Distinguishing Linear Sets and Pattern Languages

With Membership Examples

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Abstract

In Computational Learning Theory, a binary concept is often represented by a set of pairs $(x, \ell)$, where $x$ varies over a set of instances known as a universe or instance space, and $\ell$ is a label, either “+” or “−”, indicating whether or not $x$ belongs to the concept. This thesis will consider a type of machine learning task known as supervised learning, where an algorithm $M$ is fed with a set of labelled data for a target concept belonging to a given class of concepts, and $M$ produces a hypothesis function that is used to accurately predict the correct labels of unseen instances. The learning process typically involves two agents, a teacher and a learner. The task of a teacher is to provide labelled examples for a target concept to a learner, who in turn makes a hypothesis based on the seen data. The present thesis focusses on the teacher’s role; specifically, the question: given any concept class $\mathcal{C}$ and a learning algorithm $M$, what is the minimum number of labelled examples needed by $M$ to exactly identify any given concept belonging to $\mathcal{C}$? This quantity may be conceived as a measure of the information complexity or teaching complexity of $\mathcal{C}$.

Our work will study the teaching complexity of three related families of concept classes: the linear sets and some of its subclasses, the non-erasing pattern languages
and the erasing pattern languages. Linear sets and pattern languages are mathematical objects that are closely connected to automata theory and formal languages. In this thesis, the teaching complexity of a concept class will be measured mainly by two combinatorial parameters, the teaching dimension (TD) and the recursive teaching dimension (RTD). The TD of a concept \( C \) with respect to a class \( \mathcal{C} \) containing \( C \) is defined as the size of a smallest sample (called a teaching set for \( C \) with respect to \( \mathcal{C} \)) that is labelled consistently with \( C \) but not with any \( C' \) in \( \mathcal{C} \) distinct from \( C \); the TD of \( \mathcal{C} \) is the worst-case TD over all \( C \) in \( \mathcal{C} \) (or infinity if the set consisting of the teaching dimensions of all \( C \) in \( \mathcal{C} \) with respect to \( \mathcal{C} \) is unbounded). Concerning the TD, we are chiefly interested in two questions with respect to any given concept class \( \mathcal{C} \): first, is there a decidable characterisation of the concepts in \( \mathcal{C} \) that have finite TD with respect to \( \mathcal{C} \); second, given a representation for concepts in \( \mathcal{C} \), how does the TD of \( C \) with respect to \( \mathcal{C} \) vary with the size of the representation of \( C' \)?

The second main parameter studied in this work, the recursive teaching dimension (RTD), arises from the recently introduced recursive teaching model, and it is of particular interest in learning theory due to its links to other learning models as well as to the Vapnik-Chervonenkis dimension – one of the most important parameters in Statistical Learning Theory – in the context of finite concept classes. Our main finding about the RTD in this thesis is that for many classes of linear sets (resp. pattern languages), recursive teaching is significantly more sample efficient than the classical teaching protocol.
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Chapter 1

Introduction

This thesis investigates the following question for various prominent families of concept classes in automata theory and formal languages: given a class $C$ of concepts and any $C$ in $C$, what is the minimum number of membership examples needed to distinguish $C$ from every other concept in $C$? In Computational Learning Theory, and particularly in Algorithmic Teaching, the latter problem is equivalent to that of determining the *teaching complexity* of $C$ in some teaching model. Algorithmic Teaching is a branch of Computational Learning Theory that studies models in which a teacher provides well-chosen samples to a machine so that successful learning can occur. Machine teaching may be viewed as an inverse process of machine learning: while the goal of a traditional learning algorithm is to identify a hypothesis that best fits a given set of data, a teaching algorithm takes as input a hypothesis $C$ and constructs an optimal training set from which a given learning algorithm can

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$^1$A membership example (or labelled example) for $C$ is a pair $(x, \ell)$, where $\ell = +$ if $x$ belongs to $C$ and $\ell = -$ otherwise.
Chapter 1. Introduction

exactly identify $C$. The optimality criterion of a training set depends on the learning task at hand; in the present work, we shall define an ‘optimal’ training set to be one that has the minimum possible cardinality and guarantees exact learnability of the given target concept.

In order to rigorously model the process of teaching and learning, one needs to formalise the following elements:

1. a class of concepts to be learnt;

2. a learner and a teacher;

3. an environment in which the teacher passes on information about a target concept to the learner;

4. a protocol agreed on between the teacher and learner that applies to all concept classes defined over a given set of attributes, and by which the teacher selects data for any given target concept according to a certain rule, while the learner makes its hypothesis about the target concept based on information provided in the environment and an interpretation rule.

Throughout this work, a class of concepts, or simply a concept class, is a family of subsets of a set called a universe or instance space. Every element of a concept class is known as a concept; such concepts are often known as binary concepts. Given a concept class $C$ defined over a universe $\chi$, a teacher $\tau$ is a function mapping every $C \in C$ to a subset $T$ of $\chi \times \{+, -\}$ such that for all $(x, \ell) \in T$, $\ell = +$ holds if and only if $x \in C$. Intuitively, this means that the teacher presents partial information about the target concept $C$ to the learner; this partial information consists of examples, to
be thought of as data points labelled as to whether or not they belong to the target class $C$. A learner $\lambda$ is a function mapping every subset of $\chi \times \{+, -\}$ to a member of $C$. Intuitively, the learner processes the batch of examples provided by the teacher and makes a guess as to which target concept in $C$ the examples stem from. Note that we do not impose any restrictions on the computational power of either the teacher or the learner; in other words, the teacher and learner may even be uncomputable functions. The teaching and learning environment is modelled as a set of labelled examples, which are elements of $\chi \times \{+, -\}$. A protocol may be construed as a function mapping every concept class $C$ over a fixed universe $\chi$ to a pair $(\tau, \lambda)$, where $\tau$ (resp. $\lambda$) is a teacher (resp. learner) for $C$.

The performance of a learning algorithm is often evaluated on the basis of the following criteria: (a) time and space complexity, (b) sample complexity – that is, the minimum number of labelled examples the algorithm needs to guarantee learnability of any concept in a given concept class, and (c) the learning error rate. In the present work, we generally neglect computational efficiency issues and instead focus on (b): the sample efficiency of a teaching/learning algorithm. Moreover, we shall require that the learner achieve exact identification of the target concept from any sample provided by the teacher, that is, no error is allowed, and the error rate is thus zero.

The broad aim of this thesis is to determine the sample efficiency of teaching and learning three families of concept classes – the linear sets, non-erasing pattern languages and erasing pattern languages – in two teaching models, (1) the teaching dimension model due to Goldman, Rivest and Schapire \[22, 20, 38\] as well as Shinohara and Miyano \[42\], and (2) the recursive teaching model due to Zilles, Lange,
Holte and Zinkevich [48]. The sample efficiency of teaching/learning a concept class $C$ within a teaching model may be interpreted as a measure of the teaching complexity of $C$.

In the teaching dimension (or classical teaching) model, a teacher uses knowledge of the target concept $C$, which belongs to a given concept class $C$, to select a smallest batch of labelled examples for $C$ so that any consistent learner – consistent in that the learner’s hypotheses always agree with the presented data – can perfectly predict $C$ from these examples. The learner’s task in the classical teaching protocol is to identify any concept $C'$ in $C$ that is consistent with the given sample $S$, that is to say, it has to identify some $C' \in C$ such that $x \in C'$ holds for all $(x, +) \in S$ and $y \notin C'$ holds for all $(y, -) \in S$. Associated to the teaching dimension model is a combinatorial parameter known as the teaching dimension (TD), which measures the intrinsic difficulty of teaching a given concept class. The teaching dimension of a concept $C$ with respect to a class $C$ containing $C$ is defined as the size of a smallest sample (called a teaching set for $C$ with respect to $C$) that is labelled consistently with $C$ but not with any $C'$ in $C$ distinct from $C$; the teaching dimension of $C$ is the worst-case teaching dimension over all $C$ in $C$ (or infinity if the set consisting of the teaching dimensions of all $C$ in $C$ with respect to $C$ is unbounded). We mention in passing that finding a smallest teaching set for a given concept with respect to a concept class is a fairly old combinatorial problem that has been studied in extremal combinatorics under the notion of “witness sets” [28].

The requirement that a concept class be learnable by any consistent learner may seem a bit too stringent. In reality, human learners tend to be sensitive to the way observed data are sampled, and they often draw stronger inferences from
samples selected in pedagogical settings than from randomly sampled data [41, 46]. Since the introduction of the teaching dimension model, computational learning theorists have proposed alternative teaching models in which a teacher cooperates with a smart learner to reduce the sample efficiency of teaching and learning [6, 48]. Zilles, Lange, Holte and Zinkevich [48] introduced the recursive teaching model, a teaching model in which the teacher and learner cooperate by exploiting an inherent structural property of any well-defined concept class. The key idea behind the recursive teaching protocol is to partition any given concept class $C$ into families $\{F_0, F_1, F_2, \ldots\}$ in a canonical fashion, and for every concept $C$ belonging to some $F_i$, the teacher selects a teaching set for $C$ with respect to the union of $F_j$ over all $j \geq i$. The size of a largest teaching set chosen according to the recursive teaching protocol over all concepts in $C$ is called the recursive teaching dimension (RTD) of $C$.

A notable property of the TD and RTD models is that they are collusion-free for every concept class. There appears to be no universally-agreed formal definition of a collusion-free teaching-learning protocol, but very roughly, it refers to a protocol that does not merely use some encoding scheme mapping a representation of each concept in the hypothesis class to a sample. The definition of collusion-freeness adopted in the present work is based on that of Goldman and Mathias [21]: a teaching-learning protocol is collusion-free if and only if, for every concept class $C$ and any target concept $C$ in $C$, the teacher selects a sample $S$ labelled consistently with $C$, and on any input $S'$ of examples labelled consistently with $C$ such that $S'$ contains $S$, the learner will return a hypothesis representing $C$. Since every teaching set $T$ for a given target concept $C$ contains enough information for any consistent learner to
exactly identify $C$, and augmenting $T$ with new examples labelled consistently with $C$ yields another teaching set for $C$, the classical teaching protocol is collusion-free in the sense used in this work. For a similar reason, the recursive teaching protocol is also collusion-free.

The main questions pursued in this thesis can be formulated as follows: given a concept class $C$, what are the TD and RTD of $C$? Further, can one characterise the concepts in $C$ that have finite TD with respect to $C$? These questions will be studied for subclasses of linear sets and pattern languages, both of which notions we now briefly introduce.

1.1 Linear Sets

Linear sets are the building blocks of semilinear sets, which are mathematical objects studied in automata theory and formal languages. Linear sets and semilinear sets were introduced by Parikh [35] to analyse certain invariant properties of context-free languages. A linear set $L$ is a set of vectors of nonnegative integers such that each vector is obtained by taking the sum of a fixed vector $c$ (called the constant of $L$) and a nonnegative linear combination of a fixed set of vectors known as the periods of $L$; a semilinear set is simply a finite union of linear sets (of the same dimension).

For example, consider fixed $m$-tuples $c = (1, 0, \ldots, 0)$ and $p = (0, 1, \ldots, 1)$ for some $m \geq 2$. Let $L_1$ be the set of all sums $c + np$ and $L_2$ be the set of all vectors $np$, where $n$ varies over all natural numbers. $L_1$ is a linear set consisting of all vectors of the form $(1, k, \ldots, k)$ for some natural number $k$. $L_2$ is also a linear set, consisting of all vectors of the form $(0, k, \ldots, k)$ for some natural number $k$. The union of $L_1$ and $L_2$
is a semilinear set comprising all vectors of the form \((\ell, k, \ldots, k)\), where \(\ell \in \{0, 1\}\) and \(k\) is a natural number. Parikh’s theorem\(^{35}\) elucidates the relationship between semilinear sets and context-free languages: any context-free language is mapped to a semilinear set via a function known as the Parikh vector of a string. In addition to being objects of theoretical interest in formal language theory, semilinear sets have also been applied in the fields of DNA self-assembly\(^{13}\) and membrane computing\(^{25}\).

The learnability of semilinear sets has been investigated in Valiant’s PAC-learning model\(^{1}\), Gold’s learning in the limit model\(^{45}\), and Angluin’s query learning model\(^{45}\). To the best of our knowledge, however, semilinear sets, and in particular linear sets, have never been systematically studied in teaching models. One goal of the present thesis is to fill this research gap by investigating the TD and RTD of various subclasses of linear sets, in the hope that such a study will lead to further insights into the structural properties of linear and semilinear sets as well as the relationship between the teachability and learnability of these objects.

1.2 Pattern Languages

Angluin\(^{2}\) defined and introduced the non-erasing pattern languages. Throughout the present work, a pattern is a nonempty finite string made up of symbols from two disjoint alphabets \(X\) and \(\Sigma\). The elements of \(X\) and \(\Sigma\) are known as variables and constants respectively. The non-erasing pattern language generated by a pattern \(\pi\) is the set of all strings derived from \(\pi\) by substituting nonempty words over \(\Sigma\) for the variables occurring in \(\pi\), with the condition that any two occurrences of the
same variable must be replaced with the same word. Shinohara [43] observed that a more expressive class of pattern languages, which he called the extended pattern languages, are more suitable than the non-erasing pattern languages for applications to automatic data entry systems. In this thesis, extended pattern languages will be called erasing pattern languages. Erasing pattern languages have also been applied in bioinformatics to identify transmembrane domains [5]. The erasing pattern language generated by a pattern \( \pi \) is defined similarly to its non-erasing counterpart, the only difference being that variables may be replaced with the empty string.

For example, consider the class of protein sequences, which are sequences over a 20-letter alphabet \( \Sigma_p = \{ A, C, D, E, F, G, H, I, K, L, M, N, P, Q, R, S, T, V, W, Y \} \). Let \( X = \{ x_1, x_2, x_3, \ldots \} \). An example of a pattern over \( \Sigma_p \cup X \) is \( \pi = D - x_1 - T - A - G - Q - E - x_1 - L - V - G - x_2 - N - K - x_3 \). The sequence \( D - A - T - A - G - Q - E - A - L - V - G - A - C - N - K - S - T - V \) belongs to the non-erasing pattern language generated by \( \pi \), obtained by substituting \( A \) for \( x_1 \), \( A - C \) for \( x_2 \), and \( S - T - V \) for \( x_3 \). The sequence \( D - A - T - A - G - Q - E - A - L - V - G - P - Q - N - K \) belongs to the erasing pattern language generated by \( \pi \), derived from \( \pi \) by replacing \( x_1 \) with \( A \), \( x_2 \) with \( P - Q \), and \( x_3 \) with the empty string.

In the field of automatic pattern discovery, the principal goal is to find patterns that match most (if not all) of the sequences in a given set. This task is closely related to the consistency problem for patterns [32]: given a set of examples \( S \subseteq \Sigma^* \times \{ +, - \} \), is there a non-erasing (resp. erasing) pattern language \( L \) that is consistent with \( S \), in the sense that \( x \in L \) for all \( (x,+) \in S \) and \( y \notin L \) for all \( (y,-) \in S \)? The general question taken up in the present work may be viewed as a partial converse of the latter two problems: rather than finding patterns that best fit a set of data, we
aim to find a set of sequences that best describes a given pattern. Our results on
this question complement prior work on the learnability of non-erasing as well as
erasing pattern languages [3, 5, 14, 33, 36, 15], and they may shed new light on
the structural properties of pattern languages since most of our proofs establishing
the teaching complexity of a given subclass $U$ of patterns proceed by effectively
constructing teaching sets for all patterns in $U$.

Figure 1.1 summarises the relationships between the various concept classes
studied in this work. For a fixed order $(a_1, \ldots, a_m)$ over a given finite alphabet
$\Sigma = \{a_1, \ldots, a_m\}$, the Parikh vector of a finite string $s$ over $\Sigma$ is the vector
$(q_1, \ldots, q_m) \in \mathbb{N}_0^m$, where $q_i$ is the number of occurrences of the symbol $a_i$ in $s$. For
any given pattern $\pi$, the Parikh mapping maps the pattern language $L(\pi)$ (either
erasing or non-erasing) over $\Sigma$ to the class of all vectors $w$ such that $w$ is the Parikh
vector of a member of $L(\pi)$.

1.3 Overview of Results

- Chapter 2

This chapter formally introduces the teaching models studied in the present
work. We review some known properties of the TD and RTD, and describe
a graph-theoretic characterisation of the RTD. This characterisation will be
especially useful in Chapter 3 for proving upper bounds on the RTD of various
concept classes. The properties of the RTD and its relation to other learning
parameters have been investigated thoroughly in the context of finite concept
classes [12]. We shall see that the TD and RTD (as well as a few variants
of these parameters) generally behave similarly for both finite and infinite concept classes. However, there are peculiarities of the RTD for finite concept classes that do not carry over to the infinite case; moreover, even when a property of the RTD holds for both finite and infinite concept classes, it may require a more general proof for infinite classes than for finite classes.

- Chapter 3

It is shown that the minimum teaching set of any cofinite linear subset $L$ of $\mathbb{N}_0$ with constant 0 may be characterised by means of a partial order defined with respect to the set of periods of $L$. Further, we establish a connection between the RTD$^+$ (an analogue of the RTD when teaching sets are restricted to positive examples) and the notions of weak and strong spanning sets. In particular, we show that the RTD$^+$ of a concept class $C$ is upper bounded.
by the size of the largest minimum strong spanning set over all $L \in \mathcal{C}$, and lower bounded by the size of the largest minimum weak spanning set over all $L \in \mathcal{C}$. For intersection-closed concept classes such as the class of linear sets, weak spanning sets are also strong spanning sets and vice versa. We determine the exact values or almost matching upper and lower bounds of some teaching complexity parameters for various classes of linear sets. For many classes of linear sets considered in the present work, we observe that the RTD is substantially smaller than the TD.

- **Chapter 4**
  Partial results on the TD and the RTD of the class of pattern languages were obtained in Zeinab Mazadi’s thesis [30]. In particular, it was shown in [30] that the whole class of pattern languages has infinite TD over any alphabet, while its RTD is exactly 2 over any infinite alphabet. The main results in this chapter concern the TD of the class of regular pattern languages. We show that the TD of the class of regular patterns is exactly 5 over any infinite alphabet, and either 5 or 6 over any alphabet with at least 8 letters. Moreover, it will be shown that the RTD of the class of pattern languages over a unary alphabet is unbounded, and its RTD$^+$ over any finite alphabet with at least 2 letters is also unbounded.

- **Chapter 5**
  In the first part of this chapter, we study the finite distinguishability problem for the class of erasing pattern languages: given any pattern $\pi$, is there a finite set of labelled examples $T$ such that for all patterns $\pi'$, $\pi'$ is consistent with $T$ (that is, $x$ belongs to the erasing pattern language generated by $\pi'$ for all
\((x, +) \in T\) and \(y\) does not belong to the erasing pattern language generated by \(\pi'\) for all \((y, -) \in T\) if and only if \(\pi\) and \(\pi'\) generate the same erasing pattern language? It is shown that over unary alphabets and alphabets with at least 4 letters, all finitely distinguishable patterns obey a simple structure that can be effectively checked, and consequently the finite distinguishability problem is decidable for these alphabet sizes. We also give partial results on the finite distinguishability problem for alphabet sizes 2 and 3. Further, we make an explicit connection between the finite distinguishability problem and the problem of determining whether or not an erasing pattern language is regular (whose decidability is still open for alphabet sizes 2 and 3): all finitely distinguishable erasing pattern languages over any finite alphabet are regular.

In the second part of Chapter 5 we investigate the TD and RTD of two subclasses of erasing pattern languages: the 1-variable erasing pattern languages and the regular erasing pattern languages. Our results on the teaching complexity of the regular erasing pattern languages are very similar to those on the teaching complexity of the regular non-erasing pattern languages. On the other hand, we observe that the class of 1-variable erasing pattern languages is much harder to teach than the class of 1-variable non-erasing pattern languages.
Chapter 2

Teaching Models

In this chapter, we formally introduce two teaching models (as well as a few variants of these models) studied in the present work: the teaching dimension model and the recursive teaching model. We shall first review some basic notation. Additional notation and/or definitions will be introduced as needed.

The set of natural numbers will be denoted by $\mathbb{N}_0$, that is, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The set of positive natural numbers will be denoted by $\mathbb{N}$. For each integer $m \geq 1$, let $\mathbb{N}_0^m = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ ($m$ times). We shall regard $\mathbb{N}_0^m$ as a subset of the vector space of all $m$-tuples of rational numbers over the rational numbers. The set of rational numbers will be denoted by $\mathbb{Q}$, and the set of real numbers will be denoted by $\mathbb{R}$.

For each $r \in \mathbb{N}_0$, $[r]$ denotes $\{0, \ldots, r\}$. For each finite $P \subseteq \mathbb{N}_0$, $\langle P \rangle := \{q : q = n_1 p_1 + \cdots + n_k p_k, k \in \mathbb{N}, n_i \in \mathbb{N}_0, p_i \in P\}$. For brevity, we will write $\langle \{p_1, \ldots, p_k\}\rangle$ as $\langle p_1, \ldots, p_k \rangle$. For any $v = (a_1, \ldots, a_m) \in \mathbb{N}_0^m$, define $\|v\|_1 = \sum_{i=1}^{m} a_i$. The symbol 0 will denote the zero vector in $\mathbb{N}_0^m$ when there is no possibility of confusion. All
“numbers” in this work refer to natural numbers, unless otherwise stated.

Let \( A \) and \( B \) be any sets. \( |A| \) denotes the cardinality of \( A \). The power set of \( A \) will be denoted by \( \mathcal{P}(A) (= 2^A) \). \( \mathcal{P}(\mathbb{N}_0) \) denotes the power set of \( \mathbb{N}_0 \). \( (A \setminus B) \cup (B \setminus A) \) will be denoted by \( A \triangle B \). \( \mathbb{N}_0 \) denotes the power set of \( \mathbb{N}_0 \). \( \mathbb{N}_0 \) denotes the power set of \( \mathbb{N}_0 \). \( 1_A \) will denote the characteristic function of \( A \), that is, \( 1_A(x) = 1 \) if \( x \in A \), and \( 1_A(x) = 0 \) if \( x \notin A \). The set of all mappings from \( A \) to \( B \) will be denoted by \( B^A \). If \( A \) and \( B \) are subsets of \( \mathbb{N}_0 \), then we define \( A \times B := \{ (a, b) : a \in A \land b \in B \} \); further, \( A \oplus B \), the join of \( A \) and \( B \), is the set \( \{ 2x : x \in A \} \cup \{ 2y + 1 : y \in B \} \). Analogously, for any class \( \{ A_i : i \in \mathbb{N} \} \) of subsets of \( \mathbb{N}_0 \), \( \bigoplus_{i \in \mathbb{N}} A_i \) is the infinite join of the \( A_i \)’s. \( A^* \) denotes the set of all finite sequences of elements of \( A \). For any \( d \in \mathbb{N}_0 \) and \( A \subseteq \mathbb{N}_0 \), \( dA \) denotes the set \( \{ dx : x \in A \} \). For any function \( f \), \( \text{ran}(f) \) denotes the range of \( f \).

Next, we summarise some formal language notation. \( \Sigma \) denotes a nonempty and either finite or countably infinite set, called the alphabet. A language over \( \Sigma \) is a subset of \( \Sigma^* \), i.e., a set of finite words (simply called “words” or “strings” hereafter) formed over \( \Sigma \) (including the empty string \( \varepsilon \)). For any language \( L \subseteq \Sigma^* \), \( \overline{L} \) denotes \( \Sigma^* \setminus L \).

For any (finite or countably infinite) set \( \Gamma \) of symbols, \( \Gamma^* = \Gamma^* \setminus \{ \varepsilon \} \) is the set of nonempty words over \( \Gamma \). For any \( w \in \Gamma^+ \), \( |w| \) denotes the length of \( w \). For any \( p \in \{ 1, \ldots, |w| \} \), \( w[p] \) denotes the \( p \)th symbol of \( w \). Similarly, for any \( 1 \leq p \leq q \leq |w| \), \( w[p:q] \) denotes the substring \( w[p] \ldots w[q] \). For a given symbol \( a \) and any \( n \in \mathbb{N} \), \( a^n \) denotes the string equal to \( n \) concatenated copies of \( a \). Specifically, \( a^0 \) is the empty string. We often omit any reference to \( \Sigma \) (resp. \( \Gamma \) when the choice of alphabet is clear from the context. For any \( s \in \Sigma \) and any string \( w \), the term \( #(s)[w] \) denotes the total number of occurrences of the symbol \( s \) in the word \( w \).
A concept may be viewed as a function mapping a set known as an instance space (or universe) to a set of labels. Throughout this work we shall restrict ourselves to binary concepts, that is, concepts that are boolean valued functions. “Concepts” will hereafter always refer to binary concepts; furthermore, we shall always use the labels “+” and “−”. Let χ denote an instance space. Any concept C over χ is the characteristic function of a subset of χ (and vice versa); by abuse of notation, a concept and its associated subset of the instance space will often be denoted with the same symbol. A concept class over χ will be treated as a subset of ϕ(χ). The basic learning scenario is that a teacher provides a learner with a set T of examples labelled consistently with a target concept from a concept class, and based on T the learner must perfectly predict the label of any given instance.

2.1 The Teaching Dimension and Recursive Teaching Dimension

Let C be a family of subsets of χ. Let C ∈ C and T be a subset of χ × {+, −}. Furthermore, let T+ = {n: (n, +) ∈ T}, T− = {n: (n, −) ∈ T} and χ(T) = T+ ∪ T−. A subset C of χ is said to be consistent with T iff T+ ⊆ C and T− ∩ C = ∅. T is a teaching set for C with respect to C iff T is consistent with C and for all C′ ∈ C \ {C}, T is not consistent with C′. If T is a teaching set for C with respect to C, then χ(T) is known as a distinguishing set for C with respect to C. Every element of χ × {+, −} is known as a labelled example.

1A detailed study of multi-label concept learning can be found in Samei’s doctoral thesis [40].
Definition 2.1.1 Let $C$ be any family of subsets of $\chi$. Let $C \in C$ be given. The size of a smallest teaching set for $C$ with respect to $C$ is called the teaching dimension of $C$ with respect to $C$, denoted by $TD(C, C)$. The teaching dimension of $C$ is defined as $\sup \{ TD(C, C) : C \in C \}$ and is denoted by $TD(C)$. If there is a teaching set for $C$ with respect to $C$ that consists of only positive examples, then the positive teaching dimension of $C$ with respect to $C$ is defined to be the smallest possible size of such a set, and is denoted by $TD^+(C, C)$. If there is no teaching set for $C$ with respect to $C$ that consists of only positive examples, then $TD^+(C, C)$ is defined to be $\infty$. A teaching set for $C$ with respect to $C$ that consists of only positive examples is known as a positive teaching set for $C$ with respect to $C$. The positive teaching dimension of $C$, denoted by $TD^+(C)$, is defined as $\sup \{ TD^+(C, C) : C \in C \}$.

The following classical example will illustrate some of the definitions in this chapter.

Example 2.1.2 For each $n \geq 0$, let $C_n = \{\emptyset\} \cup \{\{x\} : 0 \leq x \leq n\}$ be a family of sets over the instance space $\mathbb{N}_0$. Then for each $x \in [n]$, $\{(x, +)\}$ is a teaching set for $\{x\}$ w.r.t. $C_n$ and $TD(\{x\}, C_n) = TD^+(\{x\}, C_n) = 1$. Further, $\{(x, -) : 0 \leq x \leq n\}$ is a teaching set for $\emptyset$ w.r.t. $C_n$, $TD(\emptyset, C_n) = TD(C_n) = n + 1$ and $TD^+(\emptyset, C_n) = TD^+(C_n) = \infty$. See Table 2.1 for an illustration of $C_3$. To see that $TD(\emptyset, C_3) \geq 4$ (in other words, $\emptyset$ cannot be distinguished from every other concept in $C_3$ using at most 3 labelled examples), one observes from Table 2.1 that for any $\{x_1, x_2, x_3\} \subseteq \mathbb{N}_0$ and any $y \in \{0, 1, 2, 3\} \setminus \{x_1, x_2, x_3\}$, the sample $S = \{(x_1, -), (x_2, -), (x_3, -)\}$ is consistent with both $\emptyset$ and $\{y\}$, and so $S$ cannot distinguish $\emptyset$ from $\{y\}$.

The next example is the infinite analogue of Example 2.1.2.
Table 2.1: The concept class $C_3$

<table>
<thead>
<tr>
<th>Concept/Instance</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>${0}$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>${1}$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>${2}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>${3}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Example 2.1.3 Let $C_\infty = \{\emptyset\} \cup \{\{x\} : x \in \mathbb{N}_0\}$ be a family of sets over the instance space $\mathbb{N}_0$. Then $TD(C_\infty) = TD^+(C_\infty) = \infty$. To see this, one observes that for any $\{x_1, \ldots, x_k\} \subseteq \mathbb{N}_0$ and any $y \in \mathbb{N}_0 \setminus \{x_1, \ldots, x_k\}$, the sample $S = \{(x_i, -) : 1 \leq i \leq k\}$ is consistent with both $\emptyset$ and $\{y\}$, and so $S$ cannot distinguish $\emptyset$ from $\{y\}$.

We will often consider the smallest possible TD of a concept in any given concept class $C$; this number will be denoted by $TD_{\min}(C)$, that is, $TD_{\min}(C) := \inf\{TD(C, C \setminus \bigcup_{0 \leq j < i} F_j) : C \in F_i\}$. $TD^+_{\min}$ is defined analogously.

Another complexity parameter recently studied in computational learning theory is the recursive teaching dimension. The recursive teaching dimension is typically defined as the maximum size of teaching sets in a series of nested subfamilies of the family.

Definition 2.1.4 (Based on [48, 52]) Let $C$ be any family of subsets of $\chi$. A teaching sequence for $C$ is any sequence $TS = ((F_0, d_0), (F_1, d_1), \ldots)$ where (i) the families $F_i$ form a partition of $C$ with each $F_i$ nonempty, and (ii) $d_i = \sup\{TD(C, C \setminus \bigcup_{0 \leq j < i} F_j) : C \in F_i\}$ for all $i$. $\sup\{d_i : i \in \mathbb{N}_0\}$ is called the order of $TS$, and is denoted by $\text{ord}(TS)$. The recursive teaching dimension of $C$ is defined as $\inf\{\text{ord}(TS) : TS$ is a teaching sequence for $C\}$ and is denoted by $RTD(C)$. Given
any $C \in F_i$, a recursive teaching set for $C$ w.r.t. $TS$ is any teaching set for $C$ w.r.t. $\bigcup_{j \geq i} F_j$.

**Example 2.1.5** Reverting to Examples 2.1.2 and 2.1.3, one sees that for every $n \geq 0$, $RTD(C_\infty) = RTD(C_n) = 1$ ($RTD(C_\infty) \geq 1$ and $RTD(C_n) \geq 1$ follows from the fact that $C_\infty$ and $C_n$ each contains at least 2 elements). We know from Example 2.1.2 that for all $x \in N_0$, $TD(\{x\}, C_\infty) = 1$ and $TD(\emptyset, C_\infty \setminus \{\{x\} : x \in N_0\}) = 0$ (since the only member of $C_\infty \setminus \{\{x\} : x \in N_0\}$ is the empty set). Hence $((\{\{x\} : x \in N_0\}, 1), (\{\emptyset\}, 0))$ is a teaching sequence for $C_\infty$ of order 1. Similarly, $((\{\{x\} : 0 \leq x \leq n\}, 1), (\{\emptyset\}, 0))$ is a teaching sequence for $C_n$ of order 1.

**Remark 2.1.6** In Definition 2.1.4, one could define a teaching sequence for $C$ to be any sequence $TS = ((F_0, d_0), (F_1, d_1), \ldots)$ where (i) the families $F_i$ form a partition of $C$ with each $F_i$ nonempty, and (ii) $d_i = TD(C, C \setminus \bigcup_{0 \leq j < i} F_j)$ for all $C \in F_i$ and all $i$. Note that a teaching sequence for $C$ in the latter sense can be derived from a teaching sequence for $C$ in the sense of Definition 2.1.4. Given any teaching sequence $((F_0, d_0), (F_1, d_1), \ldots)$ satisfying the conditions in Definition 2.1.4, the idea is to split each $F_i$ into a finite teaching subsequence by first teaching the subclass $F_i^0$ of $F_i$ comprising all $C \in F_i$ with $TD(C, C \setminus \bigcup_{j < i} F_j) = d_i$; any remaining member of $F_i$ must have $TD$ (w.r.t. $C \setminus \bigcup_{j < i} F_i \setminus F_i^0$) strictly less than $d_i$ (otherwise this member would belong to $F_i^0$). Continuing this process inductively, all the members of $F_i$ will eventually be taught in a finite teaching subsequence because the teaching dimensions of $F_i^0, F_i^1, F_i^2, \ldots$ (w.r.t. the relevant subclasses of $C$) are strictly decreasing. For the remainder of this work, we shall adhere to the definition of a teaching sequence in Definition 2.1.4.
In teaching a concept class $\mathcal{C}$, a teacher has in general more than one choice of a teaching sequence for $\mathcal{C}$ whose order equals RTD($\mathcal{C}$). However, there is a canonical choice of a teaching sequence that always has minimal order over all teaching sequences for $\mathcal{C}$. Define the canonical teaching sequence for $\mathcal{C}$ to be the sequence $R_{\text{can}} = ((\mathcal{C}_0, d_0), (\mathcal{C}_1, d_1), \ldots)$ such that for all $i$,

$$
\mathcal{C}_i = \{C \in \mathcal{C} \setminus \bigcup_{j<i} \mathcal{C}_j : \text{TD}(C, \mathcal{C} \setminus \bigcup_{j<i} \mathcal{C}_j) \leq \text{RTD}(\mathcal{C})\}.
$$

Observe that ord($R_{\text{can}}$) = RTD($\mathcal{C}$). Given any $C \in \mathcal{C}$, the teacher uses the canonical teaching sequence $R_{\text{can}}$ for $\mathcal{C}$ to determine the unique index $i$ with $C \in \mathcal{C}_i$. The teacher then presents the examples in a minimal teaching set for $C$ w.r.t. $\mathcal{C} \setminus \bigcup_{j<i} \mathcal{C}_j$ to the learner, who in turn will use the canonical teaching sequence for $\mathcal{C}$ to determine the target concept from the sample provided by the teacher.

An alternative definition of the canonical teaching sequence for $\mathcal{C}$ may be derived by assigning a canonical priority level, represented by a natural number, to every $C \in \mathcal{C}$. As we shall see in Proposition 2.3.2, this alternative definition is occasionally quite convenient for proving upper bounds on the RTD of a concept class. First, one defines the notion of the recursive teaching dimension of a concept $C$ w.r.t. $\mathcal{C}$ (denoted by RTD($\mathcal{C}, C$)), where $\mathcal{C}$ is any concept class containing $C$. A priority assignment on a subclass $\mathcal{C}' \subseteq \mathcal{C}$ is any function $F : \mathcal{C}' \mapsto \mathbb{N}_0$. Note that for any concept class $\mathcal{C}$,

$$
\text{RTD}(\mathcal{C}) = \inf \{\sup \{\text{TD}(C, \{C' \in \mathcal{C} : F(C') \geq F(C)\}) : C \in \mathcal{C} \} : F \in \mathbb{N}_0^{\mathcal{C}}\}. \quad (2.1)
$$

For any subclass $\mathcal{C}_0$ of $\mathcal{C}$ and any priority assignment $F$ on $\mathcal{C}_0$, the order of $(F, \mathcal{C}_0)$
with respect to $C$ \((\text{ord}(F, C_0, C))\) is defined to be

$$\sup\{\text{TD}(C', \{C'' : C'' \notin C_0 \lor (C'' \in C_0 \land F(C'') \geq F(C'))\}) : C' \in C_0\}.$$ 

\(\text{RTD}(C, C)\) is then defined as \(\inf\{\text{ord}(F_0, C_1, C) : C \in C_1 \subseteq C \land F_0 \in \mathbb{N}_{C_1^0}\}\). It is readily verified that \(\text{RTD}(C) = \sup\{\text{RTD}(C, C') : C \in C\}\).

We now let \(G_{\text{can}}\) be the priority assignment defined as follows: for all \(C \in C\),

$$G_{\text{can}}(C) := \min\{F_1(C) : (\exists C' \in C)(\exists C_2 \subseteq C)(\exists F_1 \in \mathbb{N}_{C_2^0})\{C', C\} \subseteq C_2$$

$$\land \text{ord}(F_1, C_2, C) = \text{RTD}(C', C)\}.$$ 

We shall show that among all priority assignments \(G\) defined on \(C\), \(G_{\text{can}}\) minimises the value of \(\sup\{\text{TD}(C, \{C' : G_{\text{can}}(C') \geq G(C)\}) : C \in C\}\). Suppose for the sake of a contradiction that there exists a priority assignment \(G'\) such that for some \(C^* \in C\),

$$\text{TD}(C^*, \{C' : G_{\text{can}}(C') \geq G_{\text{can}}(C^*)\}) > \sup\{\text{TD}(C', \{C'' : G'(C'') \geq G'(C^*)\}) \land G'(C^*) = G_{\text{can}}(C^*)\}.$$ (2.2) 

By the construction of \(G_{\text{can}}\), one can find some \(C \in C\), a class \(C_0 \subseteq C\) with \(\{C, C^*\} \subseteq C_0\), and an assignment \(F\) on \(C_0\) such that

$$\text{RTD}(C, C) \geq \text{TD}(C^*, C \setminus \{C' \in C_0 : F(C') < F(C^*)\}) \land F(C^*) = G_{\text{can}}(C^*).$$ (2.3) 

From the definition of \(G_{\text{can}}\), it follows that for all \(C' \in C_0\), \(G_{\text{can}}(C') \leq F(C')\).
Consequently, by (2.2) and (2.3), as well as by the definition of $\text{RTD}(C, C)$,

$$\text{RTD}(C, C) \leq \sup \{ \text{TD}(C', \{ C'' \in C : G'(C'') \geq G'(C') \}) : C' \in C \}$$

$$< \text{TD}(C^*, \{ C' \in C : G_{\text{can}}(C') \geq G_{\text{can}}(C^*) \})$$

$$= \text{TD}(C^*, C \setminus \{ C' \in C : G_{\text{can}}(C') < G_{\text{can}}(C^*) \})$$

$$\leq \text{TD}(C^*, C \setminus \{ C' \in C_0 : G_{\text{can}}(C') < G_{\text{can}}(C^*) \})$$

$$\leq \text{TD}(C^*, C \setminus \{ C' \in C_0 : F(C') < F(C^*) \})$$

$$\leq \text{RTD}(C, C),$$

a contradiction.

The priority assignment $G_{\text{can}}$ induces a teaching sequence on $C$, namely the sequence whose $i^{th}$ family is $\{ C \in C : G_{\text{can}}(C) = i \}$; this teaching sequence in fact coincides with $R_{\text{can}}$.

One can also restrict the instances of the teaching sets in a teaching sequence to positive examples; the best possible order of such a teaching sequence will be denoted by $\text{RTD}^+$. Any teaching set that does not contain negatively labelled examples will be called a positive teaching set. As we shall see in the forthcoming chapters, positive teaching sets are often easier to analyse and characterise.

**Definition 2.1.7** Let $C$ be any family of subsets of $\mathbb{N}_0$. A teaching sequence with positive examples for $C$ (or positive teaching sequence for $C$) is any sequence $TS = ((F_0, d_0), (F_1, d_1), \ldots)$ such that (i) the families $F_i$ form a partition of $C$ with each $F_i$ nonempty, and (ii) for all $i$ and all $C \in F_i$, there is a subset $S_C \subseteq C$ with $|S_C| = d_i < \infty$ such that for all $C' \in \bigcup_{i \geq 1} F_j$, it holds that $S_C \subseteq C' \Rightarrow C = C'$.
sup\{d_i : i \in \mathbb{N}_0\} is called the order of TS, and is denoted by ord(TS). If \(C\) has at least one teaching sequence with positive examples, then the positive recursive teaching dimension of \(C\) is defined as inf\{ord(TS) : TS is a teaching sequence with positive examples for \(C\)\} and is denoted by \(RTD^+(C)\). If \(C\) does not have any teaching sequence with positive examples, define \(RTD^+(C) = \infty\).

A teaching plan for \(C\) is a teaching sequence \(((\mathcal{F}_0, d_0), (\mathcal{F}_1, d_1), \ldots)\) for \(C\) such that \(|\mathcal{F}_i| = 1\) for each \(i\). A teaching plan \(((\{C_0\}, d_0), (\{C_1\}, d_1), (\{C_2\}, d_2), \ldots)\) for \(C\) will often be written as \(((C_0, S_0), (C_1, S_1), (C_2, S_2), \ldots)\), where \(S_i\) is a teaching set for \(C_i\) with respect to \(C \setminus \{C_j : 0 \leq j < i\}\). If \(C\) has at least one teaching plan, then \(RTD_1(C)\) is defined to be inf\{ord(TS) : TS is a teaching plan for \(C\)\}. If \(C\) does not have any teaching plan, define \(RTD_1(C) = \infty\).

A teaching plan with positive examples for \(C\) (or positive teaching plan for \(C\)) is defined analogously. If \(C\) has at least one teaching plan with positive examples, then \(RTD_1^+(C)\) is defined to be inf\{ord(TS) : TS is a teaching plan with positive examples for \(C\)\}. If \(C\) does not have any teaching plan with positive examples, define \(RTD_1^+(C) = \infty\).

**Remark 2.1.8**

1. The TD and recursive teaching models are collusion-free in the following sense defined by Goldman and Mathias [21]: for every concept class \(C\) and any target concept \(C\) in \(C\), the teacher selects a sample \(S\) labelled consistently with \(C\), and on any input \(S'\) of examples labelled consistently with \(C\) such that \(S'\) contains \(S\), the learner will return a hypothesis representing \(C\).

2. If \(TD(C, C) = \infty\) (resp. \(TD^+(C, C) = \infty\)) for some \(C \in C\), then \(RTD_1(C) = RTD_1^+(C) = \infty\) (resp. \(RTD_1^+(C) = \infty\)) Furthermore, if \(C\) is uncountable,
then there is no teaching plan or positive teaching plan for $\mathcal{C}$, and therefore

$$RTD_1(\mathcal{C}) = RTD_1^+(\mathcal{C}) = \infty.$$  

**Example 2.1.9** For each $n \geq 0$, define $\mathcal{C}_n$ and $\mathcal{C}_\infty$ as in Examples 2.1.2 and 2.1.3. Notice from Examples 2.1.2 and 2.1.5 that both $\mathcal{C}_n$ and $\mathcal{C}_\infty$ have positive teaching sequences of order 1; thus $RTD^+(\mathcal{C}_n) = RTD^+(\mathcal{C}_\infty) = 1$. Further, note that any teaching set for $\emptyset$ w.r.t. $\mathcal{C}_\infty$ must be a subset of $N = \{(x, -) : x \in \mathbb{N}_0\}$. Since any finite subset of $N$ is consistent with some concept $\{y\} \in \mathcal{C}_\infty$, $TD(\emptyset, \mathcal{C}_\infty) = \infty$. Hence by Remark 2.1.8.2, $RTD_1(\mathcal{C}_\infty) = RTD_1^+(\mathcal{C}_\infty) = \infty$. On the other hand, $((\{0\}, 1), (\{1\}, 1), \ldots, (\{n\}, 1), (\emptyset, 0))$ is a positive teaching plan for $\mathcal{C}_n$ of order 1, and therefore $RTD_1(\mathcal{C}_n) = RTD_1^+(\mathcal{C}_n) = 1$.

### 2.2 A Graph-Theoretic Characterisation of Teaching

To streamline the proofs of some results in the present work, particularly for results on upper bounds for the RTD and its variants, it will be useful to adopt a graph-theoretic formulation of the teaching complexity measures introduced so far. We will show that the RTD has a rather visually intuitive graph-theoretic interpretation.

Given a family $\mathcal{C}$ of sets over an instance space, let $T$ be a mapping $C \mapsto T(C)$ that assigns a set $T(C)$ of labelled examples to every concept $C \in \mathcal{C}$ such that the labels in $T(C)$ are consistent with $C$. $T$ is said to be positive iff for all $C \in \mathcal{C}$, $T(C)$ does not contain any negative examples. Define the *digraph induced by* $T$ as the graph $G = (V_G, A_G)$, where the nodes of $G$ are identified with the members $C$ of $\mathcal{C}$, i.e., $V_G = \mathcal{C}$, and a pair $(C', C) \in \mathcal{C} \times \mathcal{C}$ is included in $A_G$ iff $C'$ is consistent with $T(C)$. The *order* of $T$, denoted by $\text{ord}(T)$, is defined as $\sup_{C \in \mathcal{C}} |T(C)|$ (possibly $\infty$).
Define the depth of a node \( v \) in \( G \) as the length of the longest path ending in \( v \) (or as \( \infty \) if the paths ending in \( v \) can become arbitrarily long). The weight of a node \( v \) in \( G \) is defined as the number of vertices \( w \in V_G \) such that there exists a path from \( w \) to \( v \) in \( G \). \( T \) is said to satisfy the finite-depth condition (resp. the finite-weight condition) if every node \( v \) in \( G \) has finite depth (resp. finite weight).

Say that the mapping \( C \mapsto T(C) \) with \( C \) ranging over all members of \( C \) and the labels of \( T(C) \) consistent with \( C \) is RTD-admissible for \( C \) iff \( |T(C)| < \infty \) for all \( C \in C \), the digraph \( G \) induced by \( T \) is acyclic, and \( T \) satisfies the finite depth condition; say that \( T \) is RTD\(_1\)-admissible for \( C \) iff \( T \) is RTD-admissible and \( T \) satisfies the finite-weight condition. \( T \) is RTD\(_+\)-admissible for \( C \) (resp. RTD\(_1\)+-admissible for \( C \)) iff \( T \) is RTD-admissible for \( C \) (resp. RTD\(_1\)-admissible for \( C \)) and for all \( C \in C \), \( T(C) \) consists of only positive examples. If \( T \) is RTD-admissible for \( C \), then it will often simply be called RTD-admissible if \( C \) is clear from the context.

**Example 2.2.1** Define \( C_3 \) as in Example 2.1.2. For each \( \{x\} \in C_3 \), set \( T(\{x\}) = \{(x,+)\} \) and \( T(\emptyset) = \emptyset \). The digraph induced by \( T \) is shown in Figure 2.1. The depth (resp. weight) of the node representing \( \emptyset \) is 1 (resp. 5), while the depth (resp. weight) of each node representing \( \{x\} \) for some \( x \in \{0,1,2,3\} \) is 0 (resp. 1). (Note that for any node \( v \), \( v \) itself is included in the computation of the weight of \( v \).) The order of \( T \) is 1. Further, \( T \) is K-admissible for \( C_3 \) whenever \( K \in \{RTD,RTD_1,RTD_+,RTD_1^+\} \).

**Example 2.2.2** Consider \( C_0 = \{\emptyset,\{0\}\} \). Set \( T_1(\emptyset) = \{(1,-)\} \) and \( T_1(\{0\}) = \{(2,-)\} \). Note that the digraph induced by \( T_1 \) is strongly connected and hence \( T_1 \) is

\footnote{Note that if \( T \) satisfies the finite-weight condition, then it also satisfies the finite-depth condition.}
not RTD-admissible for \( C_0 \).

Next, let \( I = \{[n] : n \in \mathbb{N}_0\} \). For each \( I_n := [n] \in I \), set \( T_2(I_n) = \{(n,+)\} \). Let \( G \) be the digraph induced by \( T_2 \), and for each \( n \in \mathbb{N}_0 \), let \( v_n \) denote the node of \( G \) representing \( I_n \). Then for all \( n \in \mathbb{N}_0 \) and \( \ell \in \mathbb{N}_0 \), \( v_n \leftarrow v_{n+1} \leftarrow v_{n+2} \leftarrow \ldots \leftarrow v_{n+\ell} \) is a path of length \( \ell \) ending in \( v_n \). Hence \( v_n \) has infinite depth for all \( n \in \mathbb{N}_0 \), which means that \( T \) is not RTD-admissible for \( I \).

