SOME PROBABILITY PROPERTIES OF THE CRACK DISTRIBUTION

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Tannen Darnell Acoose, candidate for the degree of Master of Science in Mathematics, has presented a thesis titled, *Some Probability Properties of the Crack Distribution*, in an oral examination held on August 24, 2017. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

The three parameter Crack distribution is useful for statistical analysis in prospective studies connected with the engineering problem of fatigue crack development in a metallic plate under some kind of pressure loading. As special cases it contains such well known distributions as the Birnbaum-Saunders distribution, the Inverse Gaussian distribution, and the Length Biased Inverse Gaussian distribution. These distributions are all related and have their own characteristics.

In this thesis, in Chapter 2, I derive new, more direct and more mathematically appealing methods for obtaining the distribution function and the moment generating function for the Crack distribution. In Chapter 3 I provide new probability properties of the Crack distribution. Finally, in Chapter 4 I provide a rigorous general probabilistic model of the growth of two-sided cracks.
Acknowledgements

I would like to extend my utmost gratitude and appreciation to Dr. Edward Doolittle and Dr. Andrei Volodin for their guidance, generous help, expert advice, encouragement and endless patience in making this thesis a success.
Dedication

I would like to dedicate this thesis to my better half, Robyn Pitawanakwat.
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Notation

The following notation will be used throughout the thesis:

- BS ($\lambda, \theta$) - Birnbaum-Saunders distribution with parameters $\lambda$ and $\theta$.
- IG ($\lambda, \theta$) - Inverse Gaussian distribution with parameters $\lambda$ and $\theta$.
- LB ($\lambda, \theta$) - Length Biased Inverse Gaussian distribution with parameters $\lambda$ and $\theta$.
- CR ($\lambda, \theta, p$) - Crack distribution with parameters $\lambda$, $\theta$ and $p$.

- $F$ - cumulative distribution function.
- $f$ - probability density function.
- $\phi_X(t)$ - moment generating function of random variable $X$.

- $X$ - a random variable.
Chapter 1

Introduction

The three parameter Crack distribution is useful for statistical analysis in prospective studies connected with the engineering problem of fatigue crack development in a metallic plate under some kind of pressure loading. As special cases it contains such well known distributions as the Birnbaum-Saunders distribution, the Inverse Gaussian distribution, and the Length Biased Inverse Gaussian distribution. These distributions are all related and have their own characteristics. At the beginning of this chapter I will give a historical review of these distributions, including who invented them, who studied them first, why they are important, and their applications. I will also compare the distributions and discover their relationship.

1.1 Historical review

Survival analysis is a branch of statistics for handling the analysis of time duration until one or more events, e.g., the death of a biological organism, or a failure in a mechanical system. In the most general sense, it consists of techniques for positive
valued random variables such as time to death, time to onset (or relapse) of a disease, length of a stay in a hospital, duration of a strike, money paid by health insurance, viral load measurements, time of crack development in plastic concrete, fatigue life of aluminum, fatigue life of a spring, and fatigue limit load. Moreover, this topic is called reliability theory or reliability analysis in engineering, event history analysis in sociology, and duration analysis or duration modeling in economics. Statistical models fit for any of these topics are generically called “time-to-event” models. In reliability theory, failure is called an “event”, and the goal is to project or forecast the rate of events for a given population or the probability of an event or the frequency of an event for an individual.

In order to achieve those objectives, it is necessary to define “lifetime”. In the case of biological survival, death is unambiguous, but for mechanical reliability, failure may not be well-defined, for there may well be mechanical systems in which failure is partial, a matter of degree, or not otherwise localized in time. Even in biological problems, some events (for example, a heart attack or other organ failure) may have the same ambiguity. The theory outlined below assumes well-defined events at specific times; other cases may be better treated by models which explicitly account for ambiguous events.

There are numerous families of distributions commonly used to describe a lifetime in the field, for instance the exponential distribution, gamma distribution, normal distribution, lognormal distribution, Birnbaum-Saunders distribution, inverse Gaussian distribution, length biased inverse Gaussian distribution, etc. One of the interesting aspects of lifetime distributions in models of reliability analysis is the case when a
failure of the object under consideration appears to be due to fatigue crack develop-
ment. The common distributions used in practical applications of reliability theory
for modeling the lifetime of products with failure due to development of fatigue cracks
are Birnbaum-Saunders, Inverse Gaussian, and the Length Biased version of the In-
verse Gaussian distribution. In this section, I will give a historical review of these
distributions, including who invented them, who studied them first, why they are
important, and their applications. I will also compare these distributions and find
their similarities and differences to each other.

First, I discuss the Inverse Gaussian distribution, denoted by IG. It has come to
the attention of researchers with its usefulness in reliability theory for more than 20
years. The IG distribution is a right skewed distribution also known as the first pas-
sage time distribution of Brownian motion with positive drift, which was developed by
Schrödinger (1915). Later, Tweedie (1957) proposed the name inverse Gaussian for
this distribution since its cumulant generating function is the inverse of the cumulant
generating function of a normal (Gaussian) random variable. It has many interesting
statistical and probabilistic properties like a normal distribution. Chhikara and Folks
(1989) reviewed some statistical properties of the IG distribution which is similar to
a normal distribution. For example, they all appear in Brownian motion. The differ-
ence between the Gaussian distribution and the IG distribution is that the Gaussian
distribution describes the distance traveled by a particle in fixed time in Brownian
motion, while the IG distribution describes the distribution of the time a Brownian
motion with positive drift takes to reach a fixed positive level. Moreover, they possess
intriguing properties such as the occurrence of $\chi^2$ and $F$ distributions in sampling
theory. Mudholkar and Rajeshwari (2002) proposed that the IG distribution belongs to the exponential family, as it has the reproductive property and it possesses inferential properties similar to the normal model. Accordingly, the IG distribution is an interesting alternative to the normal distribution for modeling non-negative data with positive skewness. For additional details about the IG distribution refer to Wald (1947), Johnson et al. (1995), and Seshadri (1993, 1999).

The Inverse Gaussian distribution has many applications for instance in biology, chemical engineering, finance, and actuarial science. Besides the study of reliability, this distribution has been used in a wide range of applications, most of which are based on the idea of the first passage of time of Brownian motion for an underlying process. The reason is that the concept of Brownian motion is useful for describing the process of many phenomena in natural and physical science, and it is distributed as an Inverse Gaussian. According to Lee (1992), it is logical to use it as a lifetime model and for life testing. Additionally this distribution has many applications in the field of reliability. Examples can be found in tracer dynamics, emptiness of a dam, a purchase incidence model, the distribution of strike duration and others.

One of many interesting properties of the Inverse Gaussian distribution is that it represents a family of highly skewed distributions and is useful for statistical modelling of skewed data (Chhikara and Folk, 1989). This is so because with a skewed distribution, the researcher commonly resorts to a transformation in order to normalize the data. Instead, it is desirable to analyze the data as observed using statistical methods based on a skewed distribution. Chhikara and Folk (1989) explain that the
application of the Inverse Gaussian distribution can be useful and meet the transfor-
mational needs of skewed data analysis.

The reciprocal of the IG distribution is the so-called Length Biased Inverse Gaus-
sian distribution denoted by LB. This is the length biased version of the Inverse
Gaussian distribution, which was studied by Ahsanullah and Kirmani (1984), and
Khattree (1989). It is in fact a special weighted distribution, which was proposed by

The Length Biased Inverse Gaussian distribution has received considerable atten-
tion due to its various applications. It is appropriate for certain natural sampling
plans in reliability, biometry, and survival analysis, which were studied by Zelen and
Feinleib (1969) and Blumenthal (1967). Cnaan (1985) described an application of
length biased sampling in a cardiology study involving two phases. Gupta and Tri-
patni (1987) compared the ordinary Inverse Gaussian and its length biased version
in a unified manner. In particular, they expressed the moments of the length biased
distribution in terms of the moments of the original class, and compared the informa-
tion contained in random samples from the two classes of distributions. Sen (1987)
studied the properties of the arithmetic, geometric, and harmonic means for length
biased distributions in a nonparametric fashion. He also presented a characterization
of length biased distributions by their coefficient of variation. Gupta and Kirmani
(1987) examined the relationship between the length biased and the original random
variable in the context of reliability and life testing. Gupta and Akman (1995) apply
some results from Sen (1987) in order to develop confidence intervals and tests re-
garding the mean and the coefficient of variation of the Inverse Gaussian distribution.
based on the length biased data. For other research on the Length Biased Inverse Gaussian distribution see Ahmed and Abouammoh (1993), Vardi (1982), and Cox (1969).

Birnbaum and Saunders (1969a) proposed a failure time distribution for fatigue failure under cyclic loading. The model was established under the assumption that failure is due to the development and growth of a dominant crack. They also considered some closure properties of their distribution and compared with other families such as the lognormal distribution. This distribution is the so-called two-parameter Birnbaum-Saunders distribution (hereafter called BS). Birnbaum and Saunders (1969b) presented a theoretical and practical review of fitting this distribution to several extensive sets of fatigue data.

I start with a brief literature survey on the Birnbaum-Saunders distribution, explaining what has been done in the area. As I already mentioned above, the two-parameter Birnbaum-Saunders distribution (in Chapter 4 I call it the one-sided Birnbaum-Saunders distribution) was introduced by Birnbaum and Saunders (1969a) as a failure-time distribution for fatigue failure under cyclic loading. This distribution is widely used as a lifetime distribution in the various models of reliability theory in the case when the failure of an object under consideration appears to be due to the development of fatigue cracks. Birnbaum and Saunders (1969b) presented a comprehensive review, both theoretical and practical, of fitting this family of distributions to the solution of the problem of crack development.

