ON THE THEORY OF PRICING OF DISCRETE TIME OPTIONS OF EUROPEAN TYPE

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Abstract

The contemporary financial market appears to be a part of human activity where ideas of Stochastic Analysis, in particular Martingale Theory and Stochastic Ito’s integral, have been implemented in a most complete matter. Creating a link between the theory and practice of finance generates new problems in the Theory of Probability and the Financial Mathematics investigating them.

Financial markets consist of two main (primary) assets (securities), bonds (riskless) and shares (risky). Bonds are debentures issued by a state government or a bank with a goal to accumulate capital. As an example we can consider a bank account or government bonds. Bond owners accept a strictly defined profit, conditional on the current interest rate. Shares are equity securities which are issued by companies in order to accumulate capital for further activities. Its price is defined by the situation at the stock market and by the production activity of the company. Shareholders obtain a profit according the price of the share.

Options belong to derivative (secondary) securities. An option is a security that gives to its owner a right to sell (or to buy) some worth (shares, currency, etc.) by conditions specified in advance.

In this thesis, we investigate the random behaviour of a share price, creating an
“optimal” portfolio of securities and related various problems in Financial Mathematics. In particular, we discuss the problem of option pricing. The literature on Financial Mathematics is too large to mention in my theses, consult, for example, the lengthy list of references in the textbook [2]. In our theses, we cite only the articles and books which are important and are directly used in our research. The real breakthrough in the methods of financial calculations connected with options has been done by Black, Scholes and Merton in 1973 [1], [?]. The theory developed in these manuscripts allows for finding a “fair” price of an option and also provides a guidance to optimal stock transactions that allow for the option writer to guarantee the possible pay offs, which depend on a random behaviour of prices in a financial market.

In this thesis we discuss the theory of pricing options of European type in discrete time setting. Everything is explained from scratch, the reader needs to have only a basic knowledge of Probability Theory at an elementary level. For example, we introduce an advanced and comprehensive notion of a martingale, but we consider measurability by a finite partition, not by a general $\sigma$-algebra. This simplifies the understanding of the theory significantly.
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Chapter 1

Introduction

Financial mathematics is typically characterized as applying tools and techniques from mathematics to various topics in finance and financial markets. However, because of the vastness of this subject area, financial mathematics has grown into its own field within mathematics and in fact lies at the confluence of mathematics and statistics.

Special attention has been given to Financial Mathematics after the last financial crisis in 2007 and 2008 that resulted in the collapse of a number of large financial institutions in the world. Financial options are the most important products traded, and therefore the subject of pricing for options is of great importance for preventing future financial crises and making the work of financial institutions smooth and reliable. A main focus in economic analysis is devoted to a simultaneous combination of quantitative techniques provided by Financial Mathematics and the analytical analysis to derive option pricing by particular financial experts and advisors. Whenever this is possible, this will provide the best prediction for the value of the reward
The most important and the easiest way to achieve these goals is to implement the famous Binomial Model, developed by Cox, Ross and Rubinstein in 1973 [3]. In order to elaborate this idea we will now discuss the following topics:

- Financial options and their role in causing the financial crisis
- The importance of the Binomial Model for pricing options

1.0.1 Financial options

To define the notion of a financial option, we mention that the first options were used in ancient Greece to speculate on the olive harvest. More modern options as a type of contract have been established in the United States of America, where these contracts were dealt with in the unregulated financial market through financial intermediaries, and then moved to the regulated market through the Chicago Stock Exchange on April 26, 1791. Lately, these contracts have spread to the US stock exchanges, European exchanges, etc.

An option contract is a contract between two parties: buyer and writer. The contract gives the buyer the right to buy from or to sell to a person who has released a number of real units of origin or something at a price agreed upon at the moment of signing the contract. The contract is to be exercised at a later date, called execution or expiry date. The buyer has the right not to exercise the contract, if the execution has no advantage or benefits for him. The return for compensation paid to the writer is called a reward or premium. In addition, this bonus is payable upon contract, it is
non-refundable and it is not a part of the option value. In other words, it is an amount paid by the buyer against the right of option in the execution or non-execution of the contract. Since the buyer gets this right from the moment the contract is signed, it loses the reward that moment.

1.0.2 Binomial Model for pricing options

The process of building a hedging portfolio through the use of derivatives, especially financial options, is a crucial part of risk management. The main idea is to transfer funds to other parties with greater capacity to cope with unpredictable price fluctuations as related to the components of the portfolio.

The Binomial Model is one of the most important models in the pricing of financial options, which provides a descriptive picture of the optional processes. The simplest tools would be a graph or a decision tree that represents the different paths of the implicit price during the validity of the option, but much more complicated contemporary notions of probability theory, such as martingales, conditional expectations, stopping times and others can be involved.

The following issue is discussed in the thesis: The problem of the extent to which pricing contributed by using a Binary model to build a hedge fund for the standard European call options. The problem is divided into a set of sub-problems that can be summarized as follows:

What is the relationship between the option price and its implied share price in terms of the Binary pricing model?
What is the type of relationship between the option price and its fair price in terms of the using the Binary pricing model?

Does building a hedging portfolio using the Binomial pricing model contribute to risk reduction?

Some of our derivations are based on the paper by Shiryaev, Kabanov, Kramkov, and Mel’nikov [6], but they rely on $\sigma$-algebras, while we do not. The notion of a partition is borrowed from the textbook by Shiryaev [5].

1.0.3 Research Hypotheses

In order to answer the main problem and the related sub-problems, we first adopt the following hypotheses:

There is a direct positive relationship between the option price and its implicit share price in the Binary pricing model, which means an increase in the option price implies an increase in the share price. Similarly there is a direct positive relationship between the option price and its fair price.

1.0.4 Objectives of the study

This study aims to achieve the following objectives:

Explain how options are priced according to a two-dimensional model and how to create a hedge portfolio using this model, showing the use of mathematical and statistical techniques.

With the use of an illustrative example, find out the extent to which the theoretical
side of the study matches the actual reality for the European options. This objective
has not reached completely because it is too complicated for a Master’s study. We
leave this part of our research for our future investigation. Some ideas how we would
like to solve this problem are explained in Section 4.3.

1.0.5 Thesis structure

In order to cover all aspects and fundamentals of the research and to answer
the problems raised above, we have systematically divided our research into separate
chapters as follows:

In Chapter 2 we provide all necessary mathematical tools for derivation of an Eu-
ropean option pricing. The special attention is given to such notions from probability
theory as conditional expectation and martingale.

In Chapter 3 we provide a detailed description of functioning of financial market
from the point of view of Binomial model. We discuss such notions as bond, share,
option, etc and provide the main formulae for the Binary model.

In Chapter 4 we derive all required formulae for pricing the European option from
the point of view of Binomial model.

Finally, Chapter 5 contains a summary of our results and future research.

In Appendix A we provide some additional material and more examples for the
notions introduced in the previous chapters. The material from Appendix A is not
used in the previous chapters and hence may be omitted by the reader.
Chapter 2

Mathematical Tools

In this chapter we provide all required mathematical tools from probability theory.

In the following let \( \Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\} \) be a finite sample space.

2.1 Conditional expectation by partitions

**Definition 1.1.** A class of subsets or events of the space \( \Omega \)

\[
D = \{D_1, D_2, \ldots, D_k\}, \quad D_i \subseteq \Omega, \quad (i = 1, \ldots, k)
\]

is called a *partition* of the space \( \Omega \), if

1. The events \( D_i \) are disjoint (mutually exclusive), that is, \( D_i \cap D_j = \emptyset, (i \neq j) \), and
2. \( \bigcup_{i=1}^{k} D_i = \Omega \).

The sets \( D_i \) are called *atoms* of the partition \( D \).

Suppose \( D_1 = \{D_{11}, D_{12}, \ldots, D_{1m}\} \) and \( D_2 = \{D_{21}, D_{22}, \ldots, D_{2k}\} \) are two partitions of \( \Omega \).
Definition 1.2. The partition $D_2$ majorates the partition $D_1$, or $D_2$ is more fine than $D_1$, (denoted $D_1 \preceq D_2$ ), if each atom of the partition $D_2$ is a subset of an atom of the partition $D_1$. That is, for any $1 \leq q \leq k$ there exists $p, 1 \leq p \leq m$ such that $D_{2q} \subset D_{1p}$.

Note that if partition $D_2$ majorates partition $D_1$, then $D_2$ is simply a partition of atoms of partition $D_1$. That is, for any $1 \leq p \leq m$ there exists $J_p \subset \{1, 2, \ldots, k\}$ such that $D_{1p} = \bigcup_{q \in J_p} D_{2q}$ and index sets $J_p$ are disjoint and $\bigcup_{p=1}^{m} J_p = \{1, 2, \ldots, k\}, k \geq m$.

Also note that there is the “smallest” partition $D_{\text{min}} = \{\Omega\}$ and the “largest” partition $D_{\text{max}} = \{D_1 = \{\omega_1\}, D_2 = \{\omega_2\}, \ldots, D_n = \{\omega_n\}\}$ (recall that there are $n$ elementary outcomes in $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$). Then for any partition $D$ we have that $D_{\text{min}} \preceq D \preceq D_{\text{max}}$.

Generally speaking, we can define a probability only on a partition $D$ in the following way.

Definition 1.3. Let $D = \{D_1, D_2, \ldots, D_k\}$ be a partition of space $\Omega$. A probability $P$ on the partition $D$ is a function of $D$ into an interval $[0, 1]$, such that the following conditions are satisfied:

1. $P(D_i) > 0, \quad (i = 1, \ldots, k)$;
2. $\sum_{i=1}^{k} P(D_i) = 1$.

We can extend the notion of probability via summation on disjoint unions on the class of sets $F$ which consists of all possible unions of elements of the partition $D$, that is, if $A \in F$, then $A = \bigcup_{i \in I} D_i$, where $I$ is any collection of indexes from 1 to $k$ (including $I = \emptyset$). If $A = \bigcup_{i \in I} D_i$, then we define the probability of the event $A$ as
\[ P(A) = \sum_{i \in I} P(D_i). \]

In the following we assume that the probability function \( P \) is defined on the partition \( \mathcal{D}_{\text{max}} \). In this case, for any \( A \subset \Omega \) we then have \( P(A) = \sum_{\omega \in A} P(\omega) \).

**Definition 1.4.** Let \( \mathcal{D}_1 = \{D_{11}, D_{12}, \ldots, D_{1m}\} \) and \( \mathcal{D}_2 = \{D_{21}, D_{22}, \ldots, D_{2k}\} \) be two partitions of \( \Omega \). We say that partitions \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are independent if

\[ P(D_{1i} \cap D_{2j}) = P(D_{1i})P(D_{2j}), \quad \text{for all } 1 \leq i \leq m, 1 \leq j \leq k. \]

Having defined probability of partitions, we now introduce the important concept of a random variable.

**Definition 1.5.** A function \( \eta \) from the Sample Space into the real line, that is, \( \eta : \Omega \to (-\infty, \infty) \), is called a random variable.

**Definition 1.6.** Let \( \eta \) be a random variable taking on values \( y_1, y_2, \ldots, y_k \). Denote by \( D_j = \{\omega : \eta(\omega) = y_j\} \). Then the partition

\[ \mathcal{D}_\eta = \{D_1, D_2, \ldots, D_k\} \]

is called a partition generated by random variable \( \eta \).

In the same way we can introduce the partition \( \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_m} \) generated by a family of random variables \( \eta_1, \eta_2, \ldots, \eta_m \). This partition consists of the atoms

\[ D_{y_1, y_2, \ldots, y_m} = \{\omega : \eta_1(\omega) = y_1, \eta_2(\omega) = y_2, \ldots, \eta_m(\omega) = y_m\}. \]
Note that if a random variable \( \eta \) is given, then in the following we will consider only probabilities \( P \) defined on the partition \( D_{\eta} \) such that

\[
P\{\eta = y_j\} > 0, \quad (j = 1, 2, \ldots, k).
\]

There is one random variable that plays a special role in the following

**Definition 1.7.** Let \( A \) be an event, that is, \( A \subset \Omega \). The random variable \( I_A \) is called an *indicator function* if it takes only two values:

\[
I_A(\omega) = \begin{cases} 
1, & \text{if } \omega \in A \\
0, & \text{if } \omega \notin A.
\end{cases}
\]

We note that the partition generated by the indicator function \( I_A \) has the simple structure: \( D_{I_A} = \{ A, A^c \} \).

**Definition 1.8.** Let \( D = \{ D_1, D_2, \ldots, D_k \} \) be a partition, then random variable \( \eta = \eta(\omega) \) is *measurable with respect to the partition* \( D \), or \( D \)-measurable, if

\[
D_{\eta} \subseteq D.
\]

This means that the random variable takes constant values on atoms of the partition \( D \) and hence the random variable can be written in the form

\[
\eta(\omega) = \sum_{i=1}^{k} y_i \cdot I_{D_i}(\omega),
\]
where some of the scales \( y_i \) may be equal.

