to the group $\text{PSL}_3(q)$. The eigenvalues for the derangement graph of $\text{PSL}_3(q)$ are given in Table 5 in the appendix.

**Corollary 4.5.** The maximal size of an independent set of $\Gamma_{\text{PSL}_3(q)}$ is $q^3(q^2 - 1)/(q-1)\gcd(3,q-1)$.

**Proof.** If $\gcd(3,q-1) = 1$, then $\text{PGL}_3(q) = \text{PSL}_3(q)$ and hence from Corollary 4.4 the maximal size of an independent set of $\Gamma_{\text{PSL}_3(q)}$ is $q^3(q^2 - 1)/(q-1)$.

Suppose that $q-1$ is divisible by 3. Denote by $D_0$ the number of derangements in $\text{PSL}_3(q)$ and let $\tau_0$ be the minimum eigenvalue of $\Gamma_{\text{PSL}_3(q)}$. From Table 5 we see that

$$|D_0| = (q-1)^2(q+2)(q^2-1)q^3/9, \quad \tau_0 = -(q-1)^3(q+2)q^2/3. $$

Since $|\text{PSL}_3(q)|/(1 - |D_0|/\tau_0) = q^3(q^2-1)(q-1)/3$, the rest of the proof follows from Lemma 4.2. \hfill \Box

Next we will prove Proposition 3.2.

**Proof of Proposition 3.2.** For $\alpha, \beta, \gamma, \delta \in \mathcal{P}$, the entry $(A^T A)_{(\alpha, \beta), (\gamma, \delta)}$ equals the number of permutations of $G$ mapping $\alpha$ into $\beta$ and $\gamma$ into $\delta$. Since $G$ is 2-transitive, we get by a simple counting argument that

$$(A^T A)_{(\alpha, \beta), (\gamma, \delta)} = \begin{cases} \frac{|G|}{|\mathcal{P}|} = (q-1)(q^2-1)q^3 & \text{if } \alpha = \gamma \text{ and } \beta = \delta, \\ \frac{|G|}{|\mathcal{P}|(|\mathcal{P}|-1)} = (q-1)^2q^2 & \text{if } \alpha \neq \gamma \text{ and } \beta \neq \delta, \\ 0 & \text{otherwise}. \end{cases} $$

This shows, with a properly chosen ordering of the columns of $A$ (for any fixed ordering of the elements in $\mathcal{P}$, the lexicographic ordering on the pairs will work), that

$$A^T A = (q-1)(q^2-1)q^3I_{|\mathcal{P}|^2} + ((q-1)^2q^2(J_{|\mathcal{P}|} - I_{|\mathcal{P}|}) \otimes (J_{|\mathcal{P}|} - I_{|\mathcal{P}|})). $$

(here $I_n, J_n$ denote the identity matrix and the all-1 matrix of size $n$, respectively.

As the spectrum of the matrix $J_n$ is $(n^1, 0^{n-1})$ (the exponents represent the multiplicities), it follows that the spectrum of the matrix $(J_n - I_n) \otimes (J_n - I_n)$ is $((n-1)^2)^1, (1^{(n-1)^2}), (-(-n-1))^{2(n-1)^2})$. Thus $A^T A$ is diagonalizable with spectrum

$$\begin{pmatrix} 1 & \frac{1}{2(|\mathcal{P}|-1)} \\ ((q-1)(q^3-1)q^2) & (|\mathcal{P}|-1)^2, 0 \end{pmatrix}. $$

This shows that $A^T A$ has rank $((|\mathcal{P}|-1)^2 + 1$ and nullity $2(|\mathcal{P}|-1)$ As $A$ is a matrix with real coefficients, the right kernel of $A$ equals the kernel of $A^T A$. It follows that $A$ has rank $((|\mathcal{P}|-1)^2 + 1$ and the dimension of the right kernel of $A$ is $2(|\mathcal{P}|-1)$.

Finally it is easy to verify that the vectors in $(e_1^\alpha - e_1^\alpha, e_2^\alpha - e_2^\alpha \mid \alpha \in \mathcal{P} \setminus \{\bar{\alpha}\})$ are linearly independent and, for each $\alpha \in \mathcal{P}$, we have $A(e_1^\alpha - e_1^\alpha) = A(e_2^\alpha - e_2^\alpha) = 0$. \hfill \Box

The only property of $\text{PGL}_3(q)$ that is used in the proof of Proposition 3.2 is the 2-transitivity. In fact, this result is shown to hold for any 2-transitive group in [3].

**Proof of Proposition 3.1.** Denote by $\tau$ the minimum eigenvalue of $\Gamma_G$ and write $d = |\mathcal{D}|$. If $q = 2$, then the proof follows with a computation with the computer algebra system magma [6], and hence we assume that $q > 2$. 


Let $J$ be the subspace of $\mathbb{C}[G]$ spanned by the characteristic vectors $\chi_S$ of the independent sets $S$ of maximal size of $\Gamma_G$, and let $Z$ be the subspace of $\mathbb{C}[G]$ spanned by the columns of $A$. We have to prove that $Z = J$. To do this we introduce two auxiliary subspaces of $\mathbb{C}[G]$. Let $I$ be the ideal of $\mathbb{C}[G]$ generated by the irreducible characters $\chi_0$ and $1$ (where $\chi_0 + 1$ is the permutation character of $G$ acting on $P$), and let $W$ be the direct sum of the $\delta$-eigenspace and the $\tau$-eigenspace of $\Gamma_G$.

As each column of $A$ is the characteristic vector of a coset of the stabilizer of a point (and hence the characteristic vector of an independent set of maximal size by Corollary 4.4), the column space $Z$ of $A$ is a subspace of $J$, that is, $Z \subseteq J$. Moreover, $\dim Z = (q^2 + q)^2 + 1$ by Proposition 3.2.

By Lemma 4.2, if $S$ is an independent set of maximal size of $\Gamma_G$, then $\chi_S \in W$, that is, $J \subseteq W$. Moreover, $\dim W = (q^2 + q)^2 + \gcd(3, q - 1)$ by Lemma 4.3 (2).

If $\gcd(3, q - 1) = 1$, then $\dim W = \dim Z$ and hence $W = Z = J$. Suppose then that $\gcd(3, q - 1) = 3$. By Lemma 4.3 (3), $W$ is the ideal of $\mathbb{C}[G]$ generated by $\chi_0, 1$ and by the two non-principal linear characters of $G$, which we call $\xi$ and $\xi^2$. We will show that, for each independent set $S$ of maximal size, we have $\xi(S) = \xi^2(S) = 0$. Observe that this implies that $\chi_S$ is orthogonal to $\xi$ and $\xi^2$, and hence $\chi_S$ is contained in the ideal $I$ of $\mathbb{C}[G]$ generated by $\chi_0$ and $1$, that is, $J \subseteq I$.

Since $\dim I = \chi_0(1)^2 + 1 = (q^2 + q)^2 + 1 = \dim Z$, we have $I = Z$ and $J = Z$.

Write $G = \text{PSL}_3(q) \cup \text{PSL}_3(q)x \cup \text{PSL}_3(q)x^2$, for some $x \in G \setminus \text{PSL}_3(q)$. Let $S$ be an independent set of maximal size of $\Gamma_G$ and write $S_1 = S \cap \text{PSL}_3(q)x^i$, for $i \in \{0, 1, 2\}$. Now, $S_1x^{-i}$ is an independent set of $\Gamma_{\text{PSL}_3(q)}$ and, by Corollary 4.5, $S_1 = |S_1x^{-i}| = q^3(q^2 - 1)(q - 1)/3$.

As

$$|S_0| + |S_1| + |S_2| = |S| = q^3(q^2 - 1)(q - 1),$$

we have $|S_0| = |S_1| = |S_2|$. This shows that the independent set $S$ is equally distributed among the three cosets of $\text{PSL}_3(q)$ in $G$. Thus

$$\xi(S) = |S_0| + (\cos(2\pi/3) + i\sin(2\pi/3))|S_1| + (\cos(4\pi/3) + i\sin(4\pi/3))|S_2| = 0$$

and, similarly,

$$\xi^2(S) = |S_0| + (\cos(4\pi/3) + i\sin(4\pi/3))|S_1| + (\cos(2\pi/3) + i\sin(2\pi/3))|S_2| = 0.$$

□

5. PROOF OF PROPOSITION 3.3

The real work of this paper is proving Proposition 3.3. It is not difficult to show that each of the vectors given in the statement of Proposition 3.3 is in the right kernel of $M$, but what is difficult is showing that these vectors actually span the right kernel. We will show that the nullity of $M$ is no more than $4(q^2 + q + 1)$ by constructing sufficiently many eigenvectors of the matrix $M^TM$ with non-zero eigenvalues.