See Figure 2.2 for an illustration of these two examples.

The first proposition of this section establishes a firm connection between RTD-admissibility and teaching sequences, as well as between RTD\(_1\)-admissibility and teaching plans.

**Theorem 2.2.3** Given any family \( C \) over an instance space, let \( T \) be a mapping \( C \mapsto T(C) \) that assigns a set \( T(C) \) of labelled examples to every \( C \in C \), such that \( C \) is consistent with \( T(C) \).

(1) \( T \) is RTD-admissible iff there exists a partition of \( C \) into \( C_0, C_1, \ldots \) such that, for all \( i \) and all \( C \in C_i \), \( T(C) \) is a teaching set for \( C \) w.r.t. \( \bigcup_{j \geq i} C_j \).
Chapter 2. Teaching Models

The digraph induced by $T_1$

\[ \emptyset \xrightarrow{\{0\}} \]

The digraph induced by $T_1$

An infinite path ending in $v_0$

Figure 2.2: Two mappings that are not RTD-admissible

(ii) Let $\mathcal{C}$ be finite or countably infinite. $T$ is RTD$_1$-admissible iff there exists an ordering $C_1 < C_2 < \ldots$ of $\mathcal{C}$ such that for all $i$, $T(C_i)$ is a teaching set for $C_i$ w.r.t. $\{C_j : j \geq i\}$.

Proof. Assertion (I). Let $G = (V_G, A_G)$ be the digraph induced by $T$. Suppose that $T$ is RTD-admissible. For all $i$, define $C_i = \{v \in V_G : \text{depth}(v) = i\}$. Since every node of $G$ has a finite depth and $G$ is acyclic, $\{C_i : i \in \mathbb{N}_0\}$ is a partition of $V_G$. Furthermore, suppose for the sake of a contradiction that there are $i, j$ and $C_1 \in C_i, C_2 \in C_j$ such that $i \leq j$ and $C_2$ is consistent with $T(C_1)$. Then the depth of $C_1$ must be at least $j + 1 > i$ and so $C_1 \notin C_i$, a contradiction. Conversely, suppose that there exists a partition of $\mathcal{C}$ into $\mathcal{C}_0, \mathcal{C}_1, \ldots$ such that, for all $i$ and all $C \in \mathcal{C}_i$, it holds that $T(C)$ is a teaching set for $C$ w.r.t. $\bigcup_{j \geq i} C_j$. Note that for any distinct $v_1, v_2 \in V_G$ such that $v_1 \in C_i$ and $v_2 \in C_j$, if $(v_1, v_2) \in A_G$, so that $v_1$ is consistent with $T(v_2)$, then $i < j$. (This follows from the condition that $T(v_2)$ is a teaching set for $v_2$ w.r.t. $\bigcup_{l \geq j} C_l$.) This implies that $G$ cannot contain any cycle.
Now consider any \( v \in V_G \) such that \( v \in C_i \). By the above argument, for any \( v_1 \in C_{i_1} \) with \( (v_1, v) \in A_G, i_1 < i \). Applying this argument inductively, it follows that the length of any path ending in \( v \) is at most \( i \).

**Assertion (II).** Suppose there exists an ordering \( C_1 < C_2 < \ldots \) of \( C \) such that for all \( i \), it holds that \( T(C_i) \) is a teaching set for \( C_i \) w.r.t. \( \{C_j : j \geq i\} \). By the proof of Assertion (I), \( T \) is RTD-admissible. Furthermore, since \( T(C_i) \) is a teaching set for \( C_i \) w.r.t. almost all concepts in \( C \), there are at most finitely many concepts in \( C \) that are consistent with \( T(C_i) \), so that \( C_i \) has finite weight.

Conversely, suppose that \( T \) is RTD\(_1\)-admissible. Define the relation \( R_T = \{(C, C') \in C \times C : (C \neq C') \land C \text{ is consistent with } T(C')\} \), and let \( \text{trcl}(R_T) \) be the transitive closure of \( R_T \). It follows from the acyclicity of the digraph \( G = (V_G, A_G) \) induced by \( T \) that \( \text{trcl}(R_T) \) is asymmetric. Furthermore, by the finite-weight condition on \( G \), every element of \( C \) has only finitely many predecessors w.r.t. \( \text{trcl}(R_T) \). Hence \((C, \text{trcl}(R_T)) \) is a countable strict partial order on \( C \) such that every element of \( C \) has finitely many predecessors. By [16, Theorem 1.3(1)], one can extend \((C, \text{trcl}(R_T)) \) to a linear order \((C, \prec)\) such that every element of \( C \) has finitely many predecessors (w.r.t. \( \prec \)). In particular, there is a linear ordering \( C_1 \prec C_2 \prec \ldots \) of \( C \). Pick any \( i \in \mathbb{N} \). Since, for all \( C' \in C \) with \( C' \neq C_i \),

\[
C' \text{ is consistent with } T(C_i) \rightarrow (C', C_i) \in \text{trcl}(R_T) \rightarrow C' \prec C_i,
\]

it follows that \( T(C_i) \) is a teaching set for \( C_i \) w.r.t. \( \{C_j : j \geq i\} \).

**Corollary 2.2.4** For all finite or countably infinite families \( C \) of sets over an instance space, \( K(C) = \inf(\{\text{ord}(T) : T \text{ is } K\text{-admissible for } C \lor \text{ord}(T) = \infty\}) \), where
\(K \in \{\text{RTD}, \text{RTD}^+, \text{RTD}_1, \text{RTD}_1^+\}\).

Note that for any finite family \(C\) of sets over an instance space \(\chi\) and any mapping \(T : C \rightarrow \varphi(\chi \times \{+,-\})\) such that the labels in \(T(C)\) are consistent with \(C\) for all \(C \in C\), \(T\) is RTD\(_1\)-admissible (and hence also RTD-admissible). The next corollary is a consequence of this observation.

**Corollary 2.2.5** For any finite family \(C\) of sets over an instance space, \(\text{RTD}(C) = \text{RTD}_1(C)\).

### 2.3 Further Examples and Properties of Teaching

The following proposition sums up several useful properties of the TD and RTD as well as some of their variants. These assertions follow immediately from the definitions of the teaching parameters.

**Proposition 2.3.1** Let \(C\) be any concept class over an instance space. Then the following hold.

1. \(\text{RTD}(C) \leq \text{TD}(C)\).
2. \(\text{RTD}^+(C) \leq \text{TD}^+(C)\).
3. (Monotonicity) For all \(C' \subseteq C\) and \(K \in \{\text{TD}, \text{TD}^+, \text{RTD}, \text{RTD}^+, \text{RTD}_1, \text{RTD}_1^+\}\), \(K(C') \leq K(C)\).
4. \(\text{TD}_{\text{min}}(C) \leq \text{RTD}(C)\) and \(\text{TD}_{\text{min}}^+(C) \leq \text{RTD}^+(C)\).
5. For all \(K \in \{\text{TD}, \text{RTD}, \text{RTD}_1\}\), \(K(C) \leq K^+(C)\).
Next, we examine a natural geometric concept class for which the TD and RTD models yield distinct complexity values.

**Proposition 2.3.2** Let $\mathcal{F}_{\text{circ}}$ be the family of all circles in $\mathbb{R}^2$ with radius equal to $\frac{1}{n}$ for some $n \in \mathbb{N}_0$. Then $TD(\mathcal{F}_{\text{circ}}) = TD^+(\mathcal{F}_{\text{circ}}) = 3$ and $RTD(\mathcal{F}_{\text{circ}}) = RTD^+(\mathcal{F}_{\text{circ}}) = 2$.

**Proof.** Each circle in $\mathcal{F}_{\text{circ}}$ will be represented by a pair $(x, r)$ for some $x \in \mathbb{R}^2$ and $r \in \mathbb{N}$, where $x$ is the centre of the circle and $\frac{1}{r}$ is its radius. For any given circle $C$ in $\mathcal{F}_{\text{circ}}$, pick three noncollinear points $p_1, p_2$ and $p_3$ on $C$ (see Figure 2.3). We compute $TD(C, \mathcal{F}_{\text{circ}})$. According to Euclidean geometry, $C$ is the unique circle passing through $p_1, p_2$ and $p_3$, and therefore $\{(p_1, +), (p_2, +), (p_3, +)\}$ is a teaching set for $C$ with respect to $\mathcal{F}_{\text{circ}}$. Moreover, given any $C'$ in $\mathcal{F}_{\text{circ}}$ of radius no more than $\frac{1}{2}$ and any sample $S$ of size at most two that is labelled consistently with $C'$, there is some $C''$ in $\mathcal{F}_{\text{circ}}$ that is consistent with $S$ and distinct from $C'$: if $S$ contains two positively labelled examples, then $S$ is consistent with a circle of radius 1; if $S$ contains at least one negatively labelled example, then $S$ is consistent with infinitely many circles in $\mathcal{F}_{\text{circ}}$ with radius smaller than that of $C'$. Consequently, $TD(C, \mathcal{F}_{\text{circ}}) = 3$. 

Figure 2.3: A circle in $\mathcal{F}_{\text{circ}}$
To see that $\text{RTD}(\mathcal{F}_{\text{circ}}) \leq 2$, we apply Equation (2.1), assigning the priority level $n - 1$ to all $C$ in $\mathcal{F}_{\text{circ}}$ with radius $\frac{1}{n}$, where $n \in \mathbb{N}_0$. Also, $\text{RTD}(\mathcal{F}_{\text{circ}}) \geq 2$ follows from Proposition 2.3.1(IV) and the fact that $\text{TD}(C, \mathcal{F}_{\text{circ}}) \geq 2$ for all $C \in \mathcal{F}_{\text{circ}}$. The values of $\text{TD}^+(\mathcal{F}_{\text{circ}})$ and $\text{RTD}^+(\mathcal{F}_{\text{circ}})$ can be similarly computed.

The next proposition provides a necessary condition for any family to have a teaching sequence with positive examples. This condition will be used later in Chapter 3 to establish the non-existence of positive teaching sequences for some families of linear sets.

**Proposition 2.3.3** Let $\mathcal{C}$ be a family of subsets of $\chi$ that has at least one positive teaching sequence. Let $C \in \mathcal{C}$. Then there does not exist any infinite descending chain $H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots$ such that $\{H_0, H_1, H_2, \ldots\} \subseteq \mathcal{C}$ and $C \subseteq H_i$ for each $i$.

**Proof.** Suppose there is some $C \in \mathcal{C}$ for which there exists an infinite descending chain $H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots$ with $\{H_0, H_1, H_2, \ldots\} \subseteq \mathcal{C}$ and $C \subseteq H_i$ for each $i$. Assume by way of a contradiction that $((C_0, d_0), (C_1, d_1), \ldots)$ were a positive teaching sequence for $\mathcal{C}$. Suppose $C \in \mathcal{C}_i$ for some $i$. Note that for all $j \in \{0, \ldots, i\}$, $C \subseteq H_j$ implies that $H_j \in \mathcal{C}_{k_j}$ for some $k_j < i$. Further, for all $j \in \{0, \ldots, i - 1\}$, since $H_{j+1} \subseteq H_j$, it must hold that $k_j < k_{j+1}$. This contradicts the fact that $0 \leq k_j < i$ for all $j \in \{0, \ldots, i\}$.

Doliwa, Fan, Simon and Zilles [12] showed that the problem of determining the RTD of any finite concept class $\mathcal{C}$ can be essentially reduced to that of computing the $\text{TD}_{\min}$ of every subclass of $\mathcal{C}$.

**Proposition 2.3.4** For any finite concept class $\mathcal{C}$, $\text{RTD}(\mathcal{C}) = \max_{C' \subseteq \mathcal{C}} \text{TD}_{\min}(C')$. 
For *infinite* concept classes, on the other hand, a weaker relation holds:

**Proposition 2.3.5** For any concept class $C$, $\text{RTD}(C) \geq \sup \{ \text{TD}_{\text{min}}(C') : C' \subseteq C \}$.

The following example shows that for infinite classes, the gap between $\text{RTD}(C)$ and $\sup \{ \text{TD}_{\text{min}}(C') : C' \subseteq C \}$ can be arbitrarily large.

**Theorem 2.3.6** For each $k \geq 1$, fix a mapping $f_k : \mathbb{N} \rightarrow \wp([k])$ such that

1. For each $n \in \mathbb{N}$, $f_k(n)$ is a proper subset of $[k]$.
2. For each $D \subseteq [k]$ and each $m \in \mathbb{N}$, there exists a set $\{a_1, a_2, \ldots, a_m\}$ of $m$ distinct primes such that for each nonempty $F \subseteq \{a_1, a_2, \ldots, a_m\}$, $f_k(\prod_{x \in F} x) = D$.

For each $n \in \mathbb{N}$, set $L_{k,n} = f_k(n) \oplus \langle n \rangle$ and $H_k = [k] \oplus \{0\}$. Set $C = \{L_{k,n}\}_{n \in \mathbb{N}} \cup \{H_k\}$. Then $\text{RTD}(C) = k + 1$ and for all $C' \subseteq C$, $\text{TD}_{\text{min}}(C') \leq 1$.

**Proof.** We first show that for all $C' \subseteq C$, $\text{TD}_{\text{min}}(C') \leq 1$. We proceed with a case distinction.

**Case 1:** $C'$ contains some set $L_{k,n'}$. Let $n$ be the least $n'$ such that $L_{k,n'} \in C'$. Then $\{(2n + 1, +)\}$ is a teaching set for $L_{k,n}$ w.r.t. $C'$.

**Case 2:** $C' \subseteq \{H_k\}$. Then $\text{TD}_{\text{min}}(C') = 0$.

It remains to show that $\text{RTD}(C) = k + 1$. Note that $((\{H_k\}, d_0), (\{L_{k,1}\}, d_1), (\{L_{k,2}\}, d_2), \ldots)$, where $L_{k,n}$ is removed at the $(n + 1)^{\text{st}}$ stage for $n \geq 1$, is a teaching
sequence for $C$: $H_k$ can be taught with $\{(2x,+) : x \in [k]\}$ and $L_{k,n}$ can be taught in this sequence with $\{(2n+1,+)\}$. Thus $\text{RTD}(C) \leq k + 1$.

Now assume by way of a contradiction that $P = ((L_0,d_0),(L_1,d_1),\ldots)$ were a teaching sequence for $C$ such that $\text{ord}(P) \leq k$. Let $H_k \in L_i$ and $T$ be any minimal recursive teaching set for $H_k$ w.r.t. $P$. Since $d_i \leq k$, $T^+ \cup T^-$ contains at most $k$ even elements. Choose $m > \max\{i+1,k\}$ so that $m > \max\{x : 2x + 1 \in T^+ \cup T^-\}$. By the definition of $f_k$, there is an $n$ such that $n = a_1a_2\ldots a_{m+1}$ for some $(m+1)$ prime factors $a_1, a_2, \ldots, a_{m+1}$ (which implies that $n > m$), and $f_k(\prod_{x \in F} x) = \{x : 2x \in T^+\}$ for all nonempty $F \subseteq \{a_1, a_2, \ldots, a_{m+1}\}$. Then $L_{k,n} = f_k(n) \oplus \langle n \rangle \in L_{i_1}$ for some $i_1 < i$. Note the following:

(1) Since the teaching dimension of $L_{k,n}$ w.r.t. the class of all $L_{k,n'}$ such that $f_k(n') = f_k(n)$ and $n'$ is a product of at least $m$ distinct members of $\{a_1, a_2, \ldots, a_{m+1}\}$ equals $m$ ($> k$), there exists some $n''$ for which $n''$ is a product of $m$ distinct members of $\{a_1, a_2, \ldots, a_{m+1}\}$ and $L_{k,n''} = f_k(n) \oplus \langle n'' \rangle \in L_{i_2}$ for some $i_2 < i_1$.

By applying a line of reasoning similar to (1), one obtains from $L_{k,n''}$ some $n'''$ such that $n'''$ is a product of $(m-1)$ distinct members of $\{a_1, a_2, \ldots, a_{m+1}\}$ and $L_{k,n'''} = f_k(n) \oplus \langle n''' \rangle \in L_{i_3}$ for some $i_3 < i_2$. Repeating a similar argument $m$ times yields a descending sequence $i_1 > i_2 > \ldots > i_{m+1} \geq 0$, which is impossible as $m > i + 1$.

Remark 2.3.7 For all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, set $L_{k,n} = \bigoplus_{i < 2k} \emptyset \oplus f_k(n) \oplus \langle n \rangle \oplus \bigoplus_{i > 2k+1} \emptyset$ and $H_k = \bigoplus_{i < 2k} \emptyset \oplus [k] \oplus \{0\} \oplus \bigoplus_{i > 2k+1} \emptyset$, where $f_k$ is defined as in Theorem 2.3.6. Set $\mathcal{L} = \{L_{k,n}\}_{k \in \mathbb{N}_0,n \in \mathbb{N}} \cup \{H_k\}_{k \in \mathbb{N}_0}$. One can show, as in the proof
of Theorem 2.3.6 that $\text{RTD}(\mathcal{L}) = \infty$ and $\text{TD}_{\min}(\mathcal{L}') = 1$ for all $\mathcal{L}' \subseteq \mathcal{L}$. We omit the details.

A family $\mathcal{C}$ of subsets of $\chi$ is said to have finite thickness \[3\] iff for every $x \in \chi$, the class $\{C \in \mathcal{C} : x \in C\}$ is finite. Note that finite thickness is a sufficient condition for families that do not contain the empty set to have a teaching plan with positive examples.

**Proposition 2.3.8** Let $\mathcal{C}$ be any family of subsets of a countable set $\chi$ such that $\mathcal{C}$ has finite thickness and $\emptyset \notin \mathcal{C}$. There is a mapping $C \mapsto T(C)$ that assigns a set of positive examples to every $C \in \mathcal{C}$ such that $T$ is $\text{RTD}_1^+$-admissible; furthermore, $\text{RTD}_1^+(\mathcal{C}) = \text{RTD}^+(\mathcal{C})$.

**Proof.** Let $\mathcal{C}$ be any family of subsets of $\chi$ such that $\emptyset \notin \mathcal{C}$ and $\mathcal{C}$ has the finite thickness property. $\text{RTD}_1^+(\mathcal{C}) \geq \text{RTD}^+(\mathcal{C})$ follows from Proposition \[2.3.1(V).

First, it is shown that there is a mapping $C \mapsto T(C)$ that is $\text{RTD}_1^+$-admissible. As $\mathcal{C}$ has finite thickness, it is at most countable and so there is a one-one enumeration $C_0, C_1, C_2, \ldots$ of $\mathcal{C}$. By the finite thickness of $\mathcal{C}$, there are only finitely many sets $C_0^0, \ldots, C_m^0 \in \mathcal{C}$ such that $\min(C_0) \in C_0^0 \cap \cdots \cap C_m^0$. (Note that $C_0 \neq \emptyset$ by hypothesis.) For each $i \in \{0, \ldots, m\}$, define $T(C_0^i) = \{(y, +) : y \in S_i\}$, where $S_i$ is the smallest subset of $C_0^i$ such that $\min(C_0) \in S_i$ and $S_i \not\subseteq C_j^i$ for all $C_j^i \not\subseteq C_0^i$. Inductively, assume that $T(C_{i_k})$ has been defined for all $k \leq l$. Define $i_{k+1}$ to be the least index such that $C_{i_{k+1}} \notin \{C_{i_0}, \ldots, C_{i_l}\}$, and for all $C_j$ with $\min(C_{i_{k+1}}) \in C_j$ such that $T(C_j)$ has not been defined yet, define $T(C_j)$ in a similar way to the case when $C_{i_{k+1}} = C_0$. Let $G = (V_G, A_G)$ be the digraph induced by $T$. To reduce clutter, any concept $C$ in $\mathcal{C}$ and the vertex of $G$ representing $C$ will be denoted by the
same symbol. Note that $T$ partitions $V_G$ into finite sets such that no edge connects any two vertices belonging to distinct partitions, and so $T$ satisfies the finite-weight condition. Furthermore, $G$ is acyclic as $(C, C') \in A_G$ only if $C$ is a proper superset of $C'$ and each partition induced by $T$ is finite.

It remains to show that $\text{RTD}_1^+(\mathcal{C}) \leq \text{RTD}^+(\mathcal{C})$. By the preceding argument, there is a mapping $C \mapsto T(C)$ for all $C \in \mathcal{C}$ such that $T$ is $\text{RTD}^+$-admissible. Let $T$ be an $\text{RTD}^+$-admissible mapping for $\mathcal{C}$ such that $\text{ord}(T) = \text{RTD}^+(\mathcal{C})$, and let $G = (V_G, A_G)$ be the digraph induced by $T$. If $\mathcal{C}$ is finite, then each $C \in V_G$ has only finitely many predecessors, implying that $T$ is also $\text{RTD}_1^+$-admissible (so that $\text{RTD}_1^+(\mathcal{C}) \leq \text{RTD}^+(\mathcal{C})$). Suppose that $\mathcal{C}$ were infinite. Note that $\text{ord}(T) \geq 1$, for otherwise the infinite digraph $G$ would be cyclic, contradicting the $\text{RTD}^+$-admissibility of $T$. Thus one can define a mapping $T'$ such that for all $C \in \mathcal{C}$,

$$T'(C) = \begin{cases} T(C) & \text{if } T(C) \neq \emptyset; \\ \{(\text{min}(C), +)\} & \text{if } T(C) = \emptyset. \end{cases}$$

Note that $\text{ord}(T') = \text{ord}(T) = \text{RTD}^+(\mathcal{C})$. Furthermore, $T'$ is $\text{RTD}^+$-admissible and by the finite thickness of $\mathcal{C}$, $T'$ satisfies the finite weight condition. Hence $T'$ is $\text{RTD}_1^+$-admissible and so $\text{RTD}_1^+(\mathcal{C}) \leq \text{ord}(T') = \text{RTD}^+(\mathcal{C})$. \[\square\]

**Example 2.3.9** Let $\mathcal{C} = \{\{x\} : x \in \mathbb{N}_0\}$ be a family of sets over the instance space $\mathbb{N}_0$. Note that $\mathcal{C}$ has finite thickness (as every $x \in \mathbb{N}_0$ belongs to exactly one singleton concept) and $\text{RTD}^+(\mathcal{C}) = 1$.

We introduce some additional notation for the next theorem. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concept classes over universes $\chi_1$ and $\chi_2$ respectively such that $\chi_1 \cap \chi_2 = \emptyset$. Define
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\[ C_1 \uplus C_2 := \{ C_1 \cup C_2 : C_1 \in C_1 \land C_2 \in C_2 \} \]. The additivity of the RTD with respect to \( \uplus \) (that is, \( \text{RTD}(C_1 \uplus C_2) = \text{RTD}(C_1) + \text{RTD}(C_2) \)) for finite concept classes was established by Doliwa, Fan, Simon and Zilles [12]. It will be shown here that a similar result holds for infinite concept classes.

**Definition 2.3.10** Let \( \mathcal{C} \) be any concept class. Let \( TS = ((\mathcal{L}_0, d_0), \ldots, (\mathcal{L}_k, d_k)) \) be a finite teaching sequence segment such that \( \bigcup_{i \leq k} \mathcal{L}_i \subseteq \mathcal{C} \) and for all distinct \( i, j \in \{0, \ldots, k\} \), \( d_i = \sup_{C \in \mathcal{L}_i} \text{TD}(C, \mathcal{C} \setminus \bigcup_{j<i} \mathcal{L}_j) \) and \( \mathcal{L}_i \cap \mathcal{L}_j = \emptyset \). For any \( C \in \mathcal{C} \), \( TS \) is said to cover \( C \) iff \( C \in \bigcup_{i \leq k} \mathcal{L}_i \).

Call \( C \in \mathcal{C} \) hard to teach w.r.t. \( \mathcal{C} \) iff for all finite teaching sequence segments \( ((\mathcal{L}_0, d_0), \ldots, (\mathcal{L}_k, d_k)) \) covering \( C \), there exists some \( i \in \{0, \ldots, k\} \) such that \( d_i \geq \text{RTD}(\mathcal{C}) \).

For all \( i \in \mathbb{N}_0 \), call \( C \in \mathcal{C} \) \( i \)-hard to teach w.r.t. \( \mathcal{C} \) iff for all finite teaching sequence segments \( ((\mathcal{L}_0, d_0), \ldots, (\mathcal{L}_k, d_k)) \) covering \( C \) such that \( k \leq i \), there exists some \( j \in \{0, \ldots, k\} \) such that \( d_j \geq \text{RTD}(\mathcal{C}) \).

**Lemma 2.3.11** Let \( \mathcal{C} \) be any concept class. Then the following hold.

1. There exists some \( C \in \mathcal{C} \) such that \( C \) is hard to teach w.r.t. \( \mathcal{C} \).
2. Suppose \( C \in \mathcal{C} \) is \( i \)-hard to teach w.r.t. \( \mathcal{C} \). Let \( ((\mathcal{L}_0, d_0), \ldots, (\mathcal{L}_k, d_k)) \) be a finite teaching sequence segment such that \( k \leq i \), \( C \in \mathcal{L}_k \) and \( d_k < \text{RTD}(\mathcal{C}) \). Then there exist some \( i' < k \) and \( C' \in \mathcal{L}_{i'} \) such that \( C' \) is \( (i-1) \)-hard to teach w.r.t. \( \mathcal{C} \).
3. If \( C \in \mathcal{C} \) is hard to teach w.r.t. \( \mathcal{C} \), then \( C \) is \( i \)-hard to teach w.r.t. \( \mathcal{C} \) for any \( i \in \mathbb{N}_0 \).
(iv) If \( C \in \mathcal{C} \) is \( i \)-hard to teach w.r.t. \( \mathcal{C} \), then \( C \) is \( j \)-hard to teach w.r.t. \( \mathcal{C} \) for any \( j < i \).

**Proof.** Assertion (I). Assume otherwise. Then for each \( C \in \mathcal{C} \), there exists a finite teaching sequence segment \( S_c = ((\mathcal{F}_0(C), d_0), \ldots, (\mathcal{F}_{k-1}(C), d_{k-1}), (\{C\}, d_k)) \) covering \( C \) such that \( d_i < \text{RTD}(\mathcal{C}) \) for all \( i \leq k \). Associate to each \( C \in \mathcal{C} \) such a segment \( S_c \). One can construct a teaching sequence \( ((\mathcal{L}_0, e_0), (\mathcal{L}_1, e_1), (\mathcal{L}_2, e_2), \ldots) \) for \( \mathcal{C} \) by setting \( \mathcal{L}_i = \bigcup_{C' \in \mathcal{C}'} \mathcal{F}_i(C) \) for all \( i \in \mathbb{N}_0 \) (whenever \( \mathcal{F}_i(C) \) is defined). Note that \( e_i < \text{RTD}(\mathcal{C}) \) for all \( i \in \mathbb{N}_0 \), which is a contradiction.

Assertion (II). Assume otherwise. Then for all \( i' < k \) and \( C' \in \mathcal{L}_{i'}, C' \) is not \((i-1)\)-hard to teach w.r.t. \( \mathcal{C} \), so there exists a teaching sequence segment \( ((\mathcal{F}_0, e_0), \ldots, (\mathcal{F}_l, e_l)) \) for which \( l \leq i - 1 \), \( C' \in \mathcal{F}_l \) and \( e_j < \text{RTD}(\mathcal{C}) \) for all \( j \in \{0, \ldots, l\} \). For each \( i' < k \) and each \( C' \in \mathcal{L}_{i'} \), construct a finite teaching sequence segment of length at most \( i \) covering \( C' \). Following the construction in the proof of Assertion (I), one can build a finite teaching sequence segment covering \( C \) that has length at most \( i + 1 \) and has order smaller than \( \text{RTD}(\mathcal{C}) \), contradicting the fact that \( C \) is \( i \)-hard to teach w.r.t. \( \mathcal{C} \).

 Assertions (III) and (IV). Immediate from Definition 2.3.10.

**Theorem 2.3.12** Let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be concept classes over universes \( \chi_1 \) and \( \chi_2 \) respectively such that \( \chi_1 \cap \chi_2 = \emptyset \). Then \( \text{RTD}(\mathcal{C}_1 \uplus \mathcal{C}_2) = \text{RTD}(\mathcal{C}_1) + \text{RTD}(\mathcal{C}_2) \).

**Proof.** We first prove the upper bound. Let \( T_1 : \mathcal{C}_1 \mapsto \chi_1 \times \{+, -\} \) be an RTD-admissible mapping such that \( \sup\{|T_1(C)| : C \in \mathcal{C}_1\} = \text{RTD}(\mathcal{C}_1) \) and \( T_2 : \mathcal{C}_2 \mapsto \chi_2 \times \{+, -\} \) be an RTD-admissible mapping such that \( \sup\{|T_2(C)| : C \in \mathcal{C}_2\} = \text{RTD}(\mathcal{C}_2) \).
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Define $T : C_1 \uplus C_2 \mapsto (\chi_1 \cup \chi_2) \times \{+, -\}$ by setting $T(C_1 \cup C_2) = T_1(C_1) \cup T_2(C_2)$ for all $C_1 \in C_1$ and $C_2 \in C_2$. Since $T_1$ and $T_2$ are both RTD-admissible and $C_1, C_2$ have disjoint universes, $T$ is RTD-admissible as well. Furthermore, $\text{RTD}(C_1 \uplus C_2) \leq \sup \{|T(C_1 \cup C_2)| : C_1 \in C_1 \land C_2 \in C_2\} \leq \sup \{|T_1(C)| : C \in C_1\} + \sup \{|T_2(C)| : C \in C_2\} = \text{RTD}(C_1) + \text{RTD}(C_2)$.

Now we establish the lower bound. Assume for a contradiction that $\text{RTD}(C_1 \uplus C_2) < \text{RTD}(C_1) + \text{RTD}(C_2)$. Let $TS = ((L_0, d_0), (L_1, d_1), \ldots)$ be a teaching sequence for $C_1 \uplus C_2$ such that $\text{ord}(TS) < \text{RTD}(C_1) + \text{RTD}(C_2)$. By Lemma [2.3.11(I)], one can choose $C_1 \in C_1$ and $C_2 \in C_2$ such that $C_1$ and $C_2$ are hard to teach w.r.t. $C_1$ and $C_2$ respectively. Suppose $C_1 \cup C_2 \in L_i$. By Lemma [2.3.11(III)], $C_1$ and $C_2$ are $i$-hard to teach w.r.t. $C_1$ and $C_2$ respectively. Let $j \leq i$ be the minimum index for which $L_j$ contains $C_1^* \cup C_2^*$ such that $C_1^* \in C_1$, $C_2^* \in C_2$, and $C_1^*$ and $C_2^*$ are $i'$-hard to teach w.r.t. $C_1$ and $C_2$ respectively for some $i' \leq i$ and $j \leq i'$.

Let $L_{i_0}, L_{i_1}, L_{i_2}, \ldots$ be all the members of $\{L_i : i \in \mathbb{N}_0\}$ such that $F_j := \{C_2' : C_1^* \cup C_2' \in L_{i_j}\} \neq \emptyset$ for all $j \in \mathbb{N}_0$ and $k < l \leftrightarrow i_k < i_l$. Note that $((F_0, e_0), (F_1, e_1), \ldots)$ is a teaching sequence for $C_2$. One can similarly extract a teaching sequence for $C_1$ from $TS$ by considering the subfamilies of all the $L_i$’s consisting of members $C_1' \cup C_2'$ for some $C_1' \in C_1$. Let this teaching sequence for $C_1$ be $((G_0, f_0), (G_1, f_1), \ldots)$.

Let $T$ be a recursive minimal teaching set for $C_1^* \cup C_2^*$ w.r.t. $TS$. Define $T_1 = (T^+ \cup T^-) \cap \chi_1$ and $T_2 = (T^+ \cup T^-) \cap \chi_2$. Since $\text{ord}(TS) < \text{RTD}(C_1) + \text{RTD}(C_2)$, it must hold that $(|T_1| < \text{RTD}(C_1) \lor |T_2| < \text{RTD}(C_2))$.

Consider the following case distinction.
**Case 1:** $|T_1| < \text{RTD}(C_1)$. Suppose $C_1^* \in \mathcal{G}_k$. Then $((\mathcal{G}_0, f_0), \ldots, (\mathcal{G}_k, f_k))$ is a finite teaching sequence segment covering $C_1^*$ such that $f_k < \text{RTD}(C_1)$, $k \leq j \leq i'$ and $C_1^* \in \mathcal{G}_k$. By Lemma[2.3.11][II], there exists some $k' < k$ and $C_1^{o} \in \mathcal{F}_{k'}$ such that $C_1^o$ is $(i' - 1)$-hard to teach w.r.t. $C_1$. Furthermore, note that $C_1^o \cup C_2^* \in \mathcal{L}_{j'}$ for some $j' < j \leq i'$ (which implies that $j' \leq i' - 1$) and by Lemma[2.3.11][IV], $C_2^*$ is $(i' - 1)$-hard to teach w.r.t. $C_2$. This contradicts the choice of $C_1^*$ and $C_2^*$.

**Case 2:** $|T_2| < \text{RTD}(C_2)$. Suppose $C_2^* \in \mathcal{F}_k$. As in Case 1, there exists some $C_2^o \in \bigcup_{i < k} \mathcal{F}_i$ such that $C_2^o$ is $(i' - 1)$-hard to teach w.r.t. $C_2$. Moreover, $C_1^*$ is also $(i' - 1)$-hard to teach w.r.t. $C_1$. Thus $C_1^* \cup C_2^o \in \mathcal{L}_{j'}$ for some $j' < j$, where $C_1^*$ and $C_2^o$ are $(i' - 1)$-hard to teach w.r.t. $C_1$ and $C_2$ respectively and $i' - 1 \leq i$, $j' < j \leq i'$ (so that $j' \leq i' - 1$), which again contradicts the choice of $C_1^*$ and $C_2^*$.

Therefore $\text{RTD}(C_1 \uplus C_2) = \text{RTD}(C_1) + \text{RTD}(C_2)$, as claimed.
Chapter 3

The Teaching Complexity of Linear Sets

Semilinear sets generalise the class of ultimately periodic sets of integers to higher dimensions and have applications in formal language theory and database theory. This chapter examines a special type of semilinear set known as a linear set. A linear set $L$ is defined by a nonnegative integer (called a constant) and a finite set of non-negative integers (called periods); the members of $L$ are generated by adding to the constant an arbitrary finite sequence of the periods (allowing repetitions of the same period in the sequence). A semilinear set is a finite union of linear sets. Linear sets and semilinear sets are often defined on higher-dimensional integer lattices.\footnote{An $n$-dimensional lattice in the present context refers to the subset of $\mathbb{R}^n$ consisting of all $n$-tuples of integers.} Semilinear sets defined on integer lattices are not only objects of mathematical interest, but have also been linked to finite-state machines and formal languages. One of the
earliest and most important results on the connection between semilinear sets and
context-free languages is Parikh’s theorem \[35\], which states that any context-free
language is mapped to a semilinear set via a function known as the Parikh vector\(^2\) of
a string. Another interesting result, due to Ibarra \[24\], characterises semilinear sets
in terms of reversal-bounded multicounter machines. Moving beyond abstract the-
ory, semilinear sets have also recently been applied in the fields of DNA self-assembly
\[13\] and membrane computing \[25\].

The present chapter is primarily concerned with the sample complexity of teach-
ing classes of linear sets with a fixed dimension, which we determine mainly with
the teaching dimension and the recursive teaching dimension. Our motivation for
investigating the teaching and learning properties of linear sets arose from an earlier
study on the sample complexity of teaching pattern languages \[18\], specifically over
unary alphabets; some of these results on the pattern languages will be presented
in the next two chapters.

The learnability of semilinear sets has been investigated in Valiant’s PAC-learn-
ing model \[1\], Gold’s learning in the limit model \[45\], and Angluin’s query learning
model \[45\]. Abe \[1\] showed that when the integers are encoded in unary, the class
of semilinear sets of dimension 1 or 2 is polynomially PAC-learnable; on the other
hand, the question as to whether classes of semilinear sets of higher dimensions are
PAC-learnable is open. Takada \[45\] established that for any fixed dimension, the
family of linear sets is learnable in the limit from positive examples but the family
of semilinear sets is not learnable in the limit from only positive examples. In

\(^2\)Recall from Section 1.2 that for a fixed order \((a_1,\ldots,a_m)\) over a given finite alphabet \(\Sigma = \{a_1,\ldots,a_m\}\), the Parikh vector of a finite string \(s\) over \(\Sigma\) is the vector \((q_1,\ldots,q_m) \in \mathbb{N}_0^m\), where \(q_i\)
is the number of occurrences of the symbol \(a_i\) in \(s\).
addition, Takada showed the existence of a learning procedure via restricted subset and restricted superset queries that identifies any semilinear set over $\mathbb{N}^m_0$ (for any $m \in \mathbb{N}$) and halts; however, he proved at the same time that any such algorithm must make at least $2^m - 1$ queries in the worst case. He also proved that for any variable dimension, if for any unknown linear set $L$ and any conjectured semilinear set $U$, queries as to whether or not $L \subseteq U$ can be made, then the class of linear sets is learnable in polynomial time of the minimum size of representations (of each linear set) and the dimension.

We summarise the main results in the present chapter.

- The minimum teaching set of any cofinite linear subset $L$ of $\mathbb{N}_0$ with constant 0 may be characterised by means of a partial order defined with respect to the set of periods of $L$ (Corollary 3.2.7).

- Coinfinite linear subsets of $\mathbb{N}_0$ with constant 0 do not have finite teaching sets (Corollary 3.2.6).

- The RTD$^+$ is closely related to the notions of weak and strong spanning sets. In particular, the RTD$^+$ of a concept class $\mathcal{C}$ is bounded above by the size of the largest minimum strong spanning set over all $L \in \mathcal{C}$, and bounded below by the size of the largest minimum weak spanning set over all $L \in \mathcal{C}$ (Lemma 3.2.17). For intersection-closed concept classes such as the class of linear sets, weak spanning sets are also strong spanning sets and vice versa (Lemma 3.2.18).

- We determine the exact values or almost matching upper and lower bounds of the TD, TD$^+$, RTD and RTD$^+$ for various classes of linear sets (see Tables 3.1
and Sections 3.1 and 3.2 for the definitions of these classes). For many classes of linear sets considered in the present work, the RTD is significantly smaller than the TD (Tables 3.1 and 3.2).

Our results may be of interest from a formal language perspective as well as from a computational learning theory perspective. First, they uncover a number of structural properties of linear sets, especially in the one-dimensional case, which could be applied to study formal languages via the Parikh vector function. Consider, for example, the non-erasing pattern language $L(\pi)$ for any given pattern $\pi$ over $\{a\} \cup X$. As will be seen later, the Parikh vector maps $L(\pi)$ to a linear subset $L$ of the natural numbers such that the sum of $L$’s periods does not exceed the constant associated to $L$. Thus one could determine various teaching complexity measures of any non-erasing pattern language from the teaching complexity measures of a certain linear subset of the natural numbers. One may also investigate the teaching properties of parallel computation models via the Parikh vector function, which maps equal matrix languages [44] and weakly persistent Petri nets [47] to semilinear sets. Second, the class of linear sets affords quite a natural setting to study models of teaching and learning over infinite concept classes. Besides showing that the RTD can be significantly lower than the TD for many infinite classes of linear sets – which translates to more efficient learnability, we will consider a more stringent variant of the RTD, the RTD$^+$, which considers sequential teaching of classes using only positive examples. It will be shown that there are natural classes of linear sets that cannot even be taught sequentially using only positive examples while the RTD is finite; these examples illustrate how supplying negative information may sometimes be indispensable to successful teaching and learning.
Chapter 3. The Teaching Complexity of Linear Sets

<table>
<thead>
<tr>
<th>Class</th>
<th>TD</th>
<th>TD^+</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF–LINSET_1</td>
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<td>0</td>
</tr>
<tr>
<td>CF–LINSET_2</td>
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<td>∞</td>
</tr>
<tr>
<td>CF–LINSET_3</td>
<td>5</td>
<td>∞</td>
</tr>
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<td>∞</td>
</tr>
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<td>LINSET_1</td>
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<td>∞</td>
</tr>
<tr>
<td>LINSET_2</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
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<td>∞</td>
</tr>
<tr>
<td>LINSET</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>NE–LINSET_k, k ≥ 1</td>
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</tr>
<tr>
<td>NE–LINSET</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>LINSET^2_2</td>
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<td>∞</td>
</tr>
<tr>
<td>SL_{m,n}, m ≥ 1, n ≥ 2</td>
<td>∞</td>
<td>∞</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of results on TD, TD^+.

The results presented in this chapter have been published or accepted for publication in the following articles:


### 3.1 Linear Sets

A subset $L$ of $\mathbb{N}_0^m$ is said to be *linear* iff there exist an element $c$ and a finite subset $P$ of $\mathbb{N}_0^m$ such that $L = c + \langle P \rangle := \{ q : q = c + n_1p_1 + \ldots + n_kp_k, n_i \in \mathbb{N}_0, p_i \in P \}$. 
Chapter 3. The Teaching Complexity of Linear Sets

<table>
<thead>
<tr>
<th>Class</th>
<th>RTD</th>
<th>RTD$^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF–LINSET$_1$</td>
<td>0 (Thm 3.2.15)</td>
<td>0 (Thm 3.2.15)</td>
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<tr>
<td>CF–LINSET$_2$</td>
<td>2 (Thm 3.2.23)</td>
<td>2 (Thm 3.2.23)</td>
</tr>
<tr>
<td>CF–LINSET$_3$</td>
<td>3 (Thm 3.2.23)</td>
<td>3 (Thm 3.2.23)</td>
</tr>
<tr>
<td>CF–LINSET$_{k,k \geq 5}$</td>
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<td>$k$ (Thm 3.2.23)</td>
</tr>
<tr>
<td>LINSET$_1$</td>
<td>1 (Thm 3.2.25(II))</td>
<td>1 (Thm 3.2.25(II))</td>
</tr>
<tr>
<td>LINSET$_2$</td>
<td>$\infty$ (Thm 3.2.25(II))</td>
<td>$\infty$ (Thm 3.2.25(II))</td>
</tr>
<tr>
<td>LINSET$_3$</td>
<td>$\infty$ (Thm 3.2.25(II))</td>
<td>$\infty$ (Thm 3.2.25(II))</td>
</tr>
<tr>
<td>LINSET</td>
<td>$\infty$ (Rem 3.3.7)</td>
<td>$\infty$ (Rem 3.3.7)</td>
</tr>
<tr>
<td>NE–LINSET$_{k,k \geq 1}$</td>
<td>$(k-1,k,k+1)$ (Thm 3.3.6)</td>
<td>$k+1$ (Thm 3.3.5)</td>
</tr>
<tr>
<td>NE–LINSET</td>
<td>$\infty$ (Rem 3.3.7)</td>
<td>$\infty$ (Rem 3.3.7)</td>
</tr>
<tr>
<td>LINSET$_2^2$</td>
<td>${3,4}$ (Thm 3.4.2)</td>
<td>$\infty$ (Thm 3.4.2)</td>
</tr>
<tr>
<td>SL$_{m,n,m \geq 1,n \geq 2}$</td>
<td>$\infty$ (Thm 3.2.27)</td>
<td>$\infty$ (Thm 3.2.27)</td>
</tr>
</tbody>
</table>

Table 3.2: Summary of results on RTD and RTD$^+$. 

c is called the constant and each $p_i$ is called a period of $c + \langle P \rangle$. Denote $0 + \langle P \rangle$ by $\langle P \rangle$. For any linear set $L$, if $L = c + \langle P \rangle$, then $(c,P)$ is called a representation of $L$. Any finite $P \subset \mathbb{N}_0^m$ is independent iff for all $P' \subset P$, it holds that $\langle P' \rangle \neq \langle P \rangle$.

A representation $(c,P)$ of a linear set $L$ is canonical iff $P$ is independent. A linear subset of $\mathbb{N}_0^m$ will also be called a linear set of dimension $m$.

Our work will focus on the linear subsets of $\mathbb{N}_0$. The main classes of linear sets investigated are denoted as follows. In these definitions, $k \in \mathbb{N}$.

(i) $\text{LINSET}_k := \{ \langle P \rangle : P \subset \mathbb{N} \wedge 1 \leq |P| \leq k \}$.

(ii) $\text{LINSET} := \bigcup_{k \in \mathbb{N}} \text{LINSET}_k$.

(iii) $\text{CF–LINSET}_k := \{ \langle P \rangle : P \subset \mathbb{N} \wedge \gcd(P) = 1 \wedge 1 \leq |P| \leq k \}$.

(iv) $\text{CF–LINSET} := \bigcup_{k \in \mathbb{N}} \text{CF–LINSET}_k$.

$^3$CF stands for “cofinite.”
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(v) \( \text{NE-LINSET}_k := \{c + \langle P \rangle : c \in \mathbb{N}_0 \land P \subset \mathbb{N}_0 \land |P| \leq k \land \sum_{p \in P} p \leq c \} \)

(vi) \( \text{NE-LINSET} := \bigcup_{k \in \mathbb{N}} \text{NE-LINSET}_k \).

Note that the classes in items (I) to (IV) exclude singleton linear sets; the reason for this omission will be explained later. The motivation for studying each subfamily in items (III) to (VI) will be explained as it is introduced in the forthcoming sections.

3.2 Linear Subsets of \( \mathbb{N}_0 \) With Constant 0

This section will analyse the class LINSET of linear subsets of \( \mathbb{N}_0 \) with constant 0. Even in the apparently simple one-dimensional case, the teaching complexity measures can vary quite widely across families of linear sets. Many proofs will exploit the fact that linear sets of dimension 1 are ultimately periodic, a property that has no exact analogue for linear sets of higher dimensions. Our main results in the present section are: (1) a characterisation of the minimum teaching sets of cofinite linear sets with constant 0 (Corollary 3.2.7), (2) the exact values (or almost matching upper and lower bounds) of the TD, TD+, RTD and RTD+ for values classes of linear sets, and (3) a characterisation of the strong and weak spanning sets of cofinite linear sets with constant 0 and at most \( k \) periods (Lemma 3.2.19).

Proposition 3.2.1 \( \text{[32]} \) Let \( P \subset \mathbb{N} \) be a finite set such that \( \gcd(P) = 1 \). Then \( \mathbb{N} \setminus \langle P \rangle \) is finite. The largest number in \( \mathbb{N} \setminus \langle P \rangle \) is known as the Frobenius number of \( \langle P \rangle \).

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4NE stands for “non-erasing.”
Example 3.2.2 Let $P = \{3, 5\}$. Then $\gcd(P) = 1$ and $\langle P \rangle = \{3, 5, 6\} \cup \{x : x \geq 8\} = \mathbb{N}_0 \setminus \{1, 2, 4, 7\}$. The Frobenius number of $\langle P \rangle$ is 7.

For any $P = \{p_1, \ldots, p_k\}$ with $\gcd(P) = 1$, $F(P)$ and $F(p_1, \ldots, p_k)$ will denote the Frobenius number of $\langle P \rangle$. We will characterise the teaching sets of all linear sets $\langle P \rangle$ such that $\gcd(P) = 1$ with respect to LINSET in terms of $P$ and a certain (finite) subset of $\mathbb{N} \setminus \langle P \rangle$. The following variant of a notion from the theory of numerical semigroups will help to formulate the characterisation.

The partial ordering induced by $P$ (modified from [39]) is defined as follows: $x \leq_P y \iff \exists a \in \mathbb{N} : y - ax \in \langle P \rangle$. We write $x <_P y$ as an abbreviation of $x \leq_P y \land x \neq y$. One has $x \in \langle P \rangle \land x \leq_P y \Rightarrow y \in \langle P \rangle$, or equivalently, $y \notin \langle P \rangle \land x \leq_P y \Rightarrow x \notin \langle P \rangle$.