Sometimes it is more convenient for us to consider only distinct \( y_i \). Then we can collect all \( D_i \) for which random variable \( \eta \) takes the same values. In this case the random variable \( \eta \) can be represented as

\[
\eta = \sum_{j=1}^{l} y_j I_{A_j}, \quad A_j = \{ \omega : \eta(\omega) = y_j = \bigcup_{i \in J_j} D_i \},
\]

where the index sets \( J_j \) are disjoint and \( \bigcup_{j=1}^{l} J_j = \{1, 2, \ldots, k\}, l \leq k \).

**Definition 1.9.** Let \( \xi = \xi(\omega) \) be a random variable taking values \( x_1, x_2, \ldots, x_l \):

\[
\xi = \sum_{j=1}^{l} x_j I_{A_j}, \quad A_j = \{ \omega : \xi(\omega) = x_j \}.
\]

The *mathematical expectation* (unconditional) of the random variable \( \xi \) is defined as

\[
E(\xi) = \sum_{j=1}^{l} x_j P(A_j).
\]

We now state and prove a basic result concerning the expectation of the indicator function.

**Fact:** \( E(I_A) = P(A) \).

**Proof:** \( E = 0 \cdot P(I_A = 0) + 1 \cdot P(I_A = 1) = P(A) \).

**Proposition 1.1.** Let \( \xi = \xi(\omega) \) and \( \eta = \eta(\omega) \) be two random variables such that \( \xi \) is measurable with respect to the partition \( D_\eta \) generated by the random variable \( \eta \). Then there exists a function \( f \) such that \( \xi = f(\eta) \).
Proof. Let \( y_1, y_2, \ldots, y_k \) be the values taken by the random variable \( \eta \). Then the partition \( \mathcal{D}_\eta \) consists of atoms \( D_j = \{ \omega : \eta(\omega) = y_j \} \), \( 1 \leq j \leq k \) and \( \eta(\omega) = \sum_{j=1}^{k} y_j \cdot I_{D_j}(\omega) \).

Since \( \xi \) is measurable with respect to the partition \( \mathcal{D}_\eta \), it can be written as \( \xi(\omega) = \sum_{j=1}^{k} x_j \cdot I_{D_j}(\omega) \). The construction of the function \( f \) now is quite simple. Let \( f(y_j) = x_j, 1 \leq j \leq k \). Obviously, \( \xi(\omega) = f(\eta(\omega)) \) by the definition of the function \( f \). The proposition is proved.

Proposition 1.2 has a simple generalization. Namely, let \( \xi = \xi(\omega) \) and \( \eta_1(\omega), \eta_2(\omega), \ldots, \eta_m(\omega) \) be random variables such that \( \xi \) is measurable by the partition \( \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_m} \) generated by the random variables \( \eta_1, \eta_2, \ldots, \eta_m \). Then there exists a function \( f \) such that \( \xi = f(\eta_1, \eta_2, \ldots, \eta_m) \).

Definition 1.10. Two random variables \( \xi = \xi(\omega) \) and \( \eta = \eta(\omega) \) are said to be independent if the partitions \( \mathcal{D}_\xi \) and \( \mathcal{D}_\eta \) are independent.

Definition 1.11. Let \( \mathcal{D}_\xi \) and \( \mathcal{D}_\eta \) be two partitions. We say that partitions \( \mathcal{D}_\xi \) and \( \mathcal{D}_\eta \) are independent, if for all \( A \in \mathcal{D}_\xi \) and \( B \in \mathcal{D}_\eta \) we have that

\[
P(AB) = P(A)P(B).
\]

Let \( A \subset \Omega \) and let \( P(A \mid D_i) \) be the conditional probability of event \( A \) by event \( D_i \). That is,

\[
P(A \mid D_i) = \frac{P(A \cap D_i)}{P(D_i)}.
\]
The collection of conditional probabilities \( \{P(A \mid D_i)\}_{i=1}^k \) defines a random variable given by
\[
P(A \mid D)(\omega) = \sum_{i=1}^{k} P(A \mid D_i)I_{D_i}(\omega),
\]
which takes the values \( P(A \mid D_i) \) on the atoms \( D_i \) of the partition \( D \). The random variable (2) is called the \textit{conditional probability} of the event \( A \) with respect to the partition \( D \).

The following are some important properties of the conditional probability with respect to the partition.

1°. If two events \( A \) and \( B \) in \( \mathcal{F} \) are disjoint, then \( P(A \cup B \mid D) = P(A \mid D) + P(B \mid D) \);

2°. Recall that \( D_{\text{min}} = \{\Omega\} \) is the minimal partition. Then \( P(A \mid D_{\text{min}}) = P(A) \).

For the random variable \( P(A \mid D) \) we can define an expectation as follow.

\[
\mathbb{E}[P(A \mid D)] = \mathbb{E}\left[\sum_{i=1}^{k} P(A \mid D_i)I_{D_i}(\omega)\right] \\
= \sum_{i=1}^{k} P(A \mid D_i) \cdot P(D_i) \\
= \sum_{i=1}^{k} P(A \cdot D_i) = P(A).
\]

Therefore we have obtained the formula of Total Probability in a modified form:

3°. \( \mathbb{E}[P(A \mid D)] = P(A) \).

**Definition 1.12.** Conditional probability \( P(A \mid \mathcal{D}_\eta) \) is called the \textit{conditional probability of event} \( A \) \textit{with respect to random variable} \( \eta \) and denote \( P(A \mid \eta) \).
Denote $P(A \mid \eta = y_j) = P(A \mid D_j)$, where $D_j = \{\omega : \eta(\omega) = y_j\}$.

In the same way we can introduce the conditional probability of an event $A$ with respect to a family of random variables $\eta_1, \eta_2, \ldots, \eta_m$. The partition here is generated by the atoms

$$D_{y_1, y_2, \ldots, y_m} = \{\omega : \eta_1(\omega) = y_1, \eta_2(\omega) = y_2, \ldots, \eta_m(\omega) = y_m\}.$$

This conditional probability is then denoted by $P(A \mid \eta_1, \eta_2, \ldots, \eta_m)$.

**Definition 1.13.** The *conditional mathematical expectation of the random variable* $\xi$ by the partition $\mathcal{D}$ with respect to the formula:

$$E(\xi \mid \mathcal{D}) = \sum_{j=1}^{l} x_j \cdot P(A_j \mid \mathcal{D}). \quad (2.1.1)$$

If the partition $\mathcal{D}$ is generated by the random variables $\eta_1, \eta_2, \ldots, \eta_k$, then the conditional mathematical expectation $E(\xi \mid \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_k})$ is called the *conditional mathematical expectation of a random variable $\xi$ with respect to $\eta_1, \eta_2, \ldots, \eta_k$* and is denoted $E(\xi \mid \eta_1, \eta_2, \ldots, \eta_k)$.

According to this definition, the conditional mathematical expectation of a random variable is also a random variable, see (2.1.1). Now we approach this notion from a different perspective. Define the conditional mathematical expectation $E(\xi \mid D_i)$ of random variable $\xi$ with respect to an event $D_i$ by formula:

$$E(\xi \mid D_i) = \sum_{j=1}^{l} x_j \cdot P(A_j \mid D_i) = \frac{1}{P(D_i)} \cdot E(\xi \cdot I_{D_i}).$$
After we let

\[ E(\xi \mid D) = \sum_{i=1}^{k} E(\xi \mid D_i) \cdot I_{D_i}(\omega) = \sum_{i=1}^{k} \frac{E(\xi \cdot I_{D_i})}{P(D_i)} \cdot I_{D_i}(\omega). \] (5)

Note the the value of the conditional mathematical expectation of a random variable with respect to the partition does not depend on the representation of the random variable. Moreover, for its calculation we can follow the scheme (3), or scheme (5) and we will obtain the same result.

Now we present the fundamental properties of the conditional expectation.

1) Linearity:

\[ E(a \cdot \xi + b \cdot \eta \mid D) = a \cdot E(\xi \mid D) + b \cdot E(\eta \mid D), \]

where \( a \) and \( b \) are arbitrary constants.

**Proof.**

Let

\[ \xi = \sum_{j=1}^{L} x_j \cdot I_{A_j} \] and \( \eta = \sum_{i=1}^{k} y_i \cdot I_{D_i}. \]

Then:

\[ a \cdot \xi + b \cdot \eta = a \cdot \sum_{j=1}^{L} x_j \cdot I_{A_j} + b \cdot \sum_{i=1}^{k} y_i \cdot I_{D_i}. \]

\[ = \sum_{ij} (a \cdot x_i + b \cdot y_i) \cdot I_{A_j \cap D_i} \]
Then:

\[ E(a \cdot \xi + b \cdot \eta) = \sum_{ij} (a \cdot x_i + b \cdot y_i) + P(A_j \cap D_i). \]

\[ E(a \cdot \xi + b \cdot \eta \mid D) = \sum_{ij} (a \cdot x_i + b \cdot y_i) + P(A_j \cap D_i \mid D) \]
\[ = \sum_j a \cdot x_j \cdot P(A_j \mid D) + \sum_i b \cdot y_i \cdot P(D_i \mid D) \]
\[ = a \cdot \sum_j x_j \cdot P(A_j \mid D) + b \cdot \sum_i y_i \cdot P(D_i \mid D) \]
\[ = a \cdot E(\xi \mid D) + b \cdot E(\eta \mid D). \]

2) \( E(\xi \mid \Omega) = E\xi \).

Proof.

\[ E(\xi \mid D) = \sum_{i=1}^k \frac{E(\xi \cdot I_{D_i})}{P(D_i)} \cdot I_{D_i} \]
\[ = \frac{E(\xi \cdot \Omega)}{P(\Omega)} \cdot I_{\Omega} \]
\[ = E\xi. \]

3) Constants:

\( E(c \mid D) = c \), where \( c \) is a constant.
4). If $\xi = I_A$, then
\[ E(\xi \mid D) = P(A \mid D). \]

The next property generalizes the Law of Total Probability.

5) Expectation Law:
\[ E[E(\xi \mid D)] = E\xi. \]

**Proof.** It is enough to observe:

\[
E[E(\xi \mid D)] = E\sum_{j=1}^{L} x_j P(A_j \mid D)
= \sum_{j=1}^{L} x_j E P(A_j \mid D)
= \sum_{j=1}^{L} x_j P(A_j)
= E\xi.
\]

6) Tower Property: Let $D_1 \preceq D_2$. Then
\[ E[E(\xi \mid D_2) \mid D_1] = E(\xi \mid D_1). \]

**Proof.** Let $D_1 = \{D_{11}, D_{12}, \ldots, D_{1m}\}$, and $D_2 = \{D_{21}, D_{22}, \ldots, D_{2k}\}$. As is mentioned following Definition 1.4, since $D_1 \preceq D_2$, then for any $1 \leq p \leq m$ there exists $J_p \subset \{1, 2, \ldots, k\}$ such that $D_{1p} = \cup_{q \in J_p} D_{2q}$ and index sets $J_p$ are disjoint and $\cup_{p=1}^{m} J_p = \{1, 2, \ldots, k\}, k \geq m$. According to this we specially label or rearrange the atoms from the partition $D_2$ taking into consideration as to which atom from the
partition $D_1$ they correspond. Namely, we attach an additional label $p$, so it becomes $D^p_{2q}$ for atoms from $D_2$, such that:

$$D_{1p} = \bigcup_{q \in J_p} D^p_{2q}, 1 \leq p \leq m.$$

Note that in our “previous” notation for any event $A$:

$$P(A \mid D_2) = \sum_{q=1}^{k} P(A_j \mid D_{2q}) \cdot I_{D_{2q}},$$

while with our “new” notation:

$$P(A \mid D_2) = \sum_{p=1}^{m} \sum_{q \in J_p} P(A \mid D^p_{2q}) \cdot I_{D^p_{2q}}.$$

The main advantage of our “new” notation that we will use in this proof is that

$$P(D^i_{2q} \mid D_{1p}) = \begin{cases} 0, & i \neq p, \\ \frac{P(D^p_{2q})}{P(D_{1p})}, & i = p. \end{cases}$$

Now, if the random variable $\xi = \sum_{j=1}^{l} x_j \cdot I_{A_j}$, then

$$E(\xi \mid D_2) = \sum_{j=1}^{l} x_j \cdot P(A_j \mid D_2)$$
and

\[ E[E(\xi \mid D_2) \mid D_1] = E\left[ \sum_{j=1}^l x_j \cdot P(A_j \mid D_2) \mid D_1 \right] \]

\[ = \sum_{j=1}^l x_j E[P(A_j \mid D_2) \mid D_1]. \]

Hence it is sufficient to show that

\[ E[P(A_j \mid D_2) \mid D_1] = P(A_j \mid D_1). \]

As we already mentioned above,

\[ P(A_j \mid D_2) = \sum_{p=1}^m \sum_{q \in J_p} P(A_j \mid D_{2q}^p) \cdot I_{D_{2q}^p} \]

and hence

\[
E[P(A_j \mid D_2) \mid D_1] = \sum_{p=1}^m \sum_{q \in J_p} P(A_j \mid D_{2q}^p) \cdot P(D_{2q}^p \mid D_1)
\]

\[
= \sum_{p=1}^m \sum_{q \in J_p} P(A_j \mid D_{2q}^p) \cdot \left( \sum_{i=1}^m P(D_{2q}^p \mid D_{1i}) \cdot I_{D_{1i}} \right)
\]

\[
= \sum_{i=1}^m I_{D_{1i}} \cdot \sum_{p=1}^m \sum_{q \in J_p} P(A_j \mid D_{2q}^p) \cdot P(D_{2q}^p \mid D_{1i})
\]

\[
= \sum_{i=1}^m I_{D_{1i}} \left[ \sum_{p=1}^m \sum_{q \in J_p} P(A_j \mid D_{2q}^p) P(D_{2q}^p \mid D_{1i}) + \sum_{q \in J_i} P(A_j \mid D_{2q}^i) P(D_{2q}^i \mid D_{1i}) \right]
\]