Set $N = M^TM$. In particular, $N$ is a square $(q^2 + q + 1)^2$-matrix whose rows and columns are indexed by the ordered pairs of points of $\text{PG}_2(q)$, and

$$N_{(\alpha, \beta), (\gamma, \delta)} = |\{g \in G \mid \alpha^g = \beta, \gamma^g = \delta, g \text{ derangement}\}|,$$

for each $\alpha, \beta, \gamma, \delta \in P$. Since $N$ is symmetric, we have

$$N_{(\alpha, \beta), (\gamma, \delta)} = N_{(\gamma, \delta), (\alpha, \beta)}.$$
Moreover, it follows at once from (8) that
\[(10) \quad N_{(\alpha, \beta), (\gamma, \delta)} = N_{(\beta, \alpha), (\delta, \gamma)}.
\]

The first goal in this section is to calculate the entries of the matrix $N$ as polynomials in $q$. To do this we need to count the number of monic irreducible polynomials.

**Lemma 5.1.** The number of monic irreducible polynomials of degree 1, 2 and 3 over $\mathbb{GF}(q)$ is $q$, $(q - 1)q/2$ and $(q^2 - 1)q/3$, respectively.

**Proof.** The number of irreducible polynomials of degree $n$ over $\mathbb{GF}(q)$ is given by
\[
\frac{1}{n} \sum_{d \mid n} \mu(d)q^{\frac{n}{d}}
\]
where $\mu$ represents the Möbius $\mu$-function [10, 14.3]. Since $\mu(1) = 1$ and $\mu(2) = \mu(3) = -1$ the result follows. \hfill \Box

In the following proposition we prove that the entry $N_{(\alpha, \beta), (\gamma, \delta)}$ of the matrix $N$ is determined by the geometric position of the 4-tuple $(\alpha, \beta, \gamma, \delta)$ in $\text{PG}_2(q)$. First, we define the following two numbers:
\[(11) \quad u = (q - 2)q(q^2 - 1)/3, \quad v = (q - 1)q(q^2 - 1)/3.
\]

**Proposition 5.2.** Let $\alpha, \beta, \gamma, \delta \in \mathcal{P}$, and let $u$ and $v$ be as in (11). Then
\[
N_{(\alpha, \beta), (\gamma, \delta)} = \begin{cases}
q^2v & \text{if } \alpha \neq \beta, \alpha = \gamma \text{ and } \beta = \delta, \\
v & \text{if } \alpha, \beta, \gamma, \delta \text{ are all distinct and exactly three of } \alpha, \beta, \gamma, \delta \text{ are collinear,} \\
v & \text{if } \alpha = \delta \text{ or } \beta = \gamma \text{ and not all four of } \alpha, \beta, \gamma, \delta \text{ are collinear,} \\
u & \text{if no three of } \alpha, \beta, \gamma, \delta \text{ are collinear,} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Write $n$ for $N_{(\alpha, \beta), (\gamma, \delta)}$. If $\alpha = \beta$ or if $\gamma = \delta$, then $n = 0$. In particular, we may assume that $\alpha \neq \beta$ and $\gamma \neq \delta$.

If $\alpha = \gamma$ and $\beta = \delta$, then $n$ is the number of derangements mapping $\alpha$ to $\beta$. Since $G$ is transitive of degree $q^2 + q + 1$ and since $G$ contains $(q^2 - 1)^2q^4/3$ derangements by Lemma 4.3 (1), we have $n = ((q^2 - 1)^2q^4/3)/(q^2 + q) = q^2v$. Further, if $\alpha = \gamma$ and $\beta \neq \delta$, or $\alpha \neq \gamma$ and $\beta = \delta$, then $n = 0$ because in both of these cases, there are no permutations that map $\alpha$ to $\beta$ and $\gamma$ to $\delta$.

From this point on we assume that $\alpha \neq \gamma$ and $\beta \neq \delta$ and we will consider different cases based on the spatial arrangement of $\alpha, \beta, \gamma, \delta$.

First suppose that $\alpha, \beta, \gamma, \delta$ are collinear. We are assuming $\alpha \neq \gamma$ and $\beta \neq \delta$, and hence $\alpha \lor \gamma = \beta \lor \delta$. Assume that there is a $g \in G$ with $\alpha^g = \beta$ and $\gamma^g = \delta$, then
\[
(\alpha \lor \gamma)^g = \alpha^g \lor \gamma^g = \beta \lor \delta = \alpha \lor \gamma.
\]
In particular, $g$ fixes the line $\alpha \lor \gamma$. By the duality between points and lines of $\text{PG}_2(q)$, we get that $g$ fixes some point of $\text{PG}_2(q)$ and cannot be a derangement; hence, in this case, $n = 0$.

Suppose that exactly three of $\alpha, \beta, \gamma, \delta$ are collinear. There are two cases to consider here: first, when all of $\alpha, \beta, \gamma$ and $\delta$ are distinct and second, when $|\{\alpha, \beta, \gamma, \delta\}| = 3$. In the second case either $\alpha = \delta$ or $\beta = \gamma$ (but not both).
Assume that $|\{\alpha, \beta, \gamma, \delta\}| = 4$. From the symmetries in (9) and (10), we may assume that $\alpha, \gamma$ and $\delta$ are collinear. Let $l$ be the line spanned by $\alpha, \gamma, \delta$. Since $G$ is transitive on the lines in $L$ and since the stabilizer $G_{\ell}$ of the line $\ell$ is simultaneously 2-transitive on the points in $P \setminus \ell$, we may assume that $\ell = \langle \varepsilon_1, \varepsilon_2 \rangle$, $\alpha = \langle \varepsilon_1 \rangle$, $\gamma = \langle \varepsilon_2 \rangle$, $\delta = \langle \varepsilon_1 + \varepsilon_2 \rangle$ and $\beta = \langle \varepsilon_3 \rangle$. In particular, if $g \in G$ and $\alpha^g = \beta$, $\gamma^g = \delta$, then

$$g = \begin{bmatrix} 0 & \lambda & x \\ 0 & \lambda & y \\ 1 & 0 & z \end{bmatrix},$$

for some $\lambda, x, y, z \in \mathbb{GF}(q)$. The element $g$ is a derangement of $G$ if and only if the characteristic polynomial $p_{\lambda,x,y,z}(T)$ of $g$ has no root in $\mathbb{GF}(q)$, and since $p_{\lambda,x,y,z}(T)$ has degree 3, this happens exactly when $p_{\lambda,x,y,z}(T)$ is irreducible. A simple calculation shows that

$$p_{\lambda,x,y,z}(T) = T^3 - (\lambda + z)T^2 - (\lambda z + x)T - \lambda y - x.$$

We claim that there are exactly $v$ choices of $(\lambda, x, y, z)$ such that $p_{\lambda,x,y,z}(T)$ is irreducible. Observe that $\lambda \neq 0$ because $g$ is invertible. It is a simple computation to check that, for every $\lambda \in \mathbb{GF}(q) \setminus \{0\}$, and for every $a, b, c \in \mathbb{GF}(q)$, there exists a unique choice of $x, y, z \in \mathbb{GF}(q)$ with

$$T^3 - aT^2 - bT - c = p_{\lambda,x,y,z}(T).$$

By Lemma 5.1, there are $(q^2 - 1)q/3$ choices of $a, b, c \in \mathbb{GF}(q)$ with $T^3 - aT^2 - bT - c$ irreducible. Therefore, for every given $\lambda \in \mathbb{GF}(q) \setminus \{0\}$, there exist $(q^2 - 1)q/3$ choices for $g$. As we have $q - 1$ choices for $\lambda$, we get $(q - 1)q(q^2 - 1)/3 = v$ choices for $g$ in total.

Consider the case $|\{\alpha, \beta, \gamma, \delta\}| = 3$; we have seen that this implies that either $\alpha = \delta$ or $\beta = \gamma$. From the symmetries in (9) and (10), we may assume that $\alpha = \delta$ and that $\alpha, \beta, \gamma$ are non-collinear (otherwise all four points are collinear). Now, replacing $\alpha, \beta$ and $\gamma$ if necessary by $\alpha^h, \beta^h$ and $\gamma^h$ for $h \in G$, we get $\alpha = \langle \varepsilon_1 \rangle$, $\beta = \langle \varepsilon_3 \rangle$ and $\gamma = \langle \varepsilon_2 \rangle$. In particular, if $g \in G$ and $\alpha^g = \beta$, $\gamma^g = \delta$, then

$$g = \begin{bmatrix} 0 & \lambda & x \\ 0 & 0 & y \\ 1 & 0 & z \end{bmatrix},$$

for some $\lambda, x, y, z \in \mathbb{GF}(q)$. Arguing as above, the element $g$ is a derangement of $G$ if and only if the characteristic polynomial $p_{\lambda,x,y,z}(T)$ of $g$ is irreducible. Now,

$$p_{\lambda,x,y,z}(T) = T^3 - zT^2 - xT - \lambda y. $$

It follows at once from Lemma 5.1 that there are exactly $v$ choices of $(\lambda, x, y, z)$ such that $p_{\lambda,x,y,z}(T)$ is irreducible.