For the rest of this section, “maximal” (resp. “minimal”) always means “maximal w.r.t. $\leq_P$” (resp. “minimal w.r.t. $\leq_P$”) unless specified otherwise. Let $\text{MAX}_P$ be the set of maximal elements in $\mathbb{N} \setminus \langle P \rangle$ and let $\text{MIN}_P$ denote the set of minimal elements in $\langle P \rangle \setminus \{0\}$. The following lemma collects some useful known facts. Many of these facts are proven in [39], or may be directly deduced from related results proven in [39].

Lemma 3.2.3 (i) $\mathbb{N} \setminus \langle P \rangle$ contains an infinite ascending chain $x_0 <_P x_1 <_P x_2 <_P \ldots$ (e.g. $x_i = 1 + ip$ with an arbitrary choice of $p \in P$) iff $\gcd(P) > 1$.

(ii) If $H \subseteq \langle P \rangle$, then the following hold:

(a) $\langle H \rangle \subseteq \langle P \rangle$.

(b) The partial ordering $\leq_P$ is a refinement of the partial ordering $\leq_H$; that
is, for any \( x, y \), \( x \leq_H y \) implies \( x \leq_P y \).

(c) \( \text{MIN}_P \cap \langle H \rangle \subseteq \text{MIN}_H \).

(iii) Let \( s = a_1p_1 + \ldots + a_rp_r \in \langle P \rangle \) and let \( I = \{ i \in [r] : a_i \neq 0 \} \). Then \( p_i \leq_P s \) for each \( i \in I \). This implies that \( \text{MIN}_P \subseteq P \).

(iv) If \( p, p' \in P \) and \( p <_P p' \), then \( p' \) is superfluous, i.e., \( \langle P \rangle = \langle P \setminus \{p'\} \rangle \).

(v) If \( P \) is independent, then \( P = \text{MIN}_P \) and \( P \) is a spanning set of \( \langle P \rangle \) in the sense that \( \langle P \rangle \) is the unique smallest set of \( \text{LINSET} \) containing \( P \). Moreover, every subset of \( \langle P \rangle \) of cardinality less than \( |P| \) spans a proper subset of \( \langle P \rangle \).

For the rest of this section, it will always be assumed that \( P \) is independent.

Example 3.2.4 Let \( P = \{3, 5\} \). Then \( \mathbb{N}_0 \setminus \langle P \rangle = \{1, 2, 4, 7\} \). By Lemma 3.2.3(V), \( \text{MIN}_P = \{3, 5\} \). Note that \( 1 \leq_P 4, 2 \leq_P 7 \) and \( 4 \leq_P 7 \). Thus \( \text{MAX}_P = \{7\} \).

We now study teaching sets.

Lemma 3.2.5 (1) Let \( T \) be a teaching set for \( \langle P \rangle \) w.r.t. \( \{\langle P \rangle \} \cup \text{CF–LINSET} \). Then \( P \subseteq T^+ \).

(ii) Let \( T \) be a teaching set for \( \langle P \rangle \in \text{LINSET}_{k-1} \) w.r.t. \( \text{LINSET}_k \). Then, for each \( x \in \mathbb{N} \setminus \langle P \rangle \), there exists \( y \in T^- \) such that \( x \leq_P y \). This assertion remains valid with \( \text{CF–LINSET} \) in place of \( \text{LINSET} \).

(iii) For any finite set \( P \) and \( k \in \mathbb{N} \) with \( \langle P \rangle \in \text{CF–LINSET}_k \), let \( T \) be a teaching set for \( \langle P \rangle \) w.r.t. \( \text{CF–LINSET}_k \). Then \( |T^+| \geq k \).
Proof. Assertion (I). Since $T$ is labelled consistently with $\langle P \rangle$, it holds that $T^+ \subseteq \langle P \rangle$ and $T^- \cap \langle P \rangle = \emptyset$. Thus $\langle T^+ \rangle$ is consistent with $T$. Choose a prime $q > \max(T^+ \cup T^- \cup P)$. Then $\langle T^+ \cup \{q\} \rangle$ is also consistent with $T$ and it belongs to CF–LINSET, and since $T$ is a teaching set for $\langle P \rangle$ w.r.t. $\{\langle P \rangle\} \cup$ CF–LINSET, one has that $\langle T^+ \cup \{q\} \rangle = \langle P \rangle$. In particular, by Lemma 3.2.3(III), Lemma 3.2.3(V) and the fact that $P$ is independent, $P = \text{MIN}_P = \text{MIN}_{T^+ \cup \{q\}} \subseteq T^+ \cup \{q\}$. The choice of $q > \max(P)$ then gives that $P \subseteq T^+$.

Assertion (II). Pick an arbitrary but fixed $x \in \mathbb{N} \setminus \langle P \rangle$. We have to show that $T^-$ contains some element $y$ such that $x \leq_P y$. Let $P' = P \cup \{x\}$. Since $T$ is a teaching set for $\langle P \rangle$ w.r.t. LINSET$_k$ and $\langle P \rangle \in$ LINSET$_{k-1}$, $\langle P' \rangle$ is inconsistent with $T$. As $\langle P \rangle \subset \langle P' \rangle$, there is some $y \in T^-$ such that $y \in \langle P' \rangle \setminus \langle P \rangle$. Suppose $y = ax + \sum_{p \in P} a(p)p$ for some $a, a(p) \in \mathbb{N}_0$. Then $y \notin \langle P \rangle$ implies that $a > 0$, and therefore $x \leq_P y$. Since $\gcd(P) = 1$ implies that $\gcd(P') = 1$, the same argument holds with CF–LINSET in place of LINSET.

Assertion (III). Let $T$ be a teaching set for some $\langle P \rangle \in$ CF–LINSET$_k$ w.r.t. CF–LINSET$_k$. Suppose that $|T^+| < k$. If $\gcd(T^+) = 1$, then $\langle T^+ \rangle$ is a proper subset of $\langle P \rangle$ that is consistent with $T$ and belongs to CF–LINSET$_k$, which is impossible. If $\gcd(T^+) > 1$ or $|T^+| = 0$, then, since $\langle P \rangle$ is cofinite, there is a sufficiently large prime $p$ such that CF–LINSET$_k \ni \langle T^+ \cup \{p\} \rangle \subset \langle P \rangle$ and $\langle T^+ \cup \{p\} \rangle$ is consistent with $T$, a contradiction. 

Corollary 3.2.6 If $\gcd(P) > 1$ and $k = 1 + |P|$, then TD($\langle P \rangle$, LINSET$_k$) = $\infty$.

Proof. According to Lemma 3.2.3(I), $\mathbb{N} \setminus \langle P \rangle$ contains an infinite ascending chain $x_1 <_P x_2 <_P x_3 <_P \ldots$. According to Lemma 3.2.5(II), a teaching set for $\langle P \rangle$ must
contain infinitely many elements of this chain.

**Corollary 3.2.7** If $\text{gcd}(P) = 1$, then the set $T(P)$ given by $T(P)^+ = P$ and $T(P)^- = \text{MAX}_P$ is the unique smallest teaching set for $\langle P \rangle$ w.r.t. LINSET and also w.r.t. CF–LINSET.

**Proof.** Suppose that $\langle H \rangle$ is consistent with $T(P)$. We will show that $\langle H \rangle = \langle P \rangle$, which would imply that $T(P)$ is indeed a teaching set for $\langle P \rangle$ w.r.t. LINSET. Since $P \subseteq \langle H \rangle$, one has that $\langle P \rangle \subseteq \langle H \rangle$. Now take any $x \in \mathbb{N}_0 \setminus \langle P \rangle$. Let $x <_P y_1 <_P \ldots <_P y_m$ be a maximal chain in $(\mathbb{N}_0 \setminus \langle P \rangle, \leq_P)$ with $x$ as the minimal element (w.r.t. $\leq_P$). Such a (finite) chain must exist since $\text{gcd}(P) = 1$ implies that $\mathbb{N}_0 \setminus \langle P \rangle$ is finite. Then $y_m \in \text{MAX}_P = T(P)^-$ and so by the consistency of $\langle H \rangle$ with $T(P)$, it follows that $y_m \in \mathbb{N}_0 \setminus \langle H \rangle$. Further, there are $a \in \mathbb{N}_0$ and $x_1 \in \langle P \rangle$ such that $y_m = ax + x_1$. Since $x_1 \in \langle P \rangle \subseteq \langle H \rangle$ and $y_m \in \mathbb{N}_0 \setminus \langle H \rangle$, it must hold that $x \in \mathbb{N}_0 \setminus \langle H \rangle$. Therefore $\langle H \rangle \subseteq \langle P \rangle$, and so $\langle H \rangle = \langle P \rangle$, indeed. According to Lemma 3.2.5(I) and Lemma 3.2.5(II), any teaching set for $\langle P \rangle$ w.r.t. LINSET (resp. CF–LINSET) must contain $(p, +)$ for all $p \in P$ as well as $(q, -)$ for all $q \in \text{MAX}_P$.

**Example 3.2.8** Let $P = \{3, 5\}$ and $Q = \{2\}$. We know from Example 3.2.4 that $\text{MIN}_P = P$ and $\text{MAX}_P = \{7\}$, and so by Corollary 3.2.7, the sample $\{(3, +), (5, +), (7, -)\}$ is the unique smallest teaching set for $\langle P \rangle$ w.r.t. LINSET and also w.r.t. CF–LINSET. By Corollary 3.2.6, $\text{TD}(\langle Q \rangle, \text{LINSET}_2) = \infty$. It may be observed that $1 \leq Q 3 \leq Q 5 \leq Q \ldots \leq Q 1 + 2i \leq Q \ldots$ is an infinite ascending chain contained in $\mathbb{N}_0 \setminus \langle Q \rangle$.

**Corollary 3.2.9** If $\text{gcd}(P) = d$, then the set $T(P)$ given by $T(P)^+ = \frac{1}{d}P$ and $T(P)^- = \frac{1}{d}\text{MAX}_P$ is the unique smallest teaching set for $\frac{1}{d}\langle P \rangle$ w.r.t. LINSET and
also w.r.t. CF–LINSET.

**Proof.** Let $P' = \frac{1}{d} P$. By Corollary 3.2.7, it suffices to show that $P'$ is independent, gcd($P'$) = 1, and $\text{MAX}_{P'} = \frac{1}{d} \text{MAX}_P$. The independence of $P'$ and the fact that gcd($P'$) = 1 follows directly from the definition of $P'$. Now it is shown that $\text{MAX}_{P'} = \frac{1}{d} \text{MAX}_P$. First, for any $y \in \mathbb{N}_0 \setminus \langle P \rangle$ such that $d$ does not divide $y$, $y \leq_P y + x \in \mathbb{N}_0 \setminus \langle P \rangle$ holds for all $x \in \langle P \rangle$; hence $\text{MAX}_P \subseteq d \mathbb{N}$. Next, note that for any $y, z \in \mathbb{N}_0 \setminus \langle P \rangle$ such that $y \leq_P z$, either both $y$ and $z$ are divisible by $d$, or neither $y$ nor $z$ are divisible by $d$. Pick any $dy \in \text{MAX}_P$. Then for all $z \in \mathbb{N}_0 \setminus \langle P' \rangle$ such that $\frac{1}{d} y \leq_P z$, there are $a \in \mathbb{N}$ and $x \in \langle P \rangle$ with $z = ay + x$, or $dz = day + dx$. Since $dz \in \mathbb{N}_0 \setminus \langle P \rangle$ and $dx \in \langle P \rangle$, one has $dy \leq_P dz$. But $dy \in \text{MAX}_P$ implies that $dy = dz$, or $y = z$. Hence $y \in \text{MAX}_{P'}$. Now consider any $u \in \text{MAX}_{P'}$. For all $v \in \mathbb{N}_0 \setminus \langle P \rangle$ such that $du \leq_P v$, $v = dw$ for some $w \in \mathbb{N}$. Further, there are $a' \in \mathbb{N}$ and $dw' \in \langle P \rangle$ such that $dw = a'du + dw'$, or $w = a'u + w'$. As $w' \in \langle P' \rangle$ and $w \in \mathbb{N}_0 \setminus \langle P' \rangle$, it follows that $u \leq_{P'} w$, and so $u \in \text{MAX}_{P'}$ gives that $du = dw = v$. Therefore $du \in \text{MAX}_P$. □

The next example shows that Lemma 3.2.5(I) does not hold in general for the restricted class LINSET$_k$.

**Example 3.2.10** For any $k > 1$, let $P = \{k, k+1, \ldots, 2k-1\}$. Then $T = \{(x, -) : x < k\} \cup \{(k + i, +) : 1 \leq i \leq k + 1\}$ is a teaching set for $\langle P \rangle$ w.r.t. LINSET$_k$.

**Proof.** Suppose that $L \in \text{LINSET}_k$ is consistent with $T$. Now $L$ must be uniquely generated by an independent set $H$ of size at most $k$. Since $(k + i, +) \in T$ for all $i \in \{1, \ldots, k + 1\}$ and $y \notin L$ for all $y < k$, $k \in H$ holds. Thus $\langle P \rangle \subseteq L \subseteq \langle P \rangle$, and
so \( L = \langle P \rangle \).

**Remark 3.2.11** Let \( \mathcal{L} \) be a class of subsets of \( \mathbb{N}_0 \). Given any \( L' \in \mathcal{L} \), if \( T \) is a teaching set for \( L' \subseteq \mathbb{N}_0 \) w.r.t. \( \mathcal{L} \), then for any \( c \in \mathbb{N}_0 \), \( c + T = \{(c+x, +) : x \in T^+\} \cup \{(c+y, -) : y \in T^-\} \) is a teaching set for \( c + L' \) w.r.t. \( \mathcal{L}[n] = \{c+L : L \in \mathcal{L}\} \). Thus Lemma 3.2.5 and Corollaries 3.2.6 and 3.2.7 may be readily generalised, mutatis mutandis, to the class \( \text{LINSET}[c] = \{c + L : L \in \text{LINSET}\} \) for any \( c \in \mathbb{N}_0 \).

The proof of [31, Theorem 6] provides a construction that may be slightly modified to show that \( \text{TD}(\text{LINSET}_1) = \infty \) even though \( \text{TD}(q, \text{LINSET}_1) < \infty \) for any \( q > 0 \). By the monotonicity of \( \text{TD} \), \( \text{TD}(\text{LINSET}) = \text{TD}(\text{LINSET}_k) = \infty \) for all \( k > 0 \).

By Corollary 3.2.6, \( \text{LINSET} \) contains infinitely many members that have an infinite \( \text{TD} \) w.r.t. \( \text{LINSET} \). Thus for any \( \mathcal{L} \subseteq \text{LINSET} \), it may be difficult to interpret a value of \( \infty \) for \( \text{TD}(\mathcal{L}) \): (1) on the one hand, all cofinite subclasses of \( \mathcal{L} \) may have an infinite \( \text{TD} \) w.r.t. \( \mathcal{L} \); (2) on the other hand, there may be a cofinite subclass of \( \mathcal{L} \) that has a finite \( \text{TD} \) w.r.t. \( \mathcal{L} \). Intuitively, it seems that \( \mathcal{L} \) in Case (2) is unteachable in a weaker sense than in Case (1), but the \( \text{TD} \) makes no such distinction. It shall be shown, however, that the \( \text{RTD} \) is a bit more well-behaved when applied to \( \text{LINSET} \). In particular, for all \( \mathcal{L} \subset \text{LINSET} \), \( \text{RTD}(\mathcal{L}, \text{LINSET}) \) grows only linearly with \( \sup\{\min(P) : \langle P \rangle \in \mathcal{L} \land \min(P) > 0\} \). We will also give a finer analysis of \( \text{LINSET}_k \) for \( k \in \{1, 2, 3\} \), showing that while \( \text{LINSET}_2 \) does not have any positive teaching sequence, \( \text{RTD}(\text{LINSET}_k) < \infty \) for \( k \in \{1, 2, 3\} \). In addition, \( \text{RTD}(\text{LINSET}_k) \) grows at least linearly in \( k \), implying that \( \text{RTD}(\text{LINSET}) = \infty \). The question as to whether \( \text{RTD}(\text{LINSET}_k) < \infty \) for any \( k > 3 \) remains open.
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The following proposition explains why the singleton \{0\} was excluded from the definition of LINSET. The proof is quite similar to that of Proposition 3.4.1 which asserts that the class of linear subsets of \(\mathbb{N}_2^0\) with at most two periods has infinite RTD. First, we define the notion of the recursive teaching dimension of a subclass \(\mathcal{L}'\) of \(\mathcal{L}\) w.r.t. \(\mathcal{L}\).

For any \(\mathcal{L}' \subseteq \mathcal{L}\), \(R = ((\mathcal{F}_0, d_0), (\mathcal{F}_1, d_1), \ldots)\) is a teaching subsequence for \(\mathcal{L}\) covering \(\mathcal{L}'\) iff \(\mathcal{F}_0, \mathcal{F}_1, \ldots\) are nonempty, pairwise disjoint subsets of \(\mathcal{L}\) such that \(\mathcal{L}' \subseteq \bigcup_{i \in \mathbb{N}_0} \mathcal{F}_i\) and \(d_i = \sup\{\text{TD}(L, \mathcal{L} \setminus \bigcup_{0 \leq j < i} \mathcal{F}_j) : L \in \mathcal{F}_i\}\) for all \(i\). Define \(\text{ord}(R) = \sup\{d_i : i \in \mathbb{N}_0\}\) and \(\text{RTD}(\mathcal{L}', \mathcal{L}) = \inf\{\text{ord}(R) : R \text{ is a teaching subsequence for } \mathcal{L}\text{ covering } \mathcal{L}'\}\). A positive teaching subsequence for \(\mathcal{L}\) covering \(\mathcal{L}'\) and \(\text{RTD}^+(\mathcal{L}', \mathcal{L})\) are defined analogously. If \(\mathcal{L}' = \{L\}\) for some \(L\), then \(\text{RTD}(\mathcal{L}', \mathcal{L})\) and \(\text{RTD}^+(\mathcal{L}', \mathcal{L})\) will often be written as \(\text{RTD}(L, \mathcal{L})\) and \(\text{RTD}^+(L, \mathcal{L})\) respectively.

**Proposition 3.2.12** \(\text{RTD}(\{\{0\}\}, \{\{0\}\} \cup \text{LINSET}_1) = \infty\).

**Proof.** Assume for the sake of a contradiction that \(R = ((\mathcal{L}_0, d_0), (\mathcal{L}_1, d_1), \ldots)\) were a teaching subsequence for \(\{\{0\}\} \cup \text{LINSET}_1\) covering \(\{\{0\}\}\). Suppose \(\{0\} \in \mathcal{L}_i\), and let \(T\) be a teaching set for \(\{0\}\) w.r.t. \(\bigcup_{j \geq i} \mathcal{L}_j\). Choose \(N > \max\{d_j : j \leq i\}\) so that \(N\) is greater than \(x\) for every \(x \in X(T)\). Further, let \(p_0, \ldots, p_{N+i-1}\) be a strictly increasing sequence of \(N+i\) primes. Observe that by the choice of \(N\), \(L_0 = \langle\prod_{j=0}^{N+i-1} p_j\rangle\) is consistent with \(T\) and so it must belong to \(\mathcal{L}_{j_0}\) for some \(j_0 < i\). For any two distinct \((N+i-1)\)-subsets \(S, S'\) of \(\{p_0, \ldots, p_{N+i-1}\}\), \(L_0 \subset \langle\prod_{y \in S} y\rangle\) and \(\langle\prod_{y \in S} y\rangle \cap \langle\prod_{y \in S'} y\rangle \subseteq L_0\). Thus by the choice of \(N\), there must exist some \((N+i-1)\)-subset \(S_1\) of \(\{p_0, \ldots, p_{N+i-1}\}\) so that \(L_1 = \langle\prod_{y \in S_1} y\rangle \in \mathcal{L}_{j_1}\) for some \(j_1 < j_0\). By iterating a similar argument, one can show that for some \(N\)-subset
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$S''$ of $\{p_0, \ldots, p_{N+i-1}\}$, $(\prod_{y \in S''} y) \in \mathcal{L}_0$. But the teaching dimension of $(\prod_{y \in S''} y)$ w.r.t. $\{0\} \cup \text{LINSET}_1$ is at least $N > d_0$, a contradiction. 

Proposition 3.2.12 should be contrasted with the observation that $\text{RTD}(\text{LINSET}_1) = 1$: $((\langle 1 \rangle, 1), (\langle 2 \rangle, 1), \ldots)$ is a teaching plan with positive examples for $\text{LINSET}_1$, where the $i^{th}$ linear set in the plan is $\langle i \rangle$ and $\{(i, +)\}$ is a teaching set for $\langle i \rangle$ w.r.t. $\{\langle j \rangle : j \geq i\}$. The next theorem shows that for any finite $\mathcal{L} \subset \text{LINSET}$, $\text{RTD}(\mathcal{L}, \text{LINSET}) < \infty$; in fact, for any $\mathcal{L} \subset \text{LINSET}$, $\text{RTD}(\mathcal{L}, \text{LINSET})$ is at most linear in $\sup\{\min(P) : \langle P \rangle \in \mathcal{L} \land \min(P) > 0\}$.

**Theorem 3.2.13** Let $\mathcal{F}_n = \{\langle P \rangle : P \text{ is independent } \land \min(P) \leq n\}$. Then $\text{RTD}(\mathcal{F}_n, \text{LINSET}) \leq 2n - 1$.

**Proof.** To simplify notation, LINSET will be denoted by $\mathcal{L}$ throughout this proof. We adopt the following convention. For any $\langle p_1, \ldots, p_k \rangle \in \mathcal{L}$, it is assumed that the periods are written in strictly increasing order, that is, $p_1 < \ldots < p_k$. We shall use the following two facts: (1) if $P = \{p_1, \ldots, p_k\} \subseteq \mathbb{N}$ is independent and $U \subseteq \mathbb{N}_0$ is a set whose elements are pairwise incomparable w.r.t. $\leq_P$, then for all $u_1, u_2 \in U$, $u_1 \neq u_2 \pmod{p_1}$, so that $|U| \leq p_1$; (2) if $P = \{p_1, \ldots, p_k\} \subseteq \mathbb{N}$ is independent, then $k = |P| \leq p_1$ and $\text{MAX}_P \leq p_1 - 1$.

**Proof of (1).** Fix any distinct $u_1, u_2 \in U$ such that $u_1 < u_2$. If $u_1 \equiv u_2 \pmod{p_1}$, then there is some $k \in \mathbb{N}_0$ for which $u_2 = u_1 + kp_1$, and so $u_1 \leq_P u_2$, which contradicts the definition of $U$.

**Proof of (2).** Note that the elements of $P$ are incomparable w.r.t. $\leq_P$: otherwise, if $p_i \leq_P p_j$ for some $i, j \in \{1, \ldots, k\}$ with $i < j$, then $p_j = \sum_{\ell=1}^{j-1} a_{i\ell} p_{\ell}$ for some
a_\ell \in \mathbb{N}_0$, which would contradict the assumption that $P$ is independent. Hence by (1), $|P| \leq p_1$. For a similar reason, $|\text{MAX}_P| \leq p_1$; further, since no element of $\text{MAX}_P$ is a multiple of $p_1$ (as $\text{MAX}_P \subseteq \mathbb{N}_0 \setminus \langle P \rangle$), one has $|\text{MAX}_P| \leq p_1 - 1$.

Let $P$ range over all finite and independent subsets of $\mathbb{N}$. Define the mapping $T : \mathcal{L} \mapsto \wp(\mathbb{N}_0 \times \{+,-\})$ by $T(\langle P \rangle)^+ = P$ and $T(\langle P \rangle)^- = \text{MAX}_P$. We shall show that $T$ is RTD-admissible for $\mathcal{L}$; in other words, the digraph $G = (\mathcal{L},A)$ induced by $T$ is acyclic and every node of $G$ has finite depth. Pick $\langle H \rangle, \langle P \rangle \in \mathcal{L}$ such that $(\langle H \rangle, \langle P \rangle) \in A$. It suffices to show that $\text{gcd}(H)$ is a proper divisor of $\text{gcd}(P)$; this would show that the depth of the node representing $\langle P \rangle$ is bounded from above by the number of prime power divisors of $\text{gcd}(P)$. First, since $P = T(\langle P \rangle)^+ \subseteq \langle H \rangle$, it follows that $\text{gcd}(H)$ divides $\text{gcd}(P)$. Suppose for the sake of a contradiction that $\text{gcd}(H) = \text{gcd}(P) = d$. Define $H' = \frac{1}{d}H$ and $P' = \frac{1}{d}P$. $\langle H \rangle \neq \langle P \rangle$ implies that $\langle H' \rangle \neq \langle P' \rangle$. Further, the fact that $\langle H \rangle$ is consistent with $T$ implies both $\langle H' \rangle$ and $\langle P' \rangle$ are consistent with $T' := \left\{ \left( \frac{x}{d},+ \right) : x \in T(\langle P \rangle)^+ \right\} \cup \left\{ \left( \frac{y}{d},- \right) : y \in T(\langle P \rangle)^- \right\} = \left\{ \left( \frac{x}{d},+ \right) : x \in P \right\} \cup \left\{ \left( \frac{y}{d},- \right) : y \in \text{MAX}_P \right\}$. But Corollary 3.2.9 implies that $T'$ is a teaching set for $\langle P' \rangle$ w.r.t. $\mathcal{L}$, a contradiction.

Finally, observe from Facts (1) and (2) that for any $\langle P \rangle \in \mathcal{L}$ with $p_1 = \min(P)$, it holds that $|T(\langle P \rangle)| = |P| + |\text{MAX}_P| \leq 2p_1 - 1$. Thus, since $T$ is RTD-admissible for $\mathcal{L}$ and $\min(P) \leq n$ for all $\langle P \rangle \in \mathcal{F}_n$ with $P$ independent, one can find a teaching subsequence for $\mathcal{L}$ covering $\mathcal{F}_n$ such that the order of this teaching subsequence is at most $2n - 1$. ■

Theorem 3.2.23 (to be shown later) will imply that the order of any teaching sequence for LINSET must necessarily be infinite. Nonetheless, the preceding theo-
rem shows roughly that the growth of $\text{RTD}(\mathcal{L}, \text{LINSET})$ with $\mathcal{L}$ is relatively modest if the minimum positive periods of all $L \in \mathcal{L}$ vary only linearly.

The next series of results will present a detailed study of $\text{CF–LINSET}_k$ for each $k$ and $\text{CF–LINSET}$, which comprises all linear sets $\langle P \rangle$ such that $P (\neq \emptyset)$ is a finite subset of $\mathbb{N}$ and $\gcd(P) = 1$. By Proposition 3.2.1 this is precisely the class of cofinite linear subsets of $\mathbb{N}_0$ with constant 0. Since $\text{LINSET}_k$ is a union of classes of linear sets, each of which is isomorphic to $\text{CF–LINSET}_k$, it is hoped that investigating the teaching complexity of $\text{CF–LINSET}_k$ may lead to some insights into the question of whether $\text{RTD}(\text{LINSET}_k)$ is finite for each $k$. $\text{CF–LINSET}$ is also perhaps interesting in its own right: on the one hand, the teaching dimension of $\text{CF–LINSET}_k$ is finite for $k \leq 3$ but infinite for $k \geq 5$; on the other hand, for all $k$, $\text{CF–LINSET}_k$ has a relatively simple teaching sequence that gives it an $\text{RTD}^+$ of $k$. We shall give an almost complete analysis of $\text{TD}(\text{CF–LINSET}_k)$ for all $k$; the case $k = 4$ is left open. First, we make explicit the connection between the present work and the theory of numerical semigroups 39.

The sets in the family $\text{CF–LINSET}$ are called numerical semigroups in the literature (see, for example, [39]). Let $S = \langle P \rangle$ be a numerical semigroup. Denote by $\text{PF}(S)$ the set of pseudo-Frobenius numbers$^5$ of $S$. It is well-known [39] that $\text{PF}(S)$ coincides with the set of maximal elements of $\mathbb{N}_0 \setminus S$ w.r.t. the following partial ordering:

$$a \preceq_P b \iff b - a \in S.$$ 

Note that the partial ordering $\preceq_P$ is a refinement of $\leq_P$. It follows that maximality

$^5$An integer $x$ is a pseudo-Frobenius number of a numerical semigroup $S$ iff $x \notin S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$ [39].
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w.r.t. \( \leq_P \) implies maximality w.r.t. \( \preceq_P \); that is, \( \text{MAX}_P \subseteq \text{PF}(S) \). The following result presents a sufficient condition for \( \text{MAX}_P = \text{PF}(S) \) to hold.

**Lemma 3.2.14** Let \( S = \langle P \rangle \in \text{CF–LINSET} \). Then the following holds:

\[
(\forall a \in \mathbb{N}_0)((a \in \text{PF}(S) \land \{2a, 3a\} \subseteq S) \rightarrow (a \in \text{MAX}_P)).
\]

In particular, if \( 2a, 3a \in S \) for all \( a \in \text{PF}(S) \), then \( \text{MAX}_P = \text{PF}(S) \).

**Proof.** Let \( a \in \text{PF}(S) \). We show that if \( \{2a, 3a\} \subseteq S \), then \( a \in \text{MAX}_P \); in other words, for all \( y \in \mathbb{N}_0 \setminus \langle P \rangle \) such that \( a \preceq_P y \), it holds that \( a = y \). Consider any \( y \in \mathbb{N}_0 \setminus \langle P \rangle \) with \( a \preceq_P y \). There are \( m \in \mathbb{N} \) and \( x \in \langle P \rangle \) such that \( y = ma + x \).

If \( m \geq 2 \), then, since \( \{2a, 3a\} \subseteq S \), it would follow that \( ma + x \in S \), contradicting the choice of \( y \). Thus \( m = 1 \), and so \( a \preceq_P y \). The condition \( a \in \text{PF}(S) \) then yields \( y = a \). \( \square \)

**Theorem 3.2.15**

(i) \( TD(\text{CF–LINSET}_1) = 0 \);

(ii) \( TD(\text{CF–LINSET}_2) = 3 \);

(iii) \( TD(\text{CF–LINSET}_3) = 5 \);

(iv) for each \( k \geq 5 \), \( TD(\text{CF–LINSET}_k) = \infty \).

**Proof.** Assertion (I). Note that \( \text{CF–LINSET}_1 = \{\langle 1 \rangle\} \). The empty set is a teaching set for \( \langle 1 \rangle \) w.r.t. \( \text{CF–LINSET}_1 \).

Assertion (II). We first prove the upper bound. Note that \( \mathbb{N}_0 \) is the only member of \( \text{CF–LINSET}_2 \) that is generated by one number. \( \{\langle 1, + \rangle\} \) is a teaching set for \( \mathbb{N}_0 \),

\( \langle 1, + \rangle \) is not a teaching set for \( \mathbb{N}_0 \).

Assertion (III). \( \mathbb{N}_0 \) is the only member of \( \text{CF–LINSET}_3 \) that is generated by one number. \( \{\langle 1, + \rangle\} \) is a teaching set for \( \mathbb{N}_0 \),

\( \langle 1, + \rangle \) is not a teaching set for \( \mathbb{N}_0 \).

Assertion (IV). For each \( k \geq 5 \), \( \text{CF–LINSET}_k \) is not a teaching set for \( \mathbb{N}_0 \).

Assertion (V). Consider the case where \( k \geq 5 \).

\( \langle 1, + \rangle \) is not a teaching set for \( \mathbb{N}_0 \).

Assertion (VI). Consider the case where \( k \geq 5 \).

\( \langle 1, + \rangle \) is not a teaching set for \( \mathbb{N}_0 \).

Assertion (VII). Consider the case where \( k \geq 5 \).

\( \langle 1, + \rangle \) is not a teaching set for \( \mathbb{N}_0 \).
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w.r.t. CF–LINSET₂, and so TD(N₀, CF–LINSET₂) = 1. Now consider L = (p₁, p₂), where gcd(p₁, p₂) = 1. It is well-known \[39\] Example 2.22 that PF(S) = {p₁p₂ − p₁ − p₂}, and so it follows from Lemma \[3.2.14\] that MAXₚ = {p₁p₂ − p₁ − p₂}.

According to Corollary \[3.2.7\] the set T(P) given by T⁺ = P and T⁻ = MAXₚ is a teaching set w.r.t. LINSET. Since this teaching set consists of two positive examples and one negative example, TD(CF–LINSET₂) ≤ 3.

For the lower bound, choose primes p₁, p₂, p₃ such that 2 < p₁ < p₂ < p₃. Note that \(\langle 2, p₁p₂p₃ \rangle \subseteq \langle 2, p₁p₂ \rangle \subseteq \langle 2, p₁ \rangle\) is a chain in CF–LINSET₂. Thus if T is any teaching set for \(\langle 2, p₁p₂ \rangle\) w.r.t. CF–LINSET₂ such that |T| ≤ 2, then T must contain exactly one positive example \((x₁, +)\) and exactly one negative example \((x₂, −)\). Choose any prime \(p > max\{x₁, x₂, 2, p₁p₂\}\). Then \(\langle x₁, p \rangle\) is consistent with T but \(\langle 2, p₁p₂ \rangle \neq \langle x₁, p \rangle\). Hence |T| ≥ 3.

Assertion \(\text{(III)}\). We first show that TD(CF–LINSET₃) ≤ 5. Let P = \{p₁, p₂, p₃\} such that gcd(P) = 1 and let S = \langle P \rangle. It is well-known \[39\] Corollary 10.22 that S has at most two pseudo-Frobenius numbers; hence |MAXₚ| ≤ 2.

By Corollary \[3.2.7\] the set \{(a, +), (b, +), (c, +)\} ∪ \{(y, −) : y ∈ MAXₚ\}, which has size at most 5 by the preceding argument, is a teaching set for \langle P \rangle w.r.t. CF–LINSET₃. Combining the preceding result with Corollary \[3.2.7\] and the inequality TD(\langle P \rangle, CF–LINSET₃) ≤ TD(\langle P \rangle, LINSET) gives the upper bound of 5. To show that TD(CF–LINSET₃) ≥ 5, consider the set \(E = \langle 3, 10, 14 \rangle\). Note that \{3, 10, 14\} is independent. By Lemma \[3.2.5\] (III), any teaching set for E w.r.t. CF–LINSET₃ must contain at least 3 positive examples. In addition, note that \(E = \langle 3, 5 \rangle \cap \langle 3, 7 \rangle\). Thus T must contain at least one negative instance \((y₁, −)\) to distinguish E from
(3, 5) and at least one negative instance \((y_2, -)\) with \(y_2 \neq y_1\) to distinguish \(E\) from \((3, 7)\), and so \(T\) contains at least two negative examples. Therefore \(|T| \geq 5\).

**Assertion (IV).** We adapt a construction due to J. Backelin [17]. Consider the sequence \((S_{n,r})\) of numerical semigroups such that each \(S_{n,r}\) is given by:

\[
S_{n,r} = \langle P_{n,r} \rangle
\]

where \(n \geq 2, r \geq 3n+2\) and \(s = r(3n+2)+3\). We will show that \(\text{MAX}_{P_{n,r}}\) grows with \(n\) to infinity. As was stated in [17], the set of pseudo-Frobenius numbers of \(S_{n,r}\) satisfies

\[
\text{PF}(S_{n,r}) \supseteq M := \{(r + 1)s + i : -2 \leq i \leq 3n - 1 \land i \not\equiv 0 \pmod{3}\};
\]

The proof of the latter statement was briefly sketched in [17]; for completeness, we flesh out the main steps of the argument. One can show the following: (1) \(M \cap S_{n,r} = \emptyset\); (2) for each period \(x\) of \(S_{n,r}\) and each \(m \in M\), \(m + x \in S_{n,r}\); (3) for each \(m \in M\), \(\{2m, 3m\} \subset S_{n,r}\). It follows from the definition of a pseudo-Frobenius number (see Footnote 5) that (1) and (2) together imply that every element of \(M\) is a pseudo-Frobenius number of \(S_{n,r}\). Lemma 3.2.14 and (3) would then imply that \(|\text{MAX}_{S_{n,r}}|\) is at least as great as \(|M|\).

**Proof of (1).** First, note that for all \(m \in M\), \(m < (r + 2)s\). Consider any \(x \in S_{n,r}\) such that \(x < (r + 2)s\). Since the minimum period of \(S_{n,r}\) is \(s\), \(x\) must be a sum of at most \(\left\lfloor \frac{x}{s} \right\rfloor \leq r + 1\) periods. If \(x\) were a sum of fewer than \(\left\lfloor \frac{x}{s} \right\rfloor\) periods, then

\[
x \leq \left( \left\lfloor \frac{x}{s} \right\rfloor - 1 \right) (s+3n+2) \leq x - s + \left( \left\lfloor \frac{x}{s} \right\rfloor - 1 \right) (3n+2) \leq x - s + (r(3n+2)+3) - 3 = x - 3,
\]

a contradiction. Therefore \(x\) must be a sum of exactly \(\left\lfloor \frac{x}{s} \right\rfloor\) periods. Consider any \(m \in M\) with \(m = (r + 1)s + i\). If \(m\) were a sum of exactly \(\left\lfloor \frac{m}{s} \right\rfloor = r + 1\) periods
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$g_1, \ldots, g_{r+1}$ of $S_{n,r}$, then none of these periods is in $\{s + 3n + 1, s + 3n + 2\}$ as $i < 3n + 1$. If, on the other hand, $\{g_1, \ldots, g_{r+1}\} \subseteq \{s, s + 3\}$, then $i \equiv 0 \pmod{3}$, which is impossible. Hence $m \notin S_{n,r}$.

**Proof of (2).** We show that for each $x \in P_{n,r}$, $m + x \in S_{n,r}$ holds whenever $m = (r + 1)s + i$ for some $i$ with $-2 \leq i \leq 3n - 1 \land i \not\equiv 0 \pmod{3}$. The proof will proceed by means of a case distinction on the value of $i$.

First, we show that for all $m \in M$, $m + s \in S_{n,r}$. For any $i$ with $-1 \leq i \leq 3n$, it holds that $(r + 2)s + 3n - 2 - i = (r - i)(s + 3n + 2) + (i + 1)(s + 3n + 1) \in S_{n,r}$ (note that $r > 3n$). Hence for all $j$ with $-2 \leq j \leq 3n - 1$, $(r + 2)s + j \in S_{n,r}$.

Second, we show that for all $m \in M$, $m + s + 3 \in S_{n,r}$. Suppose $i = 3j + 2$ for some $j$ with $-1 \leq j \leq n - 1$. If $j \leq n - 2$, then a direct calculation gives that $(r + 1)s + i + (s + 3) = (r + 2)s + 3j + 5 = (3n - 3j - 6)(s + 3n + 1) + (r - 3n + 3j + 7)(s + 3n + 2) \in S_{n,r}$. If $j = n - 1$, then $(r + 1)s + 3n - 3 + s + 5 = (r + 2)s + 3n + 2 = (r + 1)s + (s + 3n + 2) \in S_{n,r}$. If $j = n$, then $(r + 1)s + 3n + s + 5 = (r + 2)s + 3n + 5 = rs + (s + 3n + 2) + (s + 3) \in S_{n,r}$. Suppose $i = 3j' + 1$ for some $j'$ with $-1 \leq j' \leq n - 1$. If $j' \leq n - 2$, then $(r + 1)s + 3j' + 1 + s + 3 = (r + 2)s + 3j' + 4 = (3n - 3j' - 5)(s + 3n + 1) + (r - 3n + 3j' + 6)(s + 3n + 2) \in S_{n,r}$. If $j' = n - 1$, then $(r + 1)s + 3n - 3 + 1 + s + 3 = (r + 1)s + (s + 3n + 1) \in S_{n,r}$.

Third, we show that for all $m \in M$, $m + s + 3n + 1 \in S_{n,r}$. Suppose $i = 3j + 2$ for some $j$ with $-1 \leq j \leq n - 1$. Then $(r + 2)s + i + 3n + 1 = (r + 2)s + 3n + 2 + 3j + 1 = (r + 2)s + 3(n + j + 1) = (n + j + 1)s + 3(n + j + 1) + (r + 1 - n - j)s = (n + j + 1)(s + 3) + (r + 1 - n - j)s \in S_{n,r}$. Suppose $i = 3j' + 1$ for some $j'$ with $-1 \leq j' \leq n - 1$. Then $(r + 2)s + i + 3n + 1 = (r + 2)s + 3j' + 3n + 2 = (r + 1 - j')s + j'(s + 3) + (s + 3n + 2) \in S_{n,r}$.
Fourth, we show that for all $m \in M$, $m + s + 3n + 2 \in S_{n,r}$. Suppose $i = 3j + 2$ for some $j$ with $-1 \leq j \leq n - 1$. Then $(r + 2)s + i + 3n + 2 = (r + 2)s + 3j + 3n + 4 = (r - j)s + (j + 1)(s + 3) + (s + 3n + 1) \in S_{n,r}$. Suppose $i = 3j' + 1$ for some $j'$ with $-1 \leq j' \leq n - 1$. Then $(r + 2)s + i + 3n + 2 = (r + 2)s + 3j' + 3n + 3 = (r - j' - n + 1)s + (j' + n + 1)(s + 3) \in S_{n,r}.$

**Proof of (3).** Now it is shown that for each $m \in M$, $2m \in S_{n,r}$. Note first that for any $i$ with $-2 \leq i \leq 3n - 1$ and $i \not\equiv 0 \pmod{3}$, it holds that $2i \not\equiv 0 \pmod{3}$. If $-2 \leq 2i \leq 3n - 1$, then $(2r + 2)s + 2i = (r + 1)s + ((r + 1)s + 2i)$; by (2), $(r + 1)s + 2i \in S_{n,r}$, and therefore $(2r + 2)s + 2i \in S_{n,r}$. Now consider any $i$ with $-2 \leq i \leq 3n - 1$ such that $2i \equiv 1 \pmod{3}$ and $2i \geq 3n + 1$. Then there exists a $j \leq n - 2$ such that $2i = 3(n + 1 + j) + 1$, and so $(2r + 2)s + 2i = (2r + 2)s + 3(n + 1 + j) + 1 = (2r + 2 - j - 1)s + (s + 3n + 1) + 3(j + 1) + s(j + 1) = (2r - j + 1)s + (j + 1)(s + 3) + (s + 3n + 1)$. A similar argument applies if $2i \equiv 2 \pmod{3}$.

To see that $3m \in S_{n,r}$ for each $m \in M$, consider again any $i$ so that $i \leq 3n - 1$. Then $3r + 3 - i \geq 0$, and therefore $(3r + 3)s + 3i = (3r + 3 - i)s + i(s + 3) \in S_{n,r}$. Thus $\{2m, 3m\} \subset S_{n,r}$ for any $m \in M$.

It follows from (1), (2) and (3) that for each $m \in M$, $m \in \text{PF}(S_{n,r})$ and $\{2m, 3m\} \subset S_{n,r}$. Consequently, by Lemma 3.2.14 $M \subseteq \text{MAX}_{P_{n,r}}$. Note that $|\text{MAX}_{P_{n,r}}| \geq |M| = 2n + 2$ grows with $n$ to infinity. According to the second assertion in Lemma 3.2.5 a teaching set for $S_{n,r}$ w.r.t. CF–LINSET$_k$ must contain all elements of $\text{MAX}_{P_{n,r}}$ as negative examples if $k \geq 5$. Thus $\text{TD}(\text{CF–LINSET}_k) = \infty$ holds for all $k \geq 5$.

**Remark 3.2.16** Note that for any family $\mathcal{L}$ such that there are $L_1, L_2 \in \mathcal{L}$ with
$L_1 \subset L_2$, $TD^+(L_1, \mathcal{L}) = TD^+(\mathcal{L}) = \infty$. In particular, $TD^+(\mathcal{L}) = \infty$ for any
$\mathcal{L} \in \{\text{LINSET}_k, \text{CF–LINSET}_{k'}\}$ whenever $k \geq 1$ and $k' \geq 2$.

To establish the next main result, we shall first relate the RTD$^+$ to certain general
structural properties of families of sets over an instance space. Recall that a family
$\mathcal{L}$ of subsets of an instance space $X$ is said to be intersection-closed iff for all
$L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 \in \mathcal{L}$. It is readily seen that LINSET and CF–LINSET are
intersection-closed families.

Say that $S$ is a \textit{weak spanning set} for $L$ w.r.t. $\mathcal{L}$ iff $S \subseteq L$ but $S \not\subseteq L'$ for any
$L' \in \mathcal{L}$ such that $L' \subset L$. That is, $L$ contains $S$ but there is no proper subset of $L$
in $\mathcal{L}$ that contains $S$ as well. Say that $S$ is a \textit{strong spanning set} for $L$ w.r.t. $\mathcal{L}$ iff
$S \subseteq L$ and for all $L' \in \mathcal{L}$, $S \subseteq L'$ holds only if $L \subseteq L'$. In other words, $L$ contains $S$
but the members of $\mathcal{L}$ do not contain $S$ unless they are supersets of $L$. Let $I_W(L, \mathcal{L})$
denote the size of the minimum weak spanning set for $L$ w.r.t. $\mathcal{L}$ (possibly $\infty$) and
let $I_W(\mathcal{L}) = \sup_{L \in \mathcal{L}} I_W(L, \mathcal{L})$ (again possibly $\infty$). Let $I_S(L, \mathcal{L})$ and $I_S(\mathcal{L})$ be the
corresponding notions for strong spanning sets. Note that $I_W$ and $I_S$ are monotonic,
that is,

$$\mathcal{L}' \subseteq \mathcal{L} \Rightarrow I_W(\mathcal{L}') \leq I_W(\mathcal{L}) \quad \text{and} \quad I_S(\mathcal{L}') \leq I_S(\mathcal{L}).$$

Say that $\mathcal{L}$ \textit{contains infinite ascending chains} if there exists a sequence $(L_i)_{i \geq 0}$ of
sets $L_i \in \mathcal{L}$ such that $L_i \subset L_{i+1}$ for all $i \geq 0$.

\textbf{Lemma 3.2.17} Let $\mathcal{L}$ be any family of sets over an instance space.

\begin{enumerate}
\item $\text{RTD}^+(\mathcal{L}) \geq I_W(\mathcal{L})$.
\end{enumerate}

\footnote{A finite weak spanning set for $L$ w.r.t. $\mathcal{L}$ is also known in the computational learning theory
literature as a \textit{tell-tale set} for $L$ w.r.t. $\mathcal{L}$ \cite{3}.}
(i) Suppose that $\mathcal{L}$ does not contain any infinite ascending chain. Then $RTD^+(\mathcal{L}) \leq I_S(\mathcal{L})$.

(ii) Suppose that $\mathcal{L}$ does not contain any infinite ascending chain. Then $RTD^+(\mathcal{L}) \leq I_S(\mathcal{L})$.

Proof. Assertion (I). Let $L \in \mathcal{L}$. In any positive teaching sequence $S$ for $\mathcal{L}$ and any $L', L'' \in \mathcal{L}$ such that $L' \subset L''$, $L''$ must be taught before $L'$. Thus the positive teaching set $S$ for $L$ that is used in $S$ must distinguish $L$ from all its proper subsets in $\mathcal{L}$. Thus $S$ is a weak spanning set for $L$ w.r.t. $\mathcal{L}$. Consequently, $RTD^+(L, \mathcal{L}) \geq I_W(L, \mathcal{L})$ and so $RTD^+(\mathcal{L}) \geq I_W(\mathcal{L})$.

Assertion (II). Let $S$ be a mapping that assigns a minimum strong spanning set $S(L)$ for $L$ w.r.t. $\mathcal{L}$ to every $L \in \mathcal{L}$. $S(L)$ distinguishes $L$ from any $L' \in \mathcal{L}$ except for supersets $L' \supseteq L$. It follows that the digraph $G_S$ induced by $S$ contains only arcs $(L', L) \in \mathcal{L} \times \mathcal{L}$ such that $L \subset L'$. $G_S$ is acyclic, and since $\mathcal{L}$ does not contain any infinite ascending chain, it follows that every node in $G_S$ has a finite depth. Thus $S$ is RTD-admissible and $RTD^+(\mathcal{L}) \leq I_S(\mathcal{L})$.