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\[
= \sum_{i=1}^{m} I_{D_{1i}} \cdot \left[ \sum_{p=1, p \neq i}^{m} \sum_{q \in J_p} \mathbb{P}(A_j \mid D_{2q}^p) \cdot 0 + \sum_{q \in J_i} \mathbb{P}(A_j \mid D_{2q}^i) \cdot \frac{\mathbb{P}(D_{2q}^i)}{\mathbb{P}(D_{1i})} \right]
\]

\[
= \sum_{i=1}^{m} I_{D_{1i}} \cdot \sum_{q \in J_i} \mathbb{P}(A_j \mid D_{2q}^i) \cdot \frac{\mathbb{P}(D_{2q}^i)}{\mathbb{P}(D_{1i})}
\]

\[
= \sum_{i=1}^{m} I_{D_{1i}} \cdot \sum_{q \in J_i} \frac{\mathbb{P}(A_j \cdot D_{2q}^i)}{\mathbb{P}(D_{1i})}
\]

\[
= \sum_{i=1}^{m} I_{D_{1i}} \cdot \frac{1}{\mathbb{P}(D_{1i})} \sum_{q \in J_i} \mathbb{P}(A_j \cdot D_{2q}^i)
\]

\[
= \sum_{i=1}^{m} I_{D_{1i}} \cdot \frac{1}{\mathbb{P}(D_{1i})} \mathbb{P}(A_j \cdot D_{1i}), \text{ since } \cup_{q \in J_i} D_{2q}^i = D_{1i}
\]

\[
= \sum_{i=1}^{m} I_{D_{1i}} \cdot \mathbb{P}(A_j \mid D_{1i}) = \mathbb{P}(A_j \mid \mathcal{D}_1).
\]

7) Suppose the random variable \( \eta \) is measurable with respect the partition \( \mathcal{D} \). Then

\[ \mathbb{E}(\eta \mid \mathcal{D}) = \eta. \]

**Proof.** Because \( \eta \) is \( \mathcal{D} \)-measurable, we have

\[
\eta = \sum_{i=1}^{k} y_i \cdot I_{D_i}
\]

where, \( \mathcal{D}_i \preceq \mathcal{D} \). Then

\[ \mathbb{E}(\eta \mid \mathcal{D}) = \sum_{i=1}^{k} (\eta \mid \mathcal{D}_i) \cdot I_{D_i} \]
\[
\begin{align*}
&= \sum_{i=1}^{k} \mathbb{E}(\sum_{j=1}^{L} y_i \cdot I_{D_i} \mid D_i) \cdot I_{D_i} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{L} \frac{\mathbb{E}(y_i \cdot I_{D_j} \cdot I_{D_i})}{\mathbb{P}(D_i)} \cdot I_{D_i} \\
&= \sum_{i=1}^{k} \mathbb{E}(y_i \cdot I_{D_i}) \cdot I_{D_i} \\
&= \sum_{i=1}^{k} y_i \frac{\mathbb{E} \cdot I_{D_i}}{\mathbb{P}(D_i)} \cdot I_{D_i} = \eta.
\end{align*}
\]

8) Independence Law: If the random variables \( \xi \) and \( \eta \) are independent, then

\[
\mathbb{E}(\xi \mid \eta) = \mathbb{E}\xi.
\]

**Proof.** Let

\[
\eta = \sum_{i=1}^{k} y_i \cdot I_{D_i}.
\]

Thus the partition generated by \( \eta \) is \( \{D_1, D_2, \ldots, D_k\} \)

Further suppose

\[
\xi = \sum_{j=1}^{L} x_j \cdot I_{A_j}.
\]

Thus the partition generated by \( \xi \) is \( \{A_1, A_2, \ldots, A_L\} \)
Since \( \xi \) and \( \eta \) are independent, this means that their partitions \( D_i \) and \( A_j \) are independent, that is,

\[
P(D_i \cap A_j) = P(D_i)P(A_j), \quad 1 \leq i \leq k, 1 \leq j \leq L.
\]

Equivalently, \( P(D_i \mid A_j) = P(A_j) \) or \( P(A_j \mid D_i) = P(D_i) \). Re-writing in terms of expectations we can express this independence as

\[
E(I_{D_i} \cdot I_{A_j}) = E(I_{D_i}) \cdot E(I_{A_j})
\]

\[
= 0 \cdot P(D_i \cap A_j)^c + 1 \cdot P(D_i \cap A_j)
\]

\[
= P(D_i \cap A_j) = P(D_i) \cdot P(A_j).
\]

Then

\[
E(\xi \mid \eta) = \sum_{i=1}^{k} \frac{E(\xi \cdot I_{D_i})}{P(D_i)} \cdot I_{D_i},
\]

where

\[
E(\xi \cdot I_{D_i}) = E \left( \sum_{j=1}^{L} x_j \cdot I_{A_j} \right) \cdot I_{D_i}
\]

\[
= \sum_{j=1}^{L} x_j E(I_{D_i} \cdot I_{A_j})
\]

\[
= \sum_{j=1}^{L} x_j E(I_{D_i}) \cdot E(I_{A_j})
\]

\[
= \sum_{j=1}^{L} x_j P(D_i) \cdot P(A_j).
\]
Returning to our original equation,

\[
E(\xi | \eta) = \sum_{i=1}^{k} \sum_{j=1}^{L} x_j P(D_i) \cdot P(A_j) \cdot I_{D_i} \\
= \sum_{j=1}^{L} x_j P(A_j) \cdot \sum_{i=1}^{k} P(D_i) = E\xi.
\]

9) \(E(\eta | \eta) = \eta\).

10) Stability: If the random variable \(\eta\) is \(D\)-measurable, then

\[
E(\eta \cdot \xi | D) = \eta \cdot E(\xi | D).
\]

**Proof.** Let

\[
\xi = \sum_{j=1}^{l} x_j \cdot I_{A_j}, \quad \eta = \sum_{i=1}^{k} y_i \cdot I_{D_i}.
\]

Then

\[
\xi \cdot \eta = \sum_{j=1}^{l} \sum_{i=1}^{k} x_j y_i \cdot I_{A_jD_i},
\]

hence

\[
E(\xi \cdot \eta | D) = \sum_{j=1}^{l} \sum_{i=1}^{k} x_j y_i \cdot P(A_jD_i | D).
\]
\[
= \sum_{j=1}^{l} \sum_{i=1}^{k} \sum_{m=1}^{l} x_j y_i \mathbb{P}(A_j | D_m) \cdot I_{D_m} \\
= \sum_{j=1}^{l} \sum_{i=1}^{k} x_j y_i \cdot \mathbb{P}(A_j D_i | D_i) \cdot I_{D_i} \\
= \sum_{j=1}^{l} \sum_{i=1}^{k} x_j y_i \cdot \mathbb{P}(A_j | D_i) \cdot I_{D_i}.
\]

On the other hand, since \(I_{D_i}^2 = I_{D_i}\), \(I_{D_i} I_{D_m} = 0\) \((i \neq m)\), we have

\[
\eta \cdot \mathbb{E}(\xi | \mathcal{D}) = \left[ \sum_{i=1}^{k} y_i I_{D_i} \right] \cdot \left[ \sum_{j=1}^{l} x_j \mathbb{P}(A_j | \mathcal{D}) \right] \\
= \left[ \sum_{i=1}^{k} y_i I_{D_i} \right] \cdot \sum_{m=1}^{k} \left[ \sum_{j=1}^{l} x_j \mathbb{P}(A_j | D_m) \right] \cdot I_{D_m} \\
= \sum_{i=1}^{k} \sum_{j=1}^{l} x_j \cdot y_i \cdot \mathbb{P}(A_j | D_i) \cdot I_{D_i}.
\]
2.2 Martingales

Let \((\Omega, \mathcal{D}, P)\) be a finite probability space, and let \(\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \ldots \subseteq \mathcal{D}_n\) be a sequence of partitions of \(\Omega\).

**Definition 2.1.** A sequence of random variables \(\xi_1, \xi_2, \ldots, \xi_n\) is called a **martingale** (with respect to the sequence of partitions \(\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \ldots \subseteq \mathcal{D}_n\)), if

1) \(\xi_k\) is \(\mathcal{D}_k\)-measurable, for each \(1 \leq k \leq n\); and

2) \(E(\xi_{k+1} | \mathcal{D}_k) = \xi_k, \quad 1 \leq k \leq n - 1\).

We denote such a martingale by

\[
\xi = (\xi_k, \mathcal{D}_k)_{k=1}^n.
\]

In the particular case, when \(\mathcal{D}_k = \mathcal{D}_{\xi_1, \xi_2, \ldots, \xi_k}\), we say that the sequence \(\xi = (\xi_k)\) is a martingale without specifying the sequence of partitions associated.

From the definition of a martingale it follows immediately that the mathematical expectation \(E \xi_k\) is constant for all \(k\):

\[
E \xi_k = E \xi_1.
\]

We now consider a classical example illustrating a basic example of martingales.

**Example 2.1.** Let \(\eta_1, \eta_2, \ldots, \eta_n\) be independent identically distributed random vari-
ables with a Bernoulli distribution

\[ P\{\eta_k = -1\} = P\{\eta_k = 1\} = \frac{1}{2}. \]

Define

\[ S_k = \eta_1 + \cdots + \eta_k, \quad \text{and} \quad \mathcal{D}_k = \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_k}. \]

The structure of the partitions associated with \( \mathcal{D}_k \) is simple:

\[ \mathcal{D}_1 = \{ D^+, D^- \}, \]

where \( D^+ = \{ \omega : \eta_1 = 1 \} \), \( D^- = \{ \omega : \eta_1 = -1 \} \); and

\[ \mathcal{D}_2 = \{ D^{++}, D^{+-}, D^{-+}, D^{--} \}, \]

where

\[ D^{++} = \{ \omega : \eta_1 = 1, \eta_2 = 1 \}, D^{--} = \{ \omega : \eta_1 = -1, \eta_2 = -1 \}, \]

and so on. Since \( \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_k} = \mathcal{D}_{S_1, S_2, \ldots, S_k} \) and each of the random variables \( S_k \) are \( \mathcal{D}_k \)-measurable. Now we show that the second assumption of a martingale is also hold. Observe that

\[
\mathbb{E}(S_{k+1} \mid \mathcal{D}_k) = \mathbb{E}(S_k + \eta_{k+1} \mid \mathcal{D}_k) \\
= \mathbb{E}(S_k \mid \mathcal{D}_k) + \mathbb{E}(\eta_{k+1} \mid \mathcal{D}_k) \\
= S_k + \mathbb{E}\eta_{k+1} = S_k.
\]
2.2.1 Lemma on martingale representation

**Definition 2.2.** Let \( D_0 \preceq D_1 \preceq \cdots \preceq D_n \) be nondecreasing sequence of partitions. A sequence of random variables \( (A_k, D_{k-1}) \) is called *predictable* if \( A_0 = 0 \) and the random variable \( A_k \) is \( D_{k-1} \)-measurable for every \( 1 \leq k \leq n \).

Let \((\Omega, \mathcal{D}, \mathbb{P})\) be a probability space, \( \rho_1, \rho_2, \ldots, \rho_N \) be a sequence of independent identically distributed random variables taking two values \( b \) and \( a \) with probabilities \( p \) and \( 1 - p \) respectively such that \(-1 < a < b, 0 < p < 1\). Suppose \( \mathbb{E} \rho_1 = r \), then \(-1 < a < r < b \) and \( p = (r - a)/(b - a) \). Set

\[
m_n = \sum_{k=1}^{n} (\rho_k - r), \quad D_n = D_{(\rho_1, \rho_2, \ldots, \rho_n)}.
\]

Note that

\[
\Delta m_n = m_n - m_{n-1} = (\rho_n - r).
\]

Let \( \mathbb{E} \) be a mathematical expectation with respect to the probability \( \mathbb{P} \). Then

\[
\mathbb{E} (\rho_1 - r) = a(1 - p) + bp - r = (b - a)p - (r - a).
\]

**Proposition 2.1** If

\[
p = p^* = \frac{r - a}{b - a},
\]

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and \( P^* \) is the corresponding probability from \( P \), then the sequence \((m_n, D_n, P^*)\) is a martingale.