Suppose that no three of $\alpha, \beta, \gamma, \delta$ are collinear (this implies that all four points are distinct). Since $G$ acts transitively on the ordered 4-tuples of non-collinear points (see Lemma 6.1) and since $N$ is $G$-invariant, we may assume that $\alpha = \langle \varepsilon_1 \rangle$, $\beta = \langle \varepsilon_3 \rangle$, $\gamma = \langle \varepsilon_2 \rangle$ and $\delta = \langle \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \rangle$. In particular, if $g \in G$ and $\alpha^g = \beta$, $\gamma^g = \delta$, then

$$g = \begin{bmatrix} 0 & \lambda & x \\ 0 & \lambda & y \\ 1 & \lambda & z \end{bmatrix},$$
for some \( \lambda, x, y, z \in \mathbb{GF}(q) \). Arguing as usual, the element \( g \) is a derangement of \( G \) if and only if the characteristic polynomial
\[
p_{\lambda,x,y,z}(T) = T^3 - (\lambda + z)T^2 - (\lambda y - \lambda z + x)T - \lambda(y - x)
\]
is irreducible. We claim that there are exactly \( u \) choices of \( (\lambda, x, y, z) \) such that \( p_{\lambda,x,y,z}(T) \) is irreducible. Observe that \( \lambda \neq 0 \) because \( g \) is invertible, and that when \( \lambda = -1 \) the polynomial \( p_{\lambda,x,y,z}(T) \) has root \(-1\). Thus \( \lambda \notin \{0, -1\} \).

Next, we need to check that, for every \( \lambda \in \mathbb{GF}(q) \setminus \{0, -1\} \), and for every \( a, b, c \in \mathbb{GF}(q) \), there exists a unique choice of \( x, y, z \in \mathbb{GF}(q) \) with
\[
T^3 - aT^2 - bT - c = p_{\lambda,x,y,z}(T).
\]
To do this, for a fixed \( \lambda \), set
\[
a = \lambda + z, \quad b = \lambda(y - z) + x, \quad c = \lambda(y - x).
\]
It is not hard to see that this system of linear equations (in \( x, y \) and \( z \)) has a unique solution if and only if \( \lambda \notin \{0, -1\} \).

By Lemma 5.1, there are \((q^2 - 1)q/3\) choices of \( a, b, c \in \mathbb{GF}(q) \) with \( T^3 - aT^2 - bT - c \) irreducible. Therefore, for every given \( \lambda \in \mathbb{GF}(q) \setminus \{0, -1\} \), there exist \((q^2 - 1)q/3\) choices for \( g \). As we have \( q - 2 \) choices for \( \lambda \), we get \((q - 2)(q - 1)^2q/3 = u \) choices for \( g \) in total.

Using Proposition 5.2 we deduce a number of properties of \( N \); the first two being that each of \( Ne^1_\alpha, Ne^2_\alpha \) and \( Ne_\ell \) (where \( e^1_\alpha, e^2_\alpha \) and \( e_\ell \) are defined in (2)) are equal to a multiple of the vector
\[
e = \sum_{\beta, \beta' \in \mathcal{P}} e_{\beta \beta'}.
\]
The \((\alpha, \beta)\)-coordinate of \( e \) is 1, unless \( \alpha = \beta \), and in this case the \((\alpha, \beta)\)-coordinate is 0.

**Lemma 5.3.** For \( \alpha \in \mathcal{P} \), we have \( Ne^1_\alpha = Ne^2_\alpha = q^2ve \) (where \( v \) is defined in (11)).

**Proof.** From (10), it suffices to show that \( Ne^1_\alpha = q^2ve \). From the definition of \( e^1_\alpha \), we have \( Ne^1_\alpha = \sum_{\beta \in \mathcal{P}} Ne_{\alpha\beta} \). In particular, by considering the \((\gamma, \delta)\)-coordinate of the vector \( Ne^1_\alpha \), it suffices to prove that
\[
\sum_{\beta \in \mathcal{P}} N_{(\gamma, \delta), (\alpha, \beta)} = \begin{cases} q^2v & \text{if } \gamma \neq \delta, \\ 0 & \text{if } \gamma = \delta. \end{cases}
\]
Clearly, the \((\gamma, \delta)\)-row of \( N \) is 0 when \( \gamma = \delta \). Next we consider three cases.

First, suppose that \( \alpha = \gamma \). Then Proposition 5.2 shows that the only non-zero summand in the left-hand side of (13) occurs for \( \beta = \delta \) with value \( q^2v \).

Second, suppose that \( \alpha \in \gamma \lor \delta \) and \( \alpha \neq \gamma \). Now Proposition 5.2 shows that the only non-zero summands in the left-hand side of (13) occur when \( \beta \notin \gamma \lor \delta \) with value \( v \). Since we have \( q^2 \) choices for \( \beta \), the equality in (13) follows.

Finally, suppose that \( \alpha \notin \gamma \lor \delta \). Observe that there are \((q - 1)^2\) choices for a point \( \beta \) with \( \alpha, \beta, \gamma, \delta \) non-collinear and hence, by Proposition 5.2, these choices of \( \beta \) contribute \((q - 1)^2u \) to the summation in (13). Next, we have \( 3(q - 1) \) choices for a point \( \beta \) with \(|\{\alpha, \beta, \gamma, \delta\}| = 4 \) and with exactly three of \( \alpha, \beta, \gamma, \delta \) collinear (namely, \( q - 1 \) choices for \( \beta \) depending on whether \( \alpha, \beta, \gamma \), or \( \alpha, \beta, \delta \), or \( \beta, \gamma, \delta \) are collinear). Each of these terms contributes \( v \). In view of Proposition 5.2, the only
remaining choice for $\beta$ that gives a non-zero contribution is when $\beta = \gamma$, and this single value contributes $v$ to the sum. Thus, in this case the left-hand side of (13) equals

$$(q - 1)^2u + 3(q - 1)v + v = q^2v.$$ 

\[\square\]

**Lemma 5.4.** For $\ell \in \mathcal{L}$, we have $Ne_{\ell} = q^2ve$ (where $v$ is defined in (11)).

**Proof.** From the definition of $e_{\ell}$, we have $Ne_{\ell} = \sum_{\alpha,\beta \in \mathcal{L}} Ne_{\alpha\beta}$. In particular, by taking the $(\gamma, \delta)$-coordinate of the vector $Ne_{\ell}$, it suffices to prove that

$$\sum_{\alpha,\beta \in \ell} N_{(\gamma, \delta), (\alpha, \beta)} = \begin{cases} q^2v & \text{if } \gamma \neq \delta, \\ 0 & \text{if } \gamma = \delta. \end{cases}$$

If $\gamma = \delta$, then the $(\gamma, \delta)$-row of $N$ is 0, and the result holds. Suppose then $\gamma \neq \delta$. If $\gamma \land \delta = \ell$, then $N_{(\gamma, \delta), (\alpha, \beta)} = 0$, unless $\alpha = \gamma$ and $\beta = \delta$, and in this case $N_{(\gamma, \delta), (\alpha, \beta)} = q^2v$.

Consider the case $\gamma \in \ell$ and $\delta \notin \ell$. The value of $N_{(\gamma, \delta), (\alpha, \beta)}$ is 0 if $\gamma = \alpha$. If $\gamma = \beta$, then $N_{(\gamma, \delta), (\alpha, \beta)} = v$ for all of the $q$ possible values of $\alpha \in \ell \setminus \{\beta\}$. There are $q(q - 1)$ values of $\alpha$ and $\beta$ so that all of $\gamma$, $\alpha$, $\beta$ are distinct. For each of these values $N_{(\gamma, \delta), (\alpha, \beta)} = v$. Thus, in this case

$$\sum_{\alpha,\beta \in \ell} N_{(\gamma, \delta), (\alpha, \beta)} = qv + q(q - 1)v = q^2v.$$ 

Similarly, the same result holds if $\delta \in \ell$ and $\gamma \notin \ell$.

The final case to consider is when neither $\gamma$ nor $\delta$ are on $\ell$. If $\alpha$ is on $\gamma \land \delta$ (so $\alpha = (\gamma \land \delta) \land \ell$), then for each of the $q$ points $\beta \in \ell \setminus \{\alpha\}$, the value of $N_{(\gamma, \delta), (\alpha, \beta)}$ is $v$. The same holds if $\beta$ is on $\gamma \land \delta$. For the remaining $q(q - 1)$ points $\alpha \neq \beta$ that are on $\ell$ but not on $\gamma \land \delta$, the value of $N_{(\gamma, \delta), (\alpha, \beta)}$ is $u$. In this case

$$\sum_{\alpha,\beta \in \ell} N_{(\gamma, \delta), (\alpha, \beta)} = qv + qv + q(q - 1)u = q^2v.$$ 

\[\square\]

**Lemma 5.5.** Let $\bar{\alpha} \in \mathcal{P}$ and let $\bar{\ell} \in \mathcal{L}$. Then the subspace

$$V_0 = \langle e_{\alpha\alpha}, e^1_{\alpha} - e^1_{\bar{\alpha}}, e^2_{\alpha} - e^2_{\bar{\alpha}}, e_{\ell} - e_{\bar{\ell}} \mid \alpha \in \mathcal{P}, \ell \in \mathcal{L} \rangle$$

of $V$ has dimension $4(q^2 + q) + 1$ and is contained in the right kernel of $M$. In particular, $V_0$ is contained in the kernel of $N$.