Assertion (III). Given the stronger condition that $\mathcal{L}$ does not contain any set $L$ that has infinitely many distinct proper supersets in $\mathcal{L}$, the conclusion that $RTD^+(\mathcal{L}) \leq I_S(\mathcal{L})$ can be strengthened to $RTD^+_1(\mathcal{L}) \leq I_S(\mathcal{L})$. Under this stronger condition, the mapping $S$ is even RTD$_1$-admissible.

Lemma 3.2.18 If $\mathcal{L}$ is intersection-closed, then every weak spanning set is a strong one, so that $I_W(\mathcal{L}) = I_S(\mathcal{L})$.

Proof. Let $S$ be a weak spanning set for $L$ w.r.t. $\mathcal{L}$. Consider any $L' \in \mathcal{L}$ such that
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$S \subseteq L'$. The intersection-closed property of $\mathcal{L}$ implies that $S \subseteq L \cap L' \in \mathcal{L}$. By the definition of a weak spanning set, $L \cap L'$ cannot be a proper subset of $L$. Thus $L \cap L' = L$ and so $L \subseteq L'$. Therefore $S$ is a strong spanning set for $L$ w.r.t. $\mathcal{L}$.

Lemma 3.2.19 Let $L = \langle P \rangle \in \text{LINSET}_k$. Then $P$ is a strong spanning set for $L$ w.r.t. LINSET (and thus also w.r.t. LINSET$_k$). Moreover, there is no minimum weak spanning set for $L$ w.r.t. LINSET$_k$ other than $P$ (and thus no minimum weak spanning set for $L$ w.r.t. LINSET other than $P$).

Proof. $P \subseteq L' \in \text{LINSET}$ implies that $L = \langle P \rangle \subseteq L'$. Thus $P$ is a strong spanning set for $L$ w.r.t. LINSET. Now let $S \subseteq L = \langle P \rangle$ be a weak spanning set for $L$ w.r.t. LINSET$_k$, so that $|S| \leq |P| = k$ and $\langle S \rangle \subseteq L = \langle P \rangle$. Since $S$ distinguishes $L$ from any $L' \in \text{LINSET}_k$ such that $L' \subset L$, it follows that $\langle S \rangle = L$. This is only possible if $P \subseteq S$, and since $|S| \leq |P|$, $S = P$.

Corollary 3.2.20 $I_W(\text{LINSET}_k) = I_S(\text{LINSET}_k) = k$.

Lemma 3.2.21 $I_W(\text{CF–LINSET}_k) = I_S(\text{CF–LINSET}_k) = k$.

Proof. Let $P$ be an independent set such that $|P| = k$ and $\gcd(P) = 1$. Let $L = \langle P \rangle$. We claim that $I_W(L|\text{CF–LINSET}_k) \geq k$. Consider any set $S$ with $|S| \leq k - 1$. It suffices to show that there is a proper subset $L' \in \text{CF–LINSET}_k$ of $L$ that contains $S$. If $\gcd(S) = 1$, set $L' = \langle S \rangle$. Suppose that $\gcd(S) > 1$. Choose any prime $p \in L \setminus S$ and set $L' = \langle S \cup \{p\} \rangle$. Note that such a prime exists because $L$ is cofinite. Further, $\gcd(S \cup \{p\}) = 1$, so that $\langle S \cup \{p\} \rangle \in \text{CF–LINSET}_k$.

Lemma 3.2.22 Let $\mathcal{L}_k = \{\langle k, p_1, \ldots, p_{k-1} \rangle : p_i \in \{k+i, 2k+i\}\}$. Then $TD_{\min}(\mathcal{L}_k) \geq k - 1$. 
Proof. Note that for every $\langle P \rangle \in \mathcal{L}_k$, where $P = \{k, p_1, \ldots, p_{k-1}\}$ for some $p_1, \ldots, p_{k-1}$ such that $p_i \in \{k+i, 2k+i\}$, $\langle P \rangle = P \cup \{x : x \geq 2k\}$. Hence if $k+i$ is neither given as a positive example nor as a negative example, then, even if the set $P' = P \setminus \{p_i\}$ is known, it is impossible to distinguish between the numerical semigroup generated by $P' \cup \{k+i\}$ and the numerical semigroup generated by $P' \cup \{2k+i\}$.

Theorem 3.2.23 For all $k \geq 1$, $\text{RTD}^+(\text{CF–LINSET}_k) \in \{k-1, k\}$ and $\text{RTD}^+(\text{CF–LINSET}_k) = \text{RTD}^+(\text{CF–LINSET}_k) = k$. Moreover, $\text{RTD}(\text{CF–LINSET}_2) = 2$.

Proof. We first prove $\text{RTD}^+(\text{CF–LINSET}_k) = k$. Every $L \in \text{CF–LINSET}_k$ is cofinite and thus has only finitely many supersets. Combining Assertions (I) and (III) of Lemma 3.2.17 with Lemma 3.2.21 gives $\text{RTD}^+(\text{CF–LINSET}_k) \geq I_W(\text{CF–LINSET}_k) = k$ and $\text{RTD}^+(\text{CF–LINSET}_k) \leq I_S(\text{CF–LINSET}_k) = k$.

The lower bound $k-1$ for $\text{RTD}(\text{CF–LINSET}_k)$ follows from $k-1 \leq \text{TD}_{\text{min}}(\mathcal{L}_k) \leq \text{RTD}(\mathcal{L}_k)$, where $\mathcal{L}_k$ is as defined in Lemma 3.2.22. The upper bound follows from $\text{RTD}(\text{CF–LINSET}_k) \leq \text{RTD}^+(\text{CF–LINSET}_k) = k$.

To see that $\text{RTD}(\text{CF–LINSET}_2) \geq 2$, consider the subfamily $\mathcal{S} = \{\langle p_1, p_2 \rangle : p_1 \neq p_2 \wedge \{p_1, p_2\} \text{ is independent} \wedge \gcd(p_1, p_2) = 1\}$ and observe that $\text{RTD}(\text{CF–LINSET}_2) \geq \text{RTD}(\mathcal{S}) \geq \min(\{\text{TD}(L, \mathcal{S}) : L \in \mathcal{S}\}) \geq 2$.

Corollary 3.2.24 $\text{TD}(\text{CF–LINSET}) = \text{RTD}(\text{CF–LINSET}) = \text{RTD}^+(\text{CF–LINSET}) = \text{RTD}(\text{LINSET}) = \infty$.

For each $k \in \{1, 2, 3\}$, the result on $\text{TD}(\text{CF–LINSET}_k)$ may be directly applied to construct a teaching sequence of finite order for LINSET$_k$. 


Theorem 3.2.25  
(1) LINSET_2 does not have a positive teaching sequence.

(ii) RTD^+(LINSET_1) = RTD(LINSET_1) = 1, RTD(LINSET_2) = 3 and 3 ≤ RTD(LINSET_3) ≤ 5.

Proof. Assertion (I). Let \( p_1, p_2, p_3, \ldots \) be a strictly increasing infinite sequence of primes with \( p_1 > 2 \). Note that for all \( j \), \( \langle 2 \rangle \subsetneq \langle 2, p_1 \ldots p_j \rangle \). Further, \( \langle 2, p_1 \rangle \supseteq \langle 2, p_1 p_2 \rangle \supseteq \ldots \supseteq \langle 2, p_1 \ldots p_j \rangle \supseteq \langle 2, p_1 \ldots p_j p_{j+1} \rangle \supseteq \ldots \) is an infinite descending chain in LINSET_2. Thus by Proposition 2.3.3 LINSET_2 does not have a positive teaching sequence.

Assertion (II). Right after Proposition 3.2.12 we described a teaching plan with positive examples of order 1 for LINSET_1. Here, we will define an RTD-admissible mapping \( T : LINSET_2 \mapsto \wp(\mathbb{N}_0 \times \{+, -\}) \) such that \( \text{ord}(T) \leq 3 \); an RTD-admissible mapping \( T' : LINSET_3 \mapsto \wp(\mathbb{N}_0 \times \{+, -\}) \) of order no more than 5 can be constructed analogously. For each \( \langle P \rangle \in LINSET_2 \) such that \( P \) is independent, let \( T(L)^+ = P \) and \( T(L)^- = \text{MAX}_P \). One can argue as in Theorem 3.2.13 that \( T \) is RTD-admissible. The proof of Assertion (II) in Theorem 3.2.15 shows that \( |P|^+ + |\text{MAX}_P| \leq 3 \). Thus by Corollary 2.2.4 RTD(LINSET_2) ≤ 3.

To see that RTD(LINSET_2) ≥ 3, assume by way of a contradiction that \( R = ((\mathcal{L}_0, d_0), (\mathcal{L}_1, d_1), (\mathcal{L}_2, d_2), \ldots) \) were a teaching sequence of order at most 2 for LINSET_2. Choose the minimum \( i \) such that \( \mathcal{L}_i \) contains some linear set \( \langle m \rangle \) with \( m \in \mathbb{N} \). Fix such an \( m \) and let \( T \) be a teaching set for \( \langle m \rangle \) w.r.t. \( \bigcup_{j \geq i} \mathcal{L}_j \). Now choose \( i + 1 \) numbers \( p_0, p_1, \ldots, p_i \) with \( \text{max}(X(T)) < p_0 < p_1 < \ldots < p_i \) such that \( \gcd(m, p_l) = 1 \) for all \( l \in \{0, 1, \ldots, i\} \) and \( \langle m, p_0 \rangle \supset \langle m, p_1 \rangle \supset \ldots \supset \langle m, p_i \rangle \). Note that for all \( l \in \{0, 1, \ldots, i\} \), \( \langle m, p_l \rangle \) is consistent with \( T \). Thus \( \langle m, p_l \rangle \in \mathcal{L}_{j_0} \) for some
We claim that \( \langle m, p_{i-1} \rangle \in L_{j_0} \) for some \( j_1 < j_0 \). Suppose not. Then, since 
\( \langle m, p_{i-1} \rangle \supset \langle m, p_i \rangle \supset \langle m \rangle \) and 
\( j_0 < i \), any teaching set \( T' \) for \( \langle m, p_i \rangle \) w.r.t. \( \bigcup_{j \geq j_0} L_j \)
must contain at least one positive example \( (m', +) \) and one negative example. Since \( \text{ord}(R) \leq 2 \), \( (m', +) \) must be the only positive example in \( T' \). Moreover, as \( \langle m' \rangle \) 
is consistent with \( T' \), it must hold that \( \langle m' \rangle \in L_i' \) for some \( i' < j_0 < i \). But this
contradicts the assumption that \( i \) is the least index \( l \) such that \( L_l \) contains a linear
set with exactly one period. Thus \( \langle m, p_{i-1} \rangle \in L_{j_1} \) for some \( j_1 < j_0 \). Repeating
a similar argument for \( \langle m, p_{i-2} \rangle, \ldots, \langle m, p_0 \rangle \) gives that \( \langle m, p_0 \rangle, \langle m, p_1 \rangle, \ldots, \langle m, p_i \rangle \)
belong to \( L_{j_1} \), \( L_{j_1-1} \), \ldots, \( L_{j_0} \) respectively for some \( j_0, j_1, \ldots, j_i \in \mathbb{N}_0 \) such that
\( j_i < j_{i-1} < \ldots < j_0 < i \), which is impossible. Therefore no such teaching sequence
\( R \) for \( \text{LINSET}_2 \) exists, and so \( \text{RTD}(\text{LINSET}_2) \geq 3 \).

A **semilinear set** is a finite union of linear sets. Semilinear sets are objects of
interest in models of parallel computation such as matrix grammars and Petri nets;
for example, the Parikh image of an equal matrix language [44], as well as any
reachability set of a weakly persistent Petri net [23], is semilinear. Unfortunately,
the next result shows that even the class of semilinear sets that are unions of at
most two linear sets generated by one period has infinite RTD.

**Definition 3.2.26** Let \( m, n \in \mathbb{N} \). The class of semilinear sets \( \{ \langle P_1 \rangle \cup \ldots \cup \langle P_n \rangle : 0 < |P_i| \leq m \} \) is denoted by \( \text{SL}_{m,n} \).

**Theorem 3.2.27** For all \( m, n \in \mathbb{N} \) with \( n \geq 2 \), \( \text{RTD}(\text{SL}_{m,n}) = \infty \).

**Proof.** By the monotonicity of RTD, it suffices to show that \( \text{RTD}(\text{SL}_{1,2}) = \infty \).
Suppose there were some mapping \( T : \text{SL}_{1,2} \mapsto \mathbb{N}_0 \times \{+, -\} \) such that \( T \) is RTD-
admissible; \( T \) induces an acyclic digraph \( G = (V_G, A_G) \). For all \( L \in \text{SL}_{1,2} \), let
Let \( v_L \) be the node in \( G \) representing \( L \). Suppose that \( \text{depth}(v_{\langle 2 \rangle}) = i < \infty \). Choose \( N > \max(\{\max(X(T(\langle 2 \rangle))), \sup\{|T(L)| : L \in \text{SL}_{1,2} \wedge \text{depth}(V_L) \leq i\}\}) \). Note that such an \( N \) can be found because \( \{\max(X(T(L))) : L \in \text{SL}_{1,2}\} \) is bounded. Consider the semilinear set \( \text{SL}_1 = \langle 2 \rangle \cup \langle p_1 \cdot \ldots \cdot p_{N+i} \rangle \), where, for all \( j \geq 2 \), \( p_{j-1} \) is the \( j^{th} \) prime number. Note that since \( \text{SL}_1 \) is consistent with \( T(\langle 2 \rangle) \), \( \text{depth}(v_{\text{SL}_1}) = i - 1 \).

Furthermore, given any two distinct \((N + i - 1)\)-subsets \( S_1, S_2 \) of \( \{p_1, \ldots, p_{N+i}\} \),
\[
\left( \langle 2 \rangle \cup \langle \Pi_{j \in S_1} p_j \rangle \right) \cap \left( \langle 2 \rangle \cup \langle \Pi_{j \in S_2} p_j \rangle \right) = \text{SL}_1.
\]
Since \( N > \sup\{|T(L)| : L \in \text{SL}_{1,2} \wedge \text{depth}(V_L) \leq i\} \), there is an \((N + i - 1)\)-subset \( S \) of \( \{p_1, \ldots, p_{N+i}\} \) such that \( \text{SL}_2 = \langle 2 \rangle \cup \langle \Pi_{j \in S} p_j \rangle \) has depth less than \( i - 1 \). Repeating a similar argument \( i \) times produces some semilinear set \( \text{SL}_{i+1} \) that has depth less than 0, a contradiction. \( \blacksquare \)

### 3.3 Linear Subsets of \( \mathbb{N}_0 \) With Bounded Period Sums

The present section will examine a special family of linear subsets of \( \mathbb{N}_0 \) that arises from studying an invariant property of the class of non-erasing pattern languages over unary alphabets. In Chapter 4 we shall establish the values of the TD and RTD for various families of non-erasing pattern languages over alphabets of arbitrary size.

Recall that the commutative image, or Parikh image, of \( w \in \{a\}^* \) is the number of times that \( a \) appears in \( w \), or the length of \( w \). Thus the commutative image of the language generated by a non-erasing pattern \( a^{k_0}x_1^{k_1} \ldots x_n^{k_n} \) is the linear subset \((k_0 + k_1 + \ldots + k_n) + \langle k_1, \ldots, k_n \rangle \) of \( \mathbb{N}_0 \), which is in NE–LINSET. Conversely, any \( L \in \text{NE–LINSET} \) is the commutative image of a non-erasing pattern language. This gives a one-to-one correspondence between the class of non-erasing pattern languages
and the class NE–LINSET, so that the two classes have equivalent teachability properties. We will establish in this section that the RTD$^+$ of NE–LINSET$^+_k$ is exactly $k + 1$. Our first observation is a consequence of Proposition 2.3.8; note that for all $k$, NE–LINSET$^+_k$ has finite thickness and $\emptyset \notin$ NE–LINSET$^+_k$.

**Proposition 3.3.1** $\text{RTD}^+(\text{NE–LINSET}_k) = \text{RTD}^+_1(\text{NE–LINSET}_k)$.

The following lemma gives an upper bound on $\text{RTD}^+(\text{NE–LINSET}_k)$.

**Lemma 3.3.2** $\text{RTD}^+(\text{NE–LINSET}_k) \leq k + 1$.

**Proof.** It suffices to show that there is a teaching plan of order $k + 1$ for NE–LINSET$^+_k[c]$ w.r.t. $\bigcup_{c' > c} \text{NE–LINSET}_k[c']$. Consider an arbitrary $L = c + \langle P \rangle \in \text{NE–LINSET}_k[c]$. The example $(c, +)$ distinguishes $L$ from all sets in $\bigcup_{c' > c} \text{NE–LINSET}_k[c']$.

Thus the presentation of $(c, +)$ collapses the teaching problem to teaching $\langle P \rangle$ w.r.t. LINSET$^+_k[c]$. It suffices therefore to show that $\text{RTD}^+(\text{LINSET}_k[c]) \leq k$. We may conclude from the third assertion of Lemma 3.2.17 that $\text{RTD}^+(\text{LINSET}_k[c]) \leq I_S(\text{LINSET}_k[c])$. An application of Corollary 3.2.20 gives that $I_S(\text{LINSET}_k[c]) = I_S(\text{LINSET}_k) = k$. Putting everything together, the inequality $\text{RTD}^+(\text{NE–LINSET}_k) \leq k + 1$ is obtained.

Next, we show that the upper bound in Lemma 3.3.2 is tight. The main argument is based on the following lemma.

**Lemma 3.3.3** Suppose that $\gcd(a_1, \ldots, a_k) = 1$, and that $\{a_1, \ldots, a_k\}$ is independent. For $i \in \{0, \ldots, a_1 - 1\}$, let $t_i = \min(\{s \in \langle a_1, \ldots, a_k \rangle : s \equiv i \pmod{a_1}\})$. Choose any integer $N > \sum_{i=0}^{a_1-1} (a_1 + t_i)$. Set $L = N + \langle a_1, \ldots, a_k \rangle$. Then the teaching dimension of $L$ with respect to $\{H \in \text{NE–LINSET}_k[N] : H \subseteq L\}$ is equal to $k$. 


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Proof. First, it is shown that $\text{TD}(L, \{H \in \text{NE–LINSET}_k[N] : H \subseteq L \}) \geq k$. Assume by way of a contradiction that $\text{TD}(L, \{H \in \text{NE–LINSET}_k[N] : H \subseteq L \}) \leq k - 1$. In particular, there is a teaching set $S$ for $L$ that contains exactly $\ell$ positive examples $(N + x_1, +), \ldots, (N + x_{\ell}, +)$ for some $\ell \leq k - 1$. Now, for every $i \in \{1, \ldots, \ell\}$, there are nonnegative integers $c_i, j_i, K^i_1, \ldots, K^i_k$ such that $x_i = c_i a_1 + t_{j_i} = \sum_{w=1}^{k} K^i_w a_w$. The assumption $\ell \leq k - 1$ implies that at least one of the following cases holds: (i) $x_i \in \langle a_2, \ldots, a_k \rangle$ for all $i \in \{1, \ldots, \ell\}$; (ii) for some $p \in \{2, \ldots, k\}$, and for all $i \in \{1, \ldots, \ell\}$ such that $K^i_p > 0$, there is some $w \in \{1, \ldots, k\} \setminus \{p\}$ for which $K^i_w > 0$. To see the reason for this case distinction, suppose that for all $q \in \{2, \ldots, k\}$, there is at least one $i_q$ such that $K^i_q > 0$ and $K^i_w = 0$ for all $w \in \{1, \ldots, k\} \setminus \{q\}$. Note that since $a_{q_1} \neq a_{q_2}$ whenever $q_1 \neq q_2$, it must hold that $x_{i_q} \neq x_{i_q'}$ for any distinct $q, q' \in \{2, \ldots, k\}$. Hence $\ell = k - 1$ and $x_i \in \langle a_2, \ldots, a_k \rangle$ for all $i \in \{1, \ldots, \ell\}$.

Suppose that (i) holds. Then $H = N + \langle \{a_1, \ldots, a_k\} \setminus \{a_1\} \rangle$ is a proper subset of $L$ that is consistent with $S$. Therefore $S$ cannot be a teaching set for $L$ with respect to $\{H \in \text{NE–LINSET}_k[N] : H \subseteq L \}$.

Suppose that (ii) holds. For each $i \in \{1, \ldots, \ell\}$, define

$$s_i = \begin{cases} a_1 + t_{j_i} & \text{if } c_i > 0; \\ t_{j_i} & \text{if } c_i = 0. \end{cases}$$

Let $H = N + \langle \{a_1\} \cup \{s_i : 1 \leq i \leq \ell\} \rangle$. By the case assumption, if $t_{j_{i'}} = a_p$ for some $i' \in \{1, \ldots, \ell\}$, then $c_{i'} > 0$ and so $s_{i'} > a_p$. As $\{a_1, \ldots, a_k\}$ is independent, it follows that $N + a_p \not\in H$ and therefore $H$ is a proper subset of $L$. Furthermore, the lower bound on $N$ in the hypothesis of the lemma ensures that $H \in \text{NE–LINSET}_k$. 
In addition, $H$ is consistent with $S$, and therefore $S$ cannot be a teaching set for $L$ with respect to $\{H \in \text{NE–LINSET}_k[N] : H \subseteq L\}$.

This completes the case distinction and establishes that $\text{TD}(L, \{H \in \text{NE–LINSET}_k[N] : H \subseteq L\}) \geq k$.

To see that $\text{TD}(L, \{H \in \text{NE–LINSET}_k[N] : H \subseteq L\}) \leq k$, consider the teaching set $S_0 = \{(N + a_i, +) : 1 \leq i \leq k\}$. Any concept $H$ in $\text{NE–LINSET}_k$ that has constant $N$ and is consistent with $S_0$ must be a superset of $L$. As $L$ is maximal (with respect to the subset relation) in $\{H \in \text{NE–LINSET}_k[N] : H \subseteq L\}$, this means that $H = L$. Thus $S_0$ is a teaching set for $L$, and so $\text{TD}(L, \{H \in \text{NE–LINSET}_k[N] : H \subseteq L\}) \leq k$, as required.

Lemma 3.3.3 immediately gives a lower bound of $k$ for $\text{RTD}^+ (\text{NE–LINSET}_k)$. Next, we show how to tighten the lower bound to $k + 1$.

**Lemma 3.3.4** For all $k \geq 1$, $\text{RTD}^+ (\text{NE–LINSET}_k) \geq k + 1$.

**Proof.** Assume by way of a contradiction that there exists a teaching sequence with positive examples for $\text{NE–LINSET}_k$ of order no more than $k$; for the present proof, it is enough to assume that the order of this teaching sequence is exactly $k$. Further, owing to the finite thickness of $\text{NE–LINSET}_k$, Proposition 2.3.8 implies that any positive teaching sequence $R$ for $\text{NE–LINSET}_k$ can be converted into a teaching plan with positive examples for $\text{NE–LINSET}$ that has the same order as $R$. So it suffices to assume that there is a teaching plan $Q = ((L_0, S_0), (L_1, S_1), \ldots)$ for $\text{NE–LINSET}_k$ such that $Q$ uses only positive examples as teaching sets and $\text{ord}(Q) = k$. The following observation will be used: for any $L_i, L_j$ such that $L_i \subseteq L_j$, $L_i$ must occur
after $L_j$ in $Q$, that is, $j < i$.

First, assume $k = 1$. Consider any $N + \langle 1 \rangle \in \text{NE-LINSET}_1$ such that $N > 1$. Suppose $L_i = N + \langle 1 \rangle$. Then $S_i$ must contain at least one positive example $(N', +)$ with $N' > N$; otherwise, $S_i$ would be consistent with $N + \langle 2 \rangle$, which is impossible since $N + \langle 2 \rangle \subsetneq L_i$ and so $N + \langle 2 \rangle$ must occur after $L_i$ in $Q$. Further, if $S_i$ contains exactly one positive example $(N', +)$ with $N' > N$, then the singleton $\{N'\}$ is consistent with $S_i$, which is impossible since $\{N'\} \subsetneq L_i$ implies that $\{N'\}$ must occur after $L_i$ in $Q$.

Second, assume $k \geq 2$. Choose some $p > 0$ such that $\{p, p + 1, \ldots, p + k\}$ is independent. Let $M_0 = F(p, p + 1)$; recall that $F(p, p + 1)$ denotes the Frobenius number of $\langle p, p + 1 \rangle$. Note that $M_0 \geq p + k$. Further, fix some $N \geq 5$ so that for all $k$-subsets $\{a_1, \ldots, a_k\} \subseteq \{p, p + 1, \ldots, M_0\}$ with $t_i = \min(\{s \in \langle a_1, \ldots, a_k \rangle : s \equiv i \ (\text{mod } a_1)\})$ for each $i \in \{0, \ldots, a_1 - 1\}$, $N > \sum_{i=0}^{a_1-1}(a_1 + t_i)$.

Now consider the concept $L_i = N + p + \langle 1 \rangle$. Note that $\{(N + p, +), (N + p + 1, +)\} \subseteq S_i$ holds; otherwise, $S_i$ would be consistent with either $d_1 = N + p + \langle 2, 3 \rangle$ or $d_2 = N + p + 1 + \langle 1 \rangle$, which is impossible as both $d_1$ and $d_2$, being proper subsets of $L_i$, must occur after $L_i$ in $Q$. Let $A$ be the set of all integers $m$ such that $(N + m, +) \in S_i$, and let $B$ be an independent set so that $\langle A \rangle = \langle B \rangle$. The choice of $M_0$ gives that $\{p, p + 1\} \subseteq B \subseteq \{p, p + 1, \ldots, M_0\}$. Furthermore, because $\{p, p + 1, \ldots, p + k\}$ is independent, there exists an independent set $G = \{g_1, \ldots, g_k\}$ of size $k$ such that $G \subseteq \{p, p + 1, \ldots, M_0\}$, $\langle B \rangle \subseteq \langle G \rangle$, and $N > \sum_{i=0}^{g_1-1}(g_1 + t_i)$, where $t_i = \min(\{s \in \langle g_1, \ldots, g_k \rangle : s \equiv i \ (\text{mod } g_1)\})$ for each $i \in \{0, \ldots, g_1 - 1\}$.

Now consider the concept $L_j = N + \langle G \rangle$. Since $S_i$ is consistent with $L_j$, $j < i$. By Lemma 3.3.3 and the fact that $\gcd(p, p + 1) = 1$, the teaching dimension of $L_j$
with respect to \( \{ H \in \text{NE–LINSET}_k[N] : H \subseteq L_j \} \) is at least \( k \). However, as every member of \( N + \langle G \rangle \) greater than \( N + p - 1 \) is contained in \( N + p + \langle 1 \rangle \), it follows that \( (N,+) \in S_j \). Thus \(|S_j| \geq k + 1\), and therefore \( \text{ord}(Q) \geq k + 1 \).

Combining Lemma 3.3.4 with Lemma 3.3.2 and Proposition 3.3.1 gives the main result of this section.

**Theorem 3.3.5** \( RTD^+(\text{NE–LINSET}_k) = RTD^+_1(\text{NE–LINSET}_k) = k + 1 \).

The lower bound on \( RTD(\text{NE–LINSET}_k) \) in the following theorem may be obtained by adapting the proof of the corresponding result for CF–LINSET_\( k \); the proof of [31 Theorem 6] immediately implies that \( \text{TD}(\text{NE–LINSET}_k) = \infty \).

**Theorem 3.3.6** For all \( k \geq 1 \), \( k - 1 \leq RTD(\text{NE–LINSET}_k) \leq k + 1 \) and \( \text{TD}(\text{NE–LINSET}_k) = \infty \).

**Remark 3.3.7** Our results on \( \text{NE–LINSET} \) may be generalised to classes of linear subsets of \( \mathbb{N}^m_0 \) for any \( m > 1 \) in the following way. For each \( m \), define

\[
(1) \quad \text{NE–LINSET}_k^m := \{ c + \langle P \rangle : c \in \mathbb{N}^m_0 \land P \subset \mathbb{N}^m_0 \land |P| \leq k \land \| \sum_{p \in P} p \|_1 \leq \| c \|_1 \}.
\]

\[
(11) \quad \text{NE–LINSET}^m := \bigcup_{k \in \mathbb{N}} \text{NE–LINSET}_k^m.
\]

Note that \( \text{NE–LINSET}_k^1 = \text{NE–LINSET}_k \) and \( \text{NE–LINSET}^1 = \text{NE–LINSET} \). Then one has \( RTD^+(\text{NE–LINSET}_k^m) = RTD^+(\text{NE–LINSET}_k) = k + 1 \), \( k - 1 \leq RTD(\text{NE–LINSET}_k^m) \leq RTD(\text{NE–LINSET}_k^m) \) and \( RTD(\text{NE–LINSET}_k) = \infty \).
3.4 Linear Subsets of $\mathbb{N}_0^2$ With Constant 0

Finally, we consider how our preceding results may be extended to general classes of linear subsets of higher dimensions. Finding teaching sequences for families of linear sets with dimension $m > 1$ seems to present a new set of challenges, as many of the proof methods for the case $m = 1$ do not carry over directly to the higher dimensional cases. The classes of linear subsets of $\mathbb{N}_0^2$ briefly studied in this section are denoted as follows. In the first definition, $k \in \mathbb{N}$.

1. $\text{LINSET}_2^k := \{\langle P \rangle : P \subset \mathbb{N}_0^2 \land \exists p \in P[p \neq 0] \land |P| \leq k\}$.

2. $\text{LINSET}_{2=2}^2 := \text{LINSET}_2^2 \setminus \text{LINSET}_1^2$.

We will show later that the RTD of $\text{LINSET}_{2=2}^2$ is either 3 or 4. The following result suggests that to identify interesting classes of linear subsets of $\mathbb{N}_0^m$ for $m > 1$ that have finite teaching complexity measures, it might be a good idea to first exclude certain linear sets.

**Proposition 3.4.1** \(\text{RTD}(\{(0,1)\}, \text{LINSET}_2^2) = \infty\).

**Proof.** Assume that $R = ((\mathcal{L}_0,d_0), (\mathcal{L}_1,d_1), \ldots)$ were a teaching subsequence for $\text{LINSET}_2^2$ covering $\{(0,1)\}$. Suppose that $\langle (0,1) \rangle \in \mathcal{L}_i$ and $T$ were a teaching set for $\langle (0,1) \rangle$ w.r.t. $\text{LINSET}_2^2 \setminus \bigcup_{j<i} \mathcal{L}_j$. Choose any $N > \max\{|d_j : j < i\}$ such that $N$ is larger than every component of any instance $(a,b) \in \mathbb{N}_0^2$ in $T$. Further, let $p_0, \ldots, p_{N+i}$ be a strictly increasing sequence of primes. Observe that by the choice of $N$, $\langle (0,1), p_0 \ldots p_{N+i}(1,1) \rangle$ is consistent with $T$. Hence this linear set
occurs in some \( L_{j_0} \) with \( j_0 < i \). In addition, since, for any two distinct \((N + i)\)-subsets \( S, S' \) of \( \{p_0, \ldots, p_{N+i}\} \), \( \langle (0,1), p_0p_1 \ldots p_{N+i}(1,1) \rangle \subseteq \langle (0,1), \prod_{x \in S} x(1,1) \rangle \) and \( \langle (0,1), \prod_{x \in S'} x(1,1) \rangle \cap \langle (0,1), \prod_{x \in S} x(1,1) \rangle \subseteq \langle (0,1), p_0 \ldots p_{N+i}(1,1) \rangle \), the choice of \( N \) again gives that for some \((N+i)\)-subset \( S_1 \) of \( \{p_0, \ldots, p_{N+i}\} \), \( \langle (0,1), \prod_{x \in S_1} x(1,1) \rangle \in L_{j_1} \) for some \( j_1 < j_0 \). The preceding line of argument can be applied again to show that for some \((N+i-1)\)-subset \( S_2 \) of \( S_1 \), \( \langle (0,1), \prod_{x \in S_2} x(1,1) \rangle \in L_{j_2} \) for some \( j_2 < j_1 \). Repeating the argument successively thus yields a chain \( S_1 \supseteq S_2 \supseteq \ldots \supseteq S_i \) of subsets of \( \{p_0, \ldots, p_{N+i}\} \) such that \( \langle (0,1), \prod_{x \in S_l} x(1,1) \rangle \in L_{j_l} \) for all \( l \in \{1, \ldots, i\} \), where \( j_i < \ldots < j_1 < j_0 < i \), which is impossible as \( j_i \geq 0 \). Hence there is no teaching subsequence of LINSET\(_{=2}^2\) covering \( \langle \langle (0,1) \rangle \rangle \).

One can define quite a meaningful subclass of LINSET\(_{=2}^2\) that does have a finite RTD. LINSET\(_{=2}^2\) consists of all linear sets in LINSET\(_{=2}^2\) that are strictly 2-generated. Examples of strictly 2-generated linear subsets include \( \langle (1,0), (0,1) \rangle \) and \( \langle (4,6), (6,9) \rangle \). \( \langle (0,1) \rangle \) is not a strictly 2-generated linear subset.

**Theorem 3.4.2**  
(i) \( TD(LINSET_{=2}^2) = \infty \).

(ii) LINSET\(_{=2}^2\) does not have a positive teaching sequence.

(iii) \( RTD(LINSET_{=2}^2) \in \{3, 4\} \).

**Proof.** Assertion (I). Observe from the proof of Proposition 3.4.1 that for any \( N \) distinct primes \( p_0, p_1, \ldots, p_{N-1} \), \( TD(\langle (0,1), p_0p_1 \ldots p_{N-1}(1,0) \rangle, LINSET_{=2}^2) \geq N \). Hence \( TD(\mathcal{L}, LINSET_{=2}^2) = \infty \) for any cofinite subclass \( \mathcal{L} \) of LINSET\(_{=2}^2\).

Assertion (II). Let \( p_1, p_2, p_3, \ldots \) be a strictly increasing infinite sequence of primes. Note that for all \( j \), \( \langle (2,0), (3,0) \rangle \subseteq \langle (1,0), p_1 \ldots p_j(0,1) \rangle \). Further, \( \langle (1,0) \rangle \)
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\[ p_1(0,1) \supseteq \langle (1,0), p_1 p_2(0,1) \rangle \supseteq \langle (1,0), p_1 p_2 p_3(0,1) \rangle \supseteq \ldots \supseteq \langle (1,0), p_1 \ldots p_j(0,1) \rangle \supseteq \langle (1,0), p_1 \ldots p_j p_{j+1}(0,1) \rangle \supseteq \ldots \] is an infinite descending chain in LINSET\[2\]_m. Thus by Proposition 2.3.3, LINSET\[2\]_m does not have a positive teaching sequence.

**Assertion (III).** The main idea is that for each strictly 2-generated linear subset \( S \) of \( \mathbb{N}_0^2 \) with canonical representation \((0, P)\), if \( M \) denotes the class of all \( S' \in \text{LINSET}^2_2 \) for which each \( S' \in M \) with canonical representation \((0, P')\) satisfies \( \| \sum_{p' \in P'} p' \|_1 \geq \| \sum_{p \in P} p \|_1 \), then TD\((S, M)\) \( \leq 4 \). The sequence \((\mathcal{L}_0, d_0), (\mathcal{L}_1, d_1), \ldots\) defined by \( \mathcal{L}_i = \{ \langle u_1, u_2 \rangle : \| u_1 + u_2 \|_1 = i+2 \} \) would then be a teaching sequence for LINSET\[2\]_m of order at most 4. To prove this assertion, it suffices to find a teaching set of size at most 4 for any \( \langle u_1, u_2 \rangle \) w.r.t. the class of all \( S' \in \text{LINSET}^2_2 \) such that if \( S' \) has the canonical representation \((0, P')\), then \( \| \sum_{p' \in P'} p' \|_1 \geq \| u_1 + u_2 \|_1 \).

Case (i): \( \{u_1, u_2\} \) is linearly independent. For a given linear set \( L \) with canonical representation \((c, P)\), call each \( p \in P \) a minimal period of \( L \). Assume that \( u_1 \) lies to the left of \( u_2 \). Consider the set \( \mathcal{A} \) of linear sets \( L \) in \( M \) such that \( \langle u_1, u_2 \rangle \subseteq L \). Since no single vector in \( \mathbb{N}_0^2 \) can generate two linearly independent vectors in \( \mathbb{N}_0^2 \), each \( L \in \mathcal{A} \) must have two linearly independent periods \( p_1 \) and \( p_2 \), neither of which lies strictly between \( u_1 \) and \( u_2 \); in addition, \( \max(\{\| p_1 \|_1, \| p_2 \|_1\}) \leq \max(\{\| u_1 \|_1, \| u_2 \|_1\}) \).

Thus \( \mathcal{A} \) is finite. Furthermore, for each \( L \in \mathcal{A} \) with canonical representation \((0, P')\), at least one of the periods in \( P' \) is not parallel to \( u_1 \) and also not parallel to \( u_2 \), for otherwise \( \| \sum_{p' \in P'} p' \|_1 < \| u_1 + u_2 \|_1 \). If \( \mathcal{A} = \emptyset \), then \( \{(u_1, +), (u_2, +)\} \) is a teaching set for \( \langle u_1, u_2 \rangle \) w.r.t. \( M \). Assume that \( \mathcal{A} \neq \emptyset \). Consider the set \( Q = \bigcup_{L \in \mathcal{A}} \{ w : w \text{ is a minimal period of } L \text{ not parallel to } u_1 \text{ and not parallel to } u_2 \} \). Choose some \( p_1 \) among the periods in \( Q \) that are closest to \( u_1 \) to the left of \( u_1 \), and choose \( p_2 \) so that \( p_2 \) is among the periods in \( Q \) that are closest to \( u_2 \) to the right of \( u_2 \) (see Figure 3.1).
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Note that at least one of \( p_1, p_2 \) exists. For every \( L \in \mathcal{A} \) with canonical representation \((0, \{v_1, v_2\})\), at least one of \( p_1 \) and \( p_2 \) lies between (not necessarily strictly) \( v_1 \) and \( v_2 \), and \( \{kp_1, k'p_2\} \cap \langle u_1, u_2 \rangle = \emptyset \) for all \( k, k' \in \mathbb{N} \). Thus there is a sufficiently large \( K \in \mathbb{N} \) such that for all \( L \in \mathcal{A} \), either \( Kp_1 \in L \setminus \langle u_1, u_2 \rangle \) or \( Kp_2 \in L \setminus \langle u_1, u_2 \rangle \). Therefore a teaching set for \( \langle u_1, u_2 \rangle \) w.r.t. \( \mathcal{M} \) is \( \{(u_1, +), (u_2, +), (Kp_1, -), (Kp_2, -)\} \). If \( p_i \) does not exist for exactly one \( i \), then remove \( (Kp_i, -) \) from this teaching set.

Case (ii): \( \{u_1, u_2\} \) is linearly dependent. Then \( u_1 \) and \( u_2 \) can be expressed as \( ma \) and \( na \) respectively for some \( a \in \mathbb{Q}^2 \) and some \( m, n > 1 \) such that \( \gcd(m, n) = 1 \).

Consider the set \( \mathcal{B} \) of linear sets \( L \) in \( \mathcal{M} \) such that \( \langle u_1, u_2 \rangle \not\subset L \). The case \( \mathcal{B} = \emptyset \) can be dealt with as in Case (i). Assume that \( \mathcal{B} \neq \emptyset \). Given any \( L \in \mathcal{B} \), let \( w_1 \) and \( w_2 \) be the periods of \( L \). Then either \( u_1 \) or \( u_2 \) cannot be written in the form \( \lambda_1 w_1 + \lambda_2 w_2 \) for any \( \lambda_1, \lambda_2 > 0 \), for otherwise the sum of the components of \( L \)'s periods would be less than \( \|u_1 + u_2\|_1 \). Thus exactly one of the periods \( \{w_1, w_2\} \), say \( w_1 \), is such that \( ma = k_1w_1 \) and \( na = k_2w_1 \) for some \( k_1, k_2 \in \mathbb{N} \). Hence \( mk_2 = nk_1 \), and as \( \gcd(m, n) = 1 \), \( k_2 = q_1n \) and \( k_1 = q_2m \) for some \( q_1, q_2 \in \mathbb{N} \). This implies that \( a = q_2w_1 = q_1w_1 \in L \setminus \langle u_1, u_2 \rangle \), so that \( \{(u_1, +), (u_2, +), (a, -)\} \) is a teaching set for \( \langle u_1, u_2 \rangle \) with respect to \( \mathcal{M} \).

Now define \( \mathcal{L}_{2i} = \{\langle u_1, u_2 \rangle : \|u_1 + u_2\|_1 = i + 2 \wedge \{u_1, u_2\} \text{ is linearly independent}\} \) and \( \mathcal{L}_{2i+1} = \{\langle u_1, u_2 \rangle : \|u_1 + u_2\|_1 = i + 2 \wedge \{u_1, u_2\} \text{ is linearly dependent}\} \) for all \( i \in \mathbb{N}_0 \). The analyses of Cases (i) and (ii) show that \( (\mathcal{L}_0, d_0), (\mathcal{L}_1, d_1), \ldots \) is indeed a teaching sequence for \( \text{LINSET}_{2}^{2} \) of order at most 4.

To prove the lower bound of 3, suppose for the sake of a contradiction that \( R = ((\mathcal{Q}_0, d_0), (\mathcal{Q}_1, d_1), \ldots) \) were a teaching sequence of order 2 for \( \text{LINSET}_{2}^{2} \). Let \( i \) be
the minimum index for which $Q_i$ contains some $L \in \text{LINSET}^2_{=2}$ generated by two linearly dependent vectors $mw$ and $nw$ for some $w \in \mathbb{N}^2$ and integers $m, n > 1$ with $\gcd(m, n) = 1$. Choose $i + 1$ primes $p_0, \ldots, p_i$ with $p_0 < \ldots < p_i$ and $v_1, v_2 \in \mathbb{N}^2$ such that $v_1$ lies to the left of $w$ and $v_2$ lies to the right of $w$. Let $p$ be any prime such that $p > \max\{m, n\}$. Since $\langle mw, npw \rangle \subseteq L$, the choice of $i$ implies that a recursive teaching set $T$ for $L$ with respect to $R$ must contain at least one positive example, say $(m'w, +)$. As $\langle p_0 \ldots p_iv_j, m'w \rangle$ is consistent with $\{(m'w, +)\}$ for any $j \in \{1, 2\}$ and $\langle p_0 \ldots p_iv_1, m'w \rangle \cap \langle p_0 \ldots p_iv_2, m'w \rangle \subseteq \langle m'w \rangle \subseteq L$, at least one of $\langle p_0 \ldots p_iv_1, m'w \rangle$ and $\langle p_0 \ldots p_iv_2, m'w \rangle$ belongs to some $Q_{j_0}$ with $j_0 < i$; suppose that $\langle p_0 \ldots p_iv_1, m'w \rangle$ satisfies this condition. As $\langle p_0 \ldots p_iv_1, m'w \rangle \not\subseteq \langle p_0 \ldots p_{i-1}v_1, m'w \rangle$ and $\{kw : k \in \mathbb{N}_0\} \cap \langle p_0 \ldots p_iv_1, m'w \rangle = \{kw : k \in \mathbb{N}_0\} \cap \langle p_0 \ldots p_{i-1}v_1, m'w \rangle$, either $\langle p_0 \ldots p_{i-1}v_1, m'w \rangle$ belongs to some $Q_{j_1}$ with $j_1 < j_0$, or any recursive teaching set $T$ for $\langle p_0 \ldots p_iv_1, m'w \rangle$ with respect to $R$ contains at least one negative example $(v_4, -)$ such that $v_4 \notin \{kw : k \in \mathbb{N}_0\}$. In the latter case, there is some prime $p' > m'$ such that $\{v_4, -\}$ is also consistent with $\langle m'w, p'w \rangle$; by the choice of $i$, $T$ must contain a positive example $(v_3, +)$ to distinguish $\langle p_0 \ldots p_iv_1, m'w \rangle$ from $\langle m'w, p'w \rangle$. Further, there exists some $L' = \langle p_0 \ldots p_iv_5, v_3 \rangle$ such that $v_3$ lies strictly between $v_4$ and $v_3$, so that $L'$ is consistent with $T$ and must belong to some $Q_{j_1}$ with $j_1 < j_0$.

One can then apply the preceding line of argument again to either $L'$ or $\langle p_0 \ldots p_{i-1}v_1, m'w \rangle$ and yield some $L'' \in \text{LINSET}^2_{=2}$ such that $L''$ has a linearly independent set of periods, one of which is equal to $p_0 \ldots p_lv'$ for some $l \geq i - 2$, $v' \in \mathbb{N}_2$, and $L'' \in Q_{j_2}$ for some $j_2 < j_1$. Repeating the argument successively gives $i + 1$ linear sets of $\text{LINSET}^2_{=2}$ contained in $Q_{j_0}, \ldots, Q_{j_i}$ such that $0 \leq j_i < j_{i-1} < \ldots < j_0 < i$, which is impossible. Hence $\text{RTD}(\text{LINSET}^2_{=2}) \geq 3$. ■
Remark 3.4.3 For any \( m, k \geq 2 \), let \( \text{LINSET}_k^m \) denote the class \( \{ \langle P \rangle : P \subset \mathbb{N}_0^m \land \exists p \in P [p \neq 0] \land |P| \leq k \} \). Note that \( \text{LINSET}_2^2 \) may be regarded as a subclass of \( \text{LINSET}_k^m \) via the mapping \( (x_1, x_2) \mapsto (x_1, x_2, 0, \ldots, 0) \). Thus by Theorem 3.4.1, \( \text{RTD}(\text{LINSET}_k^m) = \infty \) holds for all \( m, k \geq 2 \).

As mentioned earlier, the study of the TD and RTD for various subfamilies of \( \text{LINSET}_k^m \) when \( m \geq 3 \) and \( k \geq 1 \) may require proof techniques quite different from those employed in the present chapter. For example, the proof of Theorem 3.4.2(II) does not directly apply to \( \text{LINSET}_k^m \) for \( m \geq 3 \); in particular, the statement in Case (i) that at least one of \( p_1 \) and \( p_2 \) lies between (not necessarily strictly) \( v_1 \) and \( v_2 \), in the sense that at least one of \( p_1 \) and \( p_2 \) can be expressed as a linear combination of \( v_1 \) and \( v_2 \) with nonnegative rational coefficients, no longer holds when \( m \geq 3 \).
Chapter 4

The Teaching Complexity of Non-erasing Pattern Languages

Angluin [2] introduced and defined a class of formal languages known as pattern languages. A pattern is a nonempty finite string made up of symbols from a set $X$ of variables and an alphabet $\Sigma$. For any given pattern $\pi$, the set of words obtained by replacing every variable occurring in $\pi$ with a word over $\Sigma$, with the condition that any two occurrences of the same variable be replaced with the same word, is the pattern language generated by $\pi$. Angluin’s primary motivation for studying the pattern languages was to formalise the process of inductive inference – that is, the process of inferring general rules from specific examples – in a concrete setting. The study of pattern languages has since developed into a branch of formal language theory in its own right, and there has been a fair amount of work on the basic properties of pattern languages [11 26 36 37]. Pattern languages have also been
applied in bioinformatics [5], in text editing for automatic programme synthesis [34], in database theory [7], and in pattern matching [10].