Proof:

\[
E^*(m_n | D_{n-1}) = E^*[(\rho_n - r) + \sum_{k=1}^{n-1}(\rho_k - r) | D_{n-1}]
\]

\[
= E^*[(\rho_n - r) | D_{n-1}] + E^*\left[\sum_{k=1}^{n-1}(\rho_k - r) | D_{n-1}\right]
\]

\[
= E^*(\rho_n - r) + \sum_{k=1}^{n-1}(\rho_k - r) = E^*\rho_n - r + \sum_{k=1}^{n-1}(\rho_k - r)
\]

\[
= a(1 - p^*) + bp^* - r + \sum_{k=1}^{n-1}(\rho_k - r)
\]

\[
= a\left(1 - \frac{r - a}{b - a}\right) + b\frac{r - a}{b - a} - r + \sum_{k=1}^{n-1}(\rho_k - r)
\]

\[
= \sum_{k=1}^{n-1}(\rho_k - r) = m_{n-1}.
\]

Therefore

\[
E^*(m_n | D_{n-1}) = m_{n-1}.
\]

Lemma On Martingale Representation. Any martingale \((M_n, D_n, P)^N_{n=1}\) with \(EM_n = 0\) admits the following representation by the "basic" martingale \(m\):

\[
M_n = \sum_{k=1}^{n} \alpha_k \Delta m_k,
\]

where \(\alpha_k\) are predictable (that is, \(D_{k-1}\)-measurable, \(k \geq 1\)).
Proof. Since $M_n$ is $\mathcal{D}_n$-measurable, by the proceeding proof of the Proposition 2.1, there exist functions $f_n = f_n(x_1, x_2, \ldots, x_n)$, where each coordinate $x_i$ takes only two values $b$ and $a$ such that

$$M_n(\omega) = f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_n(\omega)), \omega \in \Omega.$$ 

Since $(M_n, \mathcal{D}_n, P)$ is a martingale, then

$$E[f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_n(\omega)) - f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega)) | \mathcal{D}_{n-1}] = 0,$$

which is equivalent to

$$pf_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), b) + (1 - p)f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), a) = f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega)).$$

Rewrite this expression in the form

$$\frac{f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), b) - f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega))}{1 - p} = \frac{f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega)) - f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), a)}{p}.$$
Taking into consideration that \( p = \frac{(r - a)}{(b - a)} \), we obtain

\[
\frac{f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), b) - f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega))}{b - r} = \frac{f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), a) - f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega))}{a - r}.
\]

Let \( \alpha_n(\omega) \) be equal to any of expressions in the last equality, then

\[
f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), b) - f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega)) = \alpha_n(\omega)(b - r),
\]

\[
f_n(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega), a) - f_{n-1}(\rho_1(\omega), \rho_2(\omega), \ldots, \rho_{n-1}(\omega)) = \alpha_n(\omega)(a - r),
\]

or

\[
\Delta M_n(\omega) = \alpha_n(\omega)\Delta m_k.
\]
Chapter 3

Probabilistic Analysis of a Financial Market: A Formulation of the Problem of an Investment and Hedging Concerning Option Pricing

3.1 Some Options Terminology

Options: An option gives the option holder the right not to buy or sell an asset to the option writer or seller for a specified price at a given time in the future.

European Options: An option that may only be exercised on the expiration date.

Expiration Date: The day at which the option may be exercised, denoted by $N$.

Strike Price: The amount for which the underlying asset can be bought (call option) or sold (put option), denoted by $K$.

Premium: The initial cost of the option contract.

Call Option: It gives the holder the right to buy a particular asset for an agreed
price on a specific date.

**Put Option:** It gives the holder the right to sell a particular asset for an agreed price on a specific date.

**Hedging:** A strategy that an investor uses to lower the risk of his investment with the side affect of a decreased profit.

### 3.2 Formulation of the problem

We consider a model of an \((B, S)\)-market which functions at discrete time \(n = 1, 2, \ldots, N\) \((N < \infty)\), and consisting of two assets: bank account \(B = (B_n)\) and shares \(S = (S_n)\). The terminology of Financial Mathematics we are using here is suggested by Cox, Ross and Rubinstein (1979) [3] as CRR Financial Market Model. According to this model, the dynamics of bank account follows the recurrence relations:

\[
B_{n} = (1 + r) \cdot B_{n-1}, \quad B_0 > 0,
\]

where \(r > 0\) is the fixed interest rate.

The price of the share \(S = (S_n)\) evolves according to the recurrence relations.

\[
S_{n} = (1 + \rho_n) \cdot S_{n-1}, \quad S_0 > 0,
\]

\(\rho = (\rho_n)\) is a sequence of random variables such that \(\rho_n\) takes only two values \(a\) and \(b\) with \(-1 < a < r < b\). This means the stock price can move up to \(b\) or down to
Note that if the stock goes down to $a$, then it is better for us to invest in a bank account than in shares, because the return $r > a$. If the stock goes up to $b$, then it is better to invest in shares than in the bank account because $b > r$.

On $\mathcal{D}$ we consider a family of probabilities $\mathcal{P} = \{P\}$ such that the sequence $(\rho_n) = (\rho_n(\omega), 1 \leq n \leq N)$ is a sequence of independent identically distributed random variables and with respect to each probability from this family:

$$P\{\rho_n = a\} = q, \quad P\{\rho_n = b\} = p, \quad p + q = 1, 0 < p < 1, 1 \leq n \leq N.$$ 

Note that we are talking about a family of probabilities because there is a probability for each $p, 0 < p < 1$.

We cannot be certain about the price of a share in the future, so the assumption of a random nature of the dynamics of the sequence $(\rho_n)_{n=1}^N$, which defines the change in price of the share $(S_n)$ is natural from the financial point of view. Moreover, although we may have some reliable guess for the values of $a, b$, we may have no information on a prior value of the probability $p$. Because of this, we will not fix the specified probability $P$ from the family $\mathcal{P}$, but we will assume that $P$ is some distribution from $\mathcal{P}$.

Suppose an investor has an initial capital $X_0 = x > 0$ and he wants to increase it with opportunities provided by the $(B, S)$-market.

If the investor chooses a risk free opportunity, that is, the investor invests the capital $X_0 = x$ in a bank account (or in bonds) with the interest rate $r \geq 0$, then at the time $n$ the capital will be $(1 + r)^n \cdot X_0$. Therefore, in order to achieve at time $N$
the specified amount \( f_N \), the initial capital should be

\[
x = (1 + r)^{-N} f_N.
\]

The investor may also allocate all capital into shares. In this case, at the time \( n \) the capital will be

\[
S_n = (1 + \rho_n)(1 + \rho_{n-1}) \ldots (1 + \rho_1)X_0 = x \prod_{k=1}^{n}(1 + \rho_k).
\]

If the probability \( p \) is known, then by independence and the fact that \( E(\rho_k) = b_p + a_q \), we obtain

\[
E(S_n) = (1 + (b_p + a_q))^N x.
\]

Hence, in order to achieve on average at time of \( N \) the specified amount \( f_N \), the initial capital should be

\[
x = [1 + (b_p + a_q)]^{-N} f_N.
\]

Next, the investor may have a third possibility: To invest a part of the fund in the bank account and another part into shares.

Denote by \( B_0 \) the price of one bond, \( S_0 \) the price of one share at the time \( n = 0 \). Suppose at time \( n = 0 \) the investor has \( \beta_0 \) bonds and \( \gamma_0 \) shares. We allow any values for \( \beta_0 \) and \( \gamma_0 \), for example, fractional or negative. Then the initial capital \( X_0 \) is

\[
X_0 = \beta_0 B_0 + \gamma_0 S_0.
\]
We say the pair \( \pi_0 = (\beta_0, \gamma_0) \) presents the portfolio or strategy of the investor at time \( n = 0 \). Assume that during the time interval \((0, 1)\) the investor rearranges the portfolio. Suppose by the time \( n = 1 \), that is, before the new price for the share is known, the investor has rearranged the portfolio \( \pi_0 = (\beta_0, \gamma_0) \) into a new one \( \pi_1 = (\beta_1, \gamma_1) \) based only on the information of the values \((B_0, S_0)\) and assume there is no inflow of capital “from the outside”, and there is no outflow of the capital “to the outside”. Then

\[
X_0 = \beta_1 B_0 + \gamma_1 S_0.
\]

At time \( n = 1 \) assume the new values for the pair \((B_1, S_1)\) are known and the capital becomes

\[
X_1 = \beta_1 B_1 + \gamma_1 S_1.
\]

The increment in the capital \( \Delta X_1 = X_1 - X_0 \) is represented as

\[
\Delta X_1 = \beta_1 \Delta B_1 + \gamma_1 \Delta S_1.
\]

In the same way, for any \( n \quad (n = 1, 2, \ldots, N) \); we set

\[
X_{n-1} = \beta_n B_{n-1} + \gamma_n S_{n-1}
\]

and

\[
X_n = \beta_n B_n + \gamma_n S_n.
\]

**Definition 3.1** The sequence of pairs \( \pi_n = (\beta_n, \gamma_n) \), for \( 0 \leq n \leq N \) presents the
portfolio or strategy of the investor at time \( n \).

It is assumed that the portfolio \( \pi_n = (\beta_n, \gamma_n) \) is created based only on the information of the previous prices. In mathematical terms this means that \( \beta_n, \gamma_n \) are \( D_{n-1} \)-measurable, where the partition \( D_{n-1} = D_{S_1, S_2, \ldots, S_{n-1}} \).

The increment \( \Delta X_n = X_n - X_{n-1} \) is represented in the form

\[
\Delta X_n = \beta_n \Delta B_n + \gamma_n \Delta S_n,
\]

Where, \( \beta_n \Delta B_n \) represents the part of the capital deposited in a bank account at time \( n \).

Further, \( \gamma_n \Delta S_n \) represents the investment in shares at time \( n \).

Finally, the capital by time \( n \) is given by;

\[
X_n = X_0 + \sum_{k=1}^{n} (\beta_k \Delta B_k + \gamma_k \Delta S_k).
\]

The meaning of this formula is as follows: The formation of the capital \( X_n \) is achieved only by the changes of \( (\Delta B_k, \Delta S_k) \) in prices of bonds and shares without any outflow nor inflow. If we denote \( \Delta \beta_n = \beta_n - \beta_{n-1} \) and \( \Delta \gamma_n = \gamma_n - \gamma_{n-1} \), then this assumption can be written as:

\[
B_{n-1} \Delta \beta_n + S_{n-1} \Delta \gamma_n = 0.
\]

To elaborate further on this assumption, it means that the values of the portfolios \( \pi_k = (\beta_k, \gamma_k), \ (k \leq n) \) are such that a change of capital in the bank account \( B_{n-1} \Delta \beta_n \)
may happen only as a result of the corresponding change of capital in shares \( S_{n-1} \Delta \gamma_n \)
and vice versa. The strategy \( \pi_n = (\beta_n, \gamma_n), \quad n \leq N \), is then based on the principle of self-financing and hence the strategy \( (\pi_n) \) is called a self-financed strategy \(^1\). The class of all self-financed strategies is denoted \( SF \) and in the following we will consider only these strategies.

Write the increment \( \Delta X_n \) in the form

\[
\Delta X_n = \beta_n \Delta B_n + \gamma_n \Delta S_n \\
= \beta_n r B_{n-1} + \gamma_n \rho_n S_{n-1} \\
= r X_{n-1} + \gamma_n S_{n-1} (\rho_n - r).
\]

In order to stress that the sequence \( X = (X_n) \) depends on the strategy \( \pi = (\pi_n), \quad 0 \leq n \leq N \) we write \( X^\pi = (X^\pi_n) \).

Now suppose the investor wants to solve the following problem with the help of \((B,S)\)-market ("investment problem"). At some known fixed time \( N < \infty \) in future her capital is required to be not less than \( f_N = f_N(S_0, S_1, \ldots, S_N) \), where function \( f_N \) depends, generally speaking, on the whole random realization \((S_0, S_1, \ldots, S_N)\) of the share’s prices.

Obviously, the realization of this goal depends on the value of the initial capital and on a strategy in the class \( SF \).

\(^1\)A strategy is self-financing if there is no exogenous infusion or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one.
Definition 3.2. For the given $x > 0$ and nonnegative function $f_N = f_N(S_0, S_1, \ldots, S_N)$ a self-financing strategy $\pi = (\pi_n)$ is called a $(x, f_N)$–hedge, if for all $\omega \in \Omega$,

$$X_0^\pi = x, \quad X_N^\pi(\omega) \geq f_N(S_0, S_1(\omega), \ldots, S_N(\omega)),$$

where $(S_n(\omega))_{n=0}^N$ follows the relations

$$S_n(\omega) = (1 + \rho_n(\omega))S_{n-1}(\omega), \quad n \geq 1,$$

and $S_0(\omega) = S_0$ is a positive constant.

If the hedge of the portfolio $\pi$ is not unique then we need to find a hedge $\pi^*$ with minimal value $X_n^\pi^* \leq X_n^\pi$ for any other hedge $\pi$.

Then

$$X_N^\pi(\omega) = f_N(S_0, S_1(\omega), \ldots, S_N(\omega)),$$

and the strategy $\pi = (\pi_n)$ is called minimal $(x, f_N)$–hedge.

Let $\Pi(x, f_N)$ be the set of all $(x, f_N)$–hedges, with $\pi \in SF$.

Definition 3.3. The quantity

$$C_N = \inf\{x > 0 : \Pi(x, f_N) \neq \emptyset\}$$

is called the investment cost (price) that guarantees the capital is not less than $f_N$ for all $\omega \in \Omega$.