**Proof.** As the $(\alpha, \alpha)$-column of $M$ is zero, the vector $e_{\alpha\alpha}$ is clearly in the right kernel of $M$. From Lemma 5.3, the vectors $e^1_{\alpha} - e^1_{\bar{\alpha}}$ and $e^2_{\alpha} - e^2_{\bar{\alpha}}$ are in the kernel of $N$ and since $N = MTM$, they are also in the right kernel of $M$. Similarly, by Lemma 5.4, the vector $e_{\ell} - e_{\bar{\ell}}$ is in the right kernel of $M$.

Finally we need to confirm that the dimension of $V_0$ is $4(q^2 + q) + 1$. Assume

$$\sum_{\alpha \in \mathcal{P}} a_{\alpha} e_{\alpha\alpha} + \sum_{\alpha \in \mathcal{P} \setminus \{\bar{\alpha}\}} b_{\alpha} (e^1_{\alpha} - e^1_{\bar{\alpha}}) + \sum_{\alpha \in \mathcal{P} \setminus \{\bar{\alpha}\}} c_{\alpha} (e^2_{\alpha} - e^2_{\bar{\alpha}}) + \sum_{\ell \in \mathcal{L} \setminus \{\bar{\ell}\}} d_{\ell} (e_{\ell} - e_{\bar{\ell}}) = 0,$$

for some scalars $a_{\alpha}, b_{\alpha}, c_{\alpha}$ and $d_{\ell}$. Let $\gamma$ and $\delta$ be distinct elements in $\mathcal{P} \setminus \{\bar{\alpha}\}$ with $\gamma \land \delta \neq \ell$. The $(\gamma, \delta)$-coordinate in the above linear combination yields

$$b_{\gamma} + c_{\delta} + d_{\gamma \land \delta} = 0.$$

(14) 

Thus, the dimension of $V_0$ is

$$4(q^2 + q) + 1.$$ 

\[\square\]
Observe that if \( \gamma' \in (\gamma \vee \delta) \setminus \{\alpha, \delta\} \) and \( \delta' \in (\gamma \vee \delta) \setminus \{\alpha, \gamma\} \), then (by considering the
\((\gamma', \delta')\)-coordinate and the \((\gamma, \delta')\)-coordinate) we also get the equations
\[
\beta_{\gamma'} + c_{\delta} + d_{\gamma \vee \delta} = 0, \quad \beta_{\gamma} + c_{\delta'} + d_{\gamma \vee \delta} = 0.
\]
Hence \( \beta_{\gamma} = \beta_{\gamma'} \) and \( c_{\delta} = c_{\delta'} \). From this, an easy connectedness argument yields that there exist \( b, c \in \mathbb{C} \) with \( b = b_{\alpha} \) and \( c = c_{\alpha} \), for every \( \alpha \in \mathcal{P} \setminus \{\bar{\alpha}\} \). This, in turn, implies that there exists a scalar \( d \in \mathbb{C} \) with
\( d = \delta_c \), for every \( \ell \in \mathcal{L} \setminus \{\bar{\ell}\} \).

Note that (14) gives \( b + c + d = 0 \).

Let \( \gamma \) and \( \delta \) be distinct elements in \( \mathcal{P} \setminus \{\alpha\} \) with \( \gamma \vee \delta \neq \bar{\ell} \). By considering the
\((\bar{\alpha}, \delta)\)-coordinate and the \((\gamma, \bar{\alpha})\)-coordinate, we get the equations
\[
(q^2 + q)b - c - d = 0, \quad -b + (q^2 + q)c - d = 0.
\]
Putting these two equations together with \( b + c + d = 0 \), yields that \( b = c = d = 0 \).

Finally, for \( \alpha \in \mathcal{P} \), by considering the \((\alpha, \alpha)\)-coordinate, we obtain \( a_{\alpha} = 0 \).

\[\Box\]

Lemma 5.6. The vector \( e \) is an eigenvector of \( N \) with eigenvalue \( (q^2 + q + 1)q^2v \).

Proof. Observe that \( e = \sum_{\alpha \in \mathcal{P}} (e_{\alpha}^1 - e_{\alpha \alpha}) \) and that \( Ne_{\alpha \alpha} = 0 \). From Lemma 5.3, we get
\[
Ne = \sum_{\alpha \in \mathcal{P}} Ne_{\alpha}^1 = \sum_{\alpha \in \mathcal{P}} q^2ve = \left( \sum_{\alpha \in \mathcal{P}} q^2v \right) e = (q^2 + q + 1)q^2ve.
\]

Before exhibiting other eigenvectors of \( N \), we need to define two families of
vectors. Both families are indexed by the pairs \((\alpha, \ell)\), where \( \alpha \) is a point on the line \( \ell \):
\[
(15) \quad e_{\alpha \ell} = \sum_{\beta \in \ell} e_{\alpha \beta}, \quad e_{\ell \alpha} = \sum_{\beta \in \ell} e_{\beta \alpha}.
\]
These vectors will be used to construct new eigenvectors of \( N \), thus our first step is to calculate the value of \( Ne_{\alpha \ell} \) and \( Ne_{\ell \alpha} \).

Lemma 5.7. Let \( \alpha \in \mathcal{P} \) and let \( \ell \in \mathcal{L} \) with \( \alpha \in \ell \). Then
\[
(16) \quad Ne_{\alpha \ell} = q^2v \sum_{\substack{\beta \in \ell \\ \beta \neq \alpha}} e_{\alpha \beta} + (q - 1)v \sum_{\gamma \in \ell} e_{\gamma} + qv \left( \sum_{\substack{\gamma \in \ell \\ \gamma \neq \alpha}} e_{\gamma} + \sum_{\eta \neq \gamma} e_{\eta} \right).
\]

Proof. Let \( \gamma, \delta \in \mathcal{P} \). Denote by \( w \) the vector on the right-hand side of (16). We will compare the \((\gamma, \delta)\)-coordinate of \( Ne_{\alpha \ell} \) and \( w \). First note that if \( \gamma = \delta \), then \((Ne_{\alpha \ell})(\gamma, \delta) = w_{(\gamma, \delta)} = 0 \), and hence we will suppose that \( \gamma \neq \delta \).

The \((\gamma, \delta)\)-coordinate of right-hand side of (16) can be expressed as
\[
(17) \quad w_{(\gamma, \delta)} = \begin{cases}
q^2v & \text{if } \ell = \gamma \vee \delta \text{ and } \gamma = \alpha, \\
0 & \text{if } \ell = \gamma \vee \delta \text{ and } \gamma \neq \alpha, \\
0 & \text{if } \ell \neq \gamma \vee \delta \text{ and } \gamma = \alpha, \\
(q - 1)v & \text{if } \ell \neq \gamma \vee \delta, \gamma \neq \alpha \text{ and } \ell \land (\gamma \vee \delta) \notin \{\alpha, \gamma\}, \\
qv & \text{if } \ell \neq \gamma \vee \delta, \gamma \neq \alpha \text{ and } \ell \land (\gamma \vee \delta) \in \{\alpha, \gamma\}.
\end{cases}
\]
From the definition of $e_{\alpha\ell}$ in (15) and Lemma 5.5, we get $Ne_{\alpha\ell} = \sum_{\beta \in E \setminus \{\alpha\}} Ne_{\alpha\beta}$. In particular, $(Ne_{\alpha\ell})(\gamma, \delta) = \sum_{\beta \in E \setminus \{\alpha\}} N_{(\gamma, \delta), (\alpha, \beta)}$. We now compare this number with (17).

First assume that $\ell = \gamma \lor \delta$. If $\alpha = \gamma$, then from Proposition 5.2 we see that $N_{(\gamma, \delta), (\alpha, \beta)} \neq 0$ only if $\beta = \delta$ and in this case its value is $q^2v$. If $\alpha \neq \gamma$, then Proposition 5.2 gives $\sum_{\beta \in E \setminus \{\alpha\}} N_{(\gamma, \delta), (\alpha, \beta)} = 0$.

Now assume that $\ell \neq \gamma \lor \delta$ and $\gamma = \alpha$. As $\ell \neq \gamma \lor \delta$, we get $\delta \notin \ell$, and hence $\delta \neq \beta$, for all $\beta \in \ell \setminus \{\alpha\}$. Thus $\sum_{\beta \in E \setminus \{\alpha\}} N_{(\gamma, \delta), (\alpha, \beta)} = 0$.