The teaching complexity of pattern languages was first studied in Zeinab Mazadi’s thesis [30]. It was shown [30] that the recursive teaching protocol yields a lower sample complexity than the teaching dimension protocol when both protocols are applied to certain subclasses of pattern languages, as well as to the whole class of pattern languages over any infinite alphabet. The main problems unresolved in [30] were: (1) determining the RTD of the class of arbitrary pattern languages over any finite alphabet and (2) determining the TD of the class of regular pattern languages (to be defined shortly) over any alphabet of size at least 2. In the present chapter, we show that the RTD of the class of pattern languages over any unary alphabet is infinite, and that the TD of the class of regular pattern languages is either 5 or 6 if the alphabet size is at least 8, and is exactly 5 if the alphabet is infinite. We shall also determine the TD of regular patterns with at most one block of variables or at most one block of constants over any alphabet with at least 2 letters. Further, we will show that the $RTD^+$ of the class of arbitrary pattern languages over any alphabet of size at least 2 is infinite. The question as to whether the class of arbitrary pattern languages over any finite alphabet of size at least 2 has a finite RTD remains open.

The results presented in this chapter have been published or accepted for publication with minor revisions in the following articles:

4.1 Pattern Languages

We now formally define the pattern languages and a few subclasses of pattern languages. Let $\Sigma$ be a (finite or infinite) alphabet and $X = \{x_1, x_2, \ldots\}$ a countably infinite set of variables, disjoint from $\Sigma$. A pattern $\pi$ is a nonempty string of constant symbols from $\Sigma$ and variables from $X$, i.e., $\pi \in (\Sigma \cup X)^+$. Each pattern $\pi$ generates a pattern language $L^\Sigma(\pi)$, defined as the set of words in $\Sigma^+$ obtained by substituting (nonempty) words from $\Sigma^+$ for variables, where each occurrence of the same variable must be replaced by the same word. For instance, if $\Sigma = \{a, b\}$ and $\pi = x_1ax_1x_2bx_1$, then $L^\Sigma(\pi) = \{w_1aw_1w_2b \mid w_1, w_2 \in \Sigma^+\}$. These languages are often referred to as non-erasing pattern languages to distinguish them from erasing pattern languages [43], which extend the non-erasing pattern languages by allowing the variables in a pattern to be replaced by the empty string. In this chapter, the term pattern language will always refer to a non-erasing pattern language; erasing pattern languages will be considered in the next chapter. We write $L(\pi)$ instead of $L^\Sigma(\pi)$ when $\Sigma$ is clear from the context. Furthermore, note that the TD and RTD (as well as their variants) for any given class of pattern languages depend only the size of the underlying alphabet $\Sigma$, and not on the actual symbols in $\Sigma$. For example, for all purposes of the present chapter, the language generated by the pattern $ax_1x_2bx_1$ over the alphabet $\{a, b\}$ is equivalent to the language generated by the
pattern \(0x_1x_21x_1\) over the alphabet \(\{0, 1\}\). Hence, we simply use a superscript \(z\) to refer to the alphabet size of a set of patterns or a set of pattern languages. For a set \(P\) of patterns over some fixed alphabet of size at most \(z\) \((1 \leq z \leq \infty)\), the term \(\mathcal{L}^z(P)\) denotes the family of languages over some alphabet \(\Sigma\) generated by patterns in \(P\), where \(|\Sigma| = z\) and \(\Sigma\) contains all constant symbols occurring in patterns in \(P\). Again, we drop the superscript \(z\) from \(\mathcal{L}^z\) when \(z\) is clear from the context.

An assignment is a mapping \(A : X \mapsto \Sigma^+\). By abuse of notation, we will often use the same letter \(A\) to represent the homomorphism \(A : (X \cup \Sigma)^* \mapsto \Sigma^*\) that coincides with the assignment \(A\) on individual variables and with the identity function on letters from \(\Sigma\).

A pattern \(\pi\) is (i) a constant pattern if \(\pi \in \Sigma^+\), (ii) a constant-free pattern (also called terminal-free pattern in the literature) if \(\pi \in X^+\), (iii) a \(k\)-variable pattern for any positive \(k \in \mathbb{N}\) if \(\pi\) contains at most \(k\) distinct variables (each possibly with repetitions), and (iv) a regular pattern \([43]\) if \(\pi\) has no repeated variables.

The class of arbitrary patterns over an alphabet \(\Sigma\) with \(z = |\Sigma|\) will be denoted by \(\Pi^z\). In particular, we write \(\Pi^\infty\) to denote the class of arbitrary patterns over an infinite alphabet. Similarly, the class of regular patterns over \(\Sigma\) will be denoted by \(R\Pi^z\), and \(\Pi^z_k\) \((R\Pi^z_k)\) denotes the class of patterns (regular patterns) with at most \(k\) variables. \(\Pi^z_{cf}\) denotes the class of pattern languages generated by constant-free patterns over \(\Sigma\). More generally, for every class \(P\) of patterns, \(P_{cf}\) denotes the subclass of all constant-free patterns in \(P\). Languages generated by \(k\)-variable (regular) patterns are called \(k\)-variable (regular, resp.) pattern languages. For any positive \(k \in \mathbb{N}\), \(\Pi^z_{k,cf}\) denotes the class of constant-free patterns with at most \(k\) distinct variables.
Conventions. Instead of $\text{TD}(\mathcal{L}(P))$, we henceforth simply write $\text{TD}(P)$. With a teaching set for $\pi$ w.r.t. $P$, we always mean a teaching set for $L(\pi)$ w.r.t. $\mathcal{L}(P)$. Analogous conventions apply to $\text{RTD}$ and to recursive teaching sequences.

We will often assume that a pattern $\pi \in \Pi^z_k$ is normalised in the sense that its $k' \leq k$ variables are named $x_1, \ldots, x_{k'}$ (or simply $x$ if $k' = 1$), and that whenever $i < j$, the leftmost occurrence of $x_i$ is to the left of the leftmost occurrence of $x_j$.

Given $v \in \Sigma^+$ and given a pattern $\pi$ that contains a variable $x$ (and possibly some other variables), then $\pi[x \leftarrow v]$ denotes the pattern obtained from $\pi$ by substituting $v$ for all occurrences of $x$. Note that this convention can be used iteratively. For instance, $\pi[x_i \leftarrow v, x_j \leftarrow w]$ results from $\pi$ by substituting $v$ for all occurrences of $x_i$ and by substituting $w$ for all occurrences of $x_j$. If we substitute the same string $w$ for all occurrences of any variable in $\pi$, then the resulting constant pattern is denoted $\pi(w)$.

4.2 Teaching Dimension of Regular Pattern Languages

In this section, we establish the teaching dimension of the class of regular pattern languages over any infinite alphabet, and give almost matching upper and lower bounds for the teaching dimension when the alphabet size is at least 8. The following notation will be employed in this section when we construct teaching sets for regular pattern languages.

Notation 4.2.1 Fix $c \in \Sigma^+$. For any $b \in \Sigma$, set

$$\pi_c(b) := c^\rightarrow bc^\leftarrow \text{ for } c^\rightarrow = c[1 : |c| - 1] \text{ and } c^\leftarrow = c[2 : |c|]. \quad (4.1)$$
If $|\Sigma| \geq 3$ and $a$ is a letter that differs from $c[1]$ and $c[|c|]$, then we define

$$\hat{c} := c^- ac^{-1} \text{ for } c^- = c[1 : |c| - 1] \text{ and } c^{-1} = c[2 : |c|]. \quad (4.2)$$

The notation $\hat{c}$ does not make the choice of $a$ explicit but this choice will always be clear from the context.

**Example 4.2.2** Let $\Sigma = \{a, b, c, d\}$ and $w = abcb$. Then $\pi_w(b) = \underbrace{abc}_{w^r} b \underbrace{bec}_{w^l}$ and $\pi_w(d) = \underbrace{abc}_{w^r} c \underbrace{bec}_{w^l}$. Further, one may set $\hat{w} = \underbrace{abc}_{w^r} c \underbrace{bec}_{w^l}$ (the precise choice of the letter between $w^r$ and $w^l$ is not important so long as it differs from both $w[1]$ and $w[|w|]$).

We start with the following two propositions, which will be useful for establishing lower bounds for the teaching dimension of pattern languages.

**Proposition 4.2.3** Let $|\Sigma| \in \mathbb{N} \cup \{\infty\}$. Let $P$ be any set of patterns over $\Sigma$ containing all constant patterns, and let $w \in \Sigma^+ \subseteq P$. Then every finite teaching set for $L(w) = \{w\}$ w.r.t. $L(P)$ contains at least one positively labelled example. Moreover, if $P$ contains at least one non-constant pattern $\pi$, then every teaching set for $L(\pi)$ w.r.t. $L(P)$ contains at least two positively labelled examples.

**Proof.** For any finite $W \subseteq \Sigma^* \times \{-\}$ and any $v \in \Sigma^+ \setminus X(W)$, $L(v)$ is consistent with $W$. Thus at least one positive example is needed for teaching $L(w)$ or for teaching $L(\pi)$. Further, any sample consistent with $L(\pi)$ that contains only one positively labelled example, say $(v, +)$, is also consistent with the constant pattern language $\{v\}$; hence a teaching set for $L(\pi)$ w.r.t. $L(P)$ must contain at least two positively labelled examples. □
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**Proposition 4.2.4** Let $|\Sigma| \in \mathbb{N} \cup \{\infty\}$. Let $P$ be any set of patterns over $\Sigma$ containing $x_1$, and let $\pi \in P$ be another pattern such that $L(\pi) \neq L(x_1) = \Sigma^+$. Then every teaching set for $L(\pi)$ w.r.t. $\mathcal{L}(P)$ contains at least one negatively labelled example.

**Proof.** $L(x_1)$ is consistent with any set $W \subseteq \Sigma^+ \times \{+\}$. ■

Another tool for proving lower bounds on the TD of a pattern language w.r.t. any given class is the following lemma, occasionally known as Angluin’s Sunflower Lemma [4].

**Lemma 4.2.5** [4, Lemma 1] Assume that there are $m + 1$ distinct sets $L_0, L_1, \ldots, L_m$ such that

$$\forall 1 \leq i < j \leq m : \ L_0 = L_i \cap L_j . \quad (4.3)$$

Then any teaching set $L_0$ w.r.t. $\mathcal{L} = \{L_0, L_1, \ldots, L_m\}$ contains exactly $m$ negatively labelled examples (resp. at least $m$ negatively labelled examples w.r.t. any superset of $\mathcal{L}$).

**Proof.** This lemma follows immediately from the following two observations:

- Positively labelled examples (taken from $L_0$) are useless because they do not distinguish between $L_0$ and any of the remaining languages from $\mathcal{L}$.

- A negatively labelled example (taken from $\overline{L_0}$) can belong to only one of the remaining languages from $\mathcal{L}$. ■

The next result was proven in Mazadi’s thesis [30].

**Theorem 4.2.6** [30, Lemma 4.6, Lemma 4.7] $TD(R\Pi^1) = 3$. 

We now turn our attention to the teaching dimension of regular pattern languages over alphabets of size at least 2. According to Theorem 4.2.6 and Propositions 4.2.3 and 4.2.4, any sample containing two positively labelled examples and one negatively labelled example is sufficient for teaching regular patterns over a unary alphabet. For teaching regular patterns over larger alphabets, however, at least five examples (two labelled positively and three labelled negatively) are needed.

**Lemma 4.2.7** Let \( z = |\Sigma| \geq 2 \) and \( n \geq 3 \). Let \( \pi = c_1x_1c_2x_2\ldots x_{n-1}c_n \) for some \( c_1, c_n \in \Sigma^+ \) and \( c_2, \ldots, c_{n-1} \in \Sigma^* \) such that at least one of the \( c_i \) for \( 1 < i < n \) is nonempty. Then \( TD(L(\pi), R\Pi^z) \geq 5 \).

**Proof.** First, note that by Proposition 4.2.3 any teaching set for \( L(\pi) \) w.r.t. \( R\Pi^z \) must contain at least two positively labelled examples.

Second, it is shown that any teaching set for \( L(\pi) \) w.r.t. \( L \) must have at least three negatively labelled examples. Fix \( i \in \{2, \ldots, n-1\} \) such that \( c_i \) is nonempty. For \( j = 1, i, n \), let \( \pi_j \) be the pattern derived from \( \pi \) by replacing the first symbol of \( c_j \) with a variable \( x \) that does not occur in \( \pi \). Then \( L(\pi_j) \supset L(\pi) \): if some word \( w \) is generated from \( \pi \) by substituting \( w_t \) for \( x_t \) whenever \( 1 \leq t \leq n - 1 \), then \( w \) can be generated from \( \pi_j \) by replacing \( x \) with the first symbol of \( c_j \), and substituting \( w_t \) for \( x_t \) whenever \( 1 \leq t \leq n - 1 \); further, the string obtained by substituting some constant \( a \in \Sigma \) different from the first symbol of \( c_j \) for every variable in \( \pi_j \) does not belong to \( L(\pi) \). Thus, any teaching set \( W \) for \( L(\pi) \) w.r.t. \( L \) contains some negatively labelled example \( (v_j, -) \). To show that \( W \) contains at least three negatively labelled examples, it suffices by Lemma 4.2.5 to prove \( L(\pi_j) \cap L(\pi_\ell) \subseteq L(\pi) \) for \( j, \ell \in \{1, i, n\}, j \neq \ell \). Consider any \( w \in L(\pi_j) \cap L(\pi_i) \).
Since \( w \in L(\pi_i) \), \( c_1 \) is a prefix of \( w \). Further, \( w \in L(\pi_1) \) implies that there are nonempty strings \( w_1, \ldots, w_{n-2} \) such that \( c_2w_1c_3w_3 \ldots w_{n-2}c_n \) is a suffix of \( w \), and this suffix must start at a position in \( w \) that is at least equal to \( |c_1| + 2 \). Thus \( w \) can be expressed in the form \( c_1\theta c_2w_1c_3w_3 \ldots w_{n-2}c_n \) for some nonempty string \( \theta \), and therefore \( w \in L(\pi) \). Analogous proofs apply to the cases \( \{j = 1, \ell = n\} \) and \( \{j = i, \ell = n\} \).

**Remark 4.2.8** One may also show that for any \( z \geq 2 \) and any regular patterns \( \pi' = X_1c_1 \ldots c_{n-1}X_n \), \( \pi'' = c_1X_1 \ldots c_nX_n \), and \( \pi''' = X_1c_1 \ldots X_n c_n \), where \( c_1, \ldots, c_n \in \Sigma^+ \), \( X_1, \ldots, X_n \in X^+ \) and \( n \geq 2 \), it holds that \( TD(\pi', R\Pi^z) \geq 3 \) and \( TD(\pi'', R\Pi^z) \geq 4 \). The proofs of these lower bounds are entirely analogous to the proof of Theorem 4.2.7.

In order to establish an almost matching upper bound of 6 on \( TD(R\Pi^z) \) for all \( z \geq 8 \) and a matching upper bound for \( z = \infty \), we shall proceed as follows.

1. We first consider the case of regular patterns with 0 blocks of variables (that is, the constant patterns). This case was covered in [30].

2. We next consider the case of constant-free patterns; it was shown in [30] that the teaching dimension of the class of constant-free patterns is 3 (Lemma 4.2.12). This covers the case of patterns that have 1 block of variables and neither start nor end with a constant.

3. Under the assumption \( z = |\Sigma| \geq 8 \), we show that 4 (resp. 3 if \( z = \infty \)) is an upper bound for the teaching dimension of patterns that have at least 2 blocks of variables and neither start nor end with constants (Lemmas 4.2.15.
4. We show that, loosely speaking, adding a constant prefix (resp. a constant suffix) to a pattern that originally started (resp. ended) with a variable will increase the teaching dimension w.r.t. $\mathcal{R}_\Pi^z$ by at most 1, and the teaching dimension will increase by at most 2 if both modifications (adding a prefix and a suffix) are combined (Lemma 4.2.25).

From these results, the desired upper bounds are obtained.

**Theorem 4.2.9** For any $8 \leq z < \infty$, $TD(\mathcal{R}_\Pi^z) \leq 6$. Moreover $TD(\mathcal{R}_\Pi^\infty) \leq 5$.

Theorems 4.2.7 and 4.2.9 yield $TD(\mathcal{R}_\Pi^\infty) = 5$. We now prove the various lemmas that form the building blocks of Theorem 4.2.9. First, we observe the following crucial property of the string $\hat{c}$ (see Notation 4.2.1) for any given $c \in \Sigma^*$ with $|\Sigma| \geq 3$; this property will often be applied implicitly in subsequent proofs.

**Lemma 4.2.10** Suppose $|\Sigma| \geq 3$. For any $c \in \Sigma^*$, $c$ is not a substring of $\hat{c}$.

**Proof.** Suppose $\hat{c} = c[1] \ldots c[|c| - 1]ac[2] \ldots c[|c|]$, where $a \in \Sigma$ differs from $c[1]$ and $c[|c|]$. Assume by way of a contradiction that $c$ were a substring of $\hat{c}$. By the choice of $a$, $c$ cannot start at the $|c|^{th}$ position of $\hat{c}$ or the 1$^{st}$ position of $\hat{c}$. Thus $c = c[i] \ldots c[|c| - 1]ac[2] \ldots c[i - 1]$ for some $i$ with $2 \leq i \leq |c| - 1$. But this would imply that $#(c[|c|])[c[i] \ldots c[|c| - 1]ac[2] \ldots c[i - 1]] = #(c[|c|])[c] - 1$, a contradiction. □

The TD of any constant-free or variable-free pattern language was established in [30]. To illustrate the use of the mapping $\pi_c$ defined earlier, we shall give a proof
that \( \text{TD}(c, R\Pi^x) = 2 \) for any \( c \in \Sigma^+ \) with \( |\Sigma| = z \geq 2 \).

**Lemma 4.2.11** [30, Theorem 3.6] Let \( z = |\Sigma| \geq 2 \) and let \( c \in \Sigma^+ \). Then \( \text{TD}(c, R\Pi^x) = 2 \).

**Proof.** Fix any \( b \in \Sigma \) that is different from the first letter of \( c \). Set \( \overline{c} = \pi_c(b) \) if \( |c| \geq 2 \) and \( \overline{c} = b \) if \( |c| = 1 \) (see Notation 4.2.1). We claim that \( T = \{(c, +), (\overline{c}, -)\} \) is a teaching set for the constant pattern \( c \) (by Propositions 4.2.3 and 4.2.4 at least two examples are needed). Consider a pattern \( \pi \) such that \( c \in L(\pi) \). We may decompose \( \pi \) according to \( \pi = dx' \) with \( d \in \Sigma^* \) and \( x' \) being either empty or starting with a variable. If \( x' = \varepsilon \), then \( \pi = d \). Since \( c \in L(\pi) \), this implies that \( \pi = d = c \).

Suppose now that \( x' \) is nonempty, so that it can be written in the form \( x' = x\pi'' \) for some variable \( x \) and some (possibly empty) pattern \( \pi'' \). It suffices to show that \( c \in L(\pi) \) implies that \( \overline{c} \in L(\pi) \). If \( c \) matches \( \pi = dx\pi'' \), then

- \( d \) is a proper prefix of \( c \), say \( d = c[1 : k] \) for some \( 0 \leq k \leq |c| - 1 \)
- and if \( \pi'' \) is nonempty, then it generates a proper suffix of \( c \), say \( c[\ell + 1 : |c|] \in L(\pi'') \) for some \( 1 \leq \ell \leq |c| \).

Since \( \overline{c} \) satisfies \( c = uvw \) for

\[
u = c[1 : k], \ v = c[k + 1 : |c| - 1]bc[1 : \ell] \text{ and } w = c[\ell + 1 : |c|]
\]

it follows that \( \overline{c} \) matches \( \pi = dx\pi'' \). □

**Lemma 4.2.12** [30, Example 2.1, Theorem 3.4] Let \( z = |\Sigma| \geq 2 \), \( n \geq 1 \), and
\[ \pi = x_1 \ldots x_n. \]  

Then  

\[ TD(\pi, R\Pi^z) = \begin{cases} 2 & \text{if } n = 1; \\ 3 & \text{if } n \geq 2. \end{cases} \]

To give a simple example of the method of argument used in the proofs of Lemmas 4.2.15 and 4.2.24 we shall determine the teaching dimension of another special class of regular pattern languages.

Lemma 4.2.13 Let  

\[ z = |\Sigma| \geq 2 \]  

and  

\[ \pi = c_1X_1c_2 \]  

be a regular pattern, where  

\[ c_1, c_2 \in \Sigma^+, \ X_1 \in X^+, \text{ and } |X_1| \geq 2. \]  

Then  

\[ TD(\pi, R\Pi^z) = 5. \]

Proof. Fix a variable  \( y \) that does not occur in  \( \pi \). Let  \( \tau_1, \tau_2 \) and  \( \tau_3 \) be the regular patterns derived from  \( \pi \) by the following operations:

- \( \tau_1 \): replacing the first symbol of  \( c_1 \) with  \( y \).
- \( \tau_2 \): replacing the last symbol of  \( c_2 \) with  \( y \).
- \( \tau_3 \): deleting the first symbol of  \( X_1 \).

One may verify directly that the languages  

\[ L_0 = L(\pi) \]  

and  

\[ L_i = L(\tau_i) \]  

for all  \( i \in \{1, 2, 3\} \) satisfy Condition [4.3] from Lemma 4.2.5. Thus any teaching set  \( T \) for  \( \pi \) w.r.t.  \( R\Pi^z \) must have at least three negatively labelled examples; by Proposition 4.2.3  \( T \) must also have at least two positively labelled examples. Hence  \( T \) has at least five labelled examples, as required.

To construct a teaching set for  \( \pi \) w.r.t.  \( R\Pi^z \), first choose distinct  \( a, b \in \Sigma \). Choose  \( \delta_1, \delta_2 \in \Sigma \) so that  \( \delta_1 \) is different from the last symbol of  \( c_1 \) and  \( \delta_2 \) is different from the first symbol of  \( c_2 \). Now define  \( \bar{c}_1 = \pi_{c_1}(\delta_1) \) and  \( \bar{c}_2 = \pi_{c_2}(\delta_2) \) (see [4.1], Notation...
Our teaching set $T'$ for $\pi$ w.r.t. $R\Pi^z$ contains the positively labelled examples $(\pi(a), +), (\pi(b), +)$ and the negatively labelled examples $(\overline{c_1}b^{lX_1}c_2, -), (c_1b^{lX_1}\overline{c_2}, -)$ and $(c_1b^{lX_1}-1c_2, -)$. It is readily verified that these labelled examples are indeed consistent with $L(\pi)$.

Consider any regular pattern $\rho$ such that $L(\rho)$ is consistent with $T'$ and $\rho \neq \pi$ (after $\rho$ and $\pi$ are both normalised). Let $A : X \mapsto \Sigma^+$ be the assignment that witnesses $\pi(b) \in L(\rho)$. Suppose that $A$ maps an occurrence of $b$ in $\rho$ to one of the $b$’s in $c_1b^{lX_1}c_2$ occurring between $c_1$ and $c_2$ (as indicated by braces below).

Suppose for the sake of a contradiction that $A$ maps an occurrence of $c_1$ in $\rho$ to the prefix $c_1$ of $\pi(b)$ and $A$ maps an occurrence of $c_2$ in $\rho$ to the suffix $c_2$ of $\pi(b)$. Then $\#(b)[\rho] \geq \#(b)[c_1] + \#(b)[c_2] + 1$ and since $\#(b)[\pi(a)] = \#(b)[c_1] + \#(b)[c_2]$, it follows that $\pi(a) \notin L(\rho)$, a contradiction. Thus at least one of the following cases must hold:

1. $A$ maps a variable $y$ in $\rho$ to at least one symbol occurring in $c_1$;
2. $A$ maps a variable $z$ in $\rho$ to at least one symbol occurring in $c_2$.

We illustrate Cases (I) and (II) in the same string below:

\[\pi(b) = c_1[1] \ldots \overbrace{c_1[i]}^{(I): A(y)} \ldots c_1[l] \ldots c_1[l]b^{lX_1}c_2[1] \ldots \overbrace{c_2[j]}^{(II): A(z)} \ldots c_2[l]c_2]\]

One may argue as in the proof of Lemma 4.2.11 that if Case (I) holds, then $\overline{c_1}b^{lX_1}c_2 \in L(\pi)$. 

\[\overline{c_1}b^{lX_1}c_2 \in \]

\[\overline{c_1}b^{lX_1}c_2 \in \]
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$L(\rho)$, and if Case (II) holds, then $c_1b^{|X_1|}c_2 \in L(\rho)$. Therefore $A$ maps constant symbols of $\rho$ to $c_1$ and $c_2$ and variable symbols of $\rho$ to the substring $b^{|X_1|}$ in $\pi(b)$. Hence $\rho$ must be of the form $c_1X'c_2$, where $X' \in X^+$ and $|X'| < |X_1|$. But this implies that $c_1b^{|X_1|-1}c_2 \in L(\rho)$, and so $L(\rho)$ cannot be consistent with $T'$.

Remark 4.2.14 Let $z = |\Sigma| \geq 2$. The proof of Lemma 4.2.13 implies that for any $x_1 \in X$ and $c_1, c_2 \in \Sigma^+$, the teaching dimension of $c_1x_1c_2$ w.r.t. $R_{\Pi^z}$ is 4. Furthermore, arguments quite similar to those used in the proof of Lemma 4.2.13 show that for any $X_1 \in X^+$ with $|X_1| \geq 2$, $c_1X_1$ and $X_1c_2$ both have the same teaching dimension w.r.t. $R_{\Pi^z}$ of 4, while $c_1x_1$ and $x_1c_2$ both have the same teaching dimension w.r.t. $R_{\Pi^z}$ of 3.

The next result is concerned with the teaching dimension of regular patterns over any infinite alphabet, and it is one of the two main lemmas used to establish Theorem 4.2.9 (the other being Lemma 4.2.24, which deals with the finite alphabet case).

Lemma 4.2.15 Let $z = |\Sigma| \geq 2$ and $n \geq 2$. Let $\pi = X_1c_1X_2\ldots c_{n-1}X_n$ be a regular pattern for some $c_1, c_2, \ldots, c_{n-1} \in \Sigma^+$ and $X_1, X_2, \ldots, X_n \in X^+$. Suppose there exists some $a \in \Sigma \setminus \text{Const}(\pi)$; that is, at least one letter of $\Sigma$ does not occur in $\pi$. Then $TD(L(\pi), R_{\Pi^z}) \leq 3$.

Proof. Let $a$ be an element of $\Sigma$ that does not occur in $\pi$, and choose any $b \in \Sigma \setminus \{a\}$. Denote $|\pi|$ by $m$. The required teaching set $W$ of size 3 consists of two positive examples, namely $(\pi(a), +)$ and $(\pi(b), +)$, as well as one negative example $(\sigma, -)$ defined as follows (where $\hat{c}$ is defined according to Equation (4.2) in Notation 4.2.1):

$$\sigma = w_1a^m\hat{c}_1a^m w_2a^m\hat{c}_2a^m w_3a^m \ldots \hat{c}_{n-2}a^m w_{n-1}a^m \hat{c}_{n-1}a^m w_n,$$
where

\[ w_i = \begin{cases} 
  a_1^{|X_1|-1}c_1 & \text{if } i = 1; \\
  c_{n-1}a_1^{|X_n|-1} & \text{if } i = n; \\
  c_i a_1^{|X_i|-1}c_i & \text{if } 2 \leq i \leq n - 1.
\end{cases} \]

We first show that for any \( i \in \{1, \ldots, n-1\} \) and any \( j \in \{1, \ldots, n\} \), \( \sigma \) can be expressed in two forms, \((A_i)\) and \((B_j)\):

\((A_i)\): \( \sigma = w'_1c_1w'_2 \cdots c_{i-1}w'_ic_{i+1}w'_{i+2} \cdots c_{n-1}w'_n \) for some nonempty words \( w'_1, w'_2, \ldots, w'_n \) such that for all \( k \in \{1, \ldots, n\} \), \( |w'_k| \geq |X_k| \).

\((B_j)\): \( \sigma = V_1c_1V_2 \cdots c_{j-1}V_jc_jV_{j+1}c_{j+1}V_{j+2} \cdots c_{n-1}V_n \) for some nonempty words \( V_1, V_2, \ldots, V_n \) such that \( |V_j| = |X_j| - 1 \) and for all \( k \in \{1, \ldots, n\} \setminus \{j\} \), \( |V_k| \geq |X_k| \).

We first explain why \( \sigma \) can be expressed in the form \((A_i)\) for all \( i \in \{1, \ldots, n-1\} \).

Consider the expression \( w_1a^m\hat{c}_1a^m w_2a^m\hat{c}_2a^m w_3a^m \cdots a^m w_{n-1}\hat{c}_{n-1}a^m w_n \) for \( \sigma \).

For any \( i \) with \( 1 \leq i \leq n-1 \), \( \sigma \) contains the sequence of words \( w_1, a^m, \ldots, a^m, w_i, a^m, \hat{c}_i, a^m, w_{i+1}, a^m, w_{i+2}, a^m, \ldots, a^m, w_{n-1}, a^m, w_n \). Furthermore, for all \( i \in \{2, \ldots, n-1\} \), both \( c_{i-1} \) and \( c_i \) are substrings of \( w_i \). It follows that \( \sigma \) can be expressed in the form \( w'_1c_1 \cdots w'_{i-1}c_{i-1}w'_i\hat{c}_i w'_i c_{i+1} w'_{i+1} \cdots c_{n-1}w'_{n-1} \) for some nonempty words \( w'_1, w'_2, \ldots, w'_{n-1} \) such that \( |w'_k| \geq |X_k| \) for all \( k \in \{1, \ldots, n-1\} \) whenever \( 1 \leq i \leq n - 1 \), as required.

Now we explain why \( \sigma \) can be expressed in the form \((B_j)\) for all \( j \in \{1, \ldots, n\} \).

First, it can be expressed in the form \((B_1)\) because it contains the sequence of words \( w_1, a^m, w_2, a^m, \ldots, a^m, w_{n-1}, a^m, w_n \), where \( w_1 = a_1^{|X_1|-1}c_1 \) and \( c_i \) is a substring of \( w_i \) whenever \( i \in \{2, \ldots, n\} \). Second, it can be expressed in the form
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(B_n) because it contains the sequence of words \(a^m, w_1, a^m, \ldots, a^m, w_{n-1}, a^m, w_n\) and \(c_{i-1}\) is a substring of \(w_i\) whenever \(i \in \{2, \ldots, n-1\}\). Third, it can be expressed in the form \((B_j)\) for all \(j \in \{2, \ldots, n-1\}\), because it contains the sequence of words \(w_1, a^m, w_2, a^m, \ldots, a^m, w_j, a^m, w_{j+1}, a^m, \ldots, a^m, w_{n-1}, a^m, w_n\); moreover, \(w_j\) is equal to \(c_{j-1}a^{|X_j|}c_j\) and for all \(k < j\), \(c_{k-1}\) is a substring of \(w_k\), while for all \(k > j\), \(c_k\) is a substring of \(w_k\).

The claim that \(W\) is indeed a teaching set for \(L(\pi)\) w.r.t. \(R\Pi^\infty\) can be deduced immediately from the following two assertions.

1. \(\sigma \notin L(\pi)\).

2. If a regular pattern \(\rho\) with \(L(\rho) \neq L(\pi)\) generates both \(\pi(a)\) and \(\pi(b)\), then \(\sigma' \in L(\rho)\) for every \(\sigma'\) that can be expressed in the forms \((A_i)\) and \((B_j)\) for all \(i\) with \(1 \leq i \leq n-1\) and all \(j\) with \(1 \leq j \leq n\).

Proof of Assertion 1.

Note that since \(c_i\) is not a word of \(\hat{c}_i\) for all \(i \in \{1, \ldots, n-1\}\), \(a\) does not occur in \(\pi\), and for all \(j \in \{1, \ldots, n\}\), \(w_j\) does not contain a word of the form \(c_jv_{c_{j+1}}\) for some \(v \in \Sigma^+\) with \(|v| \geq |X_j|\), it must hold that for all \(i \in \{1, \ldots, n-1\}\) the word \(w_1a^m\hat{c}_1a^m w_2 a^m \ldots a^m w_i a^m \hat{c}_i\) cannot contain a substring of the form \(v_1 c_1 v_2 \ldots v_{i-1} c_i\) such that \(|v_j| \geq |X_j|\) for all \(j \in \{1, \ldots, i-1\}\). Thus \(\sigma \notin L(\pi)\) follows inductively.

Proof of Assertion 2.

We first show that \(\rho\) can be derived from \(\pi(a)\) via a combination of exactly three types of operations: (i) either replacing at least one symbol different from \(a\) with
some variable or replacing at least one substring of the form $a^2$ with a single variable; (ii) replacing all occurrences of $a$ with (possibly non-distinct) variables; (iii) replacing all substrings of the same variable (that is, substrings of the form $xx\ldots x$ for some variable $x$) with that single variable. It will then be argued that any pattern derived from $\pi(a)$ via any combination of the three types of operations must satisfy the property given in 2.

By assumption, $\rho$ is consistent with both $(\pi(a), +)$ and $(\pi(b), +)$, and $L(\rho) \neq L(\pi)$. Denote the $p^{th}$ symbol of $\pi(a)$ by $s_p$. For $1 \leq p \leq |\pi|$, let $h_p$ be the symbol in $\rho$ that is mapped to $s_p$. Recall that $a \notin \text{Const}(\pi)$; thus, as $a \neq b$, $a$ does not occur in $\pi(b)$. Since any constant symbol in $\rho$ must occur in both $\pi(a)$ and $\pi(b)$, it follows that if $p$ is a position of $\pi(a)$ such that $s_p = a$, then $h_p$ is a variable. As $\rho$ is regular, it holds that for all distinct $p_1, p_2$ such that $p_1 < p_2$, if $h_{p_1} = h_{p_2} = x$ for some variable $x$, then $h_{p'} = x$ for all $p' \leq p' \leq p_2$. Every constant symbol in $\rho$ is mapped to the same symbol in $\pi(a)$, while every variable in $\rho$ is also mapped to at least one symbol in $\pi(a)$. Hence all the symbols of $\rho$ occur in the order $h_1, h_2, \ldots, h_{|\pi|}$, and so $\rho$ can be derived from the string $h_1h_2\ldots h_{|\pi|}$ by replacing any substring of the form $xx\ldots x$ with $x$, where $x$ is a variable. Moreover, suppose that $h_p$ is a constant for all $p$ such that $s_p \neq a$. Then $\rho$ can be expressed in the form $Y_1c_1Y_2\ldots c_{n-1}Y_{n-1}$ for some blocks of distinct variables $Y_1,Y_2,\ldots,Y_{n-1}$, but since $L(\rho) \neq L(\pi)$ and $|\rho| \leq |\pi|$, there must exist a least $i \leq n - 1$ such that $|Y_i| < |X_i|$. Consequently, $\rho$ can be derived from $\pi(a)$ via the three types of operations (i)-(iii) described earlier.

First, suppose that $\rho$ can be derived from $\pi(a)$ by a series of substitutions in which at least one constant different from $a$ is replaced with a variable. There
is a least $p''$ with $1 \leq p'' \leq n - 1$ such that if $\pi(a) = a^{X_1}c_1a^{X_2}\ldots a^{X_{p''}}c_{p''}$ $a^{X_{p''+1}}\ldots a^{X_n}$, then for some position $q'$ of $\pi(a)$ that occurs within a nonempty block $c_{p''}$, $h_{q'}$ is a variable. $\sigma$ can be expressed in the form $\alpha_1c_1\alpha_2\ldots c_{p''-1}\alpha_{p''}\hat{c}_{p''}$ $\alpha_{p''+1}c_{p''+1}\alpha_{p''+2}\ldots c_{n-1}\alpha_n$ for some nonempty strings $\alpha_1, \alpha_2, \ldots, \alpha_n$; based on this expression, one can derive $\sigma$ from $h_1h_2\ldots h_{|\sigma|}$ as follows. First, let $Q_1, Q_2, \ldots, Q_n$ be the closed non-overlapping intervals of $\{1, \ldots, m\}$ that represent all the positions of $\pi(a)$, listed in increasing order of first positions, such that for all $k \in \{1, \ldots, n\}$, the string at the interval $Q_k$ is $a^{X_k}$. Noting that for all $k \in \{1, \ldots, n\}$, it holds that $|Q_k| \leq |\alpha_k|$, replace the string of variables at the interval $Q_k$ with $\alpha_k$. Furthermore, let $q'$ be the least position of $\pi(a)$ such that $s_{q'} \neq a$ and $h_{q'}$ is a variable.

Suppose the $(q')^{th}$ position of $\pi(a)$ coincides with the $(q'')^{th}$ position of $c_{p''}$. If $|c_{p''}| = 1$, substitute $a$ for $h_{q'}$. Suppose $|c_{p''}| > 1$. If $1 < q'' < |c_{p''}|$, substitute $c_{p''}[q''], c_{p''}[q''+1] \ldots c_{p''}[|c_{p''}|-1]a c_{p''}[2]c_{p''}[3] \ldots c_{p''}[q'']$ for $h_{q'}$; if $q'' = 1$, substitute $c_{p''}[1]a c_{p''}[2] \ldots c_{p''}[|c_{p''}|-1]$ for $h_{q'}$; if $q'' = |c_{p''}|$, substitute $a c_{p''}[2]c_{p''}[3] \ldots c_{p''}[|c_{p''}|]$ for $h_{q'}$. For all $q$ such that $s_q \neq a$ and $q \neq q'$, substitute $s_q$ for $h_q$. It may be verified directly that these substitutions yield $\sigma$ from $h_1h_2\ldots h_{|\sigma|}$.

Second, suppose that $\rho$ can be derived from $\pi(a)$ by a series of substitutions in which at least one substring of the form $a^2$ is replaced with a single variable. Then either the first such replacement occurs before $c_1$, or it occurs after $c_{n-1}$, or it occurs between the blocks $c_r$ and $c_{r+1}$ for some least $r$ with $1 \leq r \leq n - 2$. Suppose the third case holds, that is, the first such replacement occurs between the blocks $c_r$ and $c_{r+1}$ for some $r$ with $1 \leq r \leq n - 2$. Let $t$ be the first position in $\pi(a)$ after $c_r$ and $t'$ be the last position in $\pi(a)$ before $c_{r+1}$. $\sigma$ can be expressed in the form $V_1c_1V_2\ldots V rc_{r+1}c_{r+1}V_{r+2}\ldots c_{n-1}V_n$ for some nonempty words $V_1, V_2, \ldots, V_n$ such
that $|V_{r+1}| = |X_{r+1}| - 1$ and for all $k \in \{1, \ldots, n\} \setminus \{r + 1\}$, $|V_k| \geq |X_k|$. To derive \(\sigma\) from $h_1h_2 \ldots h_\pi$\(|\), one may replace the string of variables \(h_t, \ldots, h_{t'}\) with $V_{r+1}$; for all other strings of variables $Y_k$ with $k \neq r + 1$, one may substitute $V_k$ for $Y_k$, while every constant symbol in $h_1h_2 \ldots h_\pi$ is replaced with the same constant. The remaining two cases (that is, the replacement occurs before $c_1$ or after $c_{n-1}$) can be dealt with similarly.

This concludes the proof of assertion 2, establishing that \(W\) is a teaching set for \(L(\pi)\) w.r.t. \(R\Pi\infty\). \(\blacksquare\)

Observe that for any infinite alphabet \(\Sigma\) and any regular pattern \(\pi\) over \(\Sigma\), there exists some \(a \in \Sigma\) that does not occur in \(\pi\). Thus Lemma 4.2.15 gives the following corollary.

**Corollary 4.2.16** Let $|\Sigma| = \infty$ and $n \geq 3$. Let $\pi = c_1X_1c_2X_2 \ldots X_{n-1}c_n$ be a regular pattern for some $c_1, c_2, \ldots, c_n \in \Sigma^+$ and $X_1, X_2, \ldots, X_{n-1} \in X^+$. Then $TD(L(\pi), R\Pi\infty) \leq 5$.

The next question we investigate is whether or not the teaching dimension of the class of regular patterns remains finite when the alphabet is finite. We give a positive answer to this question for all finite alphabets with at least 8 letters.

First, we consider the subclass of regular patterns whose maximal constant blocks are separated by variable blocks of length 1. It will be shown later how the proof for this case may be generalised to regular patterns in which the variable blocks are of any length.

**Lemma 4.2.17** Let $\pi = x_1c_1x_2c_2 \ldots c_{n-1}x_n$, where $x_1, \ldots, x_n$ are variables, $c_1, c_2,
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\[ \ldots, c_{n-1} \text{ are nonempty blocks of constants over } \Sigma, \text{ and } n \geq 2. \text{ If } z \geq 7, \text{ then } TD(\pi, \Pi^z) \leq 3. \]

**Proof.** Denote \(|\pi|\) by \(m\), and for all \(i \in \{1, \ldots, n-1\}\), denote \(|c_i|\) by \(m_i\). Pick any \(a \in \Sigma\). Let \(S = \{(\alpha, +), (\beta, +), (\sigma, -)\}\), where \(\alpha = \pi(a)\) and \(\beta, \sigma\) are defined as follows.

**Construction of \(\beta\).** Assume a fixed ordering \(<\) on \(\Sigma\). For each \(i \in \{1, \ldots, n-1\}\), define \(a_i \in \Sigma\) as follows. If there are numbers \(j_1, j_2 \in \{1, \ldots, n-1\} \setminus \{i\}\) such that \(j_1 < i, c_{j_1} \neq c_i, j_2 > i\) and \(c_{j_2} \neq c_i\) hold, let \(p_i\) be the largest possible value of \(j_1\) and \(q_i\) be the least possible value of \(j_2\). If no such \(j_1\) exists, set \(p_i = 1\); if no such \(j_2\) exists, set \(q_i = n-1\). Choose the least \(a_i \in \Sigma\) (with respect to \(<\)) such that \(a_i \neq c_{p_i}^{m_i}, c_i^{m_i}\) and \(c_{q_i}^{m_i}\). Fix some \(b \in \Sigma \setminus \{a\}\) and set

\[ \beta = ba_1^{m_i}c_1^{m_i}a_2^{m_i}c_2^{m_i}a_3^{m_i} \ldots a_{n-1}^{m_i}c_{n-1}^{m_i}a_{n-1}^{m_i}b. \]

**Construction of \(\sigma\).** Define

\[ \sigma = a_1^{m_i}c_1^{m_i}a_2^{m_i}c_2^{m_i}a_1^{m_i}a_2^{m_i}c_3^{m_i}a_3^{m_i} \ldots =: B_1 \]

\[ a_i^{m_i}c_i^{m_i}a_{i+1}^{m_i}c_{i+1}^{m_i}a_i^{m_i}w_1^{m_i}a_i^{m_i} =: B_i \]

\[ a_{n-2}^{m_i}c_{n-2}^{m_i}a_{n-2}^{m_i}c_{n-2}^{m_i}a_{n-1}^{m_i}a_{n-1}^{m_i}a_{n-1}^{m_i}w_{n-2}^{m_i}a_{n-2}^{m_i}c_{n-1}^{m_i}a_{n-1}^{m_i} \]

\[ =: B_{n-2} \]

1Such an \(a_i\) exists because \(z = |\Sigma| \geq 7\).
where, for all $i \in \{1, \ldots, n-2\}$, $w_i = c_i a_i^{2m}$ if $c_i \neq c_{i+1}$ and $w_i = \varepsilon$ otherwise.

The following facts will be used to show that $S$ is a teaching set for $\pi$.

**Fact 4.2.18** $\alpha, \beta \in L(\pi)$ and $\sigma \notin L(\pi)$.

**Fact 4.2.19** For any $\pi' \in \Pi_z$ such that $\{\alpha, \beta\} \subset L(\pi')$ holds, the first and last symbols of $\pi'$ are variables, and $|\pi'| \leq m$.

**Fact 4.2.20** Denote $|\beta|$ by $l$. Consider any $\pi' \in \Pi_z$ such that $\pi = x_1d_1x_2d_2\ldots d_{h-1}$ $x_h$, where $x_1, x_2, \ldots, x_h$ are variables and $d_1, d_2, \ldots, d_{h-1}$ are blocks of constants over $\Sigma$. Suppose that $\{\alpha, \beta\} \subset L(\pi')$. There is a sequence of non-overlapping closed intervals $I_1, \ldots, I_{h-1}$ of $\{1, \ldots, l\}$ such that for all $i \in \{1, \ldots, h-1\}$, one has $d_i = \beta(I_i)$, where, if $I_i = [q_i^1, q_i^2]$, then $\beta(I_i)$ is the substring of $\beta$ starting at the $q_i^{th}$ position and ending at the $q_i^{th}$ position.

Furthermore, let $J_1, \ldots, J_{2n-3}$ be the closed intervals corresponding to the positions of the substrings $a_1^m c_1 a_1^m, a_2^m c_2 a_2^m, a_3^m c_2 a_3^m, \ldots, a_{n-1}^m c_{n-1} a_{n-1}^m$ in $\beta$ respectively. Then for every $i \in \{1, \ldots, h-1\}$, there is some $j \in \{1, \ldots, 2n-3\}$ for which $I_i \subseteq J_j$ holds.

**Fact 4.2.21** For each $i \in \{1, \ldots, n-1\}$, there exist $s_1, \ldots, s_n, v_1, \ldots, v_{n-2} \in \Sigma^*$ such that

$$
\sigma = s_1 a_1^m c_1 a_1^m v_1 a_2^m \underbrace{s_2 c_2 a_2^m v_2 a_3^m \ldots s_i c_i a_i^m v_i a_{i+1}^m \ldots s_{n-2} c_{n-2} a_{n-2}^m v_{n-2} a_{n-1}^m}_{s_{n-1} c_{n-1} a_{n-1}^m \ldots s_n},
$$

where, for all $j \in \{1, \ldots, n-2\}$ and $j' \in \{2, \ldots, n-1\}$, $v_j$ (resp. $s_{j'}$) is either the
empty string or it contains $a_j^m$ (resp. $a_j^m$) as a suffix.

**Fact 4.2.22** For any $\pi' \in \Pi^z$ such that $\{\alpha, \beta\} \subset L(\pi')$ and $\pi' \neq \pi$, there is some $i \in \{1, \ldots, n - 1\}$ for which

$$
\sigma \in L(\pi') \setminus L(\pi)
$$

holds for all $s_1, s_2, \ldots, s_n, v_1, \ldots, v_{n-2} \in \Sigma^*$ such that for all $j \in \{1, \ldots, n-2\}$ and $j' \in \{2, \ldots, n-1\}$, $v_j$ (resp. $s_{j'}$) is either the empty string or it contains $a_j^m$ (resp. $a_{j'}^m$) as a suffix.

**Fact 4.2.23** For any $\pi' \in \Pi^z$ such that $\{\alpha, \beta\} \subset L(\pi')$ and $\pi' \neq \pi$, it holds that $\sigma \in L(\pi') \setminus L(\pi)$.

To prove Fact 4.2.18, first note that $\{\alpha, \beta\} \subset L(\pi)$ follows directly from the constructions of $\alpha$ and $\beta$. Moreover, by Lemma 4.2.10, $c_i$ is not a substring of $\hat{c}_i$ for all $i \in \{1, \ldots, n - 1\}$. Now if $c_i$ is a proper substring of $c_{i+1}$, then $c_{i+1}$ cannot be a substring of $c_i$, and so $c_{i+1}$ cannot be a substring of $w_i$. Combining the last two facts with the requirements on $a_{i+1}$ and $a_{i+2}$, it follows that $B_1 := a_1^m \hat{c}_1 a_1^m a_2^m c_2 a_2^m a_1^m w_1 a_1^m$ does not contain a substring of the form $s_1 c_1 s_2 c_2 s_3$ for some $s_1, s_2, s_3 \in \Sigma^+$. Inductively, assume that

$$
B_1 B_2 \ldots B_i = a_1^m \hat{c}_1 a_1^m a_2^m c_2 a_2^m a_1^m w_1 a_1^m \ldots a_i^m \hat{c}_i a_i^m a_{i+1}^m c_{i+1} a_{i+1}^m w_i a_i^m
$$

does not contain a substring of the form $s_1 c_1 s_2 \ldots c_{i+1} s_{i+2}$. By the construction of $w_i$, no prefix of $c_{i+1}$ is a suffix of $B_1 \ldots B_i$; consequently, as $c_{i+1}$ is not a substring
of \( a_{i+1}^m \) and \( a_{i+1}^m a_{i+2}^m a_{i+2}^m a_{i+1}^m w_{i+1} a_{i+1}^m \) does not contain a substring of the form \( c_{i+1}^v c_{i+2}^v \),

\[
B_1 B_2 \ldots B_i B_{i+1} = a_1^m c_1 a_2^m c_2 a_2^m a_1^m w_1 a_1^m \ldots a_i^m c_i a_i^m a_{i+1}^m a_{i+1}^m a_i^m w_i a_i^m
\]
cannot be expressed in the form \( s'_1 c'_1 s'_2 c_2 \ldots c_i s'_{i+1} c_{i+1} s'_{i+2} c_{i+2} s'_{i+3} \) for some \( s'_1, s'_2, \ldots, s'_{i+2}, s'_{i+3} \in \Sigma^+ \), and so it follows by induction that \( \sigma \notin L(\pi) \).