Remark. In the case of a finite $(B, S)$–market there is always an initial capital $x > 0$
such that $\Pi(x, f_N) \neq \emptyset$ and hence $C_N < \infty$.

The value of $C_N$ is interesting in connection with “fair price” for so-called European options, or contracts with options of European type. Now we clarify the essence of these contracts by an example of the so-called standard call-option of European type. A $(B, S)$–market participant (writer) can issue a security that gives a right for its buyer to purchase shares at the specified price $K$ and can then exercise them at some fixed time $N$ in future. This security is the call option of European type.

If at the specified time $N$ the situation of the $(B, S)$–market is such that $S_N > K$, then the owner of the option (the person who purchased the option from the writer) will exercise it, that is, the owner will buy shares at the price $K$. Right after the owner may immediately sell them at the current price $S_N$ for a profit of $f_N = S_N - K$. If it happens that $S_N \leq K$, then there is no need for the owner of the option to exercise it, so the profit is zero. Therefore, in both cases the payoff function (profit for the owner of the call option) $f_N$, which can also be interpreted as a payment from the option seller (the writer) to the buyer (the owner of the option) is $(S_N - K)^+ = \max(S_N - K, 0)$.

As we can see, the writer of contract with an option guarantees the owner of the option a payoff defined by the function $f_N = f_N(S_0, S_1, \ldots, S_N)$ (which can be zero or positive). Obviously, it is required to pay some bonus to the writer to compensate for probable losses. Of course, if this bonus is too large, then it will either scare away all possible buyers, or (in the case if the option is purchased) it will create a so-called arbitrage situation at the market. That is, the writer will have a profit without any risk. If the bonus is too small, then the writer will not be able to payoff the sum.
defined in the contract (and by the tools of \((B,S)\)-market).

Therefore we have a problem to determine the fair price of an option and how to calculate the price. It is simple to understand that it should be exactly \(C_N\).

We have the following two standard options of the European type:

1) \textit{Call option} with the payoff function \(f_N = (S_N - K)^+\);

2) \textit{Put option} with the payoff function \(f_N = (K - S_N)^+\). So at the time of expiration, there are three possible outcomes.

<table>
<thead>
<tr>
<th>Case</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the money</td>
<td>(S_N &gt; K)</td>
<td>(S_N &lt; K)</td>
</tr>
<tr>
<td>At the money</td>
<td>(S_N = K)</td>
<td>(S_N = K)</td>
</tr>
<tr>
<td>Out of the money</td>
<td>(S_N &lt; K)</td>
<td>(S_N &gt; K)</td>
</tr>
</tbody>
</table>

Table 3.1: Possible outcomes upon expiration

In general, a call option is profitable if you expect increasing in the stock price while the put option is profitable if you expect the stock price to fall.

\textbf{Example 3.1:} European call option is purchased by an investor with a strike price \(K = \$100\) to purchase a share. Assume that \(b = 0.1\), \(a = -0.1\) and the option will mature at the end of the month\(,\) means \(N = 1\). What is the payoff function for the call option if the stock price at the expiration date is equal to \(\$100\)?

\textbf{Solution:} Since \(b = 0.1\) we have \(S_1 = (1 + b)S_0 = (1 + 0.1)100 = \$110\) which means the price of the stock is increasing to \(\$110\). Since \(a = -0.1\), \(S_1 = (1 + a)S_0 = (1 - 0.1)100 = \$90\), which means the price of the stock is decreasing to \(\$90\). Now,
following the formula \( f_N = (S_N - K)^+ \) we find that the payoff for this call option at
the end of the month is:

\[
f_N = \max(110 - 100, 0) = \max(10, 0) = 10.
\]

Then \( K < S_N \), and the call option “in the money”, hence will be exercised. If
\( f_N = \max(90 - 100, 0) = 0 \), then \( K > S_N \) and the call option “out of the money”.
Hence it is worthless to exercise since the profit is zero.

There are many types of options, not only European. One of the most common
are options of American type, which are characterized in that they may be exercised
at an arbitrary time as a decided in advance governed by the set \( \{0, 1, 2, \ldots, N\} \). We
provide more discussion on the American option in Appendix A.
Chapter 4

A Theory of the Price Calculation and Design of Hedge Strategies for European Type Options

4.1 Main results

Recall that the \((B,S)\)-market is defined by the relations

\[
B_n = (1 + r)B_{n-1}, \quad B_0 > 0,
\]

\[
S_n = (1 + \rho_n)S_{n-1}, \quad S_0 > 0,
\]

where \(-1 < a < r < b\), \(\rho_n\) are independent identically distributed random variables defined on the sample space \(\Omega\), and \(\rho_n\) are such that for any probability \(P \in \mathcal{P}\):

\[
P\{\rho_1 = a\} = q, \quad P\{\rho_1 = b\} = p, \quad p + q = 1, \quad 0 < p < 1.
\]
Remark on the construction of the sample space $\Omega$. We would like to consider the “best choice” of the sample space $\Omega$ that suits the problem we consider. By the “best” we understand that $\Omega$ is minimal, it should contain only elements that have nonzero probabilities, that is, for any $\omega \in \Omega$ and and probability $P \in \mathcal{P}$, we have $P(\omega) > 0$.

For this, we introduce new random variables $\varepsilon_n$, by the following formula:

$$
\varepsilon_n = \frac{2\rho_n - (a + b)}{b - a}, \quad \rho_n = \frac{a + b}{2} + \frac{b - a}{2} \varepsilon_n.
$$

Obviously the random variable $\varepsilon_n$ takes the value 1 when the share price is going up, that is, $\rho_n = b$ and the random variable $\varepsilon_n$ takes the value $-1$ when the share price is going down, that is, $\rho_n = a$. Namely,

$$
\varepsilon_n = \begin{cases} +1 & \Leftrightarrow \rho = \begin{cases} b \\ a \end{cases} \\
-1 & \Leftrightarrow \rho = \begin{cases} b \\ a \end{cases} \end{cases}.
$$

Now we formulate the construction of the sample space $\Omega$ on which the sequence $\rho = (\rho_n)$ is defined in the best way. As an elementary outcome $\omega$ consider a sequence of length $N$ that consists of only $(-1)$s and $(+1)$s. For example,

$$
\omega = (+1, -1, -1, +1, \ldots - 1)._{N \text{ times}}
$$

Note that $\omega$ is realization (collection of possible values) of the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$. The set of such sequences of length $N$ that consist of only $(-1)$s and $(+1)$s is the
sample space and it is denoted $\Omega = \{+1, -1\}^N$. There are $2^N$ elementary outcomes in $\Omega$. Therefore, the sample space $\Omega$ is the space of all possible realizations of the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$. Assume that the partition $\mathcal{D} = \mathcal{D}_{\text{max}} = \{\omega, \omega \in \Omega\}$ consists of $2^N$ atoms which are elementary outcomes of $\Omega$, the most fine partition.

On $\mathcal{D}$ we consider a family of probabilities $\mathcal{P} = \{P\}$ such that for each $\omega \in \Omega$:

$$\omega = (\underbrace{+1, -1, -1, +1, \cdots -1}_{N \text{ times}}, \text{with } k \text{ of } (+1)s \text{ and } N - k \text{ of } (-1)s)$$

we have that $P(\omega) = p^k q^{N-k}, 0 \leq k \leq N$.

Let the strategy (portfolio) $\pi = (\pi_n) \in SF$ and $X^\pi = X^\pi_n$ be the corresponding capital consisting of money in the bank account $B = (B_n)$ (bonds) and in shares $S = (S_n), \ n = 0, 1, 2, \ldots, N$. Note that the values of $p$ are a priori unknown. But the hedge assumption is formulated as a property for all $\omega \in \Omega$. According to the Remark on the construction of the sample space $\Omega$ above, this is equivalent to the hedge assumption holds for all $P \in \mathcal{P}$. Therefore, if for the strategy $\pi$ we have the hedge assumption for some $P^* \in \mathcal{P}$, then it will be satisfied for all $P \in \mathcal{P}$ and all $\omega \in \Omega$.

Consider the problem of finding the fair price $C_N$ of an option. Let $\pi = (\pi_n) \in SF$ and $X^\pi = X^\pi_n$ be the corresponding capital, $X^\pi_0 = x$. Let

$$M^\pi_n = \frac{X^\pi_n}{B_n}, 0 \leq n \leq N,$$
Calculate

\[ \Delta M_n^\pi = \frac{X_n^\pi}{B_n} - \frac{X_{n-1}^\pi}{B_{n-1}} = \frac{X_n^\pi - (1 + r)X_{n-1}^\pi}{B_n} \]

\[ = \frac{X_n^\pi - X_{n-1}^\pi - rX_{n-1}^\pi}{B_n} = \frac{\Delta X_n^\pi - rX_{n-1}^\pi}{B_n} = \frac{\gamma_n S_{n-1}}{B_n} (\rho_n - r). \]

Recall that we denote

\[ m_n = \sum_{k=1}^{n} (\rho_k - r), \quad \Delta m_n = \rho_n - r. \]

Then

\[ M_n^\pi = M_0^\pi + \sum_{k=1}^{n} \gamma_k S_{k-1} \Delta m_k. \]

Let \( E \) be a mathematical expectation with respect to the probability \( P \). Then

\[ E(\rho_1 - r) = a(1 - p) + bp - r = (b - a)p - (r - a). \]

Hence, if

\[ p = p^* = \frac{r - a}{b - a}, \]

and \( P^* \) is the corresponding probability from \( P \), then in Proposition 2.2 we proved that the sequence \((m_n, D_n, P^*)\) is a martingale. Since the random variable \((\gamma_k S_{k-1})/B_k\) is \( D_{k-1} \)-measurable, the representation of \( M_n^\pi \) can be derived so that the sequence \((M_n^\pi, D_n, P^*)\) is also a martingale. Hence

\[ E^* M_n^\pi = M_0^\pi. \]
for all $\pi \in SF$. Therefore

$$E^*(1 + r)^{-N}X_N^\pi = M_0^\pi B_0 = X_0^\pi = x.$$  

Hence, if the strategy $\pi \in \Pi(x, f_N)$, then by the definition of $(x, f_N)$–hedge and the last equality we get

$$x \geq E^*(1 + r)^{-N}f_N.$$  

If further the hedge $\pi$ is minimal, then

$$x = E^*(1 + r)^{-N}f_N.$$  

We formulate this as the following lemma.

**Lemma 4.1.** (Necessity) Let for $(B, S)$–market the self-financing strategy $\pi$ is a $(x, f_N)$–hedge. Then

$$x \geq E^*(1 + r)^{-N}f_N.$$  

If further the hedge is minimal, then

$$x = E^*(1 + r)^{-N}f_N,$$

where $E^*$ is the mathematical expectation by the probability $P^*$ such that

$$P^*(\rho_n = b) = p^*, \quad P^*(\rho_n = a) = 1 - p^*, \quad p^* = \frac{r - a}{b - a}.$$
The structure of the financial \((B, S)\)-market is such that condition (***) for \(x\) and \(f_N\) is actually sufficient for the existence of the minimal hedge.

**Lemma 4.2.** (Sufficiency) Suppose the initial capital \(x > 0\) and payoff function \(f_N = f_N(S_0, S_1, \ldots, S_N)\) are chosen such that the assumption (***) is satisfied. Then there exist a minimal \((x, f_N)\)-hedge \(\pi^* \in SF\) in the class \(\Pi(x, f_N)\).

**Proof:** Define a random variable

\[
M_n^* = E^*(\frac{f_N}{B_N} | D_n), \quad 0 \leq n \leq N,
\]

where \(E^*\) is an expectation with respect to the probability \(P^*\) defined above. Then

\[
M_0^* = E^*(\frac{f_N}{B_N}), \quad M_N^* = \frac{f_N}{B_N}.
\]

It is obvious that the sequence \((M_n^*, D_n, P^*)^N_{n=0}\) is a martingale. The sequence of partitions

\[
(D_n)^N_{n=1} = (D_{\rho_1, \rho_2, \ldots, \rho_n})^N_{n=1}
\]

is generated by random variables that take only two values. By the Lemma of Martingale Representation, the martingale \((M_n^*, D_n, P^*)^N_{n=1}\) can be represented in the form

\[
M_n^* = M_0^* + \sum_{k=1}^{n} \alpha_k^*(\rho_1, \ldots, \rho_{k-1})\Delta m_k,
\]

where \(\alpha_k^* = \alpha_k^*(\rho_1, \ldots, \rho_{k-1}), (k \geq 2)\) are some predictable (that is, \(D_{k-1}\)-measurable)
random variables, $\alpha_1 = \text{constant}, \Delta m_k = \rho_k - r$. If we denote

$$\gamma_k^* = \frac{\alpha_k^* B_k}{S_{k-1}}, \quad k \geq 1,$$

then it is natural to rewrite $M_n^*$ as

$$M_n^* = M_0^* + \sum_{k=1}^{n} \frac{\gamma_k^* S_{k-1}}{B_k} \Delta m_k.$$

Suppose the assumption (***) be satisfied, since $B_N = (1 + r)B_0$, then rewrite (***) in the following way

$$\frac{x}{B_0} = \mathbf{E}^*(\frac{f_N}{B_N}).$$

We show that there exists a strategy $\pi^* \in SF$ such that the corresponding normalized capital

$$M_n^{\pi^*} \equiv \frac{X_n^{\pi^*}}{B_n}$$

is equal to $M_n^*$, $0 \leq n \leq N$.