Desmond (1985, 1986) proposed a more general derivation based on a biological model and strengthened the physical justification for the use of the distribution. The
derivation follows from considerations of renewal theory for the number of cycles needed to force a fatigue crack extension to exceed a critical value. Also Desmond (1986) considered estimation of the parameters for censored data. Ahmad (1988) proposed estimation of the scale parameter by the jackknife method to eliminate first-order bias. This estimate has the same limiting behavior as that of Birnbaum and Saunders (1969b).

Maximum likelihood estimators were first discussed by Birnbaum and Saunders (1969b), who suggested some iterative schemes to solve the resulting non-linear equation. Englehardt et al. (1981) established the asymptotic distribution of the maximum likelihood estimators. Conventional moment estimators have a drawback in that they may not always exist and, if they do exist, they may not be unique. Ng et al. (2003) considered modified moment estimators for the parameters to overcome this problem. However, Wu and Wong (2004) reported that the resulting expressions for the intervals of the estimators for $\beta$ suggested by Ng et al. (2003) are presented incorrectly. Furthermore, there is no guarantee that the upper bounds of those intervals are always positive.

Ahmad (1988) considered the estimation of the scale parameter by the jackknife method. This eliminates the first-order bias. The estimate has the same limiting behavior as that of Birnbaum and Saunders (1969b). Rieck (1995) derived an asymptotically optimal linear estimator for symmetrically type II censored samples. I refer to the monograph by Bogdanoff and Kozin (1985) for motivating examples of probabilistic models of cumulative damage. A more recent view on the problem of fatigue crack damage based on stochastic differential equations is suggested by Singpurwala
A review of these developments can be found in Johnson et al. (1995).

Lemonte et al. (2007) developed nearly unbiased estimators for the Birnbaum-Saunders distribution. They derived modified maximum likelihood estimators that are bias-free to second order. They also considered bootstrap-based bias correction. Additionally, they derived a Bartlett correction that improves the finite-sample performance of the likelihood ratio test in finite samples.


Ahmed et al. (2008) proposed a new parametrization of the Birnbaum-Saunders distribution based on recurrence relations in Birnbaum and Saunders (1969a), which present a general probability model of the growth of a one-sided crack (see the discussion in Chapter 4 of this thesis). Essentially, this re-parametrization fits the physics of the phenomena under consideration since the proposed parameters characterize the thickness of the sample and the nominal treatment loading on the sample respectively. The usual shape and scale parameters of the distribution do not allow this physical interpretation. They also presented a relationship between the usual parameters and the proposed parameters.

Balakrishnan et al. (2009) introduced a type of Birnbaum-Saunders distribution based on scale mixtures of normal models, produced a lifetime analysis, developed the EM-algorithm for maximum likelihood estimation of parameters, and illustrated the results with real data showing the robustness of the estimation procedure. Kundu
et al. (2017) presented a bivariate Birnbaum-Saunders distribution which is an absolutely continuous distribution, that is, it has a density function. They also discussed some different properties and parameter estimation of this bivariate Birnbaum-Saunders distribution. Ferreira et al. (2012) discuss the tail behavior of this generalized Birnbaum-Saunders distribution in the context of extreme value theory. The authors show that the tail properties of the generalized BS distribution are essentially governed by that of its auxiliary distribution (i.e., the standard elliptically symmetric distribution).

The Birnbaum-Saunders distribution (Birnbaum and Saunders (1969a)) is a two-parameter life time distribution originating in modeling material fatigue data. Due to its mathematical tractability and ability to fit right skewed data, the Birnbaum-Saunders model is also used for many other applications. Recently, the BS distribution has been extended to various classes of distributions. The Birnbaum-Saunders distribution has many applications in a wide variety of contexts; see, for example, Johnson et al. (1995, pp. 651–663). Specifically, some applications of the Birnbaum-Saunders distribution are in failure models in random environments described by stationary Gaussian processes. As mentioned, these models include failures due to the response process being above a pre-established level during a long period of time. Also, phenomena to be described by wear-out and cumulative damage processes can be efficiently modeled by the Birnbaum-Saunders distribution. Obviously, based on its genesis, fatigue phenomena, and more extensively lifetimes from wear, abrasion, galling, or wilting, among others, can be modeled ideally by this distribution. Some unpublished applications of the Birnbaum-Saunders distribution are the following:
1. Migration of metallic flaws in nano-circuits due to heat in a computer chip;

2. Accumulation of deleterious substances in the lungs from air pollution;

3. Ingestion by humans of toxic chemicals from industrial waste;

4. Diminution of biomass in fishing through time in a certain zone;

5. Occurrence of natural disasters such as earthquakes and tsunamis.

As another extension of the Birnbaum-Saunders distribution, the three parameter Crack distribution (Volodin and Dzhungurova, 2000) is studied as a mixture of the Inverse Gaussian distribution and the Length Biased Inverse Gaussian distribution with a weight parameter $p$. In particular, the Crack distribution with $p = 1/2$ turns into the classical two-parameter Birnbaum-Saunders distribution. By adding the mixture weight parameter $p$, the Crack distribution obtains a greater flexibility in fitting various asymmetric data sets than the Birnbaum-Saunders distribution does. However, the thin-tailedness of the standard normal distribution (i.e., the auxiliary distribution of the three-parameter Crack distribution) passes on to the Crack distribution, which imposes restrictions on the Crack distribution for modeling heavy-tailed data. The generalized Crack (GCR) distribution (Leiva et al., 2010) is constructed by replacing the auxiliary standard normal density with any symmetric density to incorporate a further flexibility in modeling data sets with various distributional shapes.

The Crack distribution is a positively skewed model, which is widely applicable to model failure times of fatiguing materials. It is also known as the Inverse Gaussian
mixture distribution, which was studied by Jorgensen et al. (1991), and Bowonrat-...

Bowonrattanaset et al. (2011) introduced the Inverse Gaussian mixture distribution based on the re-parametrization model presented in Ahmed et al. (2008), and proposed the name Crack for this distribution. It will be denoted by CR(\lambda, \theta, p). They also established some deeper results with rigorous proofs. Gupta and Kundu (2011) propose using the EM algorithm to estimate the unknown parameters of the Crack distribution for complete and censored samples. Duangsaphon (2014) studied the Crack distribution in the study of regression-quantile estimation, Bayesian estimation and confidence interval estimation. Additionally, Saengthong and Bodhisuwan (2014) proposed a two-parameter one-sided Crack distribution which is obtained by adding a new weight parameter p to the inverse Gaussian mixture distribution.

Generally speaking, there has been very little research done on the properties of the Crack distribution and the estimation of the parameters of the Crack distribution still have difficulties. Kumnadee, Volodin, and Lisawadi (2010) derived the
first three moments of the Crack distribution based on the characteristic function of the Crack distribution. Panta (2010) applied the acceptance-rejection method for Crack random number generation. Bowonrattanaseth and Budsaba (2011) compared the acceptance-rejection method and the built-in command in Wolfram Mathematica 8 for generating random numbers from the Crack distribution. They considered the method of moments and maximum likelihood methods to estimate the parameters of the Crack distribution but the estimation was unsatisfactory. The weight parameter $p$ for the maximum likelihood estimates are out of a closed interval or are far from the true value of the parameter and the moment estimates cannot be estimated or have more than one value.

1.2 Theoretical Background

1.2.1 The Inverse Gaussian Distribution

According to Chikara and Folks (1989), the density function of the Inverse Gaussian distribution with classical parametrization has the following density function:

$$f(x, \alpha, \beta) = \sqrt{\frac{\beta}{2\pi x^{\frac{3}{2}}}} \exp\left\{ -\frac{\beta(x - \alpha)^2}{2\alpha^2x} \right\}, x > 0.$$  

The parameters $\alpha$ and $\beta$ are the shape and scale parameters, respectively.

With the new parametrization introduced in Ahmed et. al. (2008), a random variable $X$ has the Inverse Gaussian distribution, denoted by $\text{IG}(\lambda, \theta)$, if its probability density function is
\[ f_{IG}(x, \lambda, \theta) = \begin{cases} \frac{\lambda}{\theta \sqrt{2\pi}} \left( \frac{\theta}{x} \right)^{\frac{3}{2}} \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{\theta}{x}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right]; x > 0 \\ 0; \text{ otherwise} \end{cases} \] 

(1.2.1)

where \( \lambda > 0, \theta > 0 \).

The corresponding distribution function is:

\[ F_{IG}(x, \theta, \lambda) = \begin{cases} \Phi \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{x}{\theta}} \right) + e^{2\lambda \Phi} \left( -\sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{x}{\theta}} \right); x > 0 \\ 0; \text{ otherwise}, \end{cases} \]

where \( \Phi \) is the standard normal distribution function.

Interrelations between the usual parameters \( \alpha, \beta \) and the new parameters \( \lambda \) and \( \theta \) are as follows:

\[ \lambda = \frac{\beta}{\alpha}, \alpha = \lambda \theta \]

\[ \theta = \frac{\alpha^2}{\beta}, \beta = \lambda^2 \theta. \]

1.2.2 The Length Biased Inverse Gaussian Distribution

Any Length Biased probability density function can be defined in terms of its original probability density function in the following way.

Let \( X \) be a non-negative random variable having an absolutely continuous probability density function \( f \). Let \( X \) have a finite first moment \( E[X] \). A non-negative random variable \( Y \) is said to be the Length Biased random variable associated with
If \( Y \) has the probability density function as follows:

\[
h(x) = \frac{xf(x)}{E[X]}, x > 0.
\]

In the thesis, I am interested in the Length Biased Inverse Gaussian distribution. Thus, I will find the density function of the Length Biased Inverse Gaussian distribution with parameters \( \lambda, \theta \).