Fact 4.2.19 is an immediate consequence of \( \{\alpha, \beta\} \subset L(\pi') \), while Fact 4.2.20 is implied by Fact 4.2.19. To establish Fact 4.2.21 it suffices to observe that since, for all \( i \in \{1, \ldots, n-2\} \), \( a_i = a_{i+1} \) holds if \( c_i = c_{i+1} \), the subsequence \( B_i a_i^m c_i^v a_{i+1}^m a_{i+1}^m a_i^m \) of \( \sigma \) must contain both \( a_i^m c_{i+1} a_{i+1}^m v_{i+1} a_{i+2}^m \) and \( a_i^m c_i a_i^m v_{i+1} a_{i+1}^m \) as (possibly overlapping) substrings if \( i \leq n-3 \), while \( B_{n-2} a_{n-1}^m c_{n-1} a_{n-1}^m \) contains both \( a_i^m c_{n-1} a_{n-1}^m \) and \( a_{n-2}^m c_{n-2} a_{n-2}^m v_{n-2} a_{n-2}^m s_{n-1} \), where, for \( j \in \{i, i+1\} \), \( v_j \) (resp. \( v_{n-2} \)) is either equal to the empty string or it contains \( a_j \) (resp. \( a_{n-2} \)) as a suffix, and \( s_{n-1} \) is either the empty string or it contains \( a_{n-1}^m \) as a suffix.

To prove Fact 4.2.22 consider any \( \pi' \) such that \( \{\alpha, \beta\} \subset L(\pi') \) and \( \pi' \neq \pi \). By Fact 4.2.19 \( \pi' \) must start and end with variables. Let \( \pi' = X_1 d_1 X_2 \ldots d_{h-1} X_h \), where \( d_1, \ldots, d_{h-1} \in \Sigma^+ \) and \( X_1, \ldots, X_h \in X^+ \). Fact 4.2.19 gives that \( |\pi'| \leq m \).

We claim that there exists a least \( i_0 \in \{1, \ldots, n-1\} \) such that no subsequence \( \langle j_1, \ldots, j_0 \rangle \) of \( \langle 1, \ldots, h-1 \rangle \) satisfies the condition that \( c_p \) is a substring of \( d_{j_p} \) for all \( p \leq i_0 \). Assume by way of a contradiction that this were not true. Then either \( |\pi'| > m \) or \( \pi' = \pi \) would hold, a contradiction.

Let \( \varphi : X \mapsto \Sigma^+ \) be a substitution witnessing \( \beta \in L(\pi') \). \( \varphi \) induces a mapping \( A_\varphi \) from the set of all intervals of positions of \( \pi' \) to the set of all intervals of positions
of \( \beta \) such that if \( A_\phi \) maps \([p_1, q_1]\) and \([q_1, r_1]\) to \([p_2, q_2]\) and \([q_3, r_2]\) respectively, then \( q_2 = q_3 \) and \( A_\phi([p_1, r_1]) = [p_2, r_2] \). There is a variable \( y \) occurring in \( \pi' \) at a position \( j_1 \) such that \( j_2 \in A_\phi(\{j_1\}) \) for some \( j_2 \in \{1, \ldots, |\beta|\} \) with \( m(2i_0 - 1) + 1 + \sum_{i<i_0} |c_i| \leq j_2 \leq m(2i_0 - 1) + |c_{i_0}| + \sum_{i<i_0} |c_i| \); in other words, the \( j_1^{th} \) symbol of \( \pi' \) is mapped to some symbol in the specific occurrence of \( c_{i_0} \) that starts at the \( \left( m(2i_0 - 1) + 1 + \sum_{i<i_0} |c_i| \right)^{th} \) position of \( \beta \). One can extend the definition of \( \phi(y) \) so that it covers the string \( \hat{c}_{i_0} \). Similarly, as \( |d_i| \leq m \) holds for all \( i \in \{1, \ldots, h-1\} \), \( \phi^{-1}(a_i^m) \) contains a variable \( y' \) for every occurrence of \( a_i^m \) in \( \beta \), and thus one can “remap” \( \phi(y') \) so that it covers an appropriate substring of

\[
\gamma := s_1^ma_1^mv_1a_2^mv_2a_3^m \cdots s_{i_0}^\hat{c}_{i_0}a_{i_0}^mv_{i_0}a_{i_0+1}^m \cdots s_{n-2c_n-2a_n-2v_n-2a_n-1}^ms_{n-1c_n-1a_n-1s_n}.
\]

In particular, fix any \( i \in \{1, \ldots, n-1\} \setminus \{i_0\} \), and consider the occurrence of \( a_i^m \) starting at the \( \left( 2mi - m + 1 + \sum_{j\leq i} |c_j| \right)^{th} \) position of \( \beta \); let \( I_i \) represent the interval of positions of \( \beta \) occupied by this occurrence of \( a_i^m \). \( J_i = A_{\phi}^{-1}(I_i) \) contains some \( p' \) such that the \( (p')^{th} \) position of \( \pi' \) is some variable \( y'' \). \( \phi(y'') \) can be redefined so that \( J_i \) is mapped to an interval of positions of

\[
s_1a_1^mc_1v_1a_2^mv_2a_3^m \cdots s_i^ca_i^mv_i^a_i+1s_{n-2c_n-2a_n-2v_n-2a_n-1s_n-1c_n-1}a_n^ms_n
\]

containing the interval corresponding to the substring indicated by braces above. \( \phi \) can be redefined similarly for other variables of \( \pi' \) so that \( \pi' \) is mapped to \( \gamma \).
Fact 4.2.23 then follows from Facts 4.2.18, 4.2.21 and 4.2.22. This establishes that $S$ is a teaching set for $\pi$.

The next result studies the case where some of the constant sections in the target regular pattern may be empty. We present a proof that is a bit more detailed than that for Lemma 4.2.17.

Lemma 4.2.24 Let $\pi = X_1c_1X_2c_2 \ldots c_{n-1}X_n$ be a regular pattern, where $X_1,\ldots,X_n \in X^+$, $c_1,c_2,\ldots,c_{n-1}$ are nonempty blocks of constants over $\Sigma$, and $n \geq 2$. If $z \geq 8$, then $TD(\pi, R\Pi^z) \leq 4$.

Proof. Let $\Sigma = \{b_1,\ldots,b_z\}$ for some $z \geq 8$. We view $\Sigma$ as an ordered set according to $b_1 < b_2 < \ldots < b_z$. Let $a, b \in \Sigma$ be two (arbitrarily chosen) distinct letters from $\Sigma$. Let $m = |\pi|$, $\ell_i = |X_i|$ and $m_i = |c_i|$. Let $\alpha = \pi(a)$, i.e.,

$$\alpha = a^{\ell_1}c_1a^{\ell_2}c_2 \ldots a^{\ell_{n-1}}c_{n-1}a^{\ell_n}. \quad (4.4)$$

It will be convenient for what follows to set $c_0 = c_n = \varepsilon$. For every $i \in \{1,\ldots,n-1\}$, let $i'$ be maximal such that $i' < i$ and $c_{i'} \neq c_i$. Similarly, let $i''$ be minimal such that $i'' > i$ and $c_{i''} \neq c_i$. Furthermore, let $a_i$ denote the smallest letter from $\Sigma$ that differs from $a$ and from the first and the last letter of the nonempty strings in the set $\{c_{i'}, c_i, c_{i''}\}$ (where only $c_{i'}$ and $c_{i''}$ are possibly empty). Note that a letter $a_i$ with the required properties must exist because $|\Sigma| \geq 8$. Given this notation, we
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define the strings $\beta$ and $\gamma$ as follows:

$$\beta = b a_1^m c_1^m a_2^m c_2^m \ldots a_{n-1}^m c_{n-1}^m a_{n-1}^m b$$

$$\gamma = a_{\ell_1-1}^\ell_1 c_1^m a_{\ell_2-1}^\ell_2 c_2^m a_{\ell_3-1}^\ell_3 \ldots a_{\ell_{n-1}-1}^\ell_{n-1} c_{n-1}^m a_{n-1}^m c_{n-1}^m.$$  \hspace{1cm} (4.5) \hspace{0.5cm} (4.6)

The following two claims are the main building blocks of the proof:

**Claim 1:** $\alpha, \beta \in L(\pi)$ and $\gamma \notin L(\pi)$.

**Claim 2:** There exists a string $\tilde{\beta} \in \Sigma^+$ such that $\tilde{\beta} \notin L(\pi)$ and such that every regular pattern $\rho$ with $\alpha, \beta \in L(\rho)$ and $L(\rho) \neq L(\pi)$ generates at least one of the strings $\gamma, \tilde{\beta}$.

These claims imply that $T = \{(\alpha, +), (\beta, +), (\gamma, -), (\tilde{\beta}, -)\}$ is a teaching set for $L(\pi)$. We now prove Claims 1 and 2. We begin with the proof of Claim 1. One has that $\alpha = \pi(a) \in L(\pi)$. The string $\beta$ is obtained from $\pi$ by choosing a variable assignment such that

$$X_1 = b a_1^m, \quad X_n = a_{n-1}^m b \quad \text{and} \quad X_i = a_{i-1}^m a_i^m \quad \text{for} \quad i = 2, \ldots, n - 1.$$ 

It follows that $\beta \in L(\pi)$. Induction over $k = 1, \ldots, n - 1$ shows that the smallest prefix of $\gamma$ matching the pattern $\pi_k := X_1 c_1 X_2 c_2 \ldots X_k c_k$ equals

$$\gamma_k := a_{\ell_1-1}^\ell_1 c_1^m a_{\ell_2-1}^\ell_2 c_2^m a_{\ell_3-1}^\ell_3 \ldots a_{\ell_{k-1}-1}^\ell_{k-1} c_k^m a_k^m c_k.$$ 

It follows that the smallest prefix of $\gamma$ matching the pattern $\pi_{n-1}$ is $\gamma_{n-1}$. Note that $\pi = \pi_{n-1} X_n$ and $\gamma = \gamma_{n-1}$. Thus, $\gamma$ does not match $\pi$, that is, $\gamma \notin L(\pi)$. This
completes the proof of Claim 1.

The proof of Claim 2 is somewhat more tedious. Suppose that $\rho$ is a regular pattern such that $\alpha, \beta \in L(\rho)$ but $L(\rho) \neq L(\pi)$. Since the first (resp. the last) letter of $\alpha$ differs from the first (resp. the last) letter of $\beta$, it follows that $\rho$ starts (resp. ends) with a nonempty block of variables. Thus $\rho$ is of the form

$$\rho = Y_1d_1Y_2d_2\ldots Y_{h-1}d_{h-1}Y_h$$

for some $h \geq 1$, some nonempty blocks $Y_1, \ldots, Y_h$ of variables and some strings $d_1, \ldots, d_{h-1} \in \Sigma^+$. Let $A : X \mapsto \Sigma^+$ be the assignment which witnesses that $\beta \in L(\rho)$. We then obtain the following decomposition of $\beta$:

$$\beta = A(Y_1)d_1A(Y_2)d_2\ldots A(Y_{h-1})d_{h-1}A(Y_h) = A(\rho) \quad (4.7)$$

For all $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, h-1\}$, we write $\kappa_i$ to denote the occurrence of $c_i$ in $\beta$ starting at the $\left(\sum_{\ell=1}^{i-1} |c_{\ell}| + (2i-1)m + 2\right)$-nd position of $\beta$ (see (4.5)); similarly, we write $\mu_j$ to denote the occurrence of $d_j$ in $\beta$ starting at the $\left(\sum_{\ell=1}^{j-1} (|Y_\ell| + |d_\ell|) + |Y_j| + 1\right)$-st position of $\beta$ (see (4.7)). Let $P_i$ denote the set of positions of $\beta$ occupied by $\kappa_i$ and $Q_j$ denote the set of positions of $\beta$ occupied by $\mu_j$. We say that the substring $\kappa_i$ of $\beta$ is covered by the substring $\mu_j$ of $\beta$ w.r.t. $A$ iff for all $p \in P_i$, there is some $q \in Q_j$ such that under $A$, the $q^{th}$ letter of $\rho$ is mapped to the $p^{th}$ letter of $\beta$. For example, suppose that $\beta = ba_1^mc_1a_1^ma_2^nc_2a_2^mb$ and $\rho = Y_1d_1Y_2d_2Y_3$, where $c_1 = c_2 = bb$, $d_1 = abb$ and $d_2 = bbbaa$. If $A'$ is a homomorphism such that $A'(Y_1) = ba_1^{m-1}$, $A'(Y_2) = a_1^ma_2^m$, $A'(Y_3) = a_2^{m-2}b$ and $A'(s) = s$ for all $s \in \Sigma$, then $\kappa_1$ is covered by $\mu_1$ w.r.t. $A'$ and $\kappa_2$ is covered by $\mu_2$. 
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w.r.t. $A'$.

We proceed with a case distinction. It will turn out that $\gamma$ matches $\rho$ if $\rho$ falls into Case 1, and a properly defined string $\tilde{\beta} \notin L(\pi)$ matches $\rho$ if $\rho$ falls into Case 2.

**Case 1:** There exists a sequence $1 \leq j(1) \leq j(2) \leq \ldots \leq j(n) \leq h - 1$ such that, for all $i = 1, \ldots, n - 1$, the string $\kappa_i$ in the decomposition (4.5) is covered by the string $\mu_{j(i)}$ in the decomposition (4.7) w.r.t. $A$.

**Case 2:** A sequence $1 \leq j(1) \leq j(2) \leq \ldots \leq j(n) \leq h - 1$ satisfying the requirements described in Case 1 does not exist.

We discuss Case 1 first. Recall from the proof of Lemma 4.2.13 that for any $s \in \Sigma$ and any $w \in \Sigma^*$, the term $\#(s)[w]$ denotes the total number of occurrences of the symbol $s$ in the word $w$. From $\alpha = \pi(a) \in L(\rho)$, we can infer that, for every $a' \in \Sigma \setminus \{a\}$, $\#(a')[d_1 \ldots d_{h-1}]$ is bounded from above by $\#(a')[c_1 \ldots c_{n-1}]$.

Recall that $b \neq a$ and $a_i \neq a$ for $i = 1, \ldots, n - 1$. Thus the string $\beta$ does not contain the letter $a$ outside its constant blocks $c_1, \ldots, c_{n-1}$. From $\beta \in L(\rho)$, we may therefore infer that $\#(a')[d_1 \ldots d_{h-1}] \leq \#(a)[c_1 \ldots c_{n-1}]$. Clearly, $\#(a')[d_{j(1)} \ldots d_{j(n-1)}] \leq \#(a')[c_1 \ldots c_{n-1}]$ holds for every letter $a' \in \Sigma$. Putting these observations together, we get

$\#(a')[d_{j(1)} \ldots d_{j(n-1)}] \leq \#(a'[d_1 \ldots d_{h-1}] \leq \#(a')[c_1 \ldots c_{n-1}]$ so that $\#(a'[d_{j(1)} \ldots d_{j(n-1)}] = \#(a'[d_1 \ldots d_{h-1}] = \#(a') [c_1 \ldots c_{n-1}]$. Moreover, $h = n$ and $j(i) = i$, $c_i = d_i$ for $i = 0, \ldots, n - 1$ because, otherwise, we would find a letter $a_i$ that occurs more often in $d_1 \ldots d_{h-1}$ than it occurs in $c_1 \ldots c_{n-1}$. The whole discussion can be summarized by saying that $\rho$ is
of the form

\[ \rho = Y_1 c_1 Y_2 c_2 \ldots Y_{n-1} c_{n-1} Y_n , \]

where no variable in \( Y_1 Y_2 \ldots Y_n \) occurs more than once. Moreover, suppose that \( B : (X \cup \Sigma)^* \rightarrow \Sigma^* \) witnesses \( \alpha \in L(\rho) \). Then \( \#(a)[\alpha] = \#(a)[c_1 \ldots c_{n-1}] + \#(a)[B(Y_1) \ldots B(Y_n)] \). Furthermore, it follows from (4.4) that \( \#(a)[\alpha] = \sum_{i=1}^{n} \ell_i \). Thus, from the preceding two expressions for \( \#(a)[\alpha] \), one has \( \sum_{i=1}^{n} |Y_i| = \sum_{i=1}^{n} |X_i| \).

Now assume by way of a contradiction that \( |Y_k| \geq |X_k| \) for all \( k \in \{1, \ldots, n-1\} \); then one has

\[ \sum_{i=1}^{n} |Y_i| \geq \sum_{i=1}^{n} |X_i|. \] (4.8)

Thus it follows from (4.8) and (4.9) that \( \sum_{i=1}^{n} |Y_i| = \sum_{i=1}^{n} |X_i| \), and therefore \( |Y_k| = |X_k| \) for all \( k \in \{1, \ldots, n-1\} \). This contradicts the assumption that \( L(\rho) \neq L(\pi) \).

Hence there must exist an index \( k \in \{1, \ldots, n-1\} \) such that \( |Y_k| < |X_k| \). Choose \( k \) minimal with this property. We now obtain \( \gamma \in L(\rho) \) because \( \gamma \) is obtained from \( \rho \) by choosing a variable assignment such that \( Y_i = a^{\ell_i-1} c_i a_i^m \) for \( i = 1, \ldots, k-1 \), \( Y_k = a^{\ell_k-1} \), \( Y_j = a_j^{m-1} c_{j-1} a_{j-1}^{m-1} \) for \( j = k+1, \ldots, n-1 \) and \( Y_n = a_n^{m-1} c_{n-1} \). Note that such a variable assignment exists because \( |Y_i| < m \) for all \( i \in \{1, \ldots, n-1\} \) and \( |Y_k| < |X_k| = \ell_k \). This completes the analysis of Case 1.

Suppose now that the pattern \( \rho \) falls into Case 2. Let \( k(\rho) \) be the maximal number \( k \) with the following property: there exists a (possibly empty) sequence \( 1 \leq j(1) \leq \)
\( j(2) \leq \ldots \leq j(k - 1) \leq h - 1 \) such that, for all \( i = 1, \ldots, k - 1 \), the string \( \kappa_i \) in the decomposition (4.5) is covered by the string \( \mu_{j(i)} \) in the decomposition (4.7) w.r.t. \( A \). We know that \( k(\rho) < n \) because, otherwise, \( \rho \) would fall into Case 1. Thus, \( 1 \leq k(\rho) \leq n - 1 \). Note that \( \rho \) has a property that it does not share with the pattern \( \pi \):

- The string \( \kappa_{k(\rho)} \) in the decomposition (4.5) of \( \beta \) is not covered by any of the strings \( \mu_1, \ldots, \mu_{h-1} \) in the decomposition (4.7) of \( \beta \) w.r.t. \( A \). Thus, there must be a variable of \( \rho \) that, via assignment \( A \), generates at least one letter of \( \kappa_{k(\rho)} \).

According to (4.1) and (4.2) from Notation 4.2.1, we get

\[
\hat{c}_i = \pi_{c_i}(a_i) = c_i[1 : |c_i| - 1]a_i c_i[2 : |c_i|]
\]

for \( i = 1, \ldots, n - 1 \). We say that a string of the form

\[
\begin{align*}
& a_1^m S_1 a_1^m c_1 a_1^m S_1' a_1^m \ldots a_{k(\rho)-1}^m S_{k(\rho)-1} a_{k(\rho)-1}^m c_{k(\rho)-1} a_{k(\rho)-1}^m S'_{k(\rho)-1} a_{k(\rho)-1}^m \times \\
& a_{k(\rho)}^m \hat{c}_{k(\rho)} a_{k(\rho)}^m c_{k(\rho)} a_{k(\rho)}^m S_{k(\rho)} a_{k(\rho)}^m + a_{k(\rho)-1}^m S_{k(\rho)-1} a_{k(\rho)-1}^m S'_{k(\rho)-1} a_{k(\rho)-1}^m + \\
& a_{n-1}^m S_{n-1} a_{n-1}^m c_{n-1} a_{n-1}^m S'_{n-1} a_{n-1}^m \ldots \\
\end{align*}
\]  

(4.10)

for some strings \( S_1, S'_1, \ldots, S_{n-1}, S'_{n-1} \in \Sigma^* \) is a \( \rho \)-relative of \( \beta \). Claim 2 is a direct consequence of the following two claims:

**Claim 3:** Every \( \rho \)-relative of \( \beta \) is a member of \( L(\rho) \).

**Claim 4:** There exists a string \( \bar{\beta} \notin L(\pi) \) which is a \( \rho \)-relative of \( \beta \) for every \( \rho \).
The proof of Claim 3 makes use of the decompositions (4.5) and (4.7) of $\beta$. Our strategy is to manipulate the assignment $A$ from (4.7) so as to obtain another assignment $\tilde{A}$ that satisfies $\tilde{A}(\rho) = \tilde{\beta}$. Note that each $\rho$-relative $\tilde{\beta}$ basically results from $\beta$ by deleting the first and last letters of $\beta$, substituting $\hat{c}_k(\rho)$ for $\kappa_k(\rho)$ within the substring $a_k^m c_k a_k^m$, and by substituting $a_i^m S_i a_i^m$ (resp. $a_i^m S'_i a_i^m$) for the occurrence of $a_i^m$ ending just before the first position of $\kappa_i$ (resp. the occurrence of $a_i^m$ starting just after the last position of $\kappa_i$) for every $i \neq k(\rho)$. The b’s at the beginning and the end of $\beta$ will not play any role in the construction of $\tilde{\beta}$; these b’s were used solely to ensure that any regular pattern consistent with $\{(\alpha, +), (\beta, +)\}$ must start and end with variables. Recall that there must be a variable of $\rho$ that, via assignment $A$, generates at least one letter of $\kappa_k(\rho)$. An analogous statement holds for the generation of the substring $a_i^m$ from $\rho$ via $A$. We may therefore modify the assignment $A$ as follows:

- Let $k = k(\rho)$ and let $y$ denote the first variable of $\rho$ which, via assignment $A$, makes a contribution to $\kappa_k$, say the contribution $\kappa_k[p : q]$ for $1 \leq p \leq q \leq m_k$, where $p$ and $q$ are chosen to be the minimum and maximum possible values respectively. Thus, there exist strings $u, u' \in \Sigma^*$ such that $A(y) = u \kappa_k[p : q] u'$. Observe that $u \neq \varepsilon$ implies that $p = 1$ and $u' \neq \varepsilon$ implies that $q = |\kappa_k|$. Choosing an assignment $\tilde{A}$ such that $\tilde{A}(y) = u c_k[p : m_k - 1] a_k c_k[2 : q] u'$ has the desired effect of substituting $\hat{c}_k$ for $\kappa_k$.

- For $i \notin \{k(\rho), 1, n - 1\}$, let $\bar{a}$ denote the occurrence of $a_i^m$ ending just before the first position of $\kappa_i$ and let $\bar{a}$ denote the occurrence of $a_i^m$ starting just after the last position of $\kappa_i$. Let $y$ (resp. $y'$) denote the first variable of $\rho$ which, via assignment $A$, makes a contribution to $\bar{a}$ (resp. $\bar{a}$). The contribution of
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This completes the proof of Claim 3.

We next verify Claim 4. For \( i = 1, \ldots, n - 2 \), let \( w_i = c_i a_i^{2m} \) if \( c_i \neq c_{i+1} \) and let \( w_i = \varepsilon \) otherwise. The string \( \tilde{\beta} \) is chosen as follows:

\[
\tilde{\beta} = \begin{cases} 
  a_1^m c_1 a_1^m a_2^{2m} c_2 a_2^m a_1^m w_1 a_1^m & \text{if } \tilde{c}_k \text{ is not a substring of } c_{k+1}^m \\
  a_1^m c_1 a_1^m a_2^{2m} c_2 a_2^m a_1^m w_1 a_1^m & \text{if } \tilde{c}_{k+1} \text{ is the shortest prefix of } c_{k+1} \text{ that contains } c_k \text{ as a substring.}
\end{cases}
\]

It follows that the smallest prefix of \( \tilde{\beta} \) matching the pattern \( \pi_{n-2} \) is \( \tilde{\beta}_{n-2} \). Note
that \( \pi = \pi_{n-2}X_{n-1}c_{n-1}X_{n} \) and \( \tilde{\beta} = \tilde{\beta}_{n-2}a_{n-2}^{3m}a_{n-1}^{m}c_{n-1}a_{n-1}^{m} \) if \( c_{n-2} \) is not a substring of \( c_{n-1} \), and \( \tilde{\beta} = \tilde{\beta}_{n-2}\tilde{c}_{n-1}a_{n-1}^{m}a_{n-2}^{m}a_{n-1}^{m}c_{n-1}a_{n-1}^{m} \) if \( \tilde{c}_{n-1} \) is the shortest prefix of \( c_{n-1} \) containing \( c_{n-2} \) as a substring and \( c_{n-1} = \tilde{c}_{n-1}c_{n-1} \) for some \( \tilde{c}_{n-1} \in \Sigma^{*} \). In the latter case, observe that \( c_{n-1} \) is not a substring of the suffix \( \tilde{c}_{n-1}a_{n-1}^{m}a_{n-2}^{m}a_{n-1}^{m} \) of \( \tilde{\beta} \) by the definitions of \( w_{n-2}, a_{n-1}, a_{n-2}, \tilde{c}_{n-1}, \) and \( \tilde{c}_{n-1} \). Similarly, since the suffix \( a_{n-2}^{m}a_{n-1}^{m} \) of \( \tilde{\beta} \) does not contain \( c_{n-1} \) as a substring, we may conclude that \( \tilde{\beta} \notin L(\pi) \), indeed.

It remains to show that \( \tilde{\beta} \) is a \( \rho \)-relative of \( \beta \). Observe first that, given the above definition of \( \tilde{\beta} \) in terms of \( B_{i} \) and \( B'_{i} \), \( \tilde{\beta} \) can be decomposed as follows:

\[
\tilde{\beta} = B_{1} \ldots B_{n-2} a_{n-1}^{m} \hat{c}_{n-1}a_{n-1}^{m} = a_{1}^{m} \hat{c}_{1}a_{1}^{m} B_{2}^{'} \ldots B_{n-1}^{'} = B_{1} \ldots B_{k-1} a_{k}^{m} \hat{c}_{k}a_{k}^{m} B_{k+1}^{'} \ldots B_{n-1}^{'}
\]

Note that this holds for all \( k = 1, \ldots, n - 1 \). Setting

\[
S_{k,i} = \begin{cases} 
\hat{c}_{i}a_{i}^{m}a_{i+1}^{2m}c_{i+1}a_{i+1}^{m} & \text{if } i \leq k - 1 \land c_{i} \neq c_{i+1} \\
\hat{c}_{i}a_{i}^{2m} & \text{if } i \leq k - 1 \land c_{i} = c_{i+1} \\
\varepsilon & \text{if } i \geq k
\end{cases}
\]

and

\[
S'_{k,i} = \begin{cases} 
a_{i}^{m} & \text{if } i \leq k - 1 \\
 a_{i-1}^{m}w_{i-1}a_{i-1}^{m}c_{i} & \text{if } i \geq k
\end{cases}
\]

for each \( k = 1, \ldots, n - 1 \), we obtain \( B_{i} = a_{i}^{m}S_{k,i}a_{i}^{m}c_{i}a_{i}^{m}S'_{k,i}a_{i}^{m} \) for all \( i \leq k - 1 \), where we made use of the fact that \( a_{i} = a_{i+1} \) whenever \( c_{i} = c_{i+1} \). Similarly, we obtain \( B'_{i} = a_{i}^{m}S_{k,i}a_{i}^{m}c_{i}a_{i}^{m}S'_{k,i}a_{i}^{m} \) for all \( i \geq k + 1 \). It follows that, for all \( k = 1, \ldots, n - 1 \),
\[ \tilde{\beta} = a_1^m S_{k,1}a_1^m c_1 a_1^m S'_{k,1}a_1^m \ldots a_{k-1}^m S_{k,k-1}a_{k-1}^m c_{k-1} a_{k-1}^m S'_{k,k-1}a_{k-1}^m \]

\[ a_k^m c_k a_k^m a_{k+1}^m S_{k,k+1} a_{k+1}^m c_{k+1} a_{k+1}^m S'_{k,k+1} a_{k+1}^m \ldots \]

\[ a_{n-1}^m S_{k,n-1} a_{n-1}^m c_{n-1} a_{n-1}^m S'_{k,n-1} a_{n-1}^m . \]

Setting \( k = k(\rho) \) and comparing the above expression for \( \tilde{\beta} \) with (4.10), it becomes clear that \( \tilde{\beta} \) is a \( \rho \)-relative of \( \beta \) for every \( \rho \), which completes the proof.  

Regular patterns starting (resp. ending) with a string from \( \Sigma^+ \) are not much harder to teach than the corresponding subpatterns that start and end with a nonempty block of variables. This is made precise in the following lemma.

**Lemma 4.2.25** Let \( z = |\Sigma| \geq 2 \). Let \( \pi \) be a regular pattern and let \( T \) be a teaching set for \( \pi \) w.r.t. \( \Pi^z \). Suppose that \( T \) contains two positive examples \((w_1, +), (w_2, +)\) such that either (i) \( \pi \) starts with a block of variables and \( w_1, w_2 \) start with the same letter, or (ii) \( \pi \) ends with a block of variables and \( w_1, w_2 \) end with the same letter. Let \( c_1, c_2 \in \Sigma^+ \). With this notation, the following hold:

1. \( TD(c_1 \pi, \Pi^z) \leq 1 + |T| \) and \( TD(\pi c_2, \Pi^z) \leq 1 + |T| \).

2. \( TD(c_1 \pi c_2, \Pi^z) \leq 2 + |T| \).

**Proof.** We verify the upper bound on the teaching dimension of \( L(c_1 \pi) \). The statements concerning \( L(\pi c_2) \) and \( L(c_1 \pi c_2) \) can be proved similarly.

In order to teach \( L(c_1 \pi) \), we make use of the teaching set \( T \) but replace every example \((u, l) \in T \) by the example \((c_1 u, l) \), yielding a new teaching set \( T' \). Note that the
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Let \( l \in \{+, -\} \) is left unchanged. Let \( d\rho \) be a regular pattern such that \( d \in \Sigma^* \), \( \rho \) starts with a variable and \( d\rho \) is consistent with the \( |T| \) examples in \( T' \) that have been presented so far. Since \( T' \) contains two positively labelled examples starting with \( c_1 \) and continuing with different letters, we may conclude that \( d \) is a prefix of \( c_1 \). Let \( m_1 = |c_1| \). Let \( a_1 \in \Sigma \) be a letter that is distinct from \( c_1[m_1] \). Define \( \pi_{c_1}(a_1) \) according to equation (4.1) in Notation 4.2.1. We pick a positively labelled example \((u', +) \in T \) and extend the current teaching set \( T' \) of size \( |T| \) by one additional example, namely by \((\pi_{c_1}(a_1)u', -) \). Now the teaching set \( T' \) for the pattern \( c_1\pi \) is complete. It may be directly verified\(^2\) that \( d\rho \) can be consistent with \((c_1u, +) \) and \((\pi_{c_1}(a_1)u, -) \) only if the prefix \( d \) of \( c_1 \) is not proper, i.e., if \( d = c_1 \). Given that \( d = c_1 \) and given \((c_1u, l) \in T' \), the fact that \( d\rho = c_1\rho \) is consistent with \((c_1u, l) \) implies that \( \rho \) is consistent with \((u, l) \). Thus \( \rho \) must be consistent with the original teaching set \( T \). It follows that \( \rho = \pi \) (up to equivalence). \( \blacksquare \)

Lemmas 4.2.11 to 4.2.25 together establish Theorem 4.2.9.

To round off our discussion of the regular pattern languages, the following proposition determines the exact value of the teaching dimension of regular patterns with exactly one nonempty block of constants and two nonempty blocks of variables over any alphabet with at least two letters.

**Proposition 4.2.26** Let \( z = |\Sigma| \geq 2 \) and \( \pi = X_1c_1X_2 \) be any regular pattern, where \( c_1 \in \Sigma^+ \) and \( X_1, X_2 \in X^+ \). Then \( \text{TD}(\pi, R\Pi^z) = 3 \).

**Proof.** Let \( |\pi| = m \). By Propositions 4.2.3 and 4.2.4 \( \text{TD}(\pi, R\Pi^z) \geq 3 \). We focus on proving the upper bound, presenting separate proofs for the cases (i) \( z = 2 \) and

\(^2\)Compare with a similar reasoning in the proof of Lemma 4.2.11.
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(ii) $z \geq 3$.

**Case (i):** Let $\Sigma = \{a, b\}$. Consider three subcases.

(a): $c_1 = \delta^{m_1}$, where $m_1 \geq 1$ and $\delta \in \{a, b\}$. Without loss of generality, assume that $\delta = a$. A teaching set for $\pi$ w.r.t. $R^{a^\infty}$ consists of the positively labelled examples $(a^{X_1}|X_2|^{m_1}, \ast), (b^{X_1}|a^{m_1}b|X_2|, \ast)$ and the negatively labelled example $(b^{X_1}|^{m_1}a, a^{m_1-1}b^{m_1}a^{m_1-1}b a^{m_1+1|X_2|-1}, \ast)$ (where the braces are included to make clear the structure of the string).

Suppose $\rho$ were a regular pattern such that $a^{X_1}|X_2|^{m_1}, b^{X_1}|a^{m_1}b|X_2| \in L(\rho)$, $b^{X_1}|^{m_1}a, a^{m_1-1}b^{m_1}a^{m_1-1}b a^{m_1+1|X_2|-1} \notin L(\rho)$ and $L(\rho) \neq L(\pi)$. From $a^{X_1}|X_2|^{m_1}$ in $L(\rho)$ it follows immediately that every constant part of $\rho$ is a substring of $a^{m_1}$. Furthermore, since $b^{X_1}|a^{m_1}b|X_2| \in L(\rho)$, $\rho$ must start and end with variable symbols. Let $A' : X \mapsto \Sigma$ be any assignment witnessing $b^{X_1}|a^{m_1}b|X_2| \in L(\rho)$. Then for every occurrence of $b$ in $b^{X_1}|a^{m_1}b|X_2|$, there must be some variable $y$ in $\rho$ such that $A'$ maps $y$ to that occurrence of $b$. Thus $\rho$ is of the shape $Y_1a^{i_1}Y_2a^{i_2}Y_3 \ldots Y_k a^{i_k}Y_{k+1}$ for some $k \geq 0$, $Y_1, \ldots, Y_{k+1} \in X^+$ and some $i_1, \ldots, i_k$ such that $\sum_{j=1}^{k} i_j \leq m_1$.

The case $k = 0$ (implying that $\rho = Y_1$) can be ruled out by the negative example $b^{X_1}|^{m_1}a, a^{m_1-1}b^{m_1}a^{m_1-1}b a^{m_1+1|X_2|-1}$.

Suppose $k = 1$, so that $\rho$ is of the shape $Y_1a^{m_1}Y_2$, where $Y_1, Y_2 \in X^+$. If $|Y_1| < |X_1|$, then one may define an assignment $A'' : X \mapsto \Sigma$ such that $A''$ maps $Y_1$ to $b^{X_1}|^{m_1}a, a^{m_1}Y_2$ to $a^{m_1}ba^{m_1-1}b^{m_1}a^{m_1-1}ba^{m_1+1|X_2|-1}$. A similar argument holds if $|Y_2| < |X_2|$.

Suppose $k \geq 2$. Then one may define an assignment $A''' : X \mapsto \Sigma$ such that $A'''$ maps $Y_1$ to $b^{X_1}|^{m_1}a, a^{i_1}Y_2 \ldots Y_k a^{i_k}$ to $a^{m_1-1}b^{m_1}a^{m_1-1}$, and $Y_{k+1}$ to $ba^{m_1+1|X_2|-1}$.
Hence for all $k \geq 1$, $b^{X_1|1}a^{m_1}ba^{m_1|1}b^{m_1}a^{m_1|1}b^{m_1+|X_2|}|1} \in \mathcal{L}(\rho)$, a contradiction.

(b): $c_1 = \delta_1^{m_1}\delta_2^{m_2}$, where $m_1, m_2 \geq 1$, $\delta_1, \delta_2 \in \{a, b\}$ and $\delta_1 \neq \delta_2$. Without loss of generality, assume that $\delta_1 = a$ and $\delta_2 = b$. A teaching set for $\pi$ w.r.t. $\text{RIP}^2$ consists of the positively labelled examples $(a|X_1|+m_1b^{m_2}+|X_2|, +)$, $(b|X_1|a^{m_1}b^{m_2}a|X_2|, +)$ and the negatively labelled example $(a|X_1|+m_1|1}^{m_1}b^{m_2}a^{m_1}b^{m_2}a^{m_1}b^{m_2+|X_2|}|1}^{|X_2|}, -)$.

Consider any regular pattern $\rho$ such that $a|X_1|+m_1b^{m_2}+|X_2|, b|X_1|a^{m_1}b^{m_2}a|X_2| \in \mathcal{L}(\rho)$ and $\mathcal{L}(\pi) \neq \mathcal{L}(\rho)$. Since $a|X_1|+m_1b^{m_2}+|X_2| \in \mathcal{L}(\rho)$, $\rho$ must be of the shape

$$X_1'|a^{i_1}X_2' \ldots a^{i_k}X_{k+1}' b^{j_{k+1}}X_{k+2}' \ldots b^iX_{l+1}'$$

where $X_1', \ldots, X_{l+1}' \in X^*$. Since $a|X_1|+m_1b^{m_2}+|X_2|, b|X_1|a^{m_1}b^{m_2}a|X_2| \in \mathcal{L}(\rho)$, $\rho$ must start and end with variable symbols, so that $X_1', X_{l+1}' \in X^*$. Furthermore, since $b|X_1|a^{m_1}b^{m_2}a|X_2| \in \mathcal{L}(\rho)$, it holds that $\sum_{p=1}^k i_p + \sum_{p=2}^k |X_p'| \leq m_1$ and $\sum_{p=k+1}^{l} i_p + \sum_{p=k+2}^{l} |X_p'| \leq m_2$ and $|\rho| \leq |\pi|$. Suppose that $A : (X \cup \Sigma)^* \mapsto \Sigma^*$ is a homomorphism witnessing $b|X_1|a^{m_1}b^{m_2}a|X_2| \in \mathcal{L}(\rho)$.

Case (b.1): There is a first variable $y$ occurring in $\rho$ such that $A$ maps $y$ to either

(i) an occurrence of $a$ in $b|X_1|a^{m_1}b^{m_2}a|X_2|$ between its ($|X_1| + 1)^{st}$ and ($|X_1| + m_1)^{st}$ positions (inclusive), or

(ii) an occurrence of $b$ in $b|X_1|a^{m_1}b^{m_2}a|X_2|$ between its ($|X_1| + m_1 + 1)^{st}$ and ($|X_1| + m_1 + m_2)$-nd positions (inclusive). Suppose (i) holds. Let $\rho'$ be the substring of $\rho$ that $A$ maps to the substring of $b|X_1|a^{m_1}b^{m_2}a|X_2|$ between its ($|X_1| + 1)^{st}$ and ($|X_1| + m_1)^{st}$ positions (inclusive). Then, since $\rho'$ contains at least one variable, one can define a homomorphism $B : (X \cup \Sigma)^* \mapsto \Sigma^*$ mapping
\(\rho'\) to \(a^{m_1-1}b^{m_2}b^ma^{m_1-1}\). Let \(\rho = \rho_1\rho'\rho_2\). One can extend \(B\) so that \(B\) maps \(\rho_1\) to \(a^{X_1}\) and \(\rho_2\) to \(b^{m_2}a^{m_1}b^{m_2-1}a^{m}a^{m_1}b^{m_2+|X_2|-1}\). Thus \(a^{X_1}+m_1-1b^{m_2}b^m a^{m_1-1}b^{m_2}a^{m_1}b^{m_2-1}a^{m}a^{m_1}b^{m_2+|X_2|-1} \in L(\rho)\). A similar argument applies if (ii) holds.

**Case (b.2):** For every \(i\) such that \(|X_1| + 1 \leq i \leq |X_1| + m_1 + m_2\), \(A\) maps an occurrence of either \(a\) or \(b\) in \(\rho\) to the \(i^{th}\) position of \(b^{X_1}a^{m_1}b^{m_2}a^{X_2}\). Then \(a^{m_1}b^{m_2}\) is a substring of \(\rho\), and since \(\sum_{p=1}^{k} i_p + \sum_{p=2}^{k} |X'_p| \leq m_1\) and \(\sum_{p=k+1}^{l} i_p + \sum_{p=k+2}^{l} |X'_p| \leq m_2\), \(\rho\) is of the shape \(X''_1a^{m_1}b^{m_2}X''_2\). Since \(|\rho| \leq |\pi|\) and \(L(\rho) \neq L(\pi)\), at least one of the following must hold: (i) \(|X''_1| < |X_1|\), or (ii) \(|X''_2| < |X_2|\). If (i) holds, then one can define a homomorphism \(B : (X \cup \Sigma)^* \mapsto \Sigma^*\) mapping \(X''_1\) to \(a^{X_1-1}\) and \(a^{m_1}b^{m_2}X''_2\) to \(a^{m_1}b^{m_2}a^{m_1-1}b^{m_2}a^{m_1}b^{m_2-1}a^{m}a^{m_1}b^{m_2+|X_2|-1}\). Thus \(a^{X_1}+m_1-1b^{m_2}b^ma^{m_1}b^{m_2-1}a^{m}a^{m_1}b^{m_2+|X_2|-1} \in L(\rho)\). A similar argument applies if (ii) holds.

**(c):** \(c_1\) is not of the shape \(\delta_1^m\delta_2^m\), where \(m_1 + m_2 \geq 1\), \(\delta_1, \delta_2 \in \{a, b\}\) and \(\delta_1 \neq \delta_2\). First, suppose that \(c_1\) starts with \(a\) and ends with \(b\). Set \(\overline{c}_1 = \pi_{c_1}(a^{m}b^{m})\) (according to (4.1) in Notation 4.2.1). A teaching set for \(\pi\) w.r.t. \(R\Upsilon^2\) consists of the positively labelled examples \((a^{X_1}|c_1b^{X_2}|, +), (b^{X_1}|c_1 a^{X_2}|, +)\) and the negatively labelled example \((b^{X_1-1}|c_1a^{m}b^{m}\overline{c}_1a^{m}b^{m}c_1 a^{X_2-1}, -)\). Note that since \(c_1\) is neither a substring of \(a^{m}b^{m}\) nor a substring of \(\overline{c}_1\) and \(\pi\) neither ends with \(a\) nor starts with \(b\), it must hold that \(b^{X_1-1}c_1a^{m}b^{m}\overline{c}_1a^{m}b^{m}c_1 a^{X_2-1} \notin L(\pi)\). It is also readily verified that \(a^{X_1}|c_1b^{X_2}|, b^{X_1}|c_1a^{X_2}| \in L(\pi)\).

Consider any regular pattern \(\rho\) such that \(L(\rho) \neq L(\pi)\) and \(a^{X_1}|c_1b^{X_2}|, b^{X_1}|c_1 a^{X_2}| \in L(\rho)\). Since \(a^{X_1}|c_1b^{X_2}| \in L(\rho)\), the number of occurrences of substrings of the shape \(aX'b\) in \(\rho\), where \(X' \in X^*\), is upper bounded by the number of times that \(ab\) occurs in \(c_1\). Together with the assumption that \(a^{X_1}|c_1b^{X_2}|, b^{X_1}|c_1 a^{X_2}| \in L(\rho)\),
one can deduce the following facts:

- $\rho$ starts and ends with variable symbols;
- if $\rho = X_1'c_1X_2'$ for some $X_1', X_2' \in X^+$, then $|X_1'| + |X_2'| < |X_1| + |X_2|$;
- if $\rho \neq X_1'c_1X_2'$ for all $X_1', X_2' \in X^+$, then some substring of $a^m b^m c_1 a^m b^m$ matches the substring of $\rho$ that starts with the first occurrence of a constant in $\rho$ and ends with the last occurrence of a constant in $\rho$.

These facts establish that $b^{X_1|X_2|} c_1 a^m b^m c_1 a^m b^m c_1 a^{|X_2|-1} \in L(\rho)$. By symmetry, it also holds that if $c_1$ starts with $b$ and ends with $a$, then $\pi$ has a teaching set w.r.t. $\Pi^\pi$ of size 3. If $c_1$ starts and ends with the same letter, say $a$, then we set $\hat{c}_1 = c_1^2 b^m c_1^2$; a teaching set for $\pi$ w.r.t. $\Pi^\pi$ consists of the positively labelled examples $(a^{X_1} c_1 a^{X_2}, +), (b^{X_1} c_1 b^{X_2}, +)$ and the negatively labelled example $(b^{X_1|X_2|} c_1 b^2 c_1 b^{X_2|X_2|} a^{|X_2|}, -)$. An argument very similar to the one in the case just discussed (that $c_1$ starts with $a$ and ends with $b$) shows that this set of labelled examples is indeed a teaching set for $\pi$ w.r.t. $\Pi^\pi$.

**Case (ii):** Choose some $a \in \Sigma$ that is different from the first symbol of $c_1$ and the last symbol of $c_1$. Next, fix some $b \in \Sigma \setminus \{a\}$. Our teaching set for $\pi$ w.r.t. $\Pi^\pi$ consists of the positively labelled examples $(b^{X_1} c_1 b^{X_2}, +), (a^{X_1} c_1 a^{X_2}, +)$ and the negatively labelled example $(a^{X_1|X_2|} c_1 a^m \hat{c}_1 a^m c_1 a^{X_2|X_2|}, -)$, where $\hat{c}_1$ is defined by (4.2) in Notation 4.2.1. Let $\rho$ be any regular pattern such that $b^{X_1} c_1 b^{X_2}, a^{X_1} c_1 a^{X_2} \in L(\rho)$ and $L(\rho) \neq L(\pi)$. Since $a \neq b$, one can deduce the following facts:

- $\rho$ starts and ends with variable symbols;

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3Compare with the proof of Claim 2 in the proof of Lemma 4.2.24
4Compare with the proof of Claim 2 in the proof of Lemma 4.2.24
• if $\rho = X'_1 c_1 X'_2$ for some $X'_1, X'_2 \in X^+$, then $|X'_1| + |X'_2| < |X_1| + |X_2|;

• if $\rho \neq X'_1 c_1 X'_2$ for all $X'_1, X'_2 \in X^+$, then some substring of $a^m c_1 a^m$ matches the substring of $\rho$ that starts with the first occurrence of a constant in $\rho$ and ends with the last occurrence of a constant in $\rho$.

These three facts show that $a^{|X_1|-1} c_1 a^m c_1 a^{|X_2|-1} \in L(\rho)$, and so $L(\rho)$ cannot be consistent with our teaching set for $\pi$.