In fact, let the initial capital be $x$. Having the value of $\gamma_1^* = (\alpha_1^* B_1)/S_0$, define the value $\beta_1^*$ from the equality

$$x = \beta_1^* B_0 + \gamma_1^* S_0.$$

Then

$$\beta_1^* = \frac{x - \gamma_1^* S_0}{B_0}.$$
Let \( \pi_1^* = (\beta_1^*, \gamma_1^*) \). Then for this portfolio the capital is

\[
X_1^{\pi^*} = \beta_1^* B_1 + \gamma_1^* S_1.
\]

Therefore

\[
M_1^{\pi^*} = \frac{X_1^{\pi^*}}{B_1} = \beta_1^* + \gamma_1^* \frac{S_1}{B_1} = \frac{x}{B_0} - \gamma_1^* \frac{S_0}{B_0} + \gamma_1^* \frac{S_1}{B_1} = \frac{x}{B_0} + \gamma_1^* \left( \frac{S_1}{B_1} - \frac{S_0}{B_0} \right) = \frac{x}{B_0} + \alpha_1^* \left( \frac{S_1}{S_0} - \frac{B_1}{B_0} \right) = \frac{x}{B_0} + \alpha_1^* (\rho_1 - r) = M_0^* + \alpha_1^* (\rho_1 - r) = M_1^*.
\]

Similarly, having the value of \( \gamma_2^* \), define \( \beta_2^* \) as

\[
\beta_2^* = \frac{X_1^{\pi^*} - \gamma_1^* S_1}{B_1},
\]

and let \( \pi_2^* = (\beta_2^*, \gamma_2^*) \). The construction of the strategy \( \pi_n^* = (\beta_n^*, \gamma_n^*) \) is obtained by the induction.
We obtain that for all \( n \)

\[ M_n^{\pi^*} = M_n^*. \]

Hence

\[ \frac{X_n^{\pi^*}}{B_n} = M_n^{\pi^*} = M_n^* = E^*\left( \frac{f_N}{B_N} \mid D_n \right), \]

or, equivalently,

\[ X_n^{\pi^*} = E^*\left( (1 + r)^{-(N-n)} f_N \mid D_n \right). \]

In particular, \( X_n^{\pi^*} \geq 0 \) for all \( n \) and \( X_N^{\pi^*} = f_N \) for all \( \omega \in \Omega \). Therefore the strategy \( \pi^* \) is a minimal \( (x, f_N) \)-hedge, and the fact that \( \pi^* \in SF \) follows from this construction.

This completes the proof of Lemma 2.

Therefore, on the basis of Lemmas 1 and 2 we can formulate the following result on the calculation of the European option price, structure of the optical strategy and the evolution of the corresponding capital.

**Theorem 4.1.** 1) For \((B, S)\)-market, the fair price \( C_N \) of the option with exercise time \( N \), payoff function \( f_N = f_N(S_0, S_1, \ldots, S_N) \), and the use of self financing strategies, if defined by the formula

\[ C_N = E^*\left[ (1 + r)^{-N} f_N \right], \]

where \( E^* \) is the expectation by the probability \( P^* \) such that

\[ P^*(\rho_1 = b) = p^* = \frac{r - a}{b - a}. \]
2) There exists the minimal self-financing \((C_N, f_N)\)-hedge \(\pi^* = (\pi^*_n)^N\_{n=1}\) such that the evolution of the corresponding capital \((X^\pi^*_n)^N\_{n=1}\) is given by the formula

\[ X^\pi^*_n = E^*\left[ (1 + r)^{-(N-n) f_N} \mid D_n \right]. \]

Moreover the predictable components \(\pi^*_n = (\beta^*_n, \gamma^*_n)\) are defined by the equalities

\[ \begin{align*}
\gamma^*_n &= \frac{\alpha^*_n B_n}{S_{n-1}}, \\
\beta^*_n &= \frac{X^\pi^*_{n-1} - \gamma^*_n S_{n-1}}{B_{n-1}}.
\end{align*} \]

**Proof.** For 1) By definition

\[ C_N = \inf\{ x > 0 : \Pi(x, f_N) \neq \emptyset \}. \]

For the fixed values of \(N\) and \(f_N\), the class \(\Pi(x, f_N)\) is not empty for at least large \(x\). Therefore \(C_N < \infty\). By Lemma 1 we have: \(C^*_N \geq E^*[(1 + r)^{-N} f_N]\). On the other hand, by Lemma 2 there exists \((x, f_N)\)-hedge with the initial capital \(x = E^*[(1 + r)^{-N} f_N]\). Hence

\[ C^*_N = E^*[(1 + r)^{-N} f_N]. \]

Statement 2) Follows directly from Lemma 2.
4.2 European options with $f_N = f(S_N)$.

Previously, we assumed that the payoff function $f_N$ depends on all values $S_0, S_1, \ldots, S_N$.

For the standard European options it is assumed that

$$f_N = f(S_N).$$

More specifically, for the call option

$$f_N = (S_N - K)^+,\,$$

where $K$ is some fixed price for the share at time 0.

Using these assumptions the formula for the fair option price and for the calculation of hedge strategies look quite simple. Consider the random variable

$$X_n^{\pi^*} = E^*( (1 + r)^{-(N-n)} f_N | D_n).$$

consider the function

$$F_n(x; p) = \sum_{k=0}^{n} f(x(1 + b)^{k}(1 + a)^{n-k}) C_n^k p^k (1 - p)^{n-k}.$$ 

Next, we need one more notation. Let $\Delta_n$ be the amount of times when the share price is going up until time $n$. Formally $\Delta_n = \sum_{k=1}^{n} \frac{\alpha_k - a}{b - a}$ and this can be explained in
the following way. Introduce new random variables $\delta_n$ using the following formulas:

$$\delta_n = \frac{\rho_n - a}{b - a} , \quad \rho_n = a + (b - a)\delta_n.$$  

Thus

$$\delta_n = \begin{cases} 1 & \Leftrightarrow \rho = b \\ 0 & \Leftrightarrow \rho = a \end{cases},$$

and $\Delta_n = \delta_1 + \cdots + \delta_n$.

Note that

$$\prod_{n<k \leq N} (1 + \rho_k) = (1 + b)^{\Delta_N - \Delta_n} (1 + a)^{(N-n)-\Delta_N-\Delta_n},$$

and hence

$$E^* f(x \prod_{n<k \leq N} (1 + \rho_k)) = F_{N-n}(x; p^*),$$

where $p^* = (r - a)/(b - a)$, with corresponding capital

$$X_n^{\pi^*} = E^* \left( (1 + r)^{-(N-n)} f(S_N) \mid D_n \right)$$

$$= E^* \left( (1 + r)^{-(N-n)} f(S_N) \mid S_n \right)$$

$$= (1 + r)^{-(N-n)} F_{N-n}(S_n; p^*).$$

In particular,

$$C_N = X_0^{\pi} = (1 + r)^{-N} F_N(S_0; p^*).$$
Taking into consideration the form of the payoff function,

\[ F_N(S_0; p^*) = \sum_{k=0}^{N} C_N^k (p^*)^k (1 - p^*)^{N-k} \times \]

\[ \times \max(0; S_0(1 + a)^N \left( \frac{1+b}{1+a} \right)^k - K). \]

we let \( k_0 = k_0(a, b, S_0, K) \) be the smallest non-negative number for which

\[ S_0(1 + a)^N \left( \frac{1+b}{1+a} \right)^{k_0} > K. \]

If \( k_0 > N \), then the option expires out of the money, see Table 3.1.

Then \( F_N(S_0; p^*) = 0 \) and hence \( C_N^* = 0 \).

Assume that \( k_0 \leq N \) (usually \( k_0 = 0 \)), the option expires in the money, see Table 3.1.

Then

\[ C_N = (1 + r)^{-N} F_N(S_0; p^*) \]

\[ = (1 + r)^{-N} \left[ \sum_{k=k_0}^{N} C_N^k (p^*)^k (1 - p^*)^{N-k} (1 + a)^N \left( \frac{1+b}{1+a} \right)^k - K \right] \]

Breaking the above expression up into two terms we have:

\[ C_N = S_0 \sum_{k=k_0}^{N} C_N^k (p^*)^k (1 - p^*)^{N-k} \left( \frac{1+a}{1+r} \right)^N \left( \frac{1+b}{1+a} \right)^k \]

\[ - K (1 + r)^{-N} \sum_{k=k_0}^{N} C_N^k (p^*)^k (1 - p^*)^{N-k}, \]
where $k_0$ is calculated by the formula

$$k_0 = 1 + \left[ \ln \frac{K}{S_0(1 + a)^N} / \ln \frac{1 + b}{1 + a} \right].$$

In a similar way we can derive formula for the form of minimal hedge $\pi^*_n = (\gamma^*_n; \beta^*_n)$.

Note here that $\gamma^*_n = \gamma^*_n(S_{n-1})$, $\beta^*_n = \beta^*_n(S_{n-1})$. Therefore

$$\gamma^*_n = (1 + r)^{-(N-n)} \cdot \frac{F_{N-n}(S_{n-1}(1 + b); p^*) - F_{N-n}(S_{n-1}(1 + a); p^*)}{S_{n-1}(b - a)};$$

$$\beta^*_n = \frac{F_{N-n+1}(S_{n-1}; p^*)}{B_N} - (1 + r)^{-(N-n)} \times$$

$$\times \frac{F_{N-n}(S_{n-1}(1 + b); p^*) - F_{N-n}(S_{n-1}(1 + a); p^*)}{B_{n-1}(b - a)}.$$

This result has been obtained by Cox, Ross and Rubinstein [3].

4.3 An example of calculation of the price of an option and minimal hedge

Assume that the situation at the $(B, S)$–market is such that the price of one share is $100$ and the price for one bond is $10$. A writer issues a call option, that is, a buyer of the option has the right to buy at the specified time $N$ by the agreed price $K = 110$. Assume that $N = 2$, $r = 0.1$, $a = -0.3$, $b = 0.4$. To simplify the
statement of the problem, we assume that the writer pays to the buyer of the option at time \( N = 2 \) the sum \( f_2 = (S_2 - K)^+ \).

We can calculate by the derived formula the fair “rational” price of the option from the point of view for the writer. Namely,

\[
C^*_N = E^*[ (1 + r)^{-N} f_N ] = 23.21
\]

Therefore, having this money, the writer should construct at the \((B, S)\)-market a strategy in such way that by the exercise time the writer will be able to accept the option, that is, to define the structure of the minimal hedge. We have

\[
\gamma_1 = 0.64, \quad \beta_1 = -4.06.
\]

This means that the writer should borrow bonds by the current price for $40.60 and invest all sum into shares. At time \( n = 1 \) the new price for a share becomes known. According to our assumptions it can become:

A) in the case the price goes down, it is $70.

B) in the case the price goes up, it is $140.

A) If the price goes down, then

\[
\gamma_2 = 0, \quad \beta_2 = 0,
\]

that is, the writer sells his shares by the new price and returns the debt \( 0.64 \cdot (70) = 40.6 \cdot (1.1) \). Even if at \( n = 2 \) if the price of the share goes up, then the price of the
share will be $98 and in this case the writer pays nothing to the buyer.

B) If at \( n = 1 \) the price of the share goes up, then

\[
\gamma_2 = 0.88, \quad \beta_2 = -7.11.
\]

Therefore, the writer borrows 3.05 bonds by the new price $11 with the sum $33.55 and invests them into shares. At \( n = 2 \) the new price of the share is announced. Here again then may there be two cases: The price goes down or up.

In the case the price goes down, the price of the share is $98. Then the writer pays nothing to the buyer. For the return of the debt the writer has shares: \( 0.88 \cdot 98 = 86.24 \).

In the case the price goes up, the price of the share is $196. Then the writer should pay to the buyer $86 and exactly such capital the buyer has after the return of the debt:

\[
X_2 = 0.88 \cdot 196 - 7.11 \cdot 12.1.
\]

Therefore the writer always can accept the option to exercise. The buyer in the “lucky” case could obtain the profit $63.24 investing for the option $23.21.
Chapter 5

Conclusions and Recommendations

5.1 Conclusions

The objectives of this research are to explain how options are priced according to a two-dimensional model and how to create a hedge portfolio using this model. We demonstrate the use of mathematical and statistical techniques and, with the use of an illustrative example, we find out the extent to which the theoretical side of the study matches the actual reality for the European options.

In this thesis, we provide an elementary probabilistic approach to the problem of pricing for European type options. Some of our derivations are based on the paper by Shiryaev, Kabanov, Kramkov, and Mel’nikov [6], but their work relies on $\sigma$-algebras, while our does not. The notion of a partition is borrowed from the textbook by Shiryaev [5]. Everything in this thesis is explained from scratch, the reader needs to have only a basic knowledge of Probability Theory at an elementary level. We introduce and prove important properties of probability, conditional expectation associated with partitions, martingales before we apply them for option pricing. The
paper by Shiryaev, Kabanov, Kramkov, and Mel’nikov [6] omits many proofs since the authors call them “elementary”. For completeness we provide their detailed rigorous proofs.