I know that the first moment of the inverse Gaussian distribution is \( E(X) = \lambda \theta \).

Hence the density function of the Length Biased Inverse Gaussian distribution can be expressed as

\[
f_{LB}(x, \lambda, \theta) = \frac{x}{\theta \sqrt{2\pi}} \left( \frac{\theta}{x} \right)^{\frac{3}{2}} \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right].
\]

Therefore, the density function of the Length Biased Inverse Gaussian distribution is given by the following formula

\[
f_{LB}(x, \lambda, \theta) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left( \frac{\theta}{x} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right]; x > 0 \\ 0; \text{ otherwise.} \end{cases}
\] (1.2.2)

Here, \( \lambda > 0 \) and \( \theta > 0 \) are the same type of parameters as for the Inverse Gaussian distribution \( IG(\lambda, \theta) \). I use the notation \( LB(\lambda, \theta) \) for this distribution.
The Birnbaum-Saunders distribution

First I provide the density function of the Birnbaum-Saunders distribution in the classical parametrization. Let a random variable $X$ have the Birnbaum-Saunders distribution. Its density function can be written as

$$f_{BS}(x, \alpha, \beta) = \begin{cases} \frac{x + \beta}{2\alpha \sqrt{2\pi} \beta x^2} \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{\beta}{x} + \frac{x}{\beta} - 2 \right) \right]; x > 0 \\ 0; \text{ otherwise,} \end{cases}$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the location parameter. The Birnbaum-Saunders distribution is a unimodal distribution with median $\beta$. The mean $\mu$ and variance $\sigma^2$ are as follows:

$$\mu = \beta \left( 1 + \frac{\alpha^2}{2} \right)$$

$$\sigma^2 = (\alpha \beta^2) \left( 1 + \frac{5\alpha^2}{4} \right).$$

The cumulative distribution function of the Birnbaum-Saunders distribution with classical parametrization is given by

$$F_{BS}(x, \alpha, \beta) = \Phi \left[ \frac{1}{\alpha} \left( \sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right) \right].$$

With the new parametrization presented in Budsaba et. al. (2008), a random variable $X$ has the Birnbaum-Saunders distribution, denoted by $BS(\lambda, \theta)$, if its probability
density function is
\[
f_{BS}(x; \lambda, \theta) = \begin{cases} 
\frac{1}{2\theta\sqrt{2\pi}} \left[ \lambda \left( \frac{x}{\theta} \right)^{2} + \left( \frac{x}{\theta} \right)^{2} \right] \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{x}{\theta}} \right)^{2} \right]; & x > 0 \\
0; & \text{otherwise,}
\end{cases}
\]  

where \( \lambda > 0, \theta > 0 \). The relation between classical parameters \( \alpha, \beta \) and new parameters \( \lambda, \theta \) are as follows
\[
\lambda = \frac{1}{\alpha^{2}}, \quad \theta = \alpha^{2}\beta,
\]

\[
\alpha = \frac{1}{\sqrt{\lambda}}, \quad \beta = \lambda\theta.
\]

In Chapter 4 of the thesis, which is devoted to the Physical Probabilistic Model of the Birnbaum-Saunders distribution, I show that the new parametrization of Birnbaum-Saunders distribution has parameters which are meaningful in a practical setting. Importantly, this re-parametrization fits the physics of the phenomena under study since the proposed parameters \( \lambda \) and \( \theta \) correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively. This is the reason why I consider all distributions with this new meaningful parametrization.

### 1.2.3 The Crack lifetime distribution

The Crack lifetime distribution depends on three parameters. It is formed by adding the weight parameter and combining the two parameter Inverse Gaussian distribution and the two parameter Length Biased Inverse Gaussian distributions. This
distribution contains as special cases three known distribution, namely the Birnbaum-Saunders distribution, the Inverse Gaussian distribution, and the Length Biased Inverse Gaussian distribution.

The calculation in terms of the probability density functions of the distributions mentioned above is as follows. Suppose $X_1$ and $X_2$ are two independent random variables such that $X_1 \sim IG(\lambda, \theta)$ and $X_2 \sim LB(\lambda, \theta)$, that is $X_1$ has the Inverse Gaussian distribution with parameters $\lambda > 0, \theta > 0$ and $X_2$ has the Length Biased Inverse Gaussian distribution with parameters $\lambda > 0, \theta > 0$. For the Crack distribution, I consider the new random variable $X$ such that

$$X = \begin{cases} 
X_1 \text{ with probability } p \\
X_2 \text{ with probability } 1 - p,
\end{cases}$$

where $0 \leq p \leq 1$. In this case I say that $X$ is a mixture of $X_1$ and $X_2$ and the density function of $X$ is given by the following formula:

$$f_{CR}(x; \lambda, \theta, p) = pf_{IG}(x, \lambda, \theta) + (1 - p)f_{LB}(x, \lambda, \theta).$$

That is the probability density function of the Crack distribution, and it can be expressed as:

$$f_{CR}(x; \lambda, \theta, p) = \begin{cases} 
\frac{1}{\theta \sqrt{2\pi}} \left[ p\lambda \left( \frac{\theta}{x} \right)^{\frac{3}{2}} + (1 - p) \left( \frac{\theta}{x} \right)^{\frac{1}{2}} \right] \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right]; x > 0 \\
0; \text{ otherwise,}
\end{cases}$$

(1.2.4)
where $\lambda > 0, \theta > 0$ and $0 \leq p \leq 1$. I use the notation $\text{CR}(\lambda, \theta, p)$.

Evidently, the Crack distribution becomes the Inverse Gaussian distribution for $p = 1$, the Length Biased Inverse Gaussian distribution for $p = 0$ and the Birnbaum-Saunders distribution by substituting $p = 0.5$.

I can provide the following engineering interpretation of a Crack distribution. Consider the time from when a crack starts to develop for a machine element until it reaches a critical value. The distribution of such times can be modelled by the Crack random variable.

1.3 The Goal and the Importance of the Research

Therefore, the objective of this thesis is the following:

1.3.1 Research Objectives

Derive new, direct and mathematically appealing methods for obtaining the distribution function and the moment generating function for the Crack distribution (Chapter 2).

Provide new probability properties of Crack distribution (Chapter 3).

Approach a construction of a rigorous general probability model of the growth of two-sided cracks by introducing a new class of distributions, so-called two-sided Birnbaum-Saunders distribution (Chapter 4).

More precisely, in Chapter 3 I derive the following new properties of the Crack
distribution.

1. I show that if $X$ has the Crack distribution, then for any constant $c > 0$, the random variable $cX$ will have a Crack distribution again, and I find expressions for its parameters.

2. Using general fact on a density function of the reciprocal random variable, I prove that if $X$ has the Crack distribution $\text{CR}(\lambda, \theta, p)$, then the reciprocal $1/X$ has also Crack distribution $\text{CR}(\lambda_1, \theta_1, p_1)$ and I find expressions for $\lambda_1, \theta_1, p_1$.

3. I investigate when a sum of two Crack distributed independent random variables has the Crack distribution, too. More rigorously, the problem can be formulated in the following way: Let $X_1$ have the Crack distribution $\text{CR}(\lambda_1, \theta_1, p_1)$ and $X_2$ have the Crack distribution $\text{CR}(\lambda_2, \theta_2, p_2)$ and assume that they are independent. I find special values of all parameters $\lambda_1, \theta_1, p_1$ and $\lambda_2, \theta_2, p_2$ such that the distribution of $X_1 + X_2$ has Crack distribution, too, and I find expressions for the parameters.
Chapter 2

The Crack distribution and its already known properties with new proofs

In this chapter I discuss the Crack distribution and provide some new proofs for some of its already known probability properties. The main result of this section is a new method for derivation of the distribution and moment generating functions for the Crack distribution.

Remind that the density function of the Crack distribution is given by

\[ f_{CR}(x; \lambda, \theta, p) = \frac{1}{\theta \sqrt{2\pi}} \left[ p \lambda \left( \frac{\theta}{x} \right)^{3/2} + (1 - p) \left( \frac{\theta}{x} \right)^{1/2} \right] \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right], \]

where \( x > 0 \).

The form of the cumulative distribution function of the \( CR(\lambda, \theta, p) \) distribution is known already; see for example Das (2015). Note that the proof presented in Das
(2015) is indirect, in the sense that it is shown that

$$\frac{d}{dx} F_{CR}(x; \lambda, \theta, p) = f_{CR}(x; \lambda, \theta, p).$$

This means that I already guessed the form of the distribution and after only confirmed the result by differentiation. It is much more interesting to derive the distribution function as an integral of the density function:

$$F_{CR}(t; \lambda, \theta, p) = \int_{-\infty}^{t} f_{CR}(x; \lambda, \theta, p) \, dx.$$ 

Moreover, it is not shown in Das (2015) that \( \lim_{x \to -\infty} F_{CR}(x; \lambda, \theta, p) = 0 \) and \( \lim_{x \to \infty} F_{CR}(x; \lambda, \theta, p) = 1 \) and it is not mentioned that \( F_{CR}(x; \lambda, \theta, p) \) is a non-decreasing function.

Hence I present the new direct proof of the form of the distribution function by integration techniques which are more natural in this situation.