Remark 4.2.27 Let $z = |\Sigma| \geq 2$. It follows from Remark 4.2.8, Lemma 4.2.25 and Proposition 4.2.26 that for any $c_1, c_2 \in \Sigma^+$ and $X_1, X_2 \in X^+$ such that $X_1, X_2$ have no repeated variables, $TD(X_1 c_1 X_2 c_2, \Pi^z) = TD(c_1 X_1 c_2 X_2, \Pi^z) = 4$.

Our results on the teaching dimensions of various regular pattern languages are summed up in Tables 4.1, 4.2, 4.3, 4.4 and 4.5. $c_1, c_2, \ldots, c_n \in \Sigma^+$, $x_1, x_2, \ldots, x_n \in X$, and $X_1, X_2, \ldots, X_n$ are nonempty strings of distinct variables over $X$. It is assumed that $n > 2$. 
### Table 4.1: TD of some regular patterns

| Pattern Form | $2 \leq |\Sigma| \leq 6$ | $|\Sigma| = 7$ |
|--------------|----------------------|------------------|
| $c_1 x_1 c_2 x_2 \ldots x_{n-1} c_n$ | $TD \geq 5$ (Lemma 4.2.7) | $TD = 5$ (Lemmas 4.2.17, 4.2.25, 4.2.7) |
| $x_1 c_1 x_2 c_2 \ldots c_{n-1} x_n$ | $TD \geq 3$ (Remark 4.2.8) | $TD = 3$ (Lemma 4.2.17, Remark 4.2.8) |
| $x_1 c_1 x_2 c_2 \ldots x_n c_n$ | $TD \geq 4$ (Remark 4.2.8) | $TD = 4$ (Lemmas 4.2.17, 4.2.25, Remark 4.2.8) |
| $c_1 x_2 c_2 \ldots c_{n-1} x_n$ | $TD \geq 4$ (Remark 4.2.8) | $TD = 4$ (Lemmas 4.2.17, 4.2.25, Remark 4.2.8) |
| $c_1 X_1 c_2 X_2 \ldots X_{n-1} c_n$ | $TD \geq 5$ (Lemma 4.2.7) | $TD \geq 5$ (Lemma 4.2.7) |
| $X_1 c_1 X_2 c_2 \ldots c_{n-1} X_n$ | $TD \geq 3$ (Remark 4.2.8) | $TD \geq 3$ (Remark 4.2.8) |
| $X_1 c_1 X_2 c_2 \ldots X_n c_n$ | $TD \geq 4$ (Remark 4.2.8) | $TD \geq 4$ (Remark 4.2.8) |
### Table 4.2: TD of some regular patterns

| Pattern Form | $8 \leq |\Sigma| \leq \infty$ |
|--------------|---------------------------------|
| $c_1x_1c_2x_2\ldots x_{n-1}c_n$ | TD = 5  |
| | (Lemmas 4.2.17/4.2.25/4.2.7) |
| $x_1c_1x_2c_2\ldots c_{n-1}x_n$ | TD = 3  |
| | (Lemma 4.2.17, Remark 4.2.8) |
| $x_1c_1x_2c_2\ldots x_{n}c_n$ | TD = 4  |
| | (Lemmas 4.2.17/4.2.25, Remark 4.2.8) |
| $c_1x_1x_2c_2\ldots c_{n-1}x_n$ | TD = 4  |
| | (Lemmas 4.2.17/4.2.25, Remark 4.2.8) |
| $c_1X_1c_2X_2\ldots X_{n-1}c_n$ | $5 \leq TD \leq 6$  |
| | (Lemmas 4.2.24/4.2.25, Lemma 4.2.7) |
| $X_1c_1X_2c_2\ldots c_{n-1}X_n$ | $3 \leq TD \leq 4$  |
| | (Lemma 4.2.24, Remark 4.2.8) |
| $X_1c_1X_2c_2\ldots X_{n}c_n$ | $4 \leq TD \leq 5$  |
| | (Lemmas 4.2.24/4.2.25, Remark 4.2.8) |
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| Pattern Form            | $2 \leq |\Sigma| \leq 6$ | $|\Sigma| = 7$ |
|-------------------------|---------------------------|---------------|
| $c_1X_1c_2\ldots c_{n-1}X_n$ | $TD \geq 4$ (Remark 4.2.8) | $TD \geq 4$ (Remark 4.2.8) |
| $X_1c_1X_2$            | $TD = 3$ (Proposition 4.2.26) | $TD = 3$ (Proposition 4.2.26) |
| $X_1c_1X_2c_2$         | $TD = 4$ (Remark 4.2.27) | $TD = 4$ (Remark 4.2.27) |
| $c_1X_1c_2X_2$         | $TD = 4$ (Remark 4.2.27) | $TD = 4$ (Remark 4.2.27) |
| $c_1X_1c_2, |X_1| \geq 2$ | $TD = 5$ (Lemma 4.2.13) | $TD = 5$ (Lemma 4.2.13) |
| $X_1c_1, |X_1| \geq 2$ | $TD = 4$ (Remark 4.2.14) | $TD = 4$ (Remark 4.2.14) |
| $c_1X_1, |X_1| \geq 2$ | $TD = 4$ (Remark 4.2.14) | $TD = 4$ (Remark 4.2.14) |
| $X_1, |X_1| \geq 2$ | $TD = 3$ (Lemma 4.2.12) | $TD = 3$ (Lemma 4.2.12) |
| $c_1x_1c_2$            | $TD = 4$ (Remark 4.2.14) | $TD = 4$ (Remark 4.2.14) |
| $x_1c_1$               | $TD = 3$ (Remark 4.2.14) | $TD = 3$ (Remark 4.2.14) |

Table 4.3: TD of some regular patterns


| Pattern Form                  | $8 \leq |\Sigma| \leq \infty$ |
|------------------------------|-------------------------------|
| $c_1 X_1 c_2 \ldots c_{n-1} X_n$ | $4 \leq \text{TD} \leq 5$     |
|                              | (Lemmas 4.2.24, 4.2.25)       |
|                              | (Remark 4.2.8)                |
| $X_1 c_1 X_2$                | TD = 3                        |
|                              | (Proposition 4.2.26)          |
| $X_1 c_1 X_2 c_2$            | TD = 4                        |
|                              | (Remark 4.2.27)               |
| $c_1 X_1 c_2 X_2$            | TD = 4                        |
|                              | (Remark 4.2.27)               |
| $c_1 X_1 c_2, |X_1| \geq 2$ | TD = 5                        |
|                              | (Lemma 4.2.13)                |
| $X_1 c_1, |X_1| \geq 2$ | TD = 4                        |
|                              | (Remark 4.2.14)               |
| $c_1 X_1, |X_1| \geq 2$ | TD = 4                        |
|                              | (Remark 4.2.14)               |
| $X_1, |X_1| \geq 2$ | TD = 3                        |
|                              | (Lemma 4.2.12)                |
| $c_1 x_1 c_2$                | TD = 4                        |
|                              | (Remark 4.2.14)               |
| $x_1 c_1$                    | TD = 3                        |
|                              | (Remark 4.2.14)               |

Table 4.4: TD of some regular patterns
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| Pattern Form | $2 \leq |\Sigma| \leq 6$ | $|\Sigma| = 7$ | $8 \leq |\Sigma| \leq \infty$ |
|--------------|-----------------|---------|-----------------|
| $c_1 x_1$    | TD = 3          | TD = 3  | TD = 3          |
|              | (Remark 4.2.14) | (Remark 4.2.14) | (Remark 4.2.14) |
| $x_1$        | TD = 2          | TD = 2  | TD = 2          |
|              | (Lemma 4.2.12)  | (Lemma 4.2.12) | (Lemma 4.2.12)  |
| $c_1$        | TD = 2          | TD = 2  | TD = 2          |
|              | (Lemma 4.2.11)  | (Lemma 4.2.11) | (Lemma 4.2.11)  |

Table 4.5: TD of some regular patterns

4.3 Recursive Teaching Dimension of Pattern Languages

We close this chapter with a short treatment of the RTD of pattern languages. Mazadi showed in her thesis that the class of regular pattern languages over any alphabet with at least 2 letters has RTD equal to 2 [30, Theorem 3.7], while the RTD of the whole class of pattern languages over any infinite alphabet is also equal to 2 [30, Theorem 5.2]. The first main result of the present section is that the RTD of the class of pattern languages over any unary alphabet is infinite, which is a rather stark contrast with the fact that $\text{RTD}(\Pi^\infty) = 2$.

**Theorem 4.3.1** $\text{RTD}(\Pi^1) = \infty$.

This result was stated without proof in Theorem 3.3.6. We now give the full details of the proof.

We first define a family of linear subsets of $\mathbb{N}_0$; this family will later be identified

5Note the one-to-one correspondence between the members of NE–LINSET and those of $\Pi^1$. 
with a family of pattern languages over \( \Sigma = \{a\} \). Let \( N, p \in \mathbb{N} \) such that \( p \geq 2 \). For \( u_0, u_1, \ldots, u_{p-1} \in \mathbb{N} \) and for \( u = (u_0, u_1, \ldots, u_{p-1}) \), define \( C_{N,p} \) as follows:

\[
C_{N,p}(u) = \left\{ N + \sum_{i=0}^{p-1} \lambda_i u_i \middle| \lambda_0, \ldots, \lambda_{p-1} \in \mathbb{N} \right\}
\]

\[
U(p) = \{ u \in \mathbb{N}^p \mid u_0 = p \text{ and } u_i \in \{p + i, 2p + i\} \text{ for } i = 1, \ldots, p-1 \}
\]

\[
C_{N,p} = \{ C_{N,p}(u) \mid u \in U(p) \}.
\]

**Lemma 4.3.2** For every \( p \geq 2 \) and every \( u \in U(p) \),

\[
C_{0,p}(u) = \{0,p\} \cup \{n \in \mathbb{N} \mid n \geq 2p\} \cup \{p + i \mid i \in \{1, \ldots, p-1\} \land u_i = p + i\}
\]

\[
= \mathbb{N}_0 \setminus \{(1, \ldots, p-1) \cup \{p + i : i \in \{1, \ldots, p-1\} \land u_i = 2p + i\}\}.
\]

**Proof.** Let \( u \in U(p) \) be arbitrary but fixed. From \( u_0 = p \), we may conclude that \( jp \in C_{0,p}(u) \) for all \( j \in \mathbb{N} \). Consider any \( n \geq 2p \) that is not a multiple of \( p \), and let \( r = n \mod p \in \{1, \ldots, p-1\} \). \( n \) can be written in the form \( \lambda_0 u_0 + \lambda_r u_r \) for properly chosen parameters \( \lambda_0 \in \mathbb{N}_0 \) and \( \lambda_r \in \mathbb{N} \). If \( u_i = p + i \), then \( p + i \) belongs to \( C_{0,p}(u) \). If, however, \( u_i = 2p+i \), then there is no way of writing \( p+i \) as a nonnegative integer combination of \( u_0, u_1, \ldots, u_{p-1} \). This completes the proof of the lemma. □

**Proof of Theorem 4.3.1.** Lemma 4.3.2 implies that \( C_{N,p}(u) = \{N,N+p\} \cup \{N+n \in \mathbb{N} \mid n \geq 2p\} \cup \{N + p + i \mid i \in \{1, \ldots, p-1\} \land u_i = p + i\} \) for every \( N,p \in \mathbb{N} \) such that \( p \geq 2 \) and for every \( u \in U(p) \). Thus, any subset of \( S = \{N + p + i \mid i = 1, \ldots, p-1\} = \{N + p + 1, \ldots, N + 2p - 1\} \) can be written as an intersection of \( S \) with a member from \( C_{N,p} \). Moreover, all sets of \( C_{N,p} \) coincide

---

\(^6\text{In the terminology of learning theory, this means that } S \text{ is shattered by } C_{N,p}.\)
on \( \mathbb{N} \setminus S \). This implies that \( TD_{\text{min}}(C_{N,p}) = |S| = p - 1 \). By identifying \( n \) with \( a^n \), one can consider \( C_{N,p} \) as a class of languages over the unary alphabet \( \Sigma = \{a\} \). We now argue that each language \( C_{N,p}(u) \) can be generated by a pattern provided that \( N \) is sufficiently large. To this end, let \( N_u = N - \sum_{i=0}^{p-1} u_i \) and observe that, for all \( N \geq 3p^2 > \sum_{i=0}^{p-1} u_i \),

\[
C_{N,p}(u) = \left\{ N + \sum_{i=0}^{p-1} \lambda_i u_i \mid \lambda_0, \ldots, \lambda_{p-1} \in \mathbb{N} \right\}
\]

The latter set can be identified with \( L(a^{N_u}x_0^{u_0}x_1^{u_1} \ldots x_{p-1}^{u_{p-1}}) \). Thus, for every \( p \geq 2 \), there exists a subclass of \( \Pi^1 \) whose \( TD_{\text{min}} \) equals \( p - 1 \). Since \( RTD(\mathcal{L}) \geq \sup_{\mathcal{L}' \subseteq \mathcal{L}} TD_{\text{min}}(\mathcal{L}') \), this implies that \( RTD(\Pi^1) \geq p - 1 \). As \( p \) can be chosen arbitrarily large, it follows that \( RTD(\Pi^1) = \infty \).

While the RTD of the whole class of pattern languages over any finite alphabet with at least 2 letters remains open, the final result of this chapter shows that positive examples alone are inadequate for teaching pattern languages in the RTD model.

**Theorem 4.3.3** Let \( z = |\Sigma| \) with \( 2 \leq z < \infty \). For each \( n \geq 1 \), define

\[
\mathcal{C}_n = \{ \pi \in X^+ : |\pi| = n \}.
\]

Then \( TD^+(\mathcal{C}_n) \geq RTD^+(\mathcal{C}_n) \geq \lceil \log_{2z}(n) \rceil \). In particular, \( TD^+(\Pi^z) = RTD^+(\Pi^z) = \infty \).

**Proof.** Let \( \pi \in \mathcal{C}_n \) be the pattern defined by \( \pi[i] = x_i \) for all \( i \in \{1, \ldots, n\} \). We
show that any weak spanning set $I$ for $L(\pi)$ w.r.t. $C_n$ must have at least $\lfloor \log_2(n) \rfloor$ elements. For each $W \subseteq \{1, \ldots, n\}$, let $\pi_W \in C_n$ be the constant-free pattern defined as follows.

- For all $i \in W$, $\pi_W[i] = x_1$.
- For all $i \in \{1, \ldots, n\} \setminus W$, $\pi_W[i] = x_{i+1}$.

Since $z \geq 2$, one has that if $W \subseteq W' \subseteq \{1, \ldots, n\}$, then $L(\pi_{W'}) \subseteq L(\pi_W)$.

We shall prove by induction on $r$ that for any set $P_r = \{\alpha_1, \ldots, \alpha_r\}$ of $r$ elements of $L(\pi)$ with $\frac{n}{(2z)^{r}} > 1$, there exists some $W_r \subseteq \{1, \ldots, n\}$ of size at least $\frac{n}{(2z)^{r}}$ such that $P_r \subseteq L(\pi_{W_r}) \subseteq L(\pi)$. This would imply that $|I| \geq \lfloor \log_2(n) \rfloor$.

For $r = 0$, it is evident that if $n \geq 2$, then one can define $W_0 = \{1, \ldots, n\} \neq \emptyset$. Furthermore, as observed earlier, one has $L(\pi_{W_0}) \subseteq L(\pi_\emptyset) = L(\pi)$. Inductively, assume there exists a subset $W_j \subseteq \{1, \ldots, n\}$ of size at least $\frac{n}{(2z)^{j}}$ such that $\{\alpha_i : i \leq j - 1\} \subseteq L(\pi_{W_j}) \subseteq L(\pi)$. Let $W_j = \{p_1, \ldots, p_l\}$, and suppose $A : X \rightarrow \Sigma^+$ is an assignment that witnesses $\alpha_j \in L(\pi)$. For each $i \in \{1, \ldots, l\}$, let $c_i$ be the first symbol of $A(x_{p_i})$. Suppose $\frac{n}{(2z)^{j+1}} > 1$. Note that $\{c_1, \ldots, c_l\} \subseteq \Sigma$, while $l \geq \left\lfloor \frac{n}{(2z)^{j}} \right\rfloor$ and $|\Sigma| = z$. Thus, by the pigeonhole principle, there exists a subset $\{p_{i_1}, \ldots, p_{i_m}\} \subseteq \{p_1, \ldots, p_l\}$ of size at least $\frac{n}{2^{j+1}}$ such that $c_{i_f} = c_{i_g}$ for all $f, g \in \{1, \ldots, m\}$. We may assume that $p_{i_1} < \ldots < p_{i_m}$. Now choose the subset of odd terms $\{p_{i_1}, p_{i_3}, p_{i_5}, \ldots, p_{i_h}\}$ of $\{p_{i_1}, \ldots, p_{i_m}\}$ such that $h = m - 1$ if $m$ is even and $h = m - 2$ if $m$ is odd. Observe that by setting $W_{j+1} = \{p_{i_1}, p_{i_3}, p_{i_5}, \ldots, p_{i_h}\}$, one has that $\{\alpha_i : i \leq j\} \subseteq L(\pi_{W_{j+1}}) \subseteq L(\pi)$. Indeed, the relation $\{\alpha_i : i \leq j - 1\} \subseteq L(\pi_{W_{j+1}})$ follows immediately from $L(\pi_{W_{j}}) \subseteq L(\pi_{W_{j+1}})$. Furthermore,
let $W_{j+1} = \{q_1, \ldots, q_u\}$, where $q_1 < \ldots < q_u$, and for all $i \in \{1, \ldots, n\}$ let $A'(x_i)$ denote the string obtained from $A(x_i)$ by deleting its first symbol. The relation $\alpha_j \in L(\pi W_{j+1})$ is witnessed by the assignment $B : X \to \Sigma^*$ given by

$$B(x_i) = \begin{cases} c_{i_1} & \text{if } i = 1 \text{ or } x_i \notin \text{Var}(\pi W_{j+1}); \\
A'(x_{q_j})A(x_{i-1}) & \text{if } i = q_j + 2 \text{ for some } j \in \{1, \ldots, u\}; \\
A(x_{i-1}) & \text{otherwise.} \end{cases}$$

Note that $B$ is well-defined due to the condition $q_i + 1 < q_j$ for all $i, j$ with $i < j$.

In addition,

$$|W_{j+1}| \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2^{2j+1}} \right\rfloor \right\rfloor = \left\lfloor \frac{n}{(2z)^{j+1}} \right\rfloor.$$

This completes the induction step, and therefore any weak spanning set for $\pi$ w.r.t. $C_n$ must indeed have at least $\lceil \log_{2z}(n) \rceil$ elements. \qed
Chapter 5

The Teaching Complexity of Erasing Pattern Languages

Shinohara extended Angluin’s original definition of a pattern language by allowing variables in a pattern to be substituted with the empty string; the corresponding pattern languages are called erasing (or extended) pattern languages. Throughout this chapter, only erasing pattern languages will be considered; in other words, the language $L(\pi)$ generated by a pattern $\pi$ over $\Sigma$ consists of all strings generated from $\pi$ when replacing variables in $\pi$ with any string (including the empty string) over $\Sigma$, where all occurrences of a single variable must be replaced by the same string. In particular, all “pattern languages” in this chapter will refer to erasing pattern languages, unless otherwise specified. Further, we shall adapt the terminology and notation of Chapter 4 to the class of erasing pattern languages.

In the first part of this chapter, we study the following decision problem: given
any pattern $\pi$, is $\pi$ finitely distinguishable w.r.t. the class of all patterns, that is, are there finite sets $T^+$ and $T^-$ of strings such that $L(\pi)$ is the only pattern language that contains all of the strings in $T^+$ and none of the strings in $T^-$?

This problem is of relevance to computational learning theory as well as to formal language theory; previously it has been studied in computational biology [9] and in a recursion-theoretic context [8]. In computational learning theory, the question of whether a given pattern $\pi$ is finitely distinguishable w.r.t. the class $\Pi^2$ of all patterns is equivalent to that of whether $\text{TD}(\pi, \Pi^2) < \infty$.

The finite distinguishability problem is trivially decidable for the class of non-erasing pattern languages: any non-erasing pattern can be distinguished with finitely many examples [2]. For erasing pattern languages, on the other hand, the finite distinguishability problem turns out to be a bit more complex. We shall show that over any unary alphabet, finite distinguishability of a given pattern can be decided solely by examining the number of occurrences of each variable in the pattern. In addition, over any alphabet of size at least 4, finitely distinguishable patterns possess a rather simple structure; for alphabets of sizes 2 and 3, the structure of a finitely distinguishable pattern can be more complicated, but there are still a number of fairly restrictive constraints on the length of its constant parts and the distribution of its variables.

The second main part of the present chapter is devoted to the study of the worst-case complexity of teaching pattern languages in the TD and RTD models. We shall see that the TD (resp. RTD) of any given subclass of erasing pattern languages is often larger than the TD (resp. RTD) of the corresponding subclass of non-erasing pattern languages. This observation is perhaps not too surprising, given the fact
that for any erasing pattern language \(L(\pi)\) and any finite set \(T\) of labelled examples for \(\pi\), it is relatively straightforward (as we shall demonstrate shortly) to construct a pattern \(\pi'\) that is distinct from \(\pi\) and yet consistent with \(T\). A notable exception is the class of regular erasing pattern languages: our results on the TD and the RTD for the erasing case are almost identical to those for the non-erasing case.

The results presented in this chapter have been accepted for publication in the following article:


### 5.1 Pattern Languages with Finite Teaching Dimension

In this section, we investigate the structural properties of patterns that are finitely distinguishable. We begin with a few preparatory definitions.

**Definition 5.1.1** Fix any alphabet \(\Sigma\) of size \(z \leq \infty\). For any \(\pi \in \Pi^z\) with \(\pi = X_1c_1X_2\ldots c_{n-1}X_n\), \(X_1,\ldots,X_n \in X^*\) and \(c_1,\ldots,c_{n-1} \in \Sigma^+\), call each nonempty block \(X_i\) a maximal variable block of \(\pi\). Call a set \(\{Y_1,\ldots,Y_k\}\) of maximal variable blocks of \(\pi\) independent with respect to \(\pi\) iff every variable \(x\) in some block \(Y_i\) does not occur in any maximal variable block \(Z \notin \{Y_1,\ldots,Y_k\}\) of \(\pi\). In particular, the set \(\{Z_1,\ldots,Z_l\}\) of all maximal variable blocks of \(\pi\) is independent w.r.t. \(\pi\). Call a variable \(x\) free w.r.t. \(\pi\) iff \(x\) occurs in \(\pi\) exactly once. A pattern \(\pi\) is called block-
regular if each of its maximal variable blocks contains a free variable w.r.t. \( \pi \).\footnote{26}

Jain, Ong and Stephan\footnote{26} showed that any block-regular pattern \( \pi \) is equivalent to the pattern obtained from \( \pi \) by dropping all the variables that occur at least twice in \( \pi \).

\textbf{Theorem 5.1.2} \footnote{26, Theorem 6(b)} Fix an alphabet \( \Sigma \), and let \( \pi = c_1 X_1 c_2 X_2 \ldots X_{n-1} c_n \) be a block-regular pattern, where \( c_1, c_n \in \Sigma^* \), \( X_1, \ldots, X_n \in X^+ \) and \( c_1, \ldots, c_{n-1} \in \Sigma^+ \). Then \( \pi \) is equivalent to the regular pattern \( \pi' = c_1 x_1 c_2 x_2 \ldots x_{n-1} c_n \).

\textbf{Definition 5.1.3} Fix an alphabet of size \( z \leq \infty \). For any \( \pi \in \Pi^z \), say that \( \pi \) is simple block-regular iff the following conditions are satisfied:

1. \( \pi \) is block-regular;

2. \( \pi \) does not contain any substring \( \alpha \in \Sigma^+ \) such that \( |\alpha| \geq 2 \);

3. \( \pi \) starts and ends with variables.

Note that by Theorem 5.1.2, every simple block-regular pattern is equivalent to a pattern \( \pi' \) of the shape \( y_1 a_1 y_2 a_2 \ldots a_k y_{k+1} \), where \( k \geq 0 \), \( a_1, a_2, \ldots, a_k \in \Sigma \) and \( y_1, y_2, \ldots, y_{k+1} \) are \( k+1 \) distinct variables.

We now present the main result of this chapter. It states that for \( z = 1 \) and \( z \geq 4 \), finite distinguishability is decidable. For \( z \in \{2, 3\} \), it shows that the finite distinguishability problem is decidable when restricted to constant-free patterns.

\textbf{Theorem 5.1.4} Let \( \pi \in \Pi^z \).
1. Let $z = 1$. Let $x_1, \ldots, x_l$ be all the distinct variables occurring in $\pi$. For all $i \in [1, l]$, let $p_i$ denote the number of times that $x_i$ occurs in $\pi$. Then $\pi$ is finitely distinguishable w.r.t. $\Pi^z$ iff $l \geq 1$ and $\gcd(p_1, \ldots, p_l) = 1$.

2. Let $z \geq 2$. If $\pi \in \Pi^z_{cf}$, then $\pi$ is finitely distinguishable w.r.t. $\Pi^z$ iff $\pi$ contains some variable exactly once.

3. Let $z \geq 4$. Then $\pi$ is finitely distinguishable w.r.t. $\Pi^z$ iff $\pi$ is simple block-regular.

**Proof.** Proof of (1). It was shown in [19] (Corollaries 9 and 10) that the linear set $\{ \vec{v}^\top \vec{x} : \vec{x} \in \mathbb{N}_0^n \}$ for any $n \geq 1$ and $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{N}_0^n$ has finite teaching dimension w.r.t. the class $\{ \{ \vec{v}^\top \vec{x} : \vec{x} \in \mathbb{N}_0^n \} : \vec{v} \in \mathbb{N}_0^n \land n \geq 1 \}$ iff $\gcd(v_1, \ldots, v_n) = 1$.

Notice that for any $c \in \mathbb{N}_0^n$, $\{ c + \vec{v}^\top \vec{x} : \vec{x} \in \mathbb{N}_0^n \}$ is the commutative image (or Parikh image) of the erasing pattern language generated by $a^c x_1^{v_1} x_2^{v_2} \ldots x_n^{v_n}$ over any unary alphabet $\{a\}$. Theorem 5.1.4.1 is thus a consequence of the following “shift lemma”.

**Claim 5.1.5** Let $\mathcal{L}$ be a class of nonempty subsets of $\mathbb{N}_0$ such that $0 \in L$ for all $L \in \mathcal{L}$. Define the shift-extension $\mathcal{L}'$ of $\mathcal{L}$ by $\mathcal{L}' = \{ c + L : (c \in \mathbb{N}_0) \land (L \in \mathcal{L}) \}$. Then for all $c \in \mathbb{N}_0$ and $L \in \mathcal{L}$, $TD(L, \mathcal{L}) \leq TD(c + L, \mathcal{L}') \leq c + 1 + TD(L, \mathcal{L})$.

**Proof of Claim 5.1.5.** We first prove $TD(L, \mathcal{L}) \leq TD(c + L, \mathcal{L}')$. Suppose for the sake of a contradiction that there exists a teaching set $T$ for $c + L$ w.r.t. $\mathcal{L}'$ that has size smaller than $TD(L, \mathcal{L})$. Define $T' = \{(x - c, +) : x \in T^+ \} \cup \{(x - c, -) : x \in T^- \land x \geq c \}$. Note that $T'$ is consistent with $L$. Since $|T'| < TD(L, \mathcal{L})$, there exists some $L' \in \mathcal{L}$ such that $L'$ is consistent with $T'$ and $L' \neq L$. Consequently, $c + L'$
is consistent with \( \{(c + y, +) : y \in T'^+\} \cup \{(c + y, -) : y \in T'^-\} \cup \{(x, -) : x \in T^- \land x < c\} = T \), a contradiction.

We next prove \( \text{TD}(c + L, \mathcal{L}') \leq c + 1 + \text{TD}(L, \mathcal{L}) \). Let \( T_1 \) be a teaching set for \( L \) w.r.t. \( \mathcal{L} \). Define \( T_2 = \{(c, +)\} \cup \{(x, -) : x < c\} \cup \{(c + x, +) : x \in T_1^+\} \cup \{(c + x, -) : x \in T_1^-\} \) (recall that \( 0 \in L \) by the definition of \( \mathcal{L} \)). Note that \( T_2 \) is consistent with \( c + \mathcal{L} \).

Suppose that for some \( c' \in \mathbb{N}_0 \) and \( L' \in L \), \( c' + L' \) is consistent with \( T_2 \). The consistency of \( c' + L' \) with \( \{(c, +)\} \cup \{(x, -) : x < c\} \) implies that \( c' = c' + \min(L') = c \). Thus \( L' \) is consistent with \( \{(x, +) : x \in T_1^+\} \cup \{(x, -) : x \in T_1^-\} = T_1 \), and therefore \( L' = L \). \( \blacksquare \) (Claim 5.1.5)

**Proof of (2).** Suppose that \( z \geq 2 \). Fix any distinct \( a, b \in \Sigma \). If \( \pi \) contains some variable exactly once, then \( L(\pi) = L(x_1) \), so that \( \{(a, +), (b, +)\} \) is a teaching set for \( \pi \) w.r.t. \( \Pi^z \). If \( \pi \) contains no variable and \( T \) is a finite set of examples labelled consistently with \( \pi \), then \( \pi' = \pi x_1^m \) is consistent with \( T \), where \( m > \max\{\lvert \alpha \rvert : \alpha \in T^+ \cup T^-\} \); i.e., \( \text{TD}(\pi, \Pi^z) = \infty \). Now suppose that \( \pi \) contains at least one variable and every variable occurring in \( \pi \) appears in \( \pi \) at least twice. Assume towards a contradiction that \( \pi \) has a finite teaching set \( T \) w.r.t. \( \Pi^z \). Choose \( m > \max\{\lvert \alpha \rvert : \alpha \in T^+ \cup T^-\} \cup \{\lvert \pi \rvert\} \). Consider the string

\[
\beta = a^{m+1} b^m a^m b^m a^m b^{m+1} a^{m+1} \cdots a^2 b^2 a^2 b^m a^m,
\]

which is a concatenation of the strings \( a^{m+i} b^{m+i} a^{m+i} \) for \( i \) increasing from 0 to \( m \).

We will show that for some appropriately chosen block \( Y \) of variables,

(I) \( \beta \pi(\epsilon) \in L(Y \pi) \setminus L(\pi) \);
(II) $L(Y\pi) \supseteq L(\pi)$;

(III) $w \in L(Y\pi) \setminus L(\pi)$ implies $|w| \geq m$.

Notice that items (I), (II) and (III) together imply that $Y\pi$ is consistent with $T$ while $L(Y\pi) \neq L(\pi)$, which contradicts the fact that $T$ is a teaching set for $\pi$ w.r.t. $\Pi^\ast$. We first prove that $\beta(\varepsilon) \notin L(\pi)$. Assume otherwise. Fix a substitution $A : (X \cup \Sigma)^* \mapsto \Sigma^*$ witnessing $\beta(\varepsilon) \in L(\pi)$. Given any strings $\alpha \in \Sigma^*$ and $\rho \in (X \cup \Sigma)^*$, say that $\rho$ covers $\alpha$ w.r.t. $A$ iff $\alpha$ is a prefix of $A(\rho)$. Our method of proof is to show by induction that for all $i \in \{-1, \ldots, m\}$ (where $\beta_{-1}$ is defined to be $\varepsilon$), the shortest prefix $\rho_i$ of $\pi$ that covers $\beta_i = a^m b^m a^m \ldots a^{m+i} b^{m+i} a^{m+i}$ w.r.t. $A$ satisfies $|\rho_i| \geq i + 1$. For $i = m$, this will imply that $|\pi| \geq |\rho_m| \geq m + 1$, a contradiction. There is nothing to prove for $i = -1$ since $\beta_{-1} = \varepsilon$. Now suppose the statement to be proven holds for $n = k$, that is, if $\rho_k$ is the shortest prefix of $\pi$ that covers $\beta_k = a^m b^m a^m \ldots a^{m+k} b^{m+k} a^{m+k}$, then $|\rho_k| \geq k + 1$. Consider $\beta_{k+1} = a^m b^m a^m \ldots a^{m+k} b^{m+k} a^{m+k+1} b^{m+k+1} a^{m+k+1}$. Let $s$ be the last symbol of $\rho_k$; note that $s$ is a variable. Suppose the string $\beta_{k+1} = \beta_k a^{m+k+1} b^{m+k+1} a^{m+k+1}$ is covered by $\rho_k$ w.r.t. $A$. Then, since no proper prefix of $\rho_k$ covers $\beta_k$ and $s$ occurs in $\pi$ at least twice, $A(\pi)$ must contain at least two copies of the string $a^{m+k+1} b^{m+k+1} a^{m+k+1}$, which is impossible. Hence there is a nonempty string $\theta$ for which the shortest prefix of $\pi$ covering $\beta_{k+1}$ w.r.t. $A$ is equal to $\rho_k \theta$, so that by the induction hypothesis, $|\rho_{k+1}| \geq k + 2$. This proves $\beta(\varepsilon) \notin L(\pi)$. Now pick distinct variables
y_1 and y_2 not occurring in \( \pi \), and set

\[
Y = y_1^m y_2^m y_1^{m+1} y_2^{m+1} y_1^{m+1} \cdots y_1^{2m} y_2^{2m} y_1^{2m}.
\]

Observe that \( \beta \pi(\varepsilon) \in L(Y \pi) \), proving (I). Further, (II) and (III) follow directly from the choice of \( m \) and \( Y \). Thus \( T \) is not a teaching set for \( \pi \) w.r.t. \( \Pi^z \), so that \( TD(\pi, \Pi^z) = \infty \).

**Proof of (3).** The proof that \( \pi \) is finitely distinguishable if it simple block-regular will be deferred to Proposition [5.1.6](#) where it will be shown, more generally, that over any finite alphabet of size at least 2, Conditions 1., 2. and 3. in Definition [5.1.3](#) together imply finite distinguishability.

It remains to show that if \( \pi \) does not satisfy either Condition 1., 2. or 3. in Definition [5.1.3](#) then \( TD(\pi, \Pi^z) = \infty \).

**Case (i):** \( \pi \) is not block-regular. Then one can fix some interval \([j_1, j_2] \) such that \( \pi[j_1] \cdots \pi[j_2] \) is a maximal variable block of \( \pi \) and for all \( j' \in [j_1, j_2] \), \( \pi[j'] \) occurs in \( \pi \) at least twice.

Suppose \( T \) were a finite teaching set for \( L(\pi) \) w.r.t. \( \Pi^z \). Choose \( m > \max\{ |\alpha| : \alpha \in T^+ \cup T^- \} \cup \{|\pi|\} \), and let \( \pi' \) be the pattern obtained from \( \pi \) by inserting

\[
Y = y_1^m y_2^m y_1^{m+1} y_2^{m+1} y_1^{m+1} \cdots y_1^{2m} y_2^{2m} y_1^{2m},
\]

which is a concatenation of \( y_1^{m+i} y_2^{m+i} y_1^{m+i} \) for \( i \) increasing from 0 to \( m \), into \( \pi \) just before the \( j_1^{th} \) symbol of \( \pi \), where \( y_1, y_2 \notin \text{Var}(\pi) \) are distinct variables. Choose distinct \( d_1, d_2 \in \Sigma \) that are different from the last constant before the \( j_1^{th} \) symbol of
\(\pi\) (suppose this occurs at the \(p_1^{th}\) position of \(\pi\); \(p_1 = 0\) if no such constant exists) and the first constant after the \(j_2^{th}\) symbol of \(\pi\) (suppose this occurs at the \(p_2^{th}\) position of \(\pi\); \(p_2 = |\pi| + 1\) if no such constant exists). Such \(d_1\) and \(d_2\) exist because \(|\Sigma| \geq 4\).

Let \(\beta\) be the string obtained from \(Y\) by substituting \(d_1\) for \(y_1\) and \(d_2\) for \(y_2\). Let \(\gamma\) be the string obtained from \(\pi\) by substituting \(d_1\) for \(y_1\), \(d_2\) for \(y_2\), and \(\varepsilon\) for every \(x \in \text{Var}(\pi)\). Then \(\gamma\) is of the form \(C_1 \beta C_2\), where \(C_1 C_2 \in \Sigma^*\) is the constant part of \(\pi\). We claim that \(\gamma \notin L(\pi)\). Suppose otherwise, and that \(A'' : (X \cup \Sigma)^* \mapsto \Sigma^*\) witnesses \(\gamma \in L(\pi)\).

**Case (i.1):** \(\pi\) contains at least one constant and \(C_1 \neq \varepsilon\). Suppose

\[
\gamma = a_1 \ldots a_i \underbrace{\beta a_{i+1} \ldots a_l}_{} ,
\]

where \(a_j \in \Sigma \cup \{\varepsilon\}\) for \(j \in [1, l]\) and \(a_i \in \Sigma\); note that \(C_1 = a_1 \ldots a_i\) and \(C_2 = a_{i+1} \ldots a_l\). \(A''\) induces a mapping \(I_{A''}\) from the set of all intervals of positions of \(\pi\) to the set of all intervals of positions of \(\gamma\) such that if \([p'_1, p'_2]\) and \([p'_2, p'_3]\) are mapped to \([q'_1, q'_2]\) and \([q'_3, q'_4]\) respectively, then \(I_{A''}\) maps \([p'_1, p'_3]\) to \([q'_1, q'_4]\). Since it is a bit more convenient to speak of mappings from a specific occurrence of a subpattern of \(\pi\) to a specific occurrence of a substring of \(\gamma\), we shall fix the convention that for any subpattern \(\pi'' = \pi[p'_1] \ldots \pi[p'_l]\) of \(\pi\) and any \(\alpha \in \{a_j : 1 \leq j \leq l\} \cup \{\beta\}\), “\(I_{A''}\) maps \(\pi''\) to \(\alpha''\)” means that \(I_{A''}\) maps \([p'_1, p'_l]\) to the interval of positions corresponding to the specific occurrence of \(\alpha\) in \(\gamma\) indicated by braces in the decomposition (5.1).

If \(I_{A''}\) maps the \(p_1^{th}\) symbol of \(\pi\) to some \(a_h\) with \(h < i\), then it must also map the second to last constant symbol before the \(j_1^{th}\) symbol of \(\pi\) to some \(a_{h'}\) with \(h' < h\); applying this argument successively then leads to a contradiction. A similar
argument shows that $I_{A'}$ cannot map the $p_1^{th}$ symbol of $\pi$ to some $a_h$ with $h > i$. Furthermore, by the choice of $d_1$ and $d_2$, $I_{A'}$ cannot map its $p_1^{th}$ position to any symbol in $\beta$. Hence $I_{A'}$ maps the $p_1^{th}$ symbol of $\pi$ to $a_i$. In particular, $I_{A'}$ maps the suffix of $\pi$ starting from its $(p_1 + 1)^{st}$ symbol to the suffix $\beta C_2$ of $\gamma$. Since $d_1$ and $d_2$ are different from the constant symbol in $\pi$’s $p_2^{th}$ position, $I_{A'}$ cannot map its $p_1^{th}$ position to any symbol in $\beta$. Hence $I_{A'}$ maps the $p_1^{th}$ symbol of $\pi$ to $a_i$. In particular, $I_{A'}$ maps the suffix of $\pi$ starting from its $(p_1 + 1)^{st}$ symbol to the suffix $\beta C_2$ of $\gamma$. Since $d_1$ and $d_2$ are different from the constant symbol in $\pi$’s $p_2^{th}$ position, $I_{A'}$ cannot map the maximal variable block of variables $\pi[j_1]...\pi[j_2]$ to $\beta$. Note that $I_{A'}$ cannot map $\pi[j_1]...\pi[j_2]$ to any proper extension of $\beta$ because otherwise $\gamma$ (as reasoned above) would not be “long enough”. By the choice of $[j_1, j_2]$, for every $j' \in [j_1, j_2]$, $\pi[j'] \in X$ and $\pi[j']$ occurs in $\pi$ at least twice. Note that for every $j' \in [j_1, j_2]$ such that $I_{A'}(j') \neq \varepsilon$, $\pi[j']$ neither occurs before the $j_1^{th}$ position of $\pi$ nor occurs after the $j_2^{th}$ position of $\pi$ because otherwise the length of $\gamma$ would have to increase by at least one. Hence the subpattern $\pi[j_1 + i_1]...\pi[j_1 + i_h]$ of $\pi$ that $I_{A'}$ maps to $\beta$ such that $I_{A'}(j_1 + i_j) \neq \varepsilon$ whenever $1 \leq j \leq h$ is of the shape $q_1^{n_1}q_2^{n_2}...q_s^{n_s}$, where $q_1, q_2, ..., q_s \in X$ and each $q_i$ occurs in $\pi[j_1 + i_1]...\pi[j_1 + i_h]$ at least twice. But an argument similar to that in the proof of statement (2) above shows that $\beta$ cannot match any such block $Q$ of variables $q_1^{n_1}q_2^{n_2}...q_s^{n_s}$, where each $q_i$ occurs in $Q$ at least twice and $|Q| < m$. Thus $\gamma \notin L(\pi)$, indeed.

**Case (i.2):** $C_1 = \varepsilon$ but $C_2 \neq \varepsilon$. This case can be argued similarly to Case 1.

**Case (i.3):** $\pi$ is constant-free. Then $\pi$ is of the shape $r_1^{n_1}r_2^{n_2}...r_s^{n_s}$, where $r_1, r_2, ..., r_s \in X$ and (since $\pi$ is not block-regular) each $r_i$ occurs in $\pi$ at least twice; hence an argument similar to that in the proof of statement (2) shows that $\gamma \notin L(\pi)$.

By construction, $\gamma \in L(\pi')$. As $\pi'$ is consistent with $T$, $TD(\pi, \Pi^z) = \infty$.

**Case (ii):** $\pi$ contains a substring of the form $ab$, where $a, b \in \Sigma$. ($a$ and $b$ are not
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necessarily distinct.) Since $|\Sigma| \geq 4$, one can fix some $c \in \Sigma$ with $c \notin \{a, b\}$. Let $j_3$ be a position of $\pi$ such that $\pi[j_3] \pi[j_3 + 1] = ab$. If $L(\pi)$ had a finite teaching set $T$ w.r.t. $\Pi^z$, then one can argue as in Case (i) that there is a positive $m$ so large that if $\pi'$ is obtained from $\pi$ by inserting $y^m$ between the $j_3^{th}$ and $(j_3 + 1)^{st}$ positions of $\pi$ for some variable $y \notin \text{Var}(\pi)$, then $\pi'$ would be consistent with $T$. On the other hand, let $\gamma$ be the string derived from $\pi'$ by substituting $c$ for $y$ and $\varepsilon$ for every other variable; note that the number of times the substring $ab$ occurs in $\gamma$ is strictly less than the number of times that $ab$ occurs in $\pi$, which implies $\gamma \notin L(\pi')$ and so $L(\pi') \neq L(\pi)$. Therefore $\text{TD}(\pi, \Pi^z) = \infty$.

Case (iii): $\pi$ starts or ends with a constant symbol (or both). Suppose $\pi$ starts with the constant symbol $a$. The proof that $L(\pi)$ has no finite teaching set w.r.t. $\Pi^z$ is very similar to that in Case (ii); the only difference here is that one chooses some $b \in \Sigma \setminus \{a\}$ and considers $\pi' = y^m \pi$ for some variable $y \notin \text{Var}(\pi)$ and a sufficiently large $m$. In this case, $b^m \pi(\varepsilon) \in L(\pi') \setminus L(\pi)$, and therefore $L(\pi') \neq L(\pi)$. An analogous argument holds if $\pi$ ends with a constant symbol.

This completes the proof of the characterisation. 

In fact, simple block-regularity is a sufficient condition for any pattern over an alphabet of size at least 2 to be finitely distinguishable.

**Proposition 5.1.6** Let $\pi \in \Pi^z$ and $z \geq 2$. Then $\pi$ is finitely distinguishable w.r.t. $\Pi^z$ if $\pi$ is equivalent to a pattern of the shape $y_1 a_1 y_2 \ldots a_k y_{k+1}$, where $a_1, \ldots, a_k \in \Sigma$ and $y_1, \ldots, y_{k+1}$ are distinct variables.

**Proof.** We start with the case $z \geq 3$. Assume that $\pi$ is of the form $y_1 a_1 y_2 \ldots a_k y_{k+1}$. 


where \( a_1, \ldots, a_k \in \Sigma \) and \( y_1, \ldots, y_{k+1} \) are distinct variables. To build a teaching set \( T \) for \( \pi \) w.r.t. \( \Pi^z \), first put \((\pi(\varepsilon), +)\) into \( T \). Next, for each \( w \in (\text{Const}(\pi(\varepsilon)))^* \) with \(|w| < |\pi(\varepsilon)|\) such that \( w = \pi(\varepsilon)[i_1] \ldots \pi(\varepsilon)[i_k] \) for some subsequence \((i_1, \ldots, i_k)\) of \((1, \ldots, |\pi(\varepsilon)|)\), put \((w, -)\) into \( T \); no more than \( 2^{|\pi|} - 1 \) of such \( w \) exist. These additional examples in \( T \) ensure that any \( \pi' \in \Pi^z \) consistent with \( T \) satisfies \( \pi'(\varepsilon) = \pi(\varepsilon) \).

Now for each \( i \in [1, k + 1] \), fix some \( b_i \in \Sigma \) that is different from all the constants adjacent to \( y_i \), and put \((\beta_i, +)\) into \( T \), where

\[
\beta_i = a_1 \ldots a_{i-1} b_i a_i \ldots a_k
\]

is obtained from \( \pi(\varepsilon) \) by inserting \( b_i \) between \( a_{i-1} \) and \( a_i \). (If \( i = 1 \), then \( b_i \) is the first symbol of \( \beta_i \); if \( i = k + 1 \), then \( b_i \) is the last symbol of \( \beta_i \).)

Suppose \( \pi' \) is consistent with the examples in \( T \) so far. Suppose \( A' : (X \cup \Sigma)^* \mapsto \Sigma^* \) witnesses \( \beta_i \in L(\pi') \). Since \( \pi'(\varepsilon) = \pi(\varepsilon) \) and \(|\beta_i| = |\pi(\varepsilon)| + 1\), there is some variable \( y \) in \( \pi' \) that occurs exactly once in \( \pi' \) such that \( A' \) maps \( y \) to exactly one symbol in \( \beta_i \) and \( A' \) maps constants in \( \pi' \) to the remaining symbols in \( \beta_i \). Suppose \( A' \) maps \( y \) to the symbol \( a_j \) in \( \beta_i \) (where the \( a_i \)'s are indicated by braces in (5.2)) for some \( j < i \). Since \( \pi'(\varepsilon) = \pi(\varepsilon) \), one has that \( a_j' = a_j \) for all \( j' \in [j, i-1] \) and \( a_{i-1} = b_i \). But \( b_i \) was chosen so that \( b_i \neq a_{i-1} \)—a contradiction. Similarly, if \( A' \) maps \( y \) to the symbol \( a_j \) in \( \beta_i \) (where the \( a_i \)'s are indicated by braces in (5.2)) for some \( j > i \), then one has \( b_i = a_i \), which again contradicts our choice of \( b_i \). Hence \( A' \) maps \( y \) to \( b_i \) in the decomposition (5.2), so that \( \pi' \) contains a variable \( y_i \) between \( a_{i-1} \) and \( a_i \) that occurs in \( \pi' \) exactly once. Repeating this argument for
each \( i \in [1, k + 1] \) implies that \( \pi' \) must be of the form

\[
X_1 y_1 X_2 a_1 X_3 y_2 X_4 a_2 \ldots a_k X_{2k+1} y_{k+1} X_{2k+2},
\]

where each \( y_i \) occurs in \( \pi' \) exactly once and \( X_1, X_2, \ldots, X_{2k+1}, X_{2k+2} \in X^* \). But \( \pi' \) is equivalent to the pattern \( y_1 a_1 y_2 a_2 \ldots a_k y_{k+1} \), and so \( L(\pi') = L(\pi) \). Hence \( \text{TD}(\pi, \Pi^z) < \infty \), indeed. \( ^1 \)

Now assume that \( z = 2 \) and let \( \Sigma = \{a, b\} \). We will use the following observation, which was shown in \[33\], Lemma 2.