5.2 Future Research

We considered only European options in my thesis. It would be very interesting to develop a similar theory for pricing American options, too. The main difficulty here is that contrary to European options, American options have a random exercise times. Hence a technique of optimal stopping times based on a partition (not $\sigma$-algebra) for finite probability spaces should be developed.

The second problem for our future research is apply in practice our theoretical findings and find out the extent to which the theoretical side of the study matches with the actual reality for both the European and American options. Some ideas how we would like to approach this problem are explained in Section 4.3.
A.1 On Conditional Probability

Here we provide a few more examples on the conditional probability calculations.

First we need one basic statement, which we state without proof.

**Proposition A.1.** Let $\xi$ and $\eta$ be two independent random variables, then

$$P\{\xi + \eta = z \mid \eta = y\} = P\{\xi + y = z\}.$$

**Example A.1.** Let $\xi$ and $\eta$ be two independent random variables with a Bernoulli distribution with parameter $p$, that is,

$$P\{\xi = 1\} = P\{\eta = 1\} = p, \quad P\{\xi = 0\} = P\{\eta = 0\} = q,$$

where $p + q = 1$. Consider an event $A = \{\xi + \eta = k\}, \ (k = 0, 1, 2)$ and find the
conditional probability $P(A \mid \eta)$ for different values of $k$.

According to Proposition A.1,

$$P\{\xi + \eta = k \mid \eta\} = P\{\xi + \eta = k \mid \eta = 0\} \cdot I_{\{\eta=0\}}(\omega)$$

$$+ P\{\xi + \eta = k \mid \eta = 1\} \cdot I_{\{\eta=1\}}(\omega)$$

$$= P\{\xi = k\} \cdot I_{\{\eta=0\}}(\omega)$$

$$+ P\{\xi = k - 1\} \cdot I_{\{\eta=1\}}(\omega).$$

Therefore we obtained

$$P\{\xi + \eta = k \mid \eta\} = \begin{cases} q \cdot I_{\{\eta=0\}}(\omega), & k = 0, \\ p \cdot I_{\{\eta=0\}}(\omega) + q \cdot I_{\{\eta=1\}}(\omega), & k = 1, \\ p \cdot I_{\{\eta=1\}}(\omega), & k = 2. \end{cases}$$

or

$$P\{\xi + \eta = k \mid \eta\} = \begin{cases} q(1 - \eta), & k = 0, \\ p(1 - \eta) + q\eta, & k = 1, \\ p\eta, & k = 2. \end{cases}$$

A.2 On Conditional Expectation

We provide more examples on conditional expectation calculations.

Example A.2. For the random variables $\xi$ and $\eta$ from Example A.1 we derive an
expression for

\[ E(\xi + \eta \mid \eta). \]

Since random variables are independent

\[ E(\xi + \eta \mid \eta) = E\xi + \eta = p + \eta. \]

The same from the definition:

\[
E(\xi + \eta \mid \eta) = \sum_{k=0}^{2} kP(\xi + \eta = k \mid \eta) \\
= q(1 - \eta) \cdot 0 + [p(1 - \eta) + q\eta] \cdot 1 + p\eta \cdot 2 = p + \eta.
\]

**Example A.3.** Let \( \xi \) and \( \eta \) be independent identically distributed random variables.

Then

\[ E(\xi \mid \xi + \eta) = E(\eta \mid \xi + \eta) = \frac{\xi + \eta}{2}. \]

Let for simplicity random variables \( \xi \) and \( \eta \) take values 1, 2, \ldots, \( m \). Then

\[
P(\xi = k \mid \xi + \eta = l) = \frac{P(\xi = k, \xi + \eta = l)}{P(\xi + \eta = l)} \\
= \frac{P(\xi = k, \eta = l - k)}{P(\xi + \eta = l)} = \frac{P(\eta = k) \cdot P(\xi = l - k)}{P(\xi + \eta = l)} \\
= \frac{P(\eta = k) \cdot P(\xi = l - k)}{P(\xi + \eta = l)} = P(\eta = k \mid \xi + \eta = l).
\]
Therefore the first equality is proved. Now we show the second one.

\[ 2 \cdot P(\xi \mid \xi + \eta) = P(\xi \mid \xi + \eta) + P(\eta \mid \xi + \eta) = P(\xi + \eta \mid \xi + \eta) = \xi + \eta. \]

One small remark for the readers which are familiar with the notion of \( \sigma \)-algebra. The notion of the conditional mathematical expectation is usually introduced for \( \sigma \)-algebras and not for partitions. But in the case of a finite sample space, algebra and \( \sigma \)-algebra are identical notions. There is only one algebra that is generated by a partition and vise virsa, each algebra is generated by a partition. Moreover, if \( \mathcal{B} \) is an algebra of subsets of a finite sample space and \( \mathcal{D} \) is a partition such that \( \sigma(\mathcal{D}) = \mathcal{B} \), then

\[ \mathbb{E}(\xi \mid \mathcal{D}) = \mathbb{E}(\xi \mid \mathcal{B}). \]

This is the reason why we do not distinguish between these two notions.

A.3 On Martingales

We start this section with more examples of martingales.

**Example A.3.** Let \( \eta_1, \eta_2, \ldots, \eta_n \) be independent identically distributed random variables with a Bernoulli distribution

\[ P\{\eta_k = -1\} = P\{\eta_k = 1\} = \frac{1}{2}. \]
Denote

\[ S_k = \eta_1 + \cdots + \eta_k, \quad \mathcal{D}_k = \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_k}. \]

The structure of the partitions \( \mathcal{D}_k \) is simple:

\[ \mathcal{D}_1 = \{ D^+, D^- \}, \]

where \( D^+ = \{ \omega : \eta_1 = 1 \} \), \( D^- = \{ \omega : \eta_1 = -1 \} \);

\[ \mathcal{D}_2 = \{ D^{++}, D^{+-}, D^{-+}, D^{--} \}, \]

where

\[ D^{++} = \{ \omega : \eta_1 = 1, \eta_2 = 1 \}, \ldots, D^{--} = \{ \omega : \eta_1 = -1, \eta_2 = -1 \}, \]

and so on. Obviously \( \mathcal{D}_{\eta_1, \eta_2, \ldots, \eta_k} = \mathcal{D}_{S_1, S_2, \ldots, S_k} \) and random variables \( S_k \) are \( \mathcal{D}_k \)-measurable. Now we show that the second assumption of a martingale also holds.

\[
\mathbb{E}(S_{k+1} \mid \mathcal{D}_k) = \mathbb{E}(S_k + \eta_{k+1} \mid \mathcal{D}_k) = \mathbb{E}(S_k \mid \mathcal{D}_k) + \mathbb{E}(\eta_{k+1} \mid \mathcal{D}_k) = S_k + \mathbb{E}\eta_{k+1} = S_k.
\]

**Example A.4.** Let \( \eta_1, \eta_2, \ldots, \eta_n \) be a sequence of independent Bernoulli random variables with the common distribution \( \mathbb{P}\{\eta_k = 1\} = p, \mathbb{P}\{\eta_k = -1\} = q, (k = \ldots) \).
1, \ldots, n). If \( p \neq q \), then the sequence \( \left( \frac{q}{p} \right)^{S_k} \) where \( S_k = \eta_1 + \eta_2 + \cdots + \eta_k \) is a martingale.

We now check martingale assumptions for the sequence \( \left( \frac{q}{p} \right)^{S_k} \).

\[
E \left[ \left( \frac{q}{p} \right)^{S_k} | D_k \right] = E \left[ \left( \frac{q}{p} \right)^{S_k + \eta_{k+1}} | D_k \right] = E \left[ \left( \frac{q}{p} \right)^{S_k} \cdot \left( \frac{q}{p} \right)^{\eta_{k+1}} | D_k \right].
\]

Since the random variable \( \left( \frac{q}{p} \right)^{S_k} \) is measurable by the partition \( D_k \), by Property 8) of the conditional mathematical expectation we have:

\[
E \left[ \left( \frac{q}{p} \right)^{S_k} | D_k \right] = \left( \frac{q}{p} \right)^{S_k} \cdot E \left[ \left( \frac{q}{p} \right)^{\eta_{k+1}} | D_k \right] = \left( \frac{q}{p} \right)^{S_k} \cdot \left( \frac{q}{p} \right)^{\eta_{k+1}} = \left( \frac{q}{p} \right)^{S_k} \cdot \left( \frac{q}{p} \right)^{\eta_{k+1}} = \left( \frac{q}{p} \right)^{S_k}.
\]

**Example A.5.** Let \( \eta \) be a random variable, \( \{D_1 \preceq D_2 \preceq \ldots \preceq D_n\} \) be a sequence of partitions. Set

\[
\xi_k = E(\eta | D_k).
\]

\( \xi_k \) is a martingale.

We claim that the sequence \( (\xi_k, D_k)_{k=1}^n \) is a martingale.

In fact, the \( D_k \)-measurability of \( \xi_k \) is obvious. Also, it is simple to see that

\[
E(\xi_{k+1} | D_k) = E(E(\eta | D_{k+1}) | D_k) = E(\eta | D_k) = \xi_k.
\]
Note that if \((\xi_k, D_k)_{k=1}^n\) is an arbitrary martingale, then by the properties of the conditional mathematical expectation it is simple to derive that

\[
\xi_k = E(\xi_{k+1} \mid D_k) = E(E(\xi_{k+2} \mid D_{k+1}) \mid D_k)
\]

\[
= E(\xi_{k+2} \mid D_k) = \ldots = E(\xi_n \mid D_k).
\]

Therefore the set of all finite martingales is exhausted by a martingale of type (\(\ast\)). Note that in the case of infinite sequences this statement is generally speaking not true.

**Example A.6. (Reverse martingale).** Let \(\eta_1, \eta_2, \ldots, \eta_n\) be a sequence of independent identically distributed random variables and

\[
S_k = \eta_1 + \eta_2 + \ldots + \eta_k, \quad D_1 = D_{S_n}, D_2 = D_{S_n, S_{n-1}}, \ldots, D_n = D_{S_n, \ldots, S_1}.
\]

We now show that the sequence \((\xi_k, D_k)_{k=1}^n\), where

\[
\xi_1 = \frac{S_n}{n}, \xi_2 = \frac{S_{n-1}}{n-1}, \ldots, \xi_k = \frac{S_{n-k+1}}{n-k+1}, \ldots, \xi_n = S_1,
\]

is a martingale.

Observe that, \(D_k \preceq D_{k+1}\), and that \(\xi_k - D_k\)-measurable. Next, because for all \(j \leq n - k + 1\)

\[
E(\eta_j \mid D_k) = E(\eta_1 \mid D_k),
\]
we have

\[(n - k + 1) \cdot E(\eta_1 \mid \mathcal{D}_k) = \sum_{j=1}^{n-k+1} E(\eta_j \mid \mathcal{D}_k) = E(S_{n-k+1} \mid \mathcal{D}_k) = S_{n-k+1}.\]

Therefore

\[\xi_k = \frac{S_{n-k+1}}{n - k + 1} = E(\eta_1 \mid \mathcal{D}_k),\]

and the martingale assumptions follow from Example 2.3.

Now we consider some applications of martingales to the theory of random walks.

**Theorem A.1** (Ballot Theorem). Let \(\eta_1, \eta_2, \ldots, \eta_n\) be a sequence of independent identically distributed random variables taking integer nonnegative values, and let

\[S_k = \eta_1 + \eta_2 + \cdots + \eta_k, \quad 1 \leq k \leq n.\]

Then

\[P\{S_k < k, \ 1 \leq k \leq n \mid S_n\} = \left(1 - \frac{S_n}{n}\right)^+,\]

where \(a^+ = \max(0, a)\).

**Proof.** Note that on the set \(\{\omega : S_n(\omega) \geq n\}\) the formula above is correct because the probability we are interested in is zero. Now we prove the formula for elementary outcomes \(\omega\) such that \(S_n(\omega) < n\).
Consider the reverse martingale \( \xi = (\xi_k, D_k)_{k=1}^n \), where

\[
\xi_k = \frac{S_{n-k+1}}{n-k+1}, \quad D_k = D_{S_{n-k+1}, \ldots, S_n}.
\]

Denote

\[
\tau = \min\{1 \leq k \leq n : \xi_k \geq 1\}.
\]

If \( \xi_k < 1 \) for all \( 1 \leq k \leq n \), or, equivalently, \( \{\max_{1 \leq l \leq n} \frac{S_l}{l} < 1\} \), then set \( \tau = n \).

On the set \( \{\max_{1 \leq l \leq n} \frac{S_l}{l} < 1\} \) we have \( \xi_\tau = \xi_n = S_1 = 0 \). Hence

\[
\left\{ \omega : \max_{1 \leq l \leq n} \frac{S_l(\omega)}{l} < 1 \right\} \subseteq \{\xi_\tau = 0\}.
\]

Consider now all elementary outcomes for which

\[
\left\{ \max_{1 \leq l \leq n} \frac{S_l}{l} \geq 1, \ S_n < n \right\}
\]

Denote \( \sigma = n - \tau + 1 \). It is obvious that

\[
\sigma = \max\{1 \leq k \leq n : S_k \geq k\}.
\]

Hence \( \sigma < n, S_\sigma \geq \sigma, S_{\sigma+1} < \sigma + 1 \). Therefore \( \eta_{\sigma+1} = S_{\sigma+1} - S_\sigma < (\sigma + 1) - \sigma = 1 \), that is, \( \eta_{\sigma+1} = 0 \).