### 2.1 Distribution function

**Proposition 2.1.** The distribution function of the Crack CR\((\lambda, \theta, p)\) distribution is

$$F_{CR}(x; \lambda, \theta, p) = \Phi \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right) + (2p - 1) e^{2\lambda} \left[ 1 - \Phi \left( \sqrt{\frac{x}{\theta}} + \lambda \sqrt{\frac{\theta}{x}} \right) \right],$$

where \( x > 0, \lambda > 0, \theta > 0, 0 \leq p \leq 1 \) and \( \Phi(x) \) is the standard normal distribution function.
Proof: To simplify our calculations, I let

\[ u = u(x) = \sqrt{\frac{x}{\theta}} + \lambda \sqrt{\frac{\theta}{x}} \quad \text{and} \quad v = v(x) = \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}}. \]

The following facts are simple to derive:

1. For any numbers \( \alpha \) and \( \beta \) I have \( (1 - p)\alpha + p\beta = \frac{1}{2}[(\alpha + \beta) + (1 - 2p)(\alpha - \beta)] \).

   If I take \( \alpha = \sqrt{\frac{p}{\theta}} \) and \( \beta = \lambda \sqrt{\frac{\theta}{x}} \), then the last identity says that

   \[ (1 - p)\sqrt{\frac{x}{\theta}} + p\lambda \sqrt{\frac{\theta}{x}} = \frac{1}{2}[u + (1 - 2p)v]. \]

2. \( v^2 = u^2 - 4\lambda \).

3. \( \frac{du}{dx} = \frac{1}{2x}v \) and \( \frac{dv}{dx} = \frac{1}{2x}u \). For integration purposes it is more convenient to write this as

   \[ du = \frac{1}{2x}v \, dx \quad \text{and} \quad dv = \frac{1}{2x}u \, dx. \]

4. \[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-w^2/2} \, dw = \Phi(s) - \frac{1}{2}, \]

   where \( s > 0 \) and \( \Phi(\cdot) \) is the standard normal distribution function.

5. \( u(0+) = +\infty, v(0+) = -\infty \).

Now I can proceed with the calculations

\[ F_{CR}(t; \lambda, \theta, p) = \int_{-\infty}^{t} f_{CR}(x; \lambda, \theta, p) \, dx \]
\[ \int_0^t \frac{1}{\theta \sqrt{2\pi}} \left[ p\lambda \left( \frac{\theta}{x} \right)^{3/2} + (1 - p) \left( \frac{\theta}{x} \right)^{1/2} \right] \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right] dx \]

Consider separately the first part of the density function, the part that does not contain the exponential function,

\[
\frac{1}{\theta} \left[ p\lambda \left( \frac{\theta}{x} \right)^{3/2} + (1 - p) \left( \frac{\theta}{x} \right)^{1/2} \right]
\]
distribute \(\theta\) and factor out \(x\)

\[ = \frac{1}{x} [(1 - p)\theta^{-1/2}x^{1/2} + p\lambda \theta^{1/2}x^{-1/2}] \]
\[ = \frac{1}{x} \left[ (1 - p)\sqrt{\frac{x}{\theta}} + p\lambda \sqrt{\frac{\theta}{x}} \right] \]
\[ = \frac{1}{2x} [u + (1 - 2p)v] \]
(by Fact 1)

Therefore, I can rewrite our integral in the form

\[ F_{CR}(t; \lambda, \theta, p) = \frac{1}{\sqrt{2\pi}} \int_{x=0}^{x=t} \frac{1}{2x} [u + (1 - 2p)v] \exp \left[ -\frac{u^2}{2} \right] \ dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{x=0}^{x=t} \exp \left[ -\frac{u^2}{2} \right] \frac{u}{2x} \ dx + (1 - 2p) \frac{1}{\sqrt{2\pi}} \int_{x=0}^{x=t} \exp \left[ -\frac{v^2}{2} \right] \frac{v}{2x} \ dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{x=0}^{x=t} \exp \left[ -\frac{u^2}{2} \right] \ exp \left[ -\frac{v^2}{2} \right] \ du + (1 - 2p) \frac{1}{\sqrt{2\pi}} \int_{x=0}^{x=t} \exp \left[ -\frac{u^2}{2} + 2\lambda \right] \ du \]
(by Fact 3)
\[ = \frac{1}{\sqrt{2\pi}} \int_{x=0}^{x=t} \exp \left[ -\frac{u^2}{2} \right] \ exp \left[ -\frac{v^2}{2} + 2\lambda \right] \ du \]
(by Fact 2)
\[\begin{align*}
&= \left( \Phi(v) - \frac{1}{2} \right) \bigg|_{x=0}^{x=t} + (1 - 2p)e^{2\lambda} \left( \Phi(u) - \frac{1}{2} \right) \bigg|_{x=0}^{x=t} \\
&= \Phi(v(t)) - \Phi(-\infty) + (2p - 1)e^{2\lambda}(\Phi(u(t)) - \Phi(+\infty)) \\
&= \Phi\left( \sqrt{\frac{t}{\theta} - \lambda\sqrt{\frac{t}{t}}} \right) + (1 - 2p)e^{2\lambda}\left( \Phi\left( \sqrt{\frac{t}{\theta} + \lambda\sqrt{\frac{t}{t}}} \right) - 1 \right) \\
&= \Phi\left( \sqrt{\frac{t}{\theta} - \lambda\sqrt{\frac{t}{t}}} \right) + (1 - 2p)e^{2\lambda}\left( 1 - \Phi\left( \sqrt{\frac{t}{\theta} + \lambda\sqrt{\frac{t}{t}}} \right) \right),
\end{align*}\]

which is exactly what I need to show.

Now, \( F_{CR}(t; \lambda, \theta, p) \) is a non-decreasing function of \( t \) because it is an integral with variable upper limit of a positive function (any density function is non-negative). The facts that \( \lim_{x \to -\infty} F_{CR}(x; \lambda, \theta, p) = F_{CR}(0; \lambda, \theta, p) = 0 \) and \( \lim_{x \to \infty} F_{CR}(x; \lambda, \theta, p) = 1 \) can now easily be established taking into consideration that \( \Phi(-\infty) = 0 \) and \( \Phi(+\infty) = 1 \).

### 2.2 The moment generating function of the Crack distribution

The form of the moment generating function of the Crack distribution is also already known; see for example Das (2015). Again, the derivation is indirect in the sense that first the moment generating functions for the Inverse Gaussian \( \phi_{IG}(t) \) and the Length Biased Inverse Gaussian \( \phi_{LB}(t) \) distributions are derived and then using the fact

\[ f_{CR}(x, \lambda \theta, p) = p f_{IG}(x, \lambda, \theta) + (1 - p) f_{LB}(x, \lambda, \theta) \]
it is shown that if \( X \sim CR(\lambda, \theta, p) \), then the moment generating function of the CR(\( \lambda, \theta, p \)) distribution is \( \phi_{CR}(t) = p\phi_{IG}(t) + (1 - p)\phi_{LB}(t) \).

Moreover, the derivation of the moment generating functions (that have been done before my thesis) for the Inverse Gaussian \( \phi_{IG}(t) \) and the Length Biased Inverse Gaussian \( \phi_{LB}(t) \) distributions is based on the following two, as they say, “useful”, integrals:

For two complex numbers \( p \) and \( q \) such that Re(\( p \)) > 0 and Re(\( q \)) > 0:

\[
\int_0^\infty x^{-1/2} \exp\left\{-px - \frac{q}{x}\right\} \, dx = \sqrt{\frac{\pi}{p}} e^{-2\sqrt{pq}},
\]

\[
\int_0^\infty x^{-3/2} \exp\left\{-px - \frac{q}{x}\right\} \, dx = \sqrt{\frac{\pi}{q}} e^{-2\sqrt{pq}}.
\]

These integral can be obtained as special cases of formula 3.472.5 from Gradshtein and Ryzhik (2015).

Contrary to Das (2015), that derived the moment generating function using these integrals, the derivation of the moment generating function of the Crack distribution that I present in this section is much more mathematically appealing in sense that it does not use any unknown formulae. Moreover, it is based only on the form of the density function of the Crack distribution.

**Proposition 2.2.** The moment generating function of the Crack CR(\( \lambda, \theta, p \)) distribution is

\[
\phi_{CR}(t; \lambda, \theta, p) = \left[p + \frac{1 - p}{\sqrt{1 - 2\theta t}}\right] \exp\left[\lambda(1 - \sqrt{(1 - 2\theta t)})\right],
\]

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where \( t < \frac{1}{2\theta} \).