Now assume that $\ell \neq \gamma \lor \delta$, $\gamma \neq \alpha$ and $\ell \land (\gamma \lor \delta) \notin \{\alpha, \gamma\}$. Here we consider two cases depending on whether $\ell \land (\gamma \lor \delta) = \delta$ or $\ell \land (\gamma \lor \delta) \neq \delta$. If $\ell \land (\gamma \lor \delta) = \delta$, then Proposition 5.2 gives that $N_{(\gamma, \delta), (\alpha, \beta)} = v$ for every $\beta \in \ell \setminus \{\alpha, \delta\}$, and $N_{(\gamma, \delta), (\alpha, \beta)} = 0$ when $\beta = \delta$. Thus $\sum_{\beta \in E \setminus \{\alpha\}} N_{(\gamma, \delta), (\alpha, \beta)} = (q - 1)v$. Now suppose that $\ell \land (\gamma \lor \delta) \neq \delta$, and hence $\ell \land (\gamma \lor \delta) \notin \{\alpha, \gamma, \delta\}$. Define $\alpha' = \ell \land (\gamma \lor \delta)$. Now, Proposition 5.2 gives that $N_{(\gamma, \delta), (\alpha, \beta)} = v$ for every $\beta \in \ell \setminus \{\alpha', \alpha\}$, and $N_{(\gamma, \delta), (\alpha, \beta)} = v$ when $\beta = \alpha'$. Thus $\sum_{\beta \in E \setminus \{\alpha\}} N_{(\gamma, \delta), (\alpha, \beta)} = (q - 1)v + v = (q - 1)v$.

Finally, assume that $\ell \neq \gamma \lor \delta$, $\gamma \neq \alpha$ and $\ell \land (\gamma \lor \delta) \in \{\alpha, \gamma\}$. Here Proposition 5.2 gives $N_{(\gamma, \delta), (\alpha, \beta)} = v$, for every $\beta \in \ell \setminus \{\alpha\}$. Thus $\sum_{\beta \in E \setminus \{\alpha\}} N_{(\gamma, \delta), (\alpha, \beta)} = qv$. □

Remark 5.8. In the proof of Lemma 5.9 we will use (16). However, for the computations there it is convenient to express the equality in (16) as a linear combination of vectors of the form $e_{\alpha\ell}v$. It is straightforward to see that

$$Ne_{\alpha\ell} = q^2v(e_{\alpha\ell} - e_{\alpha\alpha}) + (q - 1)v \left( \sum_{\gamma \neq \ell} (e^1_{\gamma} - e_{\gamma(\alpha \lor \gamma)}) \right)$$

$$+ qv \left( \sum_{\gamma \neq \ell} (e_{\gamma(\alpha \lor \gamma)} - e_{\gamma\gamma}) + \sum_{\gamma \in \ell \setminus \{\alpha\}} (e^1_{\gamma} - e_{\gamma\ell}) \right).$$

We define two new sets of vectors of $V$ that we will show are eigenvectors of $N$. Given three non-collinear points $\alpha, \beta, \gamma$ we write

$$e^1_{\alpha\beta\gamma} = (e_{\alpha(\beta \lor \gamma)} - e_{\beta(\alpha \lor \gamma)}) + (e_{\beta(\gamma \lor \alpha)} - e_{\gamma(\beta \lor \alpha)}) + (e_{\gamma(\alpha \lor \beta)} - e_{\alpha(\gamma \lor \beta)}),$$

$$e^2_{\alpha\beta\gamma} = (e_{\alpha(\beta \lor \gamma)} - e_{\alpha(\beta \lor \gamma)}) + (e_{\beta(\gamma \lor \alpha)} - e_{\gamma(\beta \lor \alpha)}) + (e_{\gamma(\alpha \lor \beta)} - e_{\gamma(\alpha \lor \beta)}).$$

Lemma 5.9. Let $\alpha \in \mathcal{P}$ and let $\ell \in \mathcal{L}$ with $\alpha \in \ell$. If $\alpha, \beta, \gamma$ are non-collinear, then $e^1_{\alpha\beta\gamma}$ and $e^2_{\alpha\beta\gamma}$ are eigenvectors of $N$ with eigenvalue $(q^2 + q + 1)v$. Moreover, the subspace $(e^1_{\alpha\beta\gamma}, e^2_{\alpha\beta\gamma} | \alpha, \beta, \gamma \text{ non-collinear points})$ of $V$ has dimension $2q^3$.

Proof. Let $\alpha, \beta, \gamma$ be non-collinear points. We start by showing that $Ne^1_{\alpha\beta\gamma} = e^1_{\alpha\beta\gamma}(q^2 + q + 1)v$. Observe that the same equality holds for $e^2_{\alpha\beta\gamma}$ from (10).
From (16) and Remark 5.8 applied with \( \ell = \alpha \vee \beta \), we obtain that

\[
N(e_{\alpha(\alpha \vee \beta)} - e_{\beta(\alpha \vee \beta)}) = q^2 v (e_{\alpha(\alpha \vee \beta)} - e_{\beta(\alpha \vee \beta)}) - q^2 v (e_{\alpha \alpha} - e_{\beta \beta})
+ (q - 1) v \left( \sum_{\zeta \in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)} - e_{\zeta(\alpha \alpha)} \right)
+ q v \left( \sum_{\zeta \in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)} - e_{\zeta(\alpha \alpha)} \right)
+ q v \left( \sum_{\eta \not\in \alpha \vee \beta} e_{\eta \eta} - e_{\alpha \eta} \right)
= q^2 v (e_{\alpha(\alpha \vee \beta)} - e_{\beta(\alpha \vee \beta)}) - q^2 v (e_{\alpha \alpha} - e_{\beta \beta})
+ v \left( \sum_{\eta \not\in \alpha \vee \beta} e_{\eta \eta} - e_{\alpha \eta} \right)
+ q v \left( \sum_{\eta \not\in \alpha \vee \beta} e_{\eta \eta} - e_{\alpha \eta} \right).
\]

Applying this formula to each of \( e_{\alpha(\alpha \vee \beta)} - e_{\beta(\alpha \vee \beta)}, e_{\beta(\beta \vee \gamma)} - e_{\gamma(\beta \vee \gamma)} \) and \( e_{\gamma(\alpha \vee \gamma)} - e_{\alpha(\alpha \vee \gamma)} \), we get that \( N e_{\alpha, \beta, \gamma} \) equals

\[
q^2 v (e_{\alpha(\alpha \vee \beta)} - e_{\beta(\alpha \vee \beta)} - e_{\beta(\beta \vee \gamma)} - e_{\gamma(\beta \vee \gamma)} - e_{\gamma(\alpha \vee \gamma)} - e_{\alpha(\alpha \vee \gamma)})
+ v \left( \sum_{\zeta \in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)} - e_{\zeta(\alpha \alpha)} \right)
+ q v \left( \sum_{\eta \not\in \alpha \vee \beta} e_{\eta \eta} - e_{\alpha \eta} \right).
\]

Write \( N e_{\alpha, \beta, \gamma} = q^2 v X_1 + v X_2 + q v X_3 \), where \( q^2 v X_1, v X_2, q v X_3 \) are the three summands in the above equation. From (18), we have \( X_1 = e_{\beta, \gamma, \gamma} \), we will show that \( X_2 \) and \( X_3 \) equal \( e_{\beta, \gamma, \gamma} \) as well.

Note that by rearranging the terms, \( X_2 \) can be rewritten as

\[
(19) \sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \alpha} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \alpha} + \sum_{\zeta \in \beta \vee \gamma} e_{\zeta \zeta \gamma} - \sum_{\zeta \not\in \beta \vee \gamma} e_{\zeta \zeta \gamma} - \sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \beta} + \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \beta} + \sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \gamma} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \gamma}.
\]

Observe now that

\[
\sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)} = \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta(\alpha \vee \beta)},
\]

and that a similar equality holds for the other two terms in (19). Therefore \( X_2 \) equals

\[
\sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \alpha} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \alpha} + \sum_{\zeta \in \beta \vee \gamma} e_{\zeta \zeta \gamma} - \sum_{\zeta \not\in \beta \vee \gamma} e_{\zeta \zeta \gamma} - \sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \beta} + \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \beta} + \sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \gamma} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \gamma}
= (\sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \alpha} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \alpha}) + (\sum_{\zeta \in \beta \vee \gamma} e_{\zeta \zeta \beta} - \sum_{\zeta \not\in \beta \vee \gamma} e_{\zeta \zeta \beta}) + (\sum_{\zeta \in \alpha \vee \beta} e_{\zeta \zeta \gamma} - \sum_{\zeta \not\in \alpha \vee \beta} e_{\zeta \zeta \gamma})
= (e_{\alpha(\alpha \vee \beta)} - e_{\beta(\alpha \vee \beta)}) + (e_{\beta(\beta \vee \gamma)} - e_{\gamma(\beta \vee \gamma)}) + (e_{\gamma(\alpha \vee \gamma)} - e_{\alpha(\alpha \vee \gamma)}) = e_{\alpha, \beta, \gamma}.
\]
By rearranging the terms, we see that $X_3$ equals
\[
\left( \sum_{\eta \notin \gamma \vee \alpha} e_{\alpha \eta} - \sum_{\eta \notin \alpha \vee \beta} e_{\alpha \eta} \right) + \left( \sum_{\eta \notin \gamma \vee \beta} e_{\beta \eta} - \sum_{\eta \notin \gamma \vee \gamma} e_{\beta \eta} \right) + \left( \sum_{\eta \notin \gamma \vee \alpha} e_{\gamma \eta} - \sum_{\eta \notin \gamma \vee \alpha} e_{\gamma \eta} \right),
\]
which reduces to $(e_{\alpha \alpha \beta} - e_{\alpha \alpha \gamma}) + (e_{\beta \beta \gamma} - e_{\beta \beta \gamma}) + (e_{\gamma \gamma \alpha} - e_{\gamma \gamma \beta}) = e_{\alpha \beta \gamma}^1$. Putting all of these together, we have that $N e_{\alpha \beta \gamma}^1 = (q^2 + q + 1) v e_{\alpha \beta \gamma}^1$.