Hence

\[
\xi_\tau = \frac{S_{n-\tau+1}}{n-\tau+1} = \frac{S_\sigma}{\sigma} = 1.
\]
Therefore
\[
\left\{ \max_{1 \leq l \leq n} \frac{S_l}{l} \geq 1, \; S_n < n \right\} \subseteq \{ \xi_\tau = 1 \}.
\]

Hence on the set \( S_n < n \) we have
\[
P\left\{ \max_{1 \leq l \leq n} \frac{S_l}{l} \geq 1 \mid S_n \right\} = P\{\xi_\tau = 1 \mid S_n\} = E(\xi_\tau \mid S_n),
\]
since \( \xi_\tau \) takes only two values 0 and 1. Note that
\[
E(\xi_\tau \mid S_n) = E(\xi_\tau \mid D_1).
\]

Then by Theorem 2.1 we have:
\[
E(\xi_\tau \mid D_1) = \xi_1 = \frac{S_n}{n}.
\]

Hence
\[
P\{S_k < k, \; 1 \leq k \leq n \mid S_n\} = 1 - \frac{S_n}{n}.
\]

Why is this theorem called the Ballot theorem?

Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent identically distributed random variables with Bernoulli distribution:
\[
P\{\xi_i = 1\} = P\{\xi_i = -1\} = 1/2.
\]

Let \( S_k = \xi_1 + \xi_2 + \cdots + \xi_k \), \( a \) and \( b \) be non negative integers: \( a - b > 0 \), \( a + b = n \).
We show that in this case

\[ P\{S_1 > 0, S_2 > 0, \ldots, S_n > 0 \mid S_n = a - b\} = \frac{a - b}{a + b}. \]

Really, by symmetry

\[ P\{S_1 > 0, S_2 > 0, \ldots, S_n > 0 \mid S_n = a - b\} = P\{S_1 < 0, S_2 < 0, \ldots, S_n < 0 \mid S_n = -(a - b)\} \]

\[ = P\{S_1 + 1 < 1, S_2 + 2 < 2, \ldots, S_n + n < n \mid S_n + n = n - (a - b)\} \]

\[ = P\left\{ \eta_1 < 1, \eta_1 + \eta_2 < 2, \ldots, \sum_{k=1}^{n} \eta_k < n \mid \sum_{k=1}^{n} \eta_k = n - (a - b) \right\} \]

\[ = \left[ 1 - \frac{n - (a - b)}{n} \right]^+ = \frac{a - b}{n} = \frac{a - b}{a + b}, \]

where \( \eta_k = \xi_k + 1 \). We may interpret the event \( \{\xi_i = +1\} \) as a ballot in favour submitted for the first candidate at an elections and \( \{\xi_i = -1\} \) submitted for the second candidate. Then the random variable \( S_k \) describes the change in difference of ballots for two candidates when \( k \) ballots are submitted and \( P\{S_1 > 0, S_2 > 0, \ldots, S_n > 0 \mid S_n = a - b\} \) is the probability that the first candidate was winning all the time under the condition that the first candidate has won \( a \) ballots in total and the second got \( b \) ballots, where \( a - b > 0, a + b = n \). By the formula obtained this probability is equal to \( (a - b)/n \).
As we already mentioned in Chapter 2, from the definition of a martingale it follows immediately that the mathematical expectation $E\xi_k$ is a constant for all $k$:

$$
E\xi_k = E\xi_1.
$$

In Theorem A.2 below we show that this property remains true if instead of fixed time $k$ we take a random time, but first we need to give the definition of stopping times.

**Definition A.1.** A random variable $\tau = \tau(\omega)$ taking values $1, 2, \ldots, n$ is called a stopping time (by a sequence of partitions $D_1 \preceq D_2 \preceq \ldots \preceq D_n$), if for each $k = 1, 2, \ldots, n$ the random variables $I_{\{\tau = k\}}(\omega)$ are $D_k$-measurable.

If we interpret the partition $D_k$ as a partition generated by observing during first $k$ steps, then $D_k$-measurability of the random variables $I_{\{\tau = k\}}(\omega)$ means that the realization of the event $\{\tau = k\}$ is defined only by observations during first $k$ steps, and does not depend on future.

As an example of stopping time we can consider the moment of first hitting of a set $A$ by a random sequence $\xi_0, \xi_1, \xi_2, \ldots, \xi_n$.

Note for the readers which are familiar with measure theory: If $B_k = \sigma(D_k)$ is the $\sigma$-algebra generated by the partition $D_k$, then $D_k$-measurability of random variables $I_{\{\tau = k\}}(\omega)$ is equivalent to the statement

$$
\{\tau = k\} \in B_k.
$$
**Theorem A.2.** Let \( \xi = (\xi_k, D_k)_{k=1}^n \) be a martingale and \( \tau = \tau(\omega) \) be a stopping time by the partition \((D_k)_{k=1}^n\). Then

\[
E(\xi_\tau | D_1) = \xi_1,
\]

where \( \xi_\tau(\omega) = \sum_{k=1}^n \xi_k I_{\{\tau=k\}}(\omega) \) and

\[
E(\xi_\tau) = E(\xi_1).
\]

**Proof.** Let \( D \in D_1 \). From the second way of defining the conditional mathematical expectation, that is, by formulae (4) and (5), we have

\[
E(\xi_\tau | D) = \frac{E(\xi_\tau \cdot I_D)}{P(D)} = \frac{1}{P(D)} \cdot \sum_{k=1}^n E(\xi_k \cdot I_{\{\tau=k\}} \cdot I_D)
\]

\[
= \frac{1}{P(D)} \cdot \sum_{k=1}^n E[E(\xi_n | D_k) \cdot I_{\{\tau=k\}} \cdot I_D]
\]

\[
= \frac{1}{P(D)} \cdot \sum_{k=1}^n E[E(\xi_n \cdot I_{\{\tau=k\}} \cdot I_D | D_k)]
\]

\[
= \frac{1}{P(D)} \cdot \sum_{k=1}^n E(\xi_n \cdot I_{\{\tau=k\}} \cdot I_D)
\]

\[
= \frac{1}{P(D)} \cdot E(\xi_n \cdot I_D) = E(\xi_n | D).
\]

Hence

\[
E(\xi_\tau | D_1) = E(\xi_n | D_1) = \xi_1.
\]
Thus

\[ E\xi_\tau = E(E(\xi_\tau | D_1)) = E\xi_1. \]

### A.4 Supermartingale and Doob’s Theorem

**Definition A.2.** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be a sequence of random variables, \( D_0 \preceq D_1 \preceq \cdots \preceq D_n \) a nondecreasing sequence of partitions. The sequence \((\xi_k, D_k)\) is called *supermartingale* if:

1) \( \xi_k \) is \( D_k \)-measurable;

2) \( E|\xi_k| < \infty \) and

\[ E(\xi_{k+1}|D_k) \leq \xi_k \) (\( E(\xi_{k+1}|D_k) \geq \xi_k \)).

**Theorem A.2.** (Doob’s Theorem ??). Let \((\xi_k, D_k, 1 \leq k \leq n)\) be a supermartingale. Then there exists a martingale \((\eta_k, D_k, 1 \leq k \leq n)\) and predictable sequence \((A_k, D_{k-1})\) such that for \( 1 \leq k \leq n \) the following representation holds:

\[ \xi_k = \eta_k - A_k, \text{ (or } A_k) \]

and this representation is unique.

**Proof.** Let \( \eta_0 = \xi_0, A_0 = 0 \) and

\[ \eta_k = \eta_0 + \sum_{j=0}^{k-1} (\xi_{j+1} - E(\xi_{j+1}|D_j)), \]

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\[ -A_k = \sum_{j=0}^{k-1} (E(\xi_{j+1}|D_j) - \xi_j). \]

The \( D_{k-1} \)-measurability of the sequence \((A_k)\) is obvious by the representation above.

Now we prove that the sequence \( \eta_k \) is a martingale. For \( 1 \leq k \leq n \):

\[
E(\xi_{k+1}|D_k) = E\left[\left(\sum_{j=0}^{k} (\xi_{j+1} - E(\xi_{j+1}|D_j))\right)|D_k\right] \\
= E\left[\left(\sum_{j=0}^{k-1} (\xi_{j+1} - E(\xi_{j+1}|D_j))\right)|D_k\right] + E\left((\xi_{k+1} - E(\xi_{k+1}|D_k))|D_k\right) \\
= \sum_{j=0}^{k-1} (\xi_{j+1} - E(\xi_{j+1}|D_j)) = \eta_k. 
\]

To verify the uniqueness of the representation consider the following argument. Let \( \xi'_k = \eta'_k - A'_k \) be another representation of the sequence \((\xi_k)\). Consider \( A'_{k+1} - A'_k = (A_{k+1} - A_k) + (\eta_{k+1} - \eta_k) - (\eta'_k - \eta'_k) \). Then
\[
E((A'_{k+1} - A'_k)|D_k) = E((A_{k+1} - A_k)|D_k) + E((A'_{k+1} - A'_k)|D_k) - E((\eta'_k - \eta'_k)|D_k). \]

Hence we obtain that \( A'_{k+1} - A'_k = A_{k+1} - A_k \) as \( D_n \)-measurable random variables. But \( A_0 = A'_0 = 0 \). Thus \( A_k = A'_k, \; \eta_k = \eta'_k \) as \( D_n \)-measurable random variables.

### A.5 American Option

There are many types of options, not only European. One of the most common are options of American type, which are characterized in that they may be exercised at an arbitrary time from a decided time in advance governed by the set \( \{0, 1, 2, \ldots, N\} \).
In contrast, for the European option the exercise time is fixed at $N$.

Let $n = 0, 1, 2, \ldots, N$ be the times when the American option may be exercised. Assume that the sequence of payoff functions at time $n$ is $f = (f_n)_{n=0}^N$, where $f_n = f_n(S_0, S_1, \ldots, S_n)$ is given and fixed in the contract. The exercise time of the option $\tau = \tau(\omega)$ is chosen arbitrary in accordance with the “history” of the $(B, S)$-market. Since the decision to exercise the option at time $n$, that is to choose $\tau = n$, or continue its activity (that is, $\tau > n$) is defined based on the information that we have by time $n$ inclusive, then it is natural to assume that $\tau$ is a stopping time (see Definition 2.2). In other words, $\tau = \tau(\omega)$ is a random variable taking values in the set $\{0, 1, 2, \ldots, N\}$. Moreover it does not depend on future, that is

$$\{\tau = n\} \in \mathcal{D}_n,$$

where $\mathcal{D}_n = \mathcal{D}_{S_0, S_1, \ldots, S_n}, 0 \leq n \leq N$.

Let $\tau = \tau(\omega)$ be the exercise time for this option. According to the contract, the writer (the person who sells the option) should be ready to pay out $f_\tau = f_\tau(S_0, S_1, \ldots, S_\tau)$ and hence the writer should choose a strategy $\pi = (\pi_n)$ such that for any time $\tau = \tau(\omega)$ the corresponding capital satisfies $X_\tau^\pi \geq f_\tau$.

**Definition 3.3.** For the given $x > 0$ and given sequence of nonnegative payoff functions $f = (f_n)_{n=0}^N, f_n = f_n(S_0, S_1, \ldots, S_n)$, a strategy $\pi = (\pi_n)_{n=0}^N$ is called $(x, f, N)$-hedge of American type, if for all $\omega \in \Omega$:

$$X_0^\pi(\omega) = x,$$
and for all $0 \leq n \leq N$

$$X_n^{\pi}(\omega) \geq f_n(S_0, S_1(\omega), \ldots, S_n(\omega)).$$

If for some random time $\tau = \tau(\omega)$ we have equality in the inequality above, that is,

$$X_\tau^{\pi}(\omega) = f_\tau(\omega)(S_0, S_1(\omega), S_\tau(\omega)), \ \omega \in \Omega,$$

then the hedge is called *minimal*.

Denote by $\Pi(x, f, N)$ the family of all $(x, f, N)$-hedges.

**Definition 3.4.** The value $C^*_N = C^*_N(f, N)$ defined by formula

$$C^*_N = \inf\{x > 0 : \Pi(x, f, N) \neq \emptyset\}$$

is called the *fair* (or rational) *price of the option of American type*.

What are the good times $\tau = \tau(\omega)$ when it is reasonable to exercise the option?

**Definition A.3.** The stopping time $\tau^* = \tau^*(\omega)$ is called a *rational exercise time* for the option of American type, if with the initial capital $C^*_N$ for any strategy $\pi \in SF$ such that

$$X^{\pi}_\tau(\omega) \geq f_\tau(\omega)(S_0, S_1(\omega), \ldots, S_\tau(\omega)), \ \omega \in \Omega,$$

equality actually holds, that is

$$X^{\pi}_\tau(\omega) = f_\tau(\omega)(S_0, S_1(\omega), \ldots, S_\tau(\omega)), \ \omega \in \Omega.$$
Bibliography