**Proof:** Let \( X \) be a CR(\( \lambda, \theta, p \)) distributed random variable; then

\[
\phi_{CR}(t; \lambda, \theta, p) = \phi_{CR}(t) = Ee^{tx} = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx
\]

\[= \int_{-\infty}^{\infty} e^{tx} f_{CR}(x; \lambda, \theta, p)dx
\]

\[= \int_{0}^{\infty} e^{tx} \frac{1}{\theta\sqrt{2\pi}} \left[ p\lambda \left( \frac{\theta}{x} \right)^{3/2} + (1-p) \left( \frac{\theta}{x} \right)^{1/2} \right] \exp \left[ -\frac{1}{2} \left( \frac{\sqrt{x}}{\theta} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right] dx
\]

\[= \int_{0}^{\infty} \frac{1}{\theta\sqrt{2\pi}} \left[ p\lambda \left( \frac{\theta}{x} \right)^{3/2} + (1-p) \left( \frac{\theta}{x} \right)^{1/2} \right] \exp \left[ tx - \frac{1}{2} \left( \frac{\sqrt{x}}{\theta} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right] dx
\]

I start with simplifying the expression in square brackets under the exponent, that is

\[
V = tx - \frac{1}{2} \left( \frac{\sqrt{x}}{\theta} - \lambda \sqrt{\frac{\theta}{x}} \right)^2
\]

\[= -\frac{1}{2} \left( -2tx + \frac{x}{\theta} - 2\lambda + \lambda^2 \frac{\theta}{x} \right)
\]

\[= -\frac{1}{2} \left( x \left( \frac{1}{\theta} - 2t \right) - 2\lambda + \lambda^2 \frac{\theta}{x} \right)
\]

\[= -\frac{1}{2} \left( \frac{x}{\theta} \frac{1-2t}{1-2t\theta} - 2\lambda + \lambda^2 \frac{\theta}{x} \right)
\]

Let \( \theta_1 = \frac{\theta}{1-2t\theta} \) and \( \lambda_1 = \lambda \sqrt{1 - 2t\theta} \). Then continuing our calculations

\[
V = -\frac{1}{2} \left( \frac{x}{\theta_1} - 2\lambda + \lambda_1^2 \frac{\theta}{1-2t\theta} \frac{1}{x} \right)
\]

\[= -\frac{1}{2} \left( \frac{x}{\theta_1} - 2\lambda_1 + \lambda_1^2 \frac{\theta_1}{x} \right) + \lambda - \lambda_1
\]

\[= -\frac{1}{2} \left( \sqrt{\frac{x}{\theta_1}} - \lambda_1 \sqrt{\frac{\theta_1}{x}} \right)^2 + \lambda(1 - \sqrt{1 - 2t\theta})
\]
Substitute this back into the formula for the moment generating function:

\[
\phi_{CR}(t) = \int_0^\infty \frac{1}{\theta \sqrt{2\pi}} \left[ p\lambda \left( \frac{\theta}{x} \right)^{3/2} + (1 - p) \left( \frac{\theta}{x} \right)^{1/2} \right] \\
\times \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda_1 \sqrt{\frac{\theta}{x}} \right)^2 + \lambda \left( 1 - \sqrt{1 - 2t\theta} \right) \right] \, dx
\]

Now consider the first term under the integral without the constant \( \frac{1}{\theta \sqrt{2\pi}} \), that is,

\[
D = \frac{1}{\theta} \left[ p\lambda \left( \frac{\theta}{x} \right)^{3/2} + (1 - p) \left( \frac{\theta}{x} \right)^{1/2} \right]
\]

Note that \( \lambda \theta^{1/2} = \lambda_1 \theta_1^{1/2} \) and \( \theta^{-1/2} = \frac{\theta_1^{-1/2}}{\sqrt{1 - 2t\theta}} \). Hence

\[
D = p\lambda \theta_1^{1/2} \left( \frac{1}{x} \right)^{3/2} + \frac{1 - p}{\sqrt{1 - 2t\theta}} \theta_1^{-1/2} \left( \frac{1}{x} \right)^{1/2}
\]

Now I use the simple identity

\[
\alpha A + \beta B = (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} A + \frac{\beta}{\alpha + \beta} B \right),
\]

for \( \alpha = p, \beta = \frac{1-p}{\sqrt{1 - 2t\theta}}, A = \lambda_1 \left( \frac{\theta_1}{x} \right)^{3/2}, B = \left( \frac{\theta_1}{x} \right)^{1/2} \).
If I let $p_1 = \frac{\alpha}{\alpha + \beta}$, then $1 - p_1 = \frac{\beta}{\alpha + \beta}$ and our expression takes the form:

$$D = (\alpha + \beta) \frac{1}{\theta_1} \left[ p_1 \lambda_1 \left( \frac{\theta_1}{x} \right)^{3/2} + (1 - p_1) \left( \frac{\theta_1}{x} \right)^{1/2} \right]$$

$$= \left[ p + \frac{1 - p}{\sqrt{1 - 2\theta t}} \right] \frac{1}{\theta_1} \left[ p_1 \lambda_1 \left( \frac{\theta_1}{x} \right)^{3/2} + (1 - p_1) \left( \frac{\theta_1}{x} \right)^{1/2} \right]$$

Returning to our integral:

$$\phi_{CR}(t) = \left[ p + \frac{1 - p}{\sqrt{1 - 2\theta t}} \right] \exp[\lambda(1 - \sqrt{1 - 2\theta t})]$$

$$\times \int_{0}^{\infty} \frac{1}{\theta_1 \sqrt{2\pi}} \left[ p_1 \lambda_1 \left( \frac{\theta_1}{x} \right)^{3/2} + (1 - p_1) \left( \frac{\theta_1}{x} \right)^{1/2} \right]$$

$$\times \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta_1}} - \lambda_1 \sqrt{\frac{\theta_1}{x}} \right)^2 \right] dx$$

Observe that the function under the integral is the density function of the $CR(\lambda_1, \theta_1, p_1)$ distribution and hence this integral equals to 1. Therefore,

$$\phi_{CR}(t; \lambda, \theta, p) = \left[ p + \frac{1 - p}{\sqrt{1 - 2\theta t}} \right] \exp \left[ \lambda(1 - \sqrt{1 - 2\theta t}) \right].$$

The assumption $t < \frac{1}{2\theta}$ is necessary because I have the expression $\sqrt{1 - 2\theta t}$ in the formula.

**Corollary 2.1.** The moment generating function of the Inverse Gaussian $IG(\lambda, \theta)$ distribution is

$$\phi_{IG}(t; \lambda, \theta) = \exp \left[ \lambda(1 - \sqrt{1 - 2\theta t}) \right],$$

where $t \leq \frac{1}{2\theta}$. 

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Proof: Take $p = 1$ in Proposition 2.2.

**Corollary 2.2.** The moment generating function of the Length Biased Inverse Gaussian $LB(\lambda, \theta)$ distribution is

$$
\phi_{LB}(t; \lambda, \theta) = \exp \left[ \lambda(1 - \sqrt{1 - 2\theta t}) \right] (1 - 2\theta t)^{-\frac{1}{2}}
$$

where $t \leq \frac{1}{2\theta}$.

Proof: Take $p = 0$ in Proposition 2.2.

**Corollary 2.3.** The moment generating function of the Birnbaum-Saunders $BS(\lambda, \theta)$ distribution is

$$
\phi_{BS}(t; \lambda, \theta) = \frac{1}{2} \left[ 1 + \frac{1}{\sqrt{1 - 2\theta t}} \right] \exp \left[ \lambda(1 - \sqrt{1 - 2\theta t}) \right],
$$

where $t < \frac{1}{2\theta}$.

Proof: Take $p = 1/2$ in Proposition 2.2.
Chapter 3

Some New Properties of the Crack Distribution

In this chapter, I propose some probabilistic properties of the three-parameter Crack lifetime distribution which have not been known before.

**Theorem 1.** If a random variable $X$ has CR($\lambda, \theta, p$) distribution, then for any $a > 0$ random variable $aX$ has CR($\lambda, a\theta, p$) distribution.

Proof. Fix $a > 0$ and consider random variable $Y = aX$. The distribution function of random variable $Y$ can be expressed in terms of the distribution function of random variable $X$ in the following way:

$$F_Y(t) = P(Y < t) = P(aX < t) = P(X < t/a) = F_X(t/a).$$

The density function of the random variable $Y$ as the derivative of the corresponding
distribution function is:

\[ f_Y(t) = F'_Y(t) = (F_X(t/a))' = f_X \left( \frac{t}{a} \right) \frac{1}{a}. \]

Because \( X \) has CR(\( \lambda, \theta, p \)) distribution, I can write:

\[
\begin{align*}
  f_Y(t) &= \frac{1}{\theta \sqrt{2\pi}} \left[ p \lambda \left( \frac{a\theta}{t} \right)^{\frac{3}{2}} + (1-p) \left( \frac{a\theta}{t} \right)^{\frac{1}{2}} \right] \exp \left[ -\frac{1}{2} \left( \frac{\sqrt{t/a}}{a\theta} - \lambda \frac{\sqrt{a\theta}}{t} \right)^2 \right] \left( \frac{1}{a} \right), \\
  &= \frac{1}{a\theta \sqrt{2\pi}} \left[ p \lambda \left( \frac{a\theta}{t} \right)^{\frac{3}{2}} + (1-p) \left( \frac{a\theta}{t} \right)^{\frac{1}{2}} \right] \exp \left[ -\frac{1}{2} \left( \frac{\sqrt{t/a}}{a\theta} - \lambda \frac{\sqrt{a\theta}}{t} \right)^2 \right] ,
\end{align*}
\]

that is, \( Y \) has CR(\( \lambda, a\theta, p \)) distribution.

**Theorem 2.** If random variable \( X \) has CR(\( \lambda, \theta, p \)) distribution, then the reciprocal random variable \( \frac{1}{X} \) has CR \( (\lambda, \frac{1}{\sqrt{\theta}}, 1-p) \) distribution.

Proof. Let \( Y = \frac{1}{X} \). Note that \( X > 0 \) as a lifetime distribution.