It remains to show that these vectors span a subspace of dimension $2q^3$. Fix $\bar{\ell}$ and $\bar{\alpha} \in \bar{\ell}$. We will show that these vectors span a subspace of dimension $2q^3$. Fix $\bar{\ell}$ and $\bar{\alpha} \in \bar{\ell}$. We will show that the vectors in
\[
T = \{ e_{\alpha \beta \gamma}^1, e_{\alpha \beta \gamma}^2 | \beta \in \mathcal{P} \setminus \bar{\ell}, \gamma \in \bar{\ell} \setminus \{ \bar{\alpha} \} \}
\]
are linearly independent. Assume that
\[
(20) \quad \sum_{\beta \in \mathcal{P} \setminus \bar{\ell}} a_{\beta \gamma} e_{\alpha \beta \gamma}^1 + \sum_{\beta \in \mathcal{P} \setminus \bar{\ell}} b_{\beta \gamma} e_{\alpha \beta \gamma}^2 = 0.
\]

Pick $\zeta$ and $\eta$ in $\mathcal{P} \setminus \bar{\ell}$ with $(\zeta \lor \eta) \land \bar{\ell} \neq \bar{\alpha}$. Then the $(\zeta, \eta)$-coordinate of $e_{\alpha \beta \gamma}^1$ is non-zero only if $\beta = \zeta$ and $\gamma = (\zeta \lor \eta) \land \bar{\ell}$. Similarly, the $(\zeta, \eta)$-coordinate of $e_{\alpha \beta \gamma}^2$ is non-zero only if $\beta = \eta$ and $\gamma = (\zeta \lor \eta) \land \bar{\ell}$. Thus, for any three collinear points $\zeta, \eta, \gamma$ with $\gamma \in \bar{\ell} \setminus \{ \bar{\alpha} \}$, we have
\[
(21) \quad a_{\zeta \gamma} + b_{\eta \gamma} = 0.
\]
Next consider $\zeta \in \bar{\ell} \setminus \{ \bar{\alpha} \}$ and $\eta \notin \bar{\ell}$. The $(\zeta, \eta)$-coordinate of $e_{\alpha \beta \gamma}^1$ is non-zero only if $\gamma = \zeta$ and $\beta \in (\zeta \lor \eta) \setminus (\zeta \lor \bar{\alpha})$. Similarly, the $(\zeta, \eta)$-coordinate of $e_{\alpha \beta \gamma}^2$ is non-zero only if $\beta = \eta$ and $\gamma = \zeta$. Thus from the $(\zeta, \eta)$-coordinate of (20), we can determine that
\[
b_{\eta \zeta} = \sum_{\beta \in (\zeta \lor \eta) \setminus (\zeta \lor \bar{\alpha})} a_{\beta \zeta} = 0.
\]
Since $\beta, \eta$ and $\zeta$ are collinear, by (21) $a_{\beta \zeta} = -b_{\eta \zeta}$, for each $\beta$. Thus $a_{\eta \gamma} = b_{\eta \gamma} = 0$, for all $\eta \notin \bar{\ell}$ and $\gamma \in \bar{\ell} \setminus \{ \bar{\alpha} \}$. \hfill \Box

We give one last family of eigenvectors of $N$, these are based on four collinear points. Given four distinct collinear points $\alpha, \beta, \gamma, \delta$, define
\[
e_{\alpha \beta \gamma \delta} = (e_{\alpha \gamma} - e_{\alpha \delta}) + (e_{\beta \delta} - e_{\beta \gamma}).
\]

**Lemma 5.10.** If $\alpha, \beta, \gamma, \delta$ are four distinct collinear points, then $e_{\alpha \beta \gamma \delta}$ is an eigenvector of $N$ with eigenvalue $q^2 v$. Moreover, for $q > 2$, the subspace
\[
\langle e_{\alpha \beta \gamma \delta} | \alpha, \beta, \gamma, \delta \text{ distinct collinear points} \rangle
\]
of $V$ has dimension at least $(q^2 + q + 1)(q^2 - q - 1)$.

**Proof.** Let $\zeta, \eta \in \mathcal{P}$ and let $\ell$ be the line containing $\alpha, \beta, \gamma, \delta$. From Proposition 5.2, we see with a case-by-case analysis that $(Ne_{\alpha \gamma})_{(\zeta, \eta)} \neq (Ne_{\alpha \delta})_{(\zeta, \eta)}$ only if one of the following occurs:

(i) $\zeta = \alpha$ and $\eta = \gamma$;

(ii) $\zeta = \alpha$ and $\eta = \delta$;

(iii) $\zeta = \delta$ and $\eta \notin \ell$;

(iv) $\zeta = \gamma$ and $\eta \notin \ell$;

(v) $\zeta, \eta \notin \ell$, $\zeta \neq \eta$ and $\{\zeta, \eta, \gamma\}$ are collinear.
Now, by considering these six cases separately, we get

\[
(N(e_{\alpha \gamma} - e_{\alpha \delta}))(\zeta, \eta) = \begin{cases}
q^2 v & \text{if } \zeta = \alpha \text{ and } \eta = \gamma, \\
-q^2 v & \text{if } \zeta = \alpha \text{ and } \eta = \delta, \\
v & \text{if } \zeta = \gamma \text{ and } \eta \notin \ell, \\
-v & \text{if } \zeta = \delta \text{ and } \eta \notin \ell, \\
v - u & \text{if } \zeta, \eta \notin \ell, \zeta \in \gamma \vee \eta \text{ and } \zeta \neq \eta, \\
-(v - u) & \text{if } \zeta, \eta \notin \ell, \zeta \in \delta \vee \eta \text{ and } \zeta \neq \eta, \\
0 & \text{otherwise}.
\end{cases}
\]

This shows that

\[
N(e_{\alpha \gamma} - e_{\alpha \delta}) = q^2 v(e_{\alpha \gamma} - e_{\alpha \delta}) + v \left( (e_1^\gamma - e_\gamma) - (e_\delta^3 - e_{\delta \gamma}) \right) + (v - u) \left( \sum_{\eta \in P \setminus \ell} (e_{\eta(\gamma \lor \eta)} - e_{\eta \eta}) - \sum_{\eta \in P \setminus \ell} (e_{\eta(\delta \lor \eta)} - e_{\eta \eta}) \right).
\]

An immediate application of this formula to \(N(e_{\alpha \gamma} - e_{\alpha \delta})\) and \(N(e_{\beta \delta} - e_{\beta \gamma})\) gives

\[
N(e_{\alpha \beta \gamma \delta}) = q^2 v e_{\alpha \beta \gamma \delta},
\]

and \(e_{\alpha \beta \gamma \delta}\) is an eigenvector of \(N\) with eigenvalue \(q^2 v\).

For \(q > 2\), it remains to give a lower bound on the dimension of

\[
W = \langle e_{\alpha \beta \gamma \delta} \mid \alpha, \beta, \gamma, \delta \text{ distinct collinear points} \rangle.
\]

For \(\ell \in L\), set \(W_\ell = \langle e_{\alpha \beta \gamma \delta} \mid \alpha, \beta, \gamma, \delta \text{ distinct points in } \ell \rangle\). From the definition of \(W\), we have \(W = \bigoplus_{\ell \in L} W_\ell\). Moreover, for \(\ell, \ell' \in L\) with \(\ell \neq \ell'\), we see that \(W_\ell\) is orthogonal (with respect to the standard scalar product) to \(W_{\ell'}\), because no two distinct points can lie in \(\ell \land \ell'\). Thus \(W = \bigoplus_{\ell \in L} W_\ell\). Since \(|L| = q^2 + q + 1\), it remains to prove that \(\dim W_\ell \geq q^2 - q - 1\).

Fix \(\ell \in L\) and write \(\ell = \{\alpha_0, \ldots, \alpha_q\}\). Consider the family of vectors

\[
F' = \langle e_{\alpha_i \alpha_j} \mid 0 \leq i, j \leq q - 2, i \neq j \rangle \cup \langle e_{\alpha_i \alpha_{q-i}} \mid 0 \leq i \leq q - 3 \rangle
\]

\[
\cup \langle e_{\alpha_{q-i} \alpha_i} \mid 0 \leq i \leq q - 3 \rangle
\]

and set \(F = F' \cup \langle e_{\alpha_{q-i} \alpha_{q-j}} \rangle\). Clearly, \(|F| = (q - 1)(q - 2) + (q - 2) + (q - 2) + 1 = q^2 - q - 1\). We claim that the vectors in \(F\) are linearly independent. To see this order the elements \((e_{\alpha_i \alpha_j})_{0 \leq i,j \leq q}\) with the lexicographic order, that is, \(e_{\alpha_i \alpha_j} < e_{\alpha_i' \alpha_j'}\) if \(i < i'\), or \(i = i'\) and \(j < j'\). By writing each \(e_{\alpha \beta \gamma \delta}\) as a \(\{0, 1\}\)-vector of length \((q + 1)^2\) with respect to the basis \((e_{\alpha_i \alpha_j})_{0 \leq i,j \leq q}\) and with respect to this ordering, we see that the elements in \(F'\) are in echelon form (specifically, when the elements of \(F'\) are ordered appropriately, the position of the first non-zero entry in each vector is strictly increasing). This implies that the elements in \(F'\) are linearly independent. Finally, it is easy to see that \(e_{\alpha_{q-i} \alpha_{q-j}}\) is linearly independent with the elements of \(F'\) as the \((q, q - 1)\)-entry is non-zero in this vector, but it is equal to zero in every vector in \(F'\).