The distribution function for \( Y \) in terms of distribution function of \( X \)

\[
F_Y(t) = P \left( \frac{1}{X} \leq t \right) = P \left( X \geq \frac{1}{t} \right) = 1 - F_X \left( \frac{1}{t} \right) ,
\]

and the density function is

\[
f_Y(t) = F'_Y(t) = -F'_X \left( \frac{1}{t} \right) = \frac{1}{t^2} f_X \left( \frac{1}{t} \right) .
\]

Because \( X \) has CR(\( \lambda, \theta, p \)) distribution, I can write:
Our goal is to represent this density function in the form

\[
f_Y(t) = \frac{1}{\theta \sqrt{2\pi}} \left[ p \lambda (t\theta)^{2/3} + (1 - p) (t\theta)^{1/2} \right] \exp \left[ -\frac{1}{2} \left( \sqrt{\frac{1}{t\theta}} - \lambda \sqrt{t\theta} \right)^2 \right] \left( \frac{1}{t^2} \right)
\]

First I consider the exponent expressions:

\[
\exp \left[ -\frac{1}{2} \left( \sqrt{\frac{t}{\theta_1}} - \lambda_1 \sqrt{\frac{\theta_1}{t}} \right)^2 \right] = \exp \left[ -\frac{1}{2} \left( \sqrt{t\lambda_1^2 - \frac{1}{t\theta}} \right)^2 \right]
\]

Equating coefficients for \( \sqrt{t} \) and \( \frac{1}{\sqrt{t}} \), I obtain

\[
\begin{align*}
\frac{1}{\sqrt{\theta_1}} &= \sqrt{\lambda_1^2} \\
\lambda_1 \sqrt{\theta_1} &= \frac{1}{\sqrt{\theta}}
\end{align*}
\]
From the first equation $\theta_1 = \frac{1}{\lambda \theta}$ and substituting this expression to the second expression, I obtain

$$\lambda_1 = \frac{1}{\theta_1} = \frac{\lambda \sqrt{\theta}}{\sqrt{\theta}} = \lambda.$$ 

Therefore, I proved that $\theta_1 = \frac{1}{\lambda \theta}$ and $\lambda_1 = \lambda$.

To find the value of $p_1$, I consider the first part of the expression for $f_Y(t)$ (without the exp function that I already considered above. I need to have:

$$\frac{1}{\theta_1 \sqrt{2\pi}} \left[ p_1 \lambda_1 \left( \frac{\theta_1}{t} \right)^{3/2} + (1 - p_1) \left( \frac{\theta_1}{t} \right)^{1/2} \right] = \frac{1}{\sqrt{2\pi}} \left[ (1 - p) \theta^{-1/2} \left( \frac{1}{t} \right)^{3/2} + p \lambda \theta^{1/2} \left( \frac{1}{t} \right)^{1/2} \right]$$

Equating coefficients for $(\frac{1}{t})^{3/2}$ and $(\frac{1}{t})^{1/2}$:

$$\begin{align*}
\frac{1}{\theta_1 \sqrt{2\pi}} p_1 \lambda_1 \theta_1^{3/2} &= \frac{1}{\sqrt{2\pi}} (1 - p) \theta^{-1/2} \\
\frac{1}{\theta_1 \sqrt{2\pi}} (1 - p_1) \theta_1^{1/2} &= \frac{1}{\sqrt{2\pi}} p \lambda \theta^{1/2}
\end{align*}$$

Substituting the expressions for $\lambda_1$ and $\theta_1$ that I obtained above:

$$\begin{align*}
\frac{\lambda^2 \theta}{\sqrt{2\pi}} p_1 \lambda \left( \frac{1}{\lambda^2 \theta} \right)^{3/2} &= \frac{1}{\sqrt{2\pi}} (1 - p) \theta^{-1/2} \\
\frac{\lambda^2 \theta}{\sqrt{2\pi}} (1 - p_1) \left( \frac{1}{\lambda^2 \theta} \right)^{1/2} &= \frac{1}{\sqrt{2\pi}} p \lambda \theta^{1/2},
\end{align*}$$

or $p_1 = 1 - p$.

Therefore, $Y$ has CR $\left( \lambda, \frac{1}{\lambda \theta}, 1 - p \right)$ distribution.

**Theorem 3.** If random variable $X$ has CR$(\lambda_1, \theta, p)$ distribution and independent random variable $Y$ has IG$(\lambda_2, \theta)$ distribution, then the sum random variable $X + Y$
has CR(λ₁ + λ₂, θ, p) distribution.

Proof. Since \( X \) and \( Y \) are independent random variables, the moment generating function \( X + Y \) is the product of the moment generating functions of each term. From Proposition 2.2, the moment generating function of \( X \) as a random variable with CR(λ₁, θ, p) distribution is

\[
\varphi_X(s) = e^{\lambda_1(1 - \sqrt{1 - 2\theta s})} \frac{1}{\sqrt{1 - 2\theta s}} \left[ 1 - p \left( 1 - \sqrt{1 - 2\theta s} \right) \right].
\]

From Corollary 2.1, the moment generating function of \( Y \) as a random variable with IG(λ₂, θ) distribution is

\[
\varphi_Y(s) = e^{\lambda_2(1 - \sqrt{1 - 2\theta s})}.
\]

Hence, the moment generating function \( X + Y \) is the product of the moment generating functions \( \varphi_{X+Y}(s) = \varphi_X(s) \times \varphi_Y(s) \) is

\[
\varphi_{X+Y}(s) = \frac{e^{\lambda_1(1 - \sqrt{1 - 2\theta s})} \left[ 1 - p \left( 1 - \sqrt{1 - 2\theta s} \right) \right]}{\sqrt{1 - 2\theta s}} \left( e^{\lambda_2(1 - \sqrt{1 - 2\theta s})} \right)
= \frac{e^{\lambda_1(1 - \sqrt{1 - 2\theta s}) + \lambda_2(1 - \sqrt{1 - 2\theta s})} \left[ 1 - p \left( 1 - \sqrt{1 - 2\theta s} \right) \right]}{\sqrt{1 - 2\theta s}}
= \frac{e^{(\lambda_1 + \lambda_2)(1 - \sqrt{1 - 2\theta s})} \left[ 1 - p \left( 1 - \sqrt{1 - 2\theta s} \right) \right]}{\sqrt{1 - 2\theta s}}.
\]

Note that this is the moment generating function for the CR(λ₁ + λ₂, θ, p) distribution.

Remark. Note that the assumption of independence of random variables \( X \) and \( Y \) in Theorem 3 is crucial. Without it I cannot state that the moment generating function \( X + Y \) is the product of the moment generating functions of each term.
Chapter 4

A Probability Model of the Growth of Two-Sided Cracks

A detailed, consistent, and rigorous general probability model of the growth of a one-sided crack for a metallic block is presented below. Until now there has been no such model for the development of fatigue cracks from the upper and lower sides of a block. Our goal is to present such detailed, consistent, and rigorous general probability model of the growth of two-sided cracks. The significance of this research follows from the fact that such investigations have numerous applications in physics, engineering, statistics, environmental studies and economics.

In this chapter, I establish and investigate a probability model for a two-sided Birnbaum-Saunders distribution. I consider different choices of impulse function which correspond to the crack development from two sides and establish how it influences the resulting probability distribution. This allows us to evaluate the performance of the proposed models for different impulse functions. Such results on models for Birnbaum-Saunders may have significant practical applications for engineering and
testing concerning reliability theory when modeling the life-time for products with failure due to the development of fatigue cracks. Another application is to predict the failure of parts on large industrial and military ships.

A general probability model of two-sided crack development will be constructed based on the new parametrization is presented by Ahmed, Budsaba, Lisawadi and Volodin (2008). I provide computer simulations for different choices of impulse functions corresponding to the crack development from two sides in order to investigate how it influences the resulting probability distribution.

4.1 Definitions of the one-sided Birnbaum-Saunders distribution

4.1.1 Definition by physical model

The following description of the general probability model of growth of a one-sided crack has been introduced in Ahmed, Budsaba, Lisawadi, and Volodin (2008). I present a modified and extended description of the construction in the present paper because it is crucial for a similar construction of two-sided crack development.

Consider a rectangular metal block which is fixed from two sides. A periodic loading is applied to its middle part and this leads to the development of a fatigue crack. Assume that at the beginning the length of the crack was $x_0 \geq 0$, and after each loading I measure the crack length and obtain a sequence of nondecreasing numbers $x_1, x_2, \ldots$. First, I am interested in the prediction of the crack length after the $n$th loading. After that, I am interested in finding a distribution of the time when the
block breaks down.

It is obvious that the crack development is achieved as a result of several factors, such as the strength of loading, the quality of metal from which the block is made, and so on. Therefore I obviously am dealing with a stochastic forecasting problem. Therefore, I should consider the measurements $x_1, x_2, \cdots$ as a realization of a sequence of random variables $X_1, X_2, \cdots$. I formalize mathematically the phenomenon of a crack development in terms of the increments $\Delta_k = X_k - X_{k-1}$ of the crack lengths.

It is natural to assume that the increment $\Delta_k \geq 0$ is achieved by the sum of all values produced by factors of the crack growth that I mentioned above. That is, under some nonnegative “impulse” $\xi_k (\geq 0)$, there exists an approximate linear relationship between $\Delta_k$ and $\xi_k$ such as $\Delta_k = \alpha_k \xi_k$, where $\alpha_k$ depends on the previous crack length $X_{k-1}$ that was achieved at the $(k-1)$th loading. Let $\alpha_k = g(X_{k-1})$ with the natural assumption that the impulse function $g(\cdot)$ is nonnegative and continuous. Therefore, I have the following recurrent relations that describe the crack development after each loading:

$$X_k - X_{k-1} = \xi_k \cdot g(X_{k-1}), \quad k = 1, 2, \ldots \quad (4.1.1)$$

Now I make some assumptions on the distributions of the random variables $\xi_{k}, k \geq 1$. Assume that these random variables are nonnegative, independent identically distributed with finite second moment and denote by $a = \mathbb{E}(\xi_k)$ their mean value and by $b^2 = \text{Var}(\xi_k)$ their variance.

Recall that I am interested in the distribution of the random variable $X_n$, whose realization $x_n$ gives the length of a particular crack after the $n$th loading. Rewrite
the first $n$ recurrent relations (4.1.1) as

$$\xi_k = \frac{X_k - X_{k-1}}{g(X_{k-1})}, \quad k = 1, \ldots, n$$

and add all of them to obtain

$$\sum_{k=1}^{n} \xi_k = \sum_{k=1}^{n} \frac{X_k - X_{k-1}}{g(X_{k-1})}.$$  

If each impulse provides an insignificant increase in the crack length, that is, all $\Delta_k = X_k - X_{k-1}$ are small, then I can interpret the right hand side of the summation as an integral sum and obtain the approximate equality

$$\sum_{k=1}^{n} \xi_k \approx \int_{X_0}^{X} \frac{dt}{g(t)}, \quad (4.1.2)$$

where $X = X_n$ is the final crack size. By the “$\approx$” sign in (4.1.2), I mean that the left hand side of the expression is a pointwise approximation of the right hand side.

Since the function $g(x)$ is positive, the integral on the right hand side of (4.1.2) represents some monotone increasing function $h(X)$. By the Central Limit Theorem applied to the left hand side of (4.1.2), I obtain the following statement: For some long time ($n >> 1$) after the crack started to grow, the distribution of its length $X$ is defined by the relations $h(X) \sim N(\mu, \sigma^2)$, where $\mu = na, \sigma^2 = nb^2$. By the monotonicity of the function $h(\cdot)$, the distribution function of the random variable $X$
is
\[ F(x) = P(X < x) = P(h(X) < h(x)) = \Phi \left( \frac{h(x) - \mu}{\sigma} \right), \]
where \( \Phi(x) \) is the standard normal distribution function.

It remains to solve the problem with a choice of the function \( g(\cdot) \). If I postulate the most practical linear form \( g(t) = t \), that is, the increase of the crack is proportional to the length achieved (this assumption is the most commonly used in the models of growth), then I obtain the lognormal distribution of the random variable \( X \).

In order to define the one-sided Birnbaum-Saunders distribution, consider the following problem. In the framework of the probability model of growth constructed above, I am not interested in finding the distribution of the crack length \( X \). Assuming the critical length \( a \) of the crack is fixed, what I am interested in is the distribution of the *moment of time (number of loadings)* at which this length will be achieved. It is interesting that in the framework of our model, this distribution does not depend on the choice of the positive function \( g(\cdot) \); the choice of \( g \) influences only the concrete values of the parameters. The distribution of the time can be obtained by the following simple observations.

Let \( \tau \) be a random variable which represents the moment when the length of the crack achieves the critical length \( a \). Then the event \( \tau > n \) is equivalent to the event \( X_n < a \) (recall that all \( \xi_k \geq 0 \) and for the moment of time \( n \) the crack length does not achieve the critical length \( a \)). Therefore

\[ P(\tau > n) = P(X_n < x) = P(h(X_n) < h(x)) = \Phi \left( \frac{h(x) - na}{b\sqrt{n}} \right). \] 

(4.1.3)
Replace the variable $n$ by a “continuous” variable $x$ and introduce new parameters $\lambda$ and $\theta$, defined as $\lambda = ah(x)/b^2, \theta = b^2/a^2$. The chain of equalities (4.1.3) helps us to write the distribution function of the random moment of time $\tau$ at which the crack achieves the critical length $a$:

$$F_{BS}(x; \theta, \lambda) = P(\tau < x) = 1 - \Phi\left(\lambda\sqrt{\frac{\theta}{x}} - \sqrt{\frac{x}{\theta}}\right) = \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right), x \geq 0.$$ 

This distribution is called the one-sided Birnbaum–Saunders distribution.

The density function for this distribution is

$$f_{BS}(x; \theta, \lambda) = \frac{1}{2\sqrt{2\pi}\theta} \left[ \lambda \left(\frac{\theta}{x}\right)^{3/2} + \left(\frac{x}{\theta}\right)^{1/2}\right] \exp\left\{ -\frac{1}{2} \left(\lambda\sqrt{\frac{\theta}{x}} - \sqrt{\frac{x}{\theta}}\right)^2 \right\}, x \geq 0.$$ 

### 4.1.2 Formal definition by Brownian motion

In this section I would like to present a formal definition of the Birnbaum-Saunders distribution. This construction will not be used in the following, but it is interesting because this most likely is the way Birnbaum-Saunders were arguing when they introduced their distribution in 1969.

Let $\{B_t\}$ be a standard Brownian motion with zero drift. I define a process $X_t = \mu t + \sigma B_t, \mu > 0, \sigma > 0$, which quantifies the level of stress or fatigue accumulated by a subject of interest up to time $t$. The failure event occurs when the accumulated stress hits a critical threshold $w > 0$. Then the failure time $T$ is the first hitting time of $X_t$ to the threshold $w$. By using quite advanced techniques of stochastic analysis, namely Girsanov’s Theorem and the reflection principle (we are not going into this...
complicated details), the distribution of $T$ is given as follows:

$$P[T \leq t] = P[M(t) > w] = 1 - \Phi\left(\frac{w - \mu t}{\sigma \sqrt{t}}\right) + e^{2\mu w/\sigma^2} \Phi\left(\frac{-w - \mu t}{\sigma \sqrt{t}}\right), \quad (4.1.4)$$

where

$$M(t) = \max_{0 \leq s \leq t} \{X_s\}$$

The Birnbaum-Saunders distribution is obtained as an approximation to (4.1.4) by ignoring the last term in the formula. Specifically, the distribution function of the Birnbaum-Saunders distribution is

$$F_{BS}(t) = 1 - \Phi\left(\frac{1}{\alpha} \left(\sqrt{\frac{\beta}{t}} - \sqrt{\frac{t}{\beta}}\right)\right),$$

where $\alpha = \sigma / \sqrt{\mu w} > 0$, $\beta = w / \mu > 0$. The parameters $\alpha$ and $\beta$ are the shape and scale parameters, respectively.

Note that the new parametrization of the Birnbaum–Saunders distribution presented in the previous section develops the new parameters which are meaningful in a practical setting. Importantly, this re-parametrization fits the physics of the phenomena under study since the proposed parameters $\lambda$ and $\theta$ correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively.
4.2 Formal definition of the Two-sided Birnbaum-Saunders distribution

4.2.1 Definition by physical model

Similar to the case of the one-sided Birnbaum-Saunders distribution, consider a rectangular metal block of size $a$ which is fixed on two sides. A periodic loading is applied first to its upper part, then immediately to its lower part (consider as one period of loading) and this leads to a development of fatigue cracks from the upper and lower sides of the block. Denote by $X_i$ the length of the upper crack and by $Y_i$ the length of the lower crack after the $i$th loading.

As above, I assume that the increases of the crack lengths obey the following recurrence relations:

$$X_i = X_{i-1} + \xi_i \cdot g_1(X_{i-1}, Y_{i-1})$$
$$Y_i = Y_{i-1} + \eta_i \cdot g_2(X_{i-1}, Y_{i-1}).$$

(4.2.1)

Here each set $\{\xi_i, i \geq 1\}$ and $\{\eta_i, i \geq 1\}$ consists of positive independent identically distributed random variables with finite second moments, and the impulse functions $g_1(\cdot, \cdot)$ and $g_2(\cdot, \cdot)$ are positive and continuous. Note that the growth of both cracks depends on the previous crack lengths in the upper as well as in the lower parts.

Rewrite the recurrence relations in the following form:

$$\frac{X_i - X_{i-1}}{g_1(X_{i-1}, Y_{i-1})} = \xi_i$$
\[
\frac{Y_i - Y_{i-1}}{g_2(X_{i-1}, Y_{i-1})} = \eta_i.
\]

Taking sums of all these equalities gives

\[
\sum_{i=1}^{n} \left[ \frac{X_i - X_{i-1}}{g_1(X_{i-1}, Y_{i-1})} + \frac{Y_i - Y_{i-1}}{g_2(X_{i-1}, Y_{i-1})} \right] = \sum_{i=1}^{n} (\xi_i + \eta_i). 
\]  (4.2.2)

Now I can investigate how the choice of functions \(g_1\) and \(g_2\) influences the distribution of the random variable \(\tau\), the moment the block breaks down under two-sided loading.

**Case 1: Reduction to the one-sided Birnbaum-Saunders distribution.**

Consider the following choice of the impulse functions \(g_1(x, y) = g_2(x, y) = g(x + y)\); that is, the impulse function depends only on the total length of the crack. Examples of such functions may be \(g_1(x, y) = g_2(x, y) = x+y\), or \(g_1(x, y) = g_2(x, y) = \exp(x+y)\), or \(g_1(x, y) = g_2(x, y) = (x+y)^2\).

If I let \(\Delta X_i = X_i - X_{i-1}\), \(\Delta Y_i = Y_i - Y_{i-1}\) and assume that these increments are sufficiently small, then I obtain the integral sum

\[
\sum_{i=1}^{n} \frac{\Delta(X_i + Y_i)}{g(X_{i-1} + Y_{i-1})} = \sum_{i=1}^{n} \frac{\Delta(Z_i)}{g(Z_{i-1})},
\]

where \(Z_i = X_i + Y_i\). Finally exchanging \(Z_{i-1}\) with the close value \(Z_i\), I obtain

\[
\sum_{i=1}^{n} \frac{\Delta(Z_i)}{g(Z_{i-1})} \approx \int_{X_0}^{X_0+Y_0} \frac{dx}{g(x)} \approx \sum_{i=1}^{n} (\xi_i + \eta_i).
\]

This integral is an increasing function of the total crack length, hence the same arguments are true as in the physical model for the one-sided Birnbaum-Saunders
distribution presented above. The moment of the break down has the distribution function

\[ P(\tau > n) = P(X_n + Y_n < a) = P(h(X_n + Y_n) < h(a)) \approx \Phi \left( \frac{h(a) - 2n\mu}{\sigma\sqrt{n}} \right). \]

If I exchange \( n \) by \( t \), then I obtain the one-sided Birnbaum-Saunders distribution. Hence in the case in which the impulse function depends only on the total length of the crack, the two-sided Birnbaum-Saunders distribution is the same as the one-sided Birnbaum-Saunders distribution up to parameter values.