___

**Proof of Proposition 3.3.** If \(q = 2\), then the proof follows with a computation with the computer algebra system **magma** [6], so we assume that \(q > 2\).

Let \(V_0\) be the subspace defined in Lemma 5.5. From Lemmas 5.6, 5.9 and 5.10, the rank of \(N\) is at least \(1 + 2q^3 + (q^2 + q + 1)(q^2 - q - 1)\) and hence the kernel of \(N\) has
dimension no more than \((q^2 + q + 1)^2 - (1 + 2q^3 + (q^2 + q + 1)(q^2 - q - 1)) = 4(q^2 + q) + 1\). Therefore Lemma 5.5 gives that the kernel of \(N\) has dimension exactly \(4(q^2 + q) + 1\) and hence equals \(V_0\). As \(N = M^T M\), we get that the right kernel \(M\) is also \(V_0\). \(\square\)

**Acknowledgment.** We would like to thank the anonymous referees, particularly the referee who carefully checked every one of our equations and offered several suggestions that improved this paper.

6. Appendix

The group \(\text{GL}_{n+1}(q)\) consists of all non-singular \((n + 1) \times (n + 1)\)-matrices with coefficients from \(\mathbb{GF}(q)\). The center of \(\text{GL}_{n+1}(q)\) is the set

\[
Z = \{cI_{n+1} \mid c \in \mathbb{GF}(q) \setminus \{0\}\},
\]

where \(I_{n+1}\) denotes the identity matrix of \(\text{GL}_{n+1}(q)\). The projective general linear group, \(\text{PGL}_{n+1}(q)\), is defined to be \(\text{GL}_{n+1}(q)/Z\). Hence the order of \(\text{PGL}_{n+1}(q)\) is

\[
|\text{PGL}_{n+1}(q)| = \frac{|\text{GL}_{n+1}(q)|}{|Z|} = \frac{\prod_{i=0}^{n} (q^{n+1} - q^i)}{q - 1}.
\]

We let \(\mathbb{GF}(q)^{n+1}\) denote the \((n + 1)\)-dimensional vector space over \(\mathbb{GF}(q)\) of row vectors. The mapping \(\tau : \text{GL}_{n+1}(q) \times \mathbb{GF}(q)^{n+1} \to \mathbb{GF}(q)^{n+1}\) defined by \(\tau(A, v) = vA\) determines a faithful group action of \(\text{GL}_{n+1}(q)\) on \(\mathbb{GF}(q)^{n+1}\). Since every element of \(Z\) fixes every 1-dimensional subspace of \(\mathbb{GF}(q)^{n+1}\), the mapping \(\tau(A, v) = vA\) also defines a faithful action of \(\text{PGL}_{n+1}(q)\) on the projective space \(\text{PG}_n(q)\).

Under this action, the stabilizer of a point is an intersecting subset of \(\text{PGL}_{n+1}(q)\) of size

\[
\frac{(q - 1)^n \prod_{i=1}^{n} (q^{n+1} - q^i)}{q - 1} = \prod_{i=1}^{n} (q^{n+1} - q^i).
\]

In this formula, the numerator counts the number of invertible linear transformations in \(\text{GL}_{n+1}(q)\) that fix a given 1-dimensional subspace of \(\mathbb{GF}(q)^{n+1}\) (this formula also follows simply from the Orbit Stabilizer Lemma [9, Theorem 1.4A]). Further, the coset of the stabilizer of a point is also an intersecting subset of the same size.

The stabilizer of a hyperplane under this action (or a coset of the stabilizer of a hyperplane) is an intersecting subset of \(\text{PGL}_{n+1}(q)\) which is also of size

\[
\frac{\left(\prod_{i=0}^{n-1} (q^n - q^i)\right) (q^{n+1} - q^n)}{q - 1} \prod_{i=1}^{n} (q^{n+1} - q^i).
\]

This is the number of invertible linear transformations in \(\text{GL}_{n+1}(q)\) that fix a given \(n\)-dimensional subspace of \(\mathbb{GF}(q)^{n+1}\) (or again, the formula follows simply from the Orbit Stabilizer Lemma).

We have conjectured in [16, Conjecture 2] that the cosets of a stabilizer of a point and the cosets of a stabilizer of a hyperplane are the only intersecting families of maximum size of \(\text{PGL}_{n+1}(q)\).

In this paper, our focus is on the group \(\text{PGL}_3(q)\), so for the remainder of the appendix we will only consider this group. It is well-known that the action of \(\text{PGL}_{n+1}(q)\) is 2-transitive on \(\text{PG}_n(q)\), but we will need the following fact about \(\text{PGL}_3(q)\).

**Lemma 6.1.** The group \(\text{PGL}_3(q)\) acts transitively on the ordered 4-tuples of non-collinear points of \(\text{PG}_2(q)\).
Proof: Write $G = \text{PGL}_3(q)$. Observe that if $(\alpha, \beta, \gamma, \delta)$ is an ordered 4-tuple of non-collinear points, then

$$\beta \neq \alpha, \quad \gamma \neq \alpha \lor \beta, \quad \delta \neq (\alpha \lor \beta) \lor (\beta \lor \gamma).$$

Therefore we have $|\text{PG}_3(q)| = q^2 + q + 1$ choices for $\alpha$, $|\text{PG}_3(q)| - |\text{PG}_1(q)| = q^2 + q$ choices for $\beta$, $|\text{PG}_3(q)| - |\text{PG}_2(q)| = q^2$ choices for $\gamma$ and (from the inclusion-exclusion principle) $|\text{PG}_3(q)| - 3|\text{PG}_2(q)| + 3|\text{PG}_1(q)| = (q-1)^2$ choices for $\delta$. Thus there are

$$\frac{(q^2 + q + 1)(q^2 + q)(q^2)(q - 1)^2}{q - 1} = |G|$$

ordered 4-tuples of non-collinear points.

Consider the 4-tuple $(\alpha, \beta, \gamma, \delta)$ with $\alpha = \langle e_1 \rangle$, $\beta = \langle e_2 \rangle$, $\gamma = \langle e_3 \rangle$ and $\delta = \langle e_1 + e_2 + e_3 \rangle$. The point-wise stabilizer of $(\alpha, \beta, \gamma, \delta)$ in $G$ is the identity (this can be seen by considering the $3 \times 3$ matrices that fix these points). As we have $|G|$ ordered 4-tuples of non-collinear points altogether, the Orbit Stabilizer Lemma yields that $G$ is transitive on ordered 4-tuples of non-collinear points. 

The character theory of $\text{PGL}_3(q)$ was developed by Robert Steinberg in [19]; Steinberg’s work was done shortly before the remarkable and pioneering work of Alexander Green [13] in which he describes the character theory of both $\text{GL}_{n+1}(q)$ and $\text{PGL}_{n+1}(q)$, for every $n$. We do not give the full details of the conjugacy classes and the irreducible complex characters of $\text{PGL}_3(q)$, rather we content ourselves to summarize the information relevant for this paper.

In [19, page 228] Steinberg partitions the elements of $\text{PGL}_3(q)$ into eight types and, for each type, determines the number of conjugacy classes and the size of each conjugacy class. It is readily seen from the description of these types that the derangements of $\text{PGL}_3(q)$ (in its action on the projective points of $\text{PG}_2(q)$) are exactly the elements of the type he calls “$C_1$”, therefore we only give details about the elements of this type.

To describe the elements of this type, we will use the fact that $\text{PGL}_3(q) \subseteq \text{PGL}_3(q^2)$. Let $\rho$ be an element of $\mathbb{GF}(q^2) \setminus \{0\}$ of multiplicative order $q^2 - 1$. The elements of type $C_1$ in $\text{PGL}_3(q)$ are conjugate, via an element of $\text{PGL}_3(q^2)$, to

$$g_a = \begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^{aq} & 0 \\ 0 & 0 & \rho^{aq^2} \end{pmatrix} \in \mathbb{Z}$$

for some $a \in \mathbb{Z}$ such that $a$ is not a multiple of $q^2 + q + 1$. In other words, the elements in $C_1$ have Jordan canonical form as above; this implies that the characteristic polynomial of these elements has splitting field $\mathbb{GF}(q^2)$. It can be inferred from [19, Table IV and page 229] that $\{g_a \mid a \in S\}$ is a set of conjugacy class representatives for the derangements of $\text{PGL}_3(q)$: the set $S$ is described in the next paragraph.