**Case 2:** Apparently reduces to the one-sided Birnbaum-Saunders distribution, too. Assume that the length of a crack from each side depends only on its previous length; for example, impulse functions \( g_1(x, y) = g(x) \) and \( g_2(x, y) = g(y) \), where the function \( g(\cdot) \) is nonnegative and continuous. Then formula (4.1.3) corresponds to two integrals:

\[
\sum_{i=1}^{n} \left[ \frac{X_i - X_{i-1}}{g_1(X_{i-1}, Y_{i-1})} + \frac{Y_i - Y_{i-1}}{g_2(X_{i-1}, Y_{i-1})} \right] \approx \int_{X_0}^{X_n} \frac{dx}{g(x)} + \int_{Y_0}^{Y_n} \frac{dy}{g(y)}.
\]

Obviously, I cannot state that the sum of these integrals produces a monotone function of the total length \( X_n + Y_n \). For example, if I take \( g_1(x, y) = x \) and \( g_2(x, y) = y \), then the sum of the integrals is a monotone function of the product \( X_n \cdot Y_n \). Hence, the integral which has approximately normal distribution will be a monotone increasing function of the product \( X_n \cdot Y_n \) and I will not be able to say that the
events $\tau > a$ and $X_n + Y_n < a$ are the same in order to establish the one-sided Birnbaum-Saunders distribution.

Therefore, for impulse functions $g_1(x, y)$ and $g_2(x, y)$ which are not monotone functions of $x + y$, the theoretical construction of the model is a completely unsolved problem. Our goal is to model it by computer simulations for the case $g_1(x, y) = x$ and $g_2(x, y) = y$ and check how it is related to the one-sided Birnbaum-Saunders distribution.

Because the problem is not solvable analytically, I simulate the recurrent relation (4.2.1) with $g_1(x, y) = x$, $g_2(x, y) = y$ and $\xi_i \sim \text{Exp}(1), \eta_i \sim \text{Exp}(1)$. Let $X_0$ and $Y_0$ be initial crack lengths. I am interested in determining the time $\tau$ when $X_i + Y_i$ becomes more than the critical length $a = 1$, that is, $\tau = \min(i : X_i + Y_i > 1)$.

Simulation plan:

1. Fix the values $X_0$ and $Y_0$ of the initial crack lengths.

2. Obtain 1000 simulated values of the random variable $\tau$ using the recurrent relation (4.2.1) with $g_1(x, y) = x$, $g_2(x, y) = y$ and $\xi_i \sim \text{Exp}(1), \eta_i \sim \text{Exp}(1)$.

3. Estimate the parameters $\theta$ and $\lambda$ of the one-sided Birnbaum-Saunders distribution by the Method of Maximum Likelihood. This is not a simple task; see Ahmed, Budsaba, Lisawadi and Volodin (2008) for a detailed discussion of this problem. Because of these difficulties with Maximum Likelihood estimates, it is interesting to try to substitute the estimates of the parameters $\theta$ and $\lambda$ of the one-sided Birnbaum-Saunders distribution by the method of minimum chi-square; see the discussion below.
4. Use the Chi-square test for the hypothesis of goodness-of-fit with the one-sided Birnbaum-Saunders distribution.

For chi-square testing I divided the data into \( r = 8 \) intervals (groups).

Note that in the case when I know completely the hypothetical distribution (the values of parameters are known) and substitute the theoretical frequencies of falling into intervals into the chi-square test statistic, then it has approximately chi-square distribution with \( r - 1 \) degrees of freedom. But in our case when I substitute the estimates of the parameters and hence the distribution of the statistics changes. Fisher proved (see the proof in Cramer (1999)) that if I substitute the estimates by the method of minimum chi-square, which are the points of the minimum (by the parameter variables in theoretical frequencies) of the chi-square statistics, then the statistics will have chi-square distribution with \( r - s - 1 \) degrees of freedom, where \( s \) is the number of parameters. In the case when I substitute the estimates by the method of maximum likelihood, then the asymptotic distribution function will be between the chi-square distribution functions with \( r - 1 \) and \( r - s - 1 \) degrees of freedom (see Chernoff and Lehmann (1954)). Hence in our case, because I substitute the estimates by maximum likelihood, \( r = 8 \) and \( s = 2 \), then the critical values should be found using quantiles of the chi-square distribution with 8-1= 7 and 8-2-1=5 degrees of freedom. From tables, the 5% critical values are \( \chi^2_{df=7} = 14.0671 \) and \( \chi^2_{df=5} = 11.0705 \).

I also note that if I consider the substitution of the estimates of parameters obtained by other methods (for example, the method of moments is exceptionally simple for the one-sided Birnbaum-Saunders distribution), then I do not know what will be
the resulting distribution. I not able to find any results in the literature for the distribution of the resulting chi-square statistics. Hence we do not follow this approach.

The simulation results show that the obtained random numbers for the two-sided Birnbaum-Saunders distribution are exceptionally consistent with the one-sided Birnbaum-Saunders distribution by chi-square distribution. Below I present the analysis of the simulated data with different values $X_0$ and $Y_0$ of the initial crack lengths. I also provide histograms of the simulated data with fitted density functions by the method of maximum likelihood.

$$\chi^2 = 5.22$$

$$\chi^2 = 4.75$$

$$\chi^2 = 1.12$$

It is still unclear why I obtain such a good fit ($p$-values are much greater than significance level $\alpha$) of the two-sided Birnbaum-Saunders distribution with the one-sided Birnbaum-Saunders distribution. Maybe this happens because of the asymptotic normality of not only one-sided, but also two-sided Birnbaum-Saunders distributions. I am working on this problem.

**Case 3.** The two–sided BS–distribution that arises from the above mentioned model with functions different from those considered in Cases 1 and 2. I have not considered this case yet, but I expect that I will obtain distributions which are strictly
different from a one–sided Birnbaum-Saunders distribution. To understand how the two-sided Birnbaum-Saunders distribution changes for different $g_1(x, y)$ and $g_2(x, y)$ is the problem I am working on now.

4.2.2 Formal definition of the two-sided Birnbaum-Saunders distribution

I should also mention that the formal definition (not based on a physical model as above) of the two–sided Birnbaum-Saunders lifetime distributions presented in Lisawadi (2008). Consider the case in which a crack develops from two sides of a metallic object. Consider a rectangular metallic block of height $a$, which is fixed from both edges. To its middle area, a periodic loading is applied which leads to the development of fatigue cracks. Consider the case when a crack develops from two sides, from the lower edge of the block and from the upper edge.

Let $\tau_U$ be a random variable with one–sided Birnbaum-Saunders distribution, that is, $\tau_U$ is the break down time for one-sided loading at the upper side of the block. Then the random variable $Y_U = a/\tau_U$ can be interpreted as the speed of the crack evolution from the upper side. If at the lower side of the block a crack is developing with the same Birnbaum-Saunders distribution, then I have two, assumed to be independent, identically distributed random variables $\tau_U$ and $\tau_L$.

Let $F_\tau(t), t > 0,$ be the distribution function of the random variable $\tau_U$ (or $\tau_L$, as they are identically distributed) and let $f(t)$ be its density function. Of course, I should consider the one-sided Birnbaum-Saunders distribution and density functions.
for \( F_\tau(t) \) and \( f(t) \). The expressions obtained after substitution the formulae of these functions are a little bit cumbersome and there are no simplifications for them, so use the notation \( F(t) \) and \( f(t) \).

The random variable \( Y_U = a/\tau_U \) has the distribution function \( F_Y(t) = 1 - F(t^{-1}) \) and density function \( f_Y(t) = t^{-2} f(t^{-1}) \).

The speed of crack evolution for this two-sided case equals \( Y_U + Y_L = a\tau_U^{-1} + a\tau_L^{-1} \) and the random variable

\[
\nu = \frac{a}{Y_U + Y_L} = [\tau_U^{-1} + \tau_L^{-1}]^{-1}
\]
corresponds to the moment the block breaks down. The distribution function of this random variable is

\[
F_\nu(z) = \int\int_{t+s>z^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{s}\right) \frac{dt ds}{t^2 s^2}
\]
and its density function is

\[
f_\nu(z) = z^{-2} \int^{z^{-1}}_{0} f\left(\frac{1}{t}\right) f\left(\frac{1}{z^{-1} - t}\right) \frac{1}{t^2 (z^{-1} - t)^2} dt.
\]

I say that the random variable \( \nu \) has the two-sided Birnbaum-Saunders distribution.

### 4.3 C++ program code

```c++
#include <fstream.h>
#include <math.h>
```

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#include <stdlib.h>

void main(){
    ofstream outfile("1.txt");
    const double a = 1;
    const double teta = 1;

    double X = X0;
    double Y = Y0;
    int i = 1;

    while ( X+Y<a ){  
        double u, ksi, eta;
        Lb1: u = (double)rand()/(double)RAND_MAX;
        if (u!=1) ksi = -teta*log(1-u); else goto Lb1;

        Lb2: u = (double)rand()/(double)RAND_MAX;
        if (u!=1) eta = -teta*log(1-u); else goto Lb2;
        X = X + ksi*X;
        Y = Y + eta*Y;
        i++;
    }
    outfile<<i<<endl;
}


Bibliography


Statistics, 43, 91-104.