We need to calculate the sizes of the conjugacy classes of the derangements, to do this first set

$$A = \mathbb{Z} \setminus (q^2 + q + 1) \mathbb{Z} \setminus \{0\}.$$ 

Clearly, $|A| = q^2 + q$. As $\gcd(q, q^2 + q + 1) = 1$ and $q^3 \equiv 1 \mod (q^2 + q + 1)$, the mapping $f : A \to A$ defined by $x \mapsto qx$ is a permutation of $A$ with order 3. If $f$ has a fixed point $x$ then $x(q - 1) \equiv 0 \mod (q^2 + q + 1)$; since $\gcd(q - 1, q^2 + q + 1) =$
gcd\( (q - 1, 3) \), this will happen if and only if \( 3 \mid q - 1 \). In fact, a simple computation shows that if \( 3 \mid q - 1 \), then \( f \) has exactly two fixed points, these are the residue classes of \((q^2 + q + 1)/3 \) and \(2(q^2 + q + 1)/3 \). If \( S \subseteq A \) is a set of representatives for the orbits of \( \langle f \rangle \) acting on \( A \), then

\[
|S| = \begin{cases} 
\frac{1}{3}(q^2 + q), & \text{if } 3 \nmid q - 1; \\
\frac{1}{3}(q^2 + q - 2) + 2, & \text{if } 3 \mid q - 1.
\end{cases}
\]

Each \( \langle f \rangle \)-orbit corresponds to a conjugacy class in \( \text{PGL}_3(q) \) and \( \{g_a \mid a \in S\} \) is a set of conjugacy class representatives for the derangements of \( \text{PGL}_3(q) \). If \( 3 \nmid q - 1 \) each \( \langle f \rangle \)-orbit corresponds to a conjugacy class in \( \text{PGL}_3(q) \) of size \( q^3(q - 1)^2(q + 1) \). When \( 3 \mid q - 1 \), the two conjugacy classes that have a representative as an element in Equation (23) with \( a \in \{(q^2 + q + 1)/3, 2(q^2 + q + 1)/3\} \) each have size \( q^2(q - 1)^2(q + 1)/3 \): these are the derangements \( g \) where \( g^3 \) fixes a point. From this we can determine the number of conjugacy classes (along with their sizes) of derangements, these are described in Table 1.

<table>
<thead>
<tr>
<th>Nr. of conj. classes</th>
<th>Conj. class sizes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \nmid q - 1 )</td>
<td>( \frac{1}{3}(q^2 + q) )</td>
</tr>
<tr>
<td>( 3 \mid q - 1 )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>

Table 1. Number and size of conjugacy classes of derangements in \( \text{PGL}_3(q) \)

Now we turn to the irreducible complex characters of \( \text{PGL}_3(q) \). Steinberg also partitions the complex irreducible characters into eight types. For each type, the degree of any character of the given type can be expressed as a fixed polynomial in \( q \), we call this polynomial the generic degree (this term is necessary since for some values of \( q \) the polynomials corresponding to different types may be equal). Two characters are of the same type if and only if they have the same generic degree.

For simplicity, we include the values in [19, Table VIII], which gives the character degrees together with their frequencies, in Table 2. This table also contains the values of the characters on the derangements, but some notation is needed to understand these entries. Let \( \varepsilon \) be a complex primitive third root of unity and let \( \eta \) be a complex primitive \((q^2 + q + 1)^{1/3}\)th root of unity. From [19, Table V, VI, VII and page 230], we see that the character values on the derangements are as given in Table 2. In this table, \( n \in \{0, \ldots, \gcd(3, q - 1) - 1\} \) and \( m \in \mathbb{Z} \setminus (q^2 + q + 1)\mathbb{Z} \).

**Lemma 6.2.** The eigenvalues, together with their multiplicities, of the derangement graph of \( \text{PGL}_3(q) \) in its action on the points of \( \text{PG}_2(q) \) are as given in Table 3.

**Proof.** We use Lemma 4.1 and the information about the conjugacy classes and the irreducible characters of \( \text{PGL}_3(q) \) that we have given so far. We will denote the conjugacy class \( \{x^g \mid g \in \text{PGL}_3(q)\} \) by \( x^{\text{PGL}_3(q)} \).

From Lemma 4.1, we see that the eigenvalues of the derangement graph are

\[
\frac{1}{\chi(1)} \sum_{a \in S} |g_a^{\text{PGL}_3(q)}| \chi(g_a),
\]

as \( \chi \) runs through the complex irreducible characters of \( \text{PGL}_3(q) \).
AN ERDŐS-KO-RADO-TYPE THEOREM FOR PGL₃(q)

Degree | Frequency | Value on Derangements
--- | --- | ---
χ(1) | 3 | 3 | ϵⁿᵃᵃ
q(q+1) | 1 | 1 | ϵⁿᵃ/
q³ | 1 | 1 | ϵⁿᵃ/
q² + q + 1 | q - 2 | q - 4 | 0
q(q² + q + 1) | q - 2 | q - 4 | 0
(q + 1)(q² + q + 1) | 1/₆(q - 2)(q - 3) | 1/₆(q² - 5q + 10) | 0
(q - 1)(q² + q + 1) | 1/₂q(q - 1) | 1/₂q(q - 1) | 0
(q - 1)²(q + 1) | 1/₃(q + 1) | 1/₃(q - 1)(q + 2) | ηᵐᵃᵃ + ηᵐᵃᵃq + ηᵐᵃᵃq²

Table 2. Character degrees of PGL₃(q) with their frequencies and value on the derangements

Suppose that χ is an irreducible character of PGL₃(q) that has generic degree 1 or q⁴. Then, from Table 2, if n = 0 the character χ gives rise to the eigenvalue

\[
\frac{1}{\chi(1)} \sum_{a \in S} |g_{PGL₃(q)}|^n = \frac{q^4(q^2-1)^2}{3\chi(1)}.
\]

This gives the first two eigenvalues in Table 3. Observe that the same argument applies for the characters of degree q(q+1) with n = 0, and we obtain the eigenvalues in the third row of Table 3.

Suppose now that χ has degree 1 or q⁴ and n ≠ 0, since n < gcd(3, q - 1), this implies that 3 | q - 1. Now using Table 1 and the description of the set S we get

\[
\frac{1}{\chi(1)} \sum_{a \in S} |g_{PGL₃(q)}|^n = \frac{1}{\chi(1)} \sum_{a \in \{(q²+q+1)/3,2(q²+q+1)/3\}} |g_{PGL₃(q)}|^n = -\frac{q³(q - 1)^2(q + 1)}{3\chi(1)}.
\]

This gives two more eigenvalues equal to \(-\frac{1}{₃}q³(q - 1)^2(q + 1)\) when 3 | q - 1 and the eigenvalues in the sixth and seventh rows of Table 3.

All the remaining eigenvalues are computed in a similar manner. Once the eigenvalues are determined, the multiplicities follow easily from Lemma 4.1. □

The character theory of PSL₃(q) can also be read (with some effort) from Steinberg’s paper [19]; this was done by Simpson and Frame [18]. In their paper, the character table for PSL₃(q) is stated explicitly and directly from [18, Table 2] we can calculate the value of every irreducible character of PSL₃(q) on the derangements of the group. We give these values in Table 4. From Table 4 it is easy to directly calculate the eigenvalues for the derangement graph for PSL₃(q). We will only include the table of eigenvalues when 3 | q - 1, since otherwise PSL₃(q) = PGL₃(q). We omit the details of these calculations since they are very similar as the calculations in the proof of Lemma 6.2.
Lemma 6.3. Assume that $3 \mid q - 1$. The eigenvalues, together with their multiplicities, of the derangement graph of $\text{PSL}_{3}(q)$ in its action on the points of $\text{PG}_{2}(q)$ are as given in Table 5.

In the introduction we pointed out that one of our main problems in generalizing this work to all $\text{PGL}_{n+1}(q)$ is Lemma 4.3, that is, pinning down the minimum eigenvalue of the derangement graph of $\text{PGL}_{n+1}(q)$. Recently, using the theory of Deligne-Lusztig for the irreducible characters of reductive algebraic groups, Alex Zalesski has proven an analogue of Lemma 4.3 for $\text{PGL}_{n+1}(q)$ when $n+1$ is a prime number [22].

References

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{9}q^2(q^2 + q - 2)(q - 1)^2(q + 1)$</td>
<td>1</td>
</tr>
<tr>
<td>$- \frac{1}{9}q^2(q^2 + q - 2)(q - 1)^2$</td>
<td>$q^2(q + 1)^2$</td>
</tr>
<tr>
<td>$\frac{1}{9}(q^2 + q - 2)(q - 1)^2(q + 1)$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{5}(q^2 + q + 1)^2(2q^4 - 3q^3 - 7q^2 + 9q - 7)$</td>
</tr>
<tr>
<td>$-q^3$</td>
<td>$\frac{1}{5}(q - 1)^5(q + 1)^2(q + 2)$</td>
</tr>
</tbody>
</table>

Table 5. Spectrum for the derangement graph of $\text{PSL}_3(q)$ when $3 \mid q - 1$