\textbf{Claim 5.1.7} Let \( \Sigma = \{a, b\} \) and \( \pi \) be any pattern over \( \Sigma \cup X \). Given any substring of \( \pi \) that has one of the following shapes: \( x_i a x_j b^m x_k, x_i b^m x_j a x_k, x_i b x_j a^m x_k \) or \( x_i a^m x_j b x_k \) where \( m \in \mathbb{N} \), \( \pi \) is equivalent to the regular pattern \( \pi' \) obtained from \( \pi \) by deleting \( x_j \).

To keep the proof of Proposition 5.1.6 self-contained, we shall prove Claim 5.1.7. Suppose that \( s = x_i a x_j b^m x_k \) is a substring of \( \pi \); if \( s \) has one of the shapes \( x_i b^m x_j a x_k, x_i b x_j a^m x_k \) or \( x_i a^m x_j b x_k \), then a similar proof applies. Since \( \pi' = \pi[x_j \mapsto \varepsilon] \), \( L(\pi') \subseteq L(\pi) \). Thus it suffices to show that for any \( w \in L(\pi) \) such that \( w \) is derived from \( \pi \) by substituting a nonempty string for \( x_j \), \( w \in L(\pi') \). Suppose \( \phi : (\Sigma \cup X)^* \mapsto \Sigma^* \) is a substitution witnessing \( w \in L(\pi) \). We define \( \phi : (\Sigma \cup X)^* \mapsto \Sigma^* \) so that \( \phi(\pi') = w \). Consider three cases.

\textbf{Case (a):} \( \phi(x_j) = wa^n \), where \( w \in \Sigma^* \) and \( n \in \mathbb{N} \). Define \( \phi(x_i) = \phi(x_i)awa^{n-1} \) and \( \phi(x_l) = \phi(x_l) \) for all \( x_l \in \text{Var}(\pi') \setminus \{x_i\} \).

\( ^1 \)Note that the size of the teaching set for \( \pi \) w.r.t. \( \Pi^* \) constructed in this proof is \( O(2^{|\pi|}) \).
Case (b): \( \varphi(x_j) = wab^n \), where \( w \in \Sigma^* \) and \( n \in \mathbb{N} \). Define \( \phi(x_i) = \varphi(x_i)aw \), \( \phi(x_k) = b^n\varphi(x_k) \) and \( \phi(x_l) = \varphi(x_l) \) for all \( x_l \in \text{Var}(\pi') \setminus \{x_i, x_k\} \).

Case (c): \( \varphi(x_j) = b^n \), where \( n \in \mathbb{N} \). Define \( \phi(x_k) = b^n\varphi(x_k) \) and \( \phi(x_l) = \varphi(x_l) \) for all \( x_l \in \text{Var}(\pi') \setminus \{x_k\} \).

\[ \text{(Claim 5.1.7)} \]

By Theorem 5.1.2, it may be assumed that \( \pi \) is of the form \( y_1a_1y_2 \ldots a_ky_{k+1} \), where \( a_1, \ldots, a_k \in \Sigma \) and \( y_1, \ldots, y_{k+1} \) are distinct variables. To build a teaching set \( T \) for \( \pi \) w.r.t. \( \Pi^2 \), first put \( (\pi(\varepsilon), +) \) into \( T \). Next, for each \( w \in (\text{Const}(\pi(\varepsilon))^* \setminus \pi(\varepsilon) \) and \( |w| < |\pi(\varepsilon)| \), put \( (w, -) \) into \( T \); no more than \( 2|\pi| - 1 \) of such \( w \) exist. These additional examples in \( T \) ensure that any \( \pi' \in \Pi^2 \) consistent with \( T \) satisfies \( \pi'(\varepsilon) = \pi(\varepsilon) \).

Pick \( b_1, b_{k+1} \in \Sigma \) such that \( b_1 \neq a_1 \) and \( b_{k+1} \neq a_k \). For each \( i \in \{1, k\} \) such that \( a_{i-1} = a_i \), fix \( b_i \in \Sigma \) such that \( b_i \neq a_i \) (\( = a_{i-1} \)). Define

\[ \beta_i = a_1 \ldots a_{i-1} b_i a_i \ldots a_k \]  

whenever \( b_i \) is defined, and put \( (\beta_i, +) \) into \( T \). For each \( i \in \{1, k\} \) such that \( a_{i-1} \neq a_i \), put both \( (a_1 \ldots a_{i-1} a a_i \ldots a_k, +) \) and \( (a_1 \ldots a_{i-1} b a_i \ldots a_k, +) \) into \( T \).

Suppose \( \pi' \) is consistent with the labelled examples in \( T \) so far. One can argue as in the proof for the case \( z \geq 3 \) that \( \pi'(\varepsilon) = \pi(\varepsilon) \) and for each \( i \) such that \( i \in \{1, k\} \) or \( a_{i-1} = a_i \), the consistency of \( \pi' \) with \( (\beta_i, +) \) implies that there is a free variable of \( \pi' \) between \( a_{i-1} \) and \( a_i \). Now consider any \( i \in \{1, k\} \) such that \( a_{i-1} \neq a_i \). By symmetry,
it may be assumed that \(a_{i-1} = a\) and \(a_i = b\). Suppose \(A : (X \cup \Sigma)^* \mapsto \Sigma^*\) witnesses
\[
\gamma_i = a_1 \ldots a_{i-1} a a_i \ldots a_k \in L(\pi^'). \tag{5.4}
\]

As was argued in the proof for the case \(z \geq 3\), there is some free variable \(y\) in \(\pi'\) such that \(A\) maps \(y\) to exactly one symbol in \(\gamma_i\). Suppose \(A\) maps \(y\) to the symbol \(a_j\) in \(\gamma_i\) (the specific occurrence of \(a_j\) in \(\gamma_i\) indicated by the sequence of braces in (5.4)) for some \(j < i\). If \(a_{i-2} = a\), then (as was argued above) \(\pi'\) contains a free variable between \(a_{i-2}\) and \(a_{i-1}\). If \(i = 2\), then (as argued above) \(\pi'\) contains a free variable just before \(a_{i-1}\). If \(a_{i-2} = b\), then an argument very similar to that in the proof for the case \(z \geq 3\) shows that a free variable of \(\pi'\) occurs either between \(a_{i-2}\) and \(a_{i-1}\) or between \(a_{i-1}\) and \(a_i\). Further, it may be argued as in the proof for the case \(z \geq 3\) that \(A\) cannot map \(y\) to any \(a_j\) in \(\gamma_i\) with \(j \geq i\).

Suppose \(B : (X \cup \Sigma)^* \mapsto \Sigma^*\) witnesses
\[
\underline{a_1} \ldots \underline{a_{i-1}} b \underline{a_i} \ldots a_k \in L(\pi^'). \tag{5.5}
\]

One can apply an argument parallel to that in the previous paragraph to show that a free variable of \(\pi'\) occurs either between \(a_i\) and \(a_{i+1}\) or between \(a_{i-1}\) and \(a_i\). Thus it holds that either a free variable of \(\pi'\) occurs between \(a_{i-1}\) and \(a_i\), or there exist free variables \(x, y\) of \(\pi'\) such that \(x\) occurs just before \(a_{i-1}\) and \(y\) occurs just after \(a_i\); in the latter case, an application of Claim 5.1.7 shows that a free variable may be inserted between \(a_{i-1}\) and \(a_i\) in \(\pi'\), yielding a pattern that is equivalent to \(\pi'\).

**Corollary 5.1.8** If \(z = 1\) or \(z \geq 4\), there is a polynomial-time decider for the set
\{ \pi \in \Pi^k : TD(\pi, \Pi^k) < \infty \}. Furthermore, if \( z \geq 2 \), there is a polynomial-time decider for the set \{ \pi \in \Pi_{cf}^k : TD(\pi, \Pi^k) < \infty \}.

**Proof.** Note that there are polynomial-time algorithms to (i) determine whether or not the greatest common divisor of a set of positive integers is equal to 1, (ii) determine whether or not a given pattern \( \pi \in \Pi_{cf}^k \) contains a variable that occurs exactly once, and (iii) determine whether or not any given \( \pi \in \Pi^k \) satisfies Conditions 1., 2. and 3. in Definition 5.1.3. For (iii), note that \( \pi \) is block-regular iff every maximal block \( Y \) of \( \pi \) contains a free variable, and this condition can be checked in \( O(|\pi|) \) steps. Further, it takes \( O(|\pi|) \) steps to check whether or not \( \pi \) contains a substring \( \alpha \in \Sigma^+ \) such that \( |\alpha| = 2 \) and another \( O(|\pi|) \) steps to determine whether or not \( \pi \) starts and ends with variables. Thus for any given \( z \geq 2 \), the set \{ \pi \in \Pi_{cf}^k : TD(\pi, \Pi^k) < \infty \} has a polynomial-time decider; similarly, for \( z \not\in \{2, 3\} \), the set \{ \pi \in \Pi^k : TD(\pi, \Pi^k) < \infty \} is polynomial-time decidable.

Jain, Ong and Stephan [26] showed that for every pattern \( \pi \) over any alphabet with at least 4 letters, \( L(\pi) \) is a regular language iff \( \pi \) is block-regular. This yields the following corollary.

**Corollary 5.1.9** Let \( z \geq 4 \) and \( \pi \in \Pi^k \). Then \( \pi \) is finitely distinguishable w.r.t. \( \Pi^k \) iff all of the following conditions are satisfied:

1. \( L(\pi) \) is regular;

2. \( \pi \) does not contain any substring \( \alpha \in \Sigma^+ \) such that \( |\alpha| \geq 2 \);

3. \( \pi \) starts and ends with variables.
Example 5.1.10 Let $\Sigma = \{a, b, c, d\}$. The patterns $x_1ax_2ax_3bx_4dx_5cx_6$, $x_1^2x_3ax_1x_24bx_5x_2^2$ and $x_1^2x_4bx_3ax_4x_5$ are simple block-regular and are thus finitely distinguishable w.r.t. $\Pi^4$ by Theorem 5.1.4.3. The pattern $x_1x_2^2ax_1x_3$, on the other hand, is not finitely distinguishable w.r.t. $\Pi^4$.

The remaining part of this section is devoted to the question of whether Theorem 5.1.4.3 (or some slight variation) extends to alphabets $\Sigma$ with $|\Sigma| \in \{2, 3\}$. We shall illustrate with examples the failure of Theorem 5.1.4.3 for alphabets that have exactly two or three letters. In particular, over such alphabets, it will be seen that the structure of finitely distinguishable patterns can be fairly complex, which suggests that the problem of deciding finite distinguishability of $\pi$ w.r.t. $\Pi^z$ for $z \in \{2, 3\}$ and any $\pi \in \Pi^z$ may be more difficult than for the case $z \geq 4$.

Example 5.1.11 Let $\Sigma = \{a, b\}$ and $\pi = x_1ax_2^2bx_3$. Note that $\pi$ is not block-regular. Let $\pi' = x_1abx_2$. We claim that $L(\pi') = L(\pi)$. $L(\pi') \subseteq L(\pi)$ is immediate. Consider any $\beta \in \Sigma^*$ obtained from $\pi$ by substituting $\alpha_i$ for $x_i$, where $i \in \{1, 2, 3\}$. Observe that $a\alpha_2^2b$ must contain the substring $ab$. Hence $\beta$ is of the shape $\gamma_1ab\gamma_2$, where $\gamma_1, \gamma_2 \in \Sigma^*$, and so $\beta \in L(\pi')$. Therefore $L(\pi') = L(\pi)$. Furthermore, observe that $\{(ab,+),(a,-),(b,-),(baba,+)\}$ is a (finite) teaching set for $\pi'$ w.r.t. $\Pi^2$. Thus the characterisation obtained in Theorem 5.1.4.3 does not apply to alphabets with exactly two letters.

The next example shows that Theorem 5.1.4.3 does not apply to the class of erasing pattern languages over any alphabet of size 3.

Example 5.1.12 Let $\Sigma = \{a, b, c\}$ and $\pi = x_1x_2x_3ax_2x_4^2x_5bx_7x_6x_8$. Then $\pi$ is finitely distinguishable w.r.t. $\Pi^3$ but $L(\pi)$ cannot be generated by any regular pattern.
Proof. Suppose \( L(\pi) \) were equal to \( L(\tau) \) for some regular pattern \( \tau \). Since \(|\Sigma| \geq 3\), it follows from a result in [27] that \( \pi \) and \( \tau \) are similar, that is, the constant parts of \( \pi \) and \( \tau \) are identical and occur in the same order in the patterns, so that (after normalisation) \( \tau = x_1ax_2bx_3 \). But \( acb \in L(\tau) \setminus L(\pi) \), and so \( L(\tau) \neq L(\pi) \).

Now we show that TD(\( \pi, \Pi^3 \)) is finite. We claim that \( T = \{ (ab,+), (a,-), (b,-), (ac^2b,+), (acb,-), (bca^2cb,+), (acb^2ca,+), (acb^2,+) \} \) is a teaching set for \( \pi \) w.r.t. \( \Pi^3 \). Let \( \pi' \) be any pattern that is consistent with \( T \). Note that the consistency of \( \pi' \) with \( (ab,+), (a,-) \) and \( (b,-) \) implies that \( \pi' \) is of the shape \( X_1aX_2bX_3 \), where \( X_1, X_2, X_3 \in X^* \). Furthermore, \( \pi' \) must fulfil the following conditions:

1. \( \pi' \) contains a variable \( y_1 \) such that \( y_1 \) occurs in \( X_2 \) exactly twice and does not occur in any other maximal variable block of \( \pi' \).
2. \( \pi' \) contains a variable \( y_2 \) such that \( y_2 \) occurs in \( X_2 \) exactly thrice and does not occur in any other maximal variable block of \( \pi' \).
3. Every variable that \( X_2 \) contains occurs in \( \pi' \) at least twice.
4. There is a variable \( y_3 \) that occurs in \( X_1 \) exactly once, occurs in \( X_2 \) exactly once, does not occur in \( X_3 \), and there are variables \( y_5 \) and \( y_6 \), each of which occurs in \( \pi' \) exactly once, such that \( X_1 = Y_1y_5y_2y_3y_6y_4 \) for some \( Y_1, Y_2, Y_3, Y_4 \in X^* \).
5. There is a variable \( y_4 \) that occurs in \( X_3 \) exactly once, occurs in \( X_2 \) exactly once, does not occur in \( X_1 \), and there are variables \( y_7 \) and \( y_8 \), each of which occurs in \( \pi' \) exactly once, such that \( X_3 = Z_1y_7Z_2y_4Z_3y_8Z_4 \), where \( Z_1, Z_2, Z_3, Z_4 \in X^* \).

Note that Condition 1. is implied by the consistency of \( \pi' \) with \( \{ (acb,-), (ac^2b,+) \} \), Condition 2. by the consistency of \( \pi' \) with \( \{ (acb,-), (ac^3,+) \} \), Condition 3.
by the consistency of $\pi'$ with \{$(acb, -)$\}, Condition 4. by the consistency of $\pi'$ with \{$(acb, -), (bca^2 cb, +)$\} and Condition 5. by the consistency of $\pi'$ with \{$(acb, -), (acb^2 ca, +)$\}. We claim further that any $\pi'$ satisfying the preceding set of conditions generates the same language as $\pi = x_1 x_2 x_3 ax_2 x_3 x_5 x_6 b x_7 x_6 x_8$. It will be shown that $L(\pi') \subseteq L(\pi)$; the reverse inclusion may be proved similarly.

Consider any $\beta \in L(\pi')$, and let $A : (X \cup \Sigma)^* \mapsto \Sigma^*$ be a substitution witnessing $\beta \in L(\pi')$. Note that $aA(X_2)b$ must contain a substring of the shape $a c b^k b$ for some least $k \geq 0$. In each of the following cases, we specify a substitution $\sigma : X \mapsto \Sigma^*$ that witnesses $\beta \in L(\pi)$.

**Case 1:** $k = 0$. Let $\beta = \gamma_1 ab \gamma_2$, where $\gamma_1, \gamma_2 \in \Sigma^*$. Define

$$
\sigma(x_i) = \begin{cases} 
\gamma_1 & \text{if } i = 3; \\
\gamma_2 & \text{if } i = 8; \\
\varepsilon & \text{if } i \notin \{3, 8\}.
\end{cases}
$$

**Case 2:** $k = 1$. Since every variable of $X_2$ occurs in $\pi'$ at least twice (Condition 3.), at least one of the following cases must hold.

**Case 2.1:** $\beta$ is of the shape $\gamma_1 c \gamma_2 abb \gamma_3$, where $\gamma_1, \gamma_2, \gamma_3 \in \Sigma^*$. Define

$$
\sigma(x_i) = \begin{cases} 
\gamma_1 & \text{if } i = 1; \\
c & \text{if } i = 2; \\
\gamma_2 & \text{if } i = 3; \\
\gamma_3 & \text{if } i = 7; \\
\varepsilon & \text{if } i \notin \{1, 2, 3, 7\}.
\end{cases}
$$
Case 2.2: $\beta$ is of the shape $\gamma_1 \textit{acb}\gamma_2 c\gamma_3$, where $\gamma_1, \gamma_2, \gamma_3 \in \Sigma^*$. Define

$$\sigma(x_i) = \begin{cases} 
\gamma_1 & \text{if } i = 3; \\
c & \text{if } i = 6; \\
\gamma_2 & \text{if } i = 7; \\
\gamma_3 & \text{if } i = 8; \\
\varepsilon & \text{if } i \notin \{3, 6, 7, 8\}.
\end{cases}$$

Case 3: $k > 1$. Given any $k > 1$, there are nonnegative integers $m_k$ and $n_k$ such that $2m_k + 3n_k = k$. Let $\beta = \gamma_1 \textit{acb}^k b\gamma_2$, where $\gamma_1, \gamma_2 \in \Sigma^*$. Define

$$\sigma(x_i) = \begin{cases} 
\gamma_1 & \text{if } i = 3; \\
\textit{acb}^m & \text{if } i = 4; \\
\textit{acb}^n & \text{if } i = 5; \\
\gamma_2 & \text{if } i = 7; \\
\varepsilon & \text{if } i \notin \{3, 4, 5, 7\}.
\end{cases}$$

This completes the case distinction, showing that $\beta \in L(\pi)$.

Remark 5.1.13 Note that while the pattern $\pi$ in Example 5.1.12 is not regular, it generates a regular language. In fact, $L(\pi) = \Sigma^*ab\Sigma^* \cup \Sigma^*c\Sigma^*acb\Sigma^* \cup \Sigma^*acb\Sigma^*c\Sigma^* \cup \Sigma^*ac^2c^*b\Sigma^*$, where $\Sigma = \{a, b, c\}$.

The next example shows that over any alphabet of size exactly 2, there is a pattern $\pi$ that is finitely distinguishable w.r.t. $\Pi^2$ while $L(\pi)$ cannot be generated by any regular pattern.
Example 5.1.14 Let $\Sigma = \{a, b\}$ and $\pi = x_1x_2ax_3x_3^2x_4^3x_5ax_5x_6$. Then $\pi$ is finitely distinguishable w.r.t. $\Pi^2$ but $L(\pi)$ cannot be generated by any regular pattern.

Proof. If $L(\pi)$ were generated by some regular pattern $\tau$, then $\tau$ must be of the shape $x_1ax_2x_3$ and $aba \not\in L(x_1ax_2ax_3) \setminus L(\pi)$, and so $L(\pi) \neq L(\tau)$. It remains to show that $\pi$ is finitely distinguishable w.r.t. $\Pi^2$. We claim that $\{(aa, +), (a, -), (baa, +), (aab, +), (ab^2a, +), (ab^3a, +), (aba, -), (abab, +), (ababa, +), (baba, +)\}$ is a teaching set for $\pi$ w.r.t. $\Pi^2$.

Claim 5.1.15 For all patterns $\pi'$, $\pi'$ is consistent with $T$ iff $L(\pi')$ consists of all finite strings $s = b^{m_1}a^{m_2}b^{m_3}a^{m_4}b^{m_5} \ldots$ such that

1. $m_2, m_4 > 0$;

2. if $m_3 = 1$, then ($b$ occurs at least twice in $s$ or $a^2$ is a substring of $s$).

Proof of Claim 5.1.15. Let $\pi'$ be any pattern. If $L(\pi')$ consists of all finite strings $s = b^{m_1}a^{m_2}b^{m_3}a^{m_4}b^{m_5} \ldots$ satisfying Conditions 1. and 2., then one may directly verify that $\{aa, baa, aab, ab^2a, ab^3a, abab, ababa, baba\} \subset L(\pi')$ while $L(\pi') \cap \{a, aba\} = \emptyset$. Thus $\pi'$ is consistent with $T$. Now suppose that $\pi'$ is consistent with $T$. Then the following hold:

(i) $(aa \in L(\pi') \land a \notin L(\pi')) \rightarrow \pi' = X_1aX_2aX_3$ for some $X_1, X_2, X_3 \in X^*$.

(ii) $baa \in L(\pi') \rightarrow X_1$ contains a free variable.

(iii) $aab \in L(\pi') \rightarrow X_3$ contains a free variable.

(iv) $(ab^2a \in L(\pi') \land aba \notin L(\pi')) \rightarrow \pi'$ contains a variable occurring exactly twice in $X_2$ and not occurring in any other maximal variable block.
(v) \((ab\delta a \in L(\pi') \land aba \notin L(\pi')) \rightarrow \pi'\) contains a variable occurring exactly thrice in \(X_2\) and not occurring in any other maximal variable block.

(vi) \(aba \notin L(\pi') \rightarrow X_2\) does not contain any free variable.

(vii) \((baba \in L(\pi') \land aba \notin L(\pi')) \rightarrow \pi'\) contains a variable \(y\) occurring once in \(X_1\), once in \(X_2\) and not occurring in any other maximal variable block.

(viii) \((abab \in L(\pi') \land aba \notin L(\pi')) \rightarrow \pi'\) contains a variable \(y\) occurring once in \(X_2\), once in \(X_3\) and not occurring in any other maximal variable block.

(ix) \((ababa \in L(\pi') \land aba \notin L(\pi')) \rightarrow (\pi'\) contains a variable \(y\) occurring exactly once in \(X_2\), exactly once in \(X_3\) and occurring in no other maximal variable block, and a free variable occurs in \(X_3\) after the occurrence of \(y\) in \(X_3\) \(\lor (\pi'\) contains a variable \(y\) occurring exactly once in \(X_1\), exactly once in \(X_2\) and not occurring in any other maximal variable block, and a free variable occurs in \(X_1\) before the occurrence of \(y\) in \(X_1\)).

First, consider any \(\alpha \in L(\pi')\). By (i), \(\alpha\) has the shape \(b^{m_1}a^{m_2}b^{m_3}a^{m_4}b^{m_5} \ldots\), where \(m_2, m_4 > 0\). Furthermore, if \(m_3 = 1\), then (vi) implies that \((b\) occurs at least twice in \(\alpha \lor a^2\) is a substring of \(\alpha\)). Now suppose \(s\) is a string of the shape \(b^{m_1}a^{m_2}b^{m_3}a^{m_4}b^{m_5} \ldots \delta^{m_k}\) satisfying Conditions 1. and 2., where \(\delta \in \{a, b\}\) and \(m_i > 0\) for all \(i \in \{1, \ldots, k\} \setminus \{1, 3\}\). We show that \(s \in L(\pi')\) by means of the following case distinction.

**Case (a):** \(a^2\) is a substring of \(s\). Let \(s = \beta_1a^2\beta_2\), where \(\beta_1, \beta_2 \in \Sigma^*\). By (ii) and (iii), one may substitute \(\beta_1\) for the free variable occurring in \(X_1\) and \(\beta_2\) for the free variable occurring in \(X_3\).
Case (b): $a^2$ is not a substring of $s$ and $m_{2j−1} ≥ 2$ for some $j$ such that $2j−1 ≤ k$.

First, suppose $m_{2j−1} ≥ 2$ for some $j$ such that $2j−1 ∉ \{1, k\}$. Then $m_{2j−2}, m_{2j} ≥ 1$. Let $n_1$ and $n_2$ be nonnegative integers such that $2n_1 + 3n_2 = m_{2j−1}$. By (iv) and (v), one may substitute $b^{n_1}$ for the variable occurring twice in $X_2$ (and occurring in no other maximal variable block) and $b^{n_2}$ for the variable occurring thrice in $X_2$ (and occurring in no other maximal variable block). By (ii) and (iii), one may substitute $b^{m_1} \ldots a^{m_{2j−2}−1}$ for the free variable occurring in $X_1$ and $a^{m_{2j}−1} \ldots \delta^{m_k}$ for the free variable occurring in $X_3$.

Second, suppose $m_{2j−1} = 1$ for all $j$ such that $2j−1 ∉ \{1, k\}$ and $m_1 ≥ 2$. By (vii), one may substitute $b$ for the variable occurring once in $X_1$, once in $X_2$ and occurring in no other maximal variable block. By (ii) and (iii), one may substitute $b^{m_1−1}$ for the free variable occurring in $X_1$ and $b^{m_5} \ldots \delta^{m_k}$ for the free variable occurring in $X_3$.

Third, suppose that $m_k ≥ 2$ and $k$ is odd. By (viii), one may substitute $b$ for the variable occurring once in $X_2$, once in $X_3$ and occurring in no other maximal variable block. By (ii) and (iii), one may substitute $a^{m_1} \ldots b^{m_k−4}$ for the free variable occurring in $X_1$ and substitute $b^{m_k−1}$ for the free variable occurring in $X_3$.

Case (c): $s$ has the shape $(ba)^i b^l$ for some $i ≥ 2$ and $l ∈ \mathbb{N}_0$. By (vii), one may substitute $b$ for the variable occurring once in $X_1$, once in $X_2$ and occurring in no other maximal variable block. By (iii), one may substitute $b^{m_5} \ldots \delta^{m_k}$ for the free variable occurring in $X_3$.

Case (d): $s$ has the shape $(ab)^i a$ for some $i ≥ 2$. By (ix), at least one of the
The following holds: (1) one may substitute $b$ for the variable $y$ occurring once in $X_2$, once in $X_3$ and occurring in no other maximal variable block, and substitute $a^{m_6}\ldots\delta^{m_k}$ for the free variable in $X_3$ occurring after the occurrence of $y$ in $X_3$, or (2) one may substitute $b$ for the variable $y$ occurring once in $X_1$, once in $X_2$ and occurring in no other maximal variable block, and substitute $a^{m_2}\ldots a^{m_{k-4}}$ for the free variable in $X_1$ occurring before the occurrence of $y$ in $X_1$.

**Case (e):** $s$ has the shape $(ab)^i$ for some $i \geq 2$. By (viii), one may substitute $b$ for the variable occurring once in $X_2$, once in $X_3$ and not occurring in any other maximal variable block. By (ii), one may substitute $a^{m_2}\ldots b^{m_{k-4}}$ for the free variable occurring in $X_1$.

This completes the case distinction, showing that $L(\pi')$ consists of all strings $s$ of the shape $b^{m_1}a^{m_2}b^{m_3}a^{m_4}b^{m_5}\ldots$ satisfying Conditions 1. and 2.  

It may be directly verified that $\pi$ is consistent with $T$. Consequently, by Claim 5.1.15, $T$ is indeed a teaching set for $\pi$ w.r.t. $\Pi^2$.

Theorem 5.1.4.1, Example 5.1.12 and Example 5.1.14 together imply that one direction of the characterisation in Theorem 5.1.4.3 – that $\text{TD}(\pi, \Pi^z) < \infty \Rightarrow \pi$ is simple block regular – applies only to the class of erasing pattern languages over any alphabet with at least four letters. The next two examples from [26] and [37] show that the reverse direction of Theorem 5.1.4.3 fails for alphabets with exactly two or three letters as well if one relaxes condition (a) by only requiring that $L(\pi)$ be a regular language.

**Example 5.1.16** [37] Let $\Sigma = \{a, b\}$ and $\pi = x_1ax_2a x_3$. Then (a) $L(\pi)$ is regular,
(b) $\pi$ does not contain any substring $\alpha \in \Sigma^+$ such that $|\alpha| \geq 2$, and (c) $\pi$ starts and ends with variables, but $\pi$ is not finitely distinguishable w.r.t. $\Pi^2$.

**Proof.** According to [37, Proposition 9], $L(\pi)$ is regular; it also follows directly from the definition of $\pi$ that $\pi$ satisfies conditions (b) and (c). It remains to show that $TD(\pi, \Pi^2) = \infty$. Suppose otherwise, and that $T$ were a finite teaching set for $L(\pi)$ w.r.t. $\Pi^2$. Then there is an $m$ sufficiently large so that for all $m' \geq m$, the erasing pattern language generated by $\pi' = x_1ax_4x_2x_5bx_6x_5x_7$ is consistent with $T$. Choose any odd $m' \geq m$. One has $ab^{m'}a \in L(\pi')$ via the assignment $x_1, x_2, x_3 \mapsto \varepsilon$ and $x_4 \mapsto b$. However, if $ab^{m'}a \in L(\pi)$ via some assignment $B : X \mapsto \Sigma^*$, then one has $B(x_1) = B(x_3) = \varepsilon$, and so $B(x_2^2) = b^{2k} = b^{m'}$ for some $k \geq 1$, which is impossible since $m'$ is odd.

**Example 5.1.17** [26] Let $\Sigma = \{a, b, c\}$ and $\pi = x_1x_2x_3ax_2x_4x_5bx_6x_5x_7$. Then (a) $L(\pi)$ is regular, (b) $\pi$ does not contain a substring of the form $\delta_1\delta_2$, where $\delta_1, \delta_2 \in \Sigma$, and (c) $\pi$ starts and ends with variables, but $\pi$ is not finitely distinguishable w.r.t. $\Pi^3$.

**Proof.** According to [26, Theorem 2], $L(\pi)$ is regular; it also follows directly from the definition of $\pi$ that $\pi$ satisfies conditions (b) and (c). Now assume by way of a contradiction that $T$ were a finite teaching set for $L(\pi)$ w.r.t. $\Pi^3$. As in Example 5.1.16, there is an $m$ large enough so that whenever $m' \geq m$, $\pi' = x_1x_2x_3ax_8x_2x_4x_5b$ is consistent with $T$. Fix some odd $m' \geq m$. Note that the assignment $x_1, x_2, x_3, x_4, x_5, x_6, x_7 \mapsto \varepsilon$ witnesses $ac^{m'b} \in L(\pi')$. If, however, there were some assignment $B : X \mapsto \Sigma^*$ witnessing $ac^{m'b}$, then it must hold that $B(x_1) = B(x_2) = B(x_3) = B(x_5) = B(x_6) = B(x_7) = \varepsilon$ and
\[ B(x^2) = c^{2k} = c^{m'} \text{ for some } k \geq 1, \text{ contradicting the fact that } m' \text{ is odd.} \]

According to Examples 5.1.12 and 5.1.14, a pattern language over any alphabet of size 2 or 3 may be finitely distinguishable without being generable by a block-regular pattern. Our next result shows, on the other hand, that over any finite alphabet, a finitely distinguishable pattern language must necessarily be regular. The converse of the latter statement (even with restrictions on the length of every constant block of the pattern and on the first as well as last symbols of the pattern) is false in general, as we have seen in Examples 5.1.16 and 5.1.17.

**Theorem 5.1.18** Let \( 1 \leq z < \infty \) and \( \pi \in \Pi^z \). If \( \pi \) is finitely distinguishable w.r.t. \( \Pi^z \), then \( L(\pi) \) is regular.

**Proof.** Let \( \Sigma = \{a_1, \ldots, a_z\} \). For each \( \delta \in \Sigma \) and \( w \in (X \cup \Sigma)^* \), let \( \#(\delta)[w] \) denote the number of occurrences of \( \delta \) in \( w \). Further, for any \( \beta, \gamma \in \Sigma^* \), recall that the shuffle product of \( \beta \) and \( \gamma \), denoted by \( \beta \uplus \gamma \), is the set \( \{\beta_1 \gamma_1 \beta_2 \gamma_2 \ldots \beta_k \gamma_k : \beta_i, \gamma_i \in \Sigma^* \land \beta_1 \beta_2 \ldots \beta_k = \beta \land \gamma_1 \gamma_2 \ldots \gamma_k = \gamma\} \), and the shuffle product of two sets \( S \) and \( T \), denoted by \( S \uplus T \), is the set \( \bigcup_{s \in S \land t \in T} s \uplus t \).

Suppose \( T \) were a finite teaching set for \( \pi \) w.r.t. \( \Pi^z \). Fix some \( m > \max\{|\alpha| : \alpha \in T^+ \cup T^- \lor |\alpha| = |\pi|\} \). Consider any pair \( (I, J) \in \wp([1, z]) \times \wp([1, z]) \) such that \( I \cap J = \emptyset \). Let \( I = \{i_1, \ldots, i_k\} \) and \( J = \{j_1, \ldots, j_\ell\} \). Define

\[
S_I = \{ w \in L(\pi) : (\forall 1 \leq d \leq k)[(a_{i_d})[\pi] + 1 \leq (a_{i_d})[w] \leq (a_{i_d})[\pi] + m - 1] \land (\forall e \in [1, z] \setminus I)[(a_e)[w] = (a_e)[\pi]]\},
\]

\[
T_J = \{ v \in \{a_{j_1}, \ldots, a_{j_\ell}\}^* : (\forall 1 \leq d \leq \ell)[(a_{j_d})[v] = m]\}.
\]
Given $S_I$ and $T_J$, set $E_{I,J} = (S_I \uplus T_J) \uplus \{a_{j_1}, \ldots, a_{j_\ell}\}^*$. Observe that $S_I$ and $T_J$ are both finite and hence regular, while $\{a_{j_1}, \ldots, a_{j_\ell}\}^*$ is also regular. As the shuffle operation preserves regularity, it follows that $E_{I,J}$ is regular. Further, since the regular languages are closed under the union operation, the required result follows immediately from the next claim.

**Claim 5.1.19** $L(\pi) = \bigcup_{I, J \subseteq [1, z] \land I \cap J = \emptyset} E_{I,J}$.

**Proof of Claim 5.1.19** We first show that $L(\pi) \subseteq \bigcup_{I, J \subseteq [1, z] \land I \cap J = \emptyset} E_{I,J}$. Consider any $\alpha \in L(\pi)$. Define $I = \{d : \#(a_d)[\pi] + 1 \leq \#(a_d)[\alpha] \leq \#(a_d)[\pi] + m - 1\}$ and $J = \{e : \#(a_e)[\alpha] \geq \#(a_e)[\pi] + m\}$. Then $\alpha \in E_{I,J}$.

Now it is shown that $\bigcup_{I, J \subseteq [1, z] \land I \cap J = \emptyset} E_{I,J} \subseteq L(\pi)$. Choose any $I, J \subseteq [1, z]$ such that $I \cap J = \emptyset$. Let $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_\ell\}$. Pick any $\alpha \in S_I, \beta \in T_J$ and $\gamma \in \{a_{j_1}, \ldots, a_{j_\ell}\}^*$. One has to show that for any $w \in (\alpha \uplus \beta) \uplus \gamma$, $w \in L(\pi)$. Let $\varphi : X \mapsto \Sigma^*$ be a substitution witnessing $\alpha \in L(\pi)$. Since $w \in (\alpha \uplus \beta) \uplus \gamma$, there is some $v \in \{a_{j_1}, \ldots, a_{j_\ell}\}^*$ such that whenever $1 \leq d \leq \ell$, $a_{j_d}$ occurs at least $m$ times in $v$ and

$$w = v_1 \alpha_1 v_2 \alpha_2 \ldots v_{n-1} \alpha_{n-1} v_n$$

for some $v_1, \ldots, v_n \in \{a_{j_1}, \ldots, a_{j_\ell}\}^*$ and $\alpha_1, \ldots, \alpha_{n-1} \in \Sigma^*$ with $\alpha = \alpha_1 \alpha_2 \ldots \alpha_{n-1}$ and $v = v_1 \ldots v_n$.

One now derives a pattern $\tau$ from the decomposition (5.6) of $w$ as follows. Let $\pi_1, \ldots, \pi_n \in (X \cup \Sigma)^*$ be strings such that $\pi = \pi_1 \ldots \pi_n$ and $\varphi(\pi_i) = \alpha_i$ for all $i \in [1, n - 1]$. Replace each $\alpha_i$ (here we are referring to the specific occurrence of $\alpha_i$ starting at the $|v_1 \alpha_1 \ldots v_i| + 1$st position of $w$) with $\pi_i$. Next, choose distinct
variables \( y_1, \ldots, y_\ell \notin \text{Var}(\pi) \). For each \( d \in [1, \ell] \), substitute \( y_d \) for every occurrence of \( a_{jd} \) in \( v_1, v_2, \ldots, v_n \) (as before, for every \( i \in [1, n] \), we are referring to the specific occurrence of \( v_i \) starting at the \((|v_1 \ldots \alpha_i-1|+1)^{st}\) position of \( w \)). Note that \( \tau \) can be derived from \( \pi \) by interleaving \( \pi \) with a string consisting of the variables \( y_1, \ldots, y_\ell \), and therefore \( L(\pi) \subseteq L(\tau) \). Further, \( \tau \) is consistent with \( T \) because every additional variable \( y_i \) occurs at least \( m \) times in \( \tau \). Thus \( L(\tau) \subseteq L(\pi) \), and as \( w \in L(\tau) \), it follows that \( w \in L(\pi) \). \( \blacksquare \)

The following theorem provides necessary conditions for a pattern to be finitely distinguishable w.r.t. the whole class of patterns over any alphabet of size 2 or 3.

**Theorem 5.1.20** Let \( z \in \{2, 3\} \), \( \Sigma_1 = \{a, b\} \), \( \Sigma_2 = \{a, b, c\} \) and \( \pi = X_1c_1X_2c_2 \ldots X_{n-1}c_{n-1}X_n \), where \( X_2, \ldots, X_{n-1} \in X^+ \), \( c_1, \ldots, c_{n-1} \in \Sigma_1^+ \) if \( z = 2 \), \( c_1, \ldots, c_{n-1} \in \Sigma_2^+ \) if \( z = 3 \), and \( X_1, X_n \in X^* \). If \( \pi \) is finitely distinguishable w.r.t. \( \Pi^z \), then the following conditions hold for all \( i \in [1, n-1] \).

1. If \( z = 2 \), then \( c_i \in \{a, b, ab, ba\} \); if \( z = 3 \), then \( c_i \in \Sigma_2 \).
2. If \( z = 2 \), then for all \( \alpha \in \{X_1, X_n, \delta X_i \delta, \delta X_i \delta X_{i+1} \delta, \delta \delta X_i \delta, \delta X_i \delta \delta\} \) such that \( \alpha \) is a substring of \( \pi \), where \( \delta, \delta \in \Sigma \) and \( \delta \neq \delta \), there is a \( k \geq 1 \) for which \( \alpha \) contains variables \( y_1, \ldots, y_k \) such that for all \( j \in [1, k] \), \( y_j \) occurs \( q_j \) times in \( \alpha \) for some \( q_j \geq 1 \), \( y_j \) does not occur outside the block \( \alpha \) and \( \gcd(q_1, \ldots, q_k) = 1 \).

If \( z = 3 \), then the latter statement holds for \( \alpha = X_i \).
3. If \( z = 2 \), then \( \pi \) contains at least one free variable; if \( z = 3 \), then \( X_1 \) and \( X_n \) each contains at least one free variable.

**Proof.** Let \( T \) be a finite teaching set for \( L(\pi) \) w.r.t. \( \Pi^z \) and fix any \( m > \max(|\gamma| : \gamma \in T) \).
\[ \gamma \in T^+ \cup T^- \cup \{|\pi|\}. \]

**Proof of (1).** Let \( z = 2 \). Suppose \( \pi[i] \pi[i + 1] = aa \) for some \( i \in [1, |\pi| - 1] \). Choose some variable \( y \notin \text{Var}(\pi) \), and let \( \pi' \) be the pattern obtained from \( \pi \) by inserting \( y^m \) between the \( i^{th} \) and \( (i + 1)^{st} \) positions of \( \pi \). Note that \( \pi' \) is consistent with \( T \). Furthermore, let \( \beta \) be the string derived from \( \pi' \) by substituting \( b \) for \( y \) and \( \varepsilon \) for every other variable. Since the number of times that \( aa \) occurs in \( \beta \) is strictly less than the number of times it occurs in \( \pi \), one has \( \beta \in L(\pi') \setminus L(\pi) \), a contradiction.

Now suppose \( \pi[i] \pi[i + 1] \pi[i + 2] = aba \) for some \( i \in [1, |\pi| - 2] \). Let \( \pi'' \) be the pattern obtained from \( \pi \) by inserting \( y^m \) between the \( i^{th} \) and \( (i + 1)^{st} \) positions of \( \pi \), and let \( \theta \) be the string derived from \( \pi'' \) by substituting \( b \) for \( y \) and \( \varepsilon \) for every other variable. One may verify as in the earlier case that \( \pi'' \) is consistent with \( T \) but \( \theta \in L(\pi'') \setminus L(\pi) \).

If \( z = 3 \), then the proof that \( c_i \in \Sigma_2 \) is similar to the preceding proof.

**Proof of (2).** Let \( z = 2 \). First consider the case \( \alpha = X_1 \). Choose \( \delta \in \Sigma \) so that \( \delta \) is different from the first symbol of \( c_1 \). As before, choose a variable \( y \notin \text{Var}(\pi) \), and note that for all \( j \geq m \), \( y^j \pi \) is consistent with \( T \). Thus \( \delta^j \pi(\varepsilon) \in L(\pi) \) for all \( j \geq m \).

This implies that \( X_1 \neq \varepsilon \), and that there exist variables \( y_1, \ldots, y_k \) occurring only in \( X_1 \) such that for all \( j \geq m \), there are nonnegative integers \( m_1, \ldots, m_k \) for which \( \sum_{i=1}^k m_i q_i = j \), where \( q_i \) is the number of times that \( y_i \) occurs in \( X_1 \). Therefore \( \gcd(q_1, \ldots, q_k) = 1 \). The case \( \alpha = X_n \) can be handled similarly.

Now suppose \( \alpha = aX_1a = \pi[j] \pi[j + 1] \ldots \pi[j + l] \). Choose some variable \( y \notin \text{Var}(\pi) \), and for any \( m' \geq m \) let \( \pi_{m'} \) be the pattern obtained from \( \pi \) by inserting \( y^{m'} \) between the \( j^{th} \) and \( (j + 1)^{st} \) positions of \( \pi \). Let \( \beta_{m'} \) be the string derived from
\( \pi_{m'} \) by substituting \( b \) for \( y \) and \( \varepsilon \) for all other variables. As in the previous case, note that \( \pi_{m'} \) is consistent with \( T \) and so \( \beta_{m'} \in L(\pi) \), which means that there exist variables \( y_1, \ldots, y_k \) occurring only in \( X_i \) such that if \( q_i \) is the number of times that \( y_i \) occurs in \( X_i \), then \( \gcd(q_1, \ldots, q_k) = 1 \).

Finally, let \( \alpha = aX_ibX_{i+1}a = \pi[j_1] \ldots \pi[j_1+l_1] \). The proof is very similar to that of the previous case; here one defines for every \( m' \geq m \) the pattern \( \pi_{m'} \) obtained from \( \pi \) by inserting \( y^{m'} \) between the \( j_1^{th} \) and \( (j_1+1)^{st} \) positions of \( \pi \) and setting \( \beta_{m'} \) to be the string derived from \( \pi_{m'} \) by replacing \( y \) with \( b \) and every other variable with \( \varepsilon \). The remaining cases in (2) (including the case \( z = 3 \)) can be dealt with similarly.

**Proof of (3).** Let \( z = 2 \). Choose two distinct variables \( y_1, y_2 \notin \text{Var}(\pi) \), and define

\[
\tau = \pi y_1^m y_2^m y_1^{m+1} y_2^{m+1} y_1^{m+1} \ldots y_1^4 y_2^4 y_1^4.
\] (5.7)

Let \( \beta \) be the string derived from \( \tau \) by substituting \( a \) for \( y_1 \), \( b \) for \( y_2 \), and \( \varepsilon \) for all other variables; that is,

\[
\beta = \pi(\varepsilon) \underbrace{a_{m}b_{m}a_{m}b_{m+1}a_{m+1}b_{m+1}a_{m+1} \ldots a_{4m}b_{4m}}_{\text{whose specific occurrence in } \beta \text{ is indicated by braces in (5.8)}} a_{4m}.
\] (5.8)

Since \( \tau \) is consistent with \( T \), one has that \( \beta \in L(\pi) \). Let \( A : (X \cup \Sigma)^* \to \Sigma^* \) be a substitution witnessing \( \beta \in L(\pi) \). By statement (1), each constant block of \( \pi \) overlaps with at most one substring of the form \( a^{m+i}b^{m+i}a^{m+i} \). Further, there is some \( j \in [0, 3m] \) such that for some \( z \in \text{Var}(\pi) \), \( A \) maps an occurrence of \( z \) in \( \pi \) to a substring \( \beta' \) of \( \beta \) such that \( a^{m+j}b^{m+j}a^{m+j} \) (whose specific occurrence in \( \beta \) is indicated by braces in (5.8)) is a substring of \( \beta' \); otherwise, for each
occurrence of a variable $z'$ in $\pi'$, $A$ maps this occurrence of $z'$ to a substring of $a^{m+i}b^{m+i}a^{m+i}b^{m+i}a^{m+i+1}a^{m+i+1}$ (whose specific occurrence in $\beta$ is indicated by braces in (5.8)) for at most one $i \in [0, 3m - 1]$, and so $|A(\pi)| < \beta$, a contradiction. Since $\beta$ cannot contain two copies of $a^{m+j}b^{m+j}a^{m+j}$, $z$ must be a free variable of $\pi$, as required. The fact that $X_1$ and $X_n$ each contains at least one free variable if $z = 3$ can be proven similarly.

5.2 Interesting Subclasses of Pattern Languages

This section investigates the finite distinguishability problem for two subclasses of pattern languages: (i) regular pattern languages and (ii) 1-variable pattern languages. We recall that a regular pattern is a pattern that has no repeated variables, and a 1-variable pattern is a pattern that contains at most 1 variable (possibly with repetitions). Characterising finite distinguishability for these subclasses turns out to be relatively simple; in particular, the finite distinguishability problem for each of these classes is decidable.

The class of regular erasing pattern languages is learnable with polynomially many membership queries (i.e., questions of the kind "does the string $w$ match the unknown pattern?") iff the learner is initially given a string from the target language \cite{33}. Note that the membership query complexity is also an upper bound on the teaching dimension. The next theorem gives a linear upper bound on $TD(\pi, R\Pi^z)$ for any regular pattern $\pi$.

**Theorem 5.2.1** Let $z \in \mathbb{N} \cup \{\infty\}$ and let $\pi$ be a regular pattern over $\Sigma$. Then $TD(\pi, R\Pi^z) \leq 2|\pi| + 1$. 
Proof. It will be shown later (Theorem 5.3.2) for all regular patterns $\pi$, $TD(\pi, R_{\Pi}^z) \leq 3$ when $z = 1$ and $TD(\pi, R_{\Pi}^z) \leq 5$ when $z \geq 7$. We shall therefore assume that $2 \leq z \leq 6$. A teaching set $T$ for $\pi$ w.r.t. $R_{\Pi}^z$ may be constructed as follows. Let $w = \pi(\varepsilon)$. First, put $(w, +)$ into $T$. Second, for each $i \in [1, |w|]$, fix some $a_i \in \Sigma$ such that $a_i \neq w[i]$ (which is possible because $z \geq 2$), let $w_i$ be the string derived from $w$ by replacing $w[i]$ with $a_i$, and put $(w_i, -)$ into $T$. Let $\tau$ be any regular pattern that is consistent with the labelled examples put into $T$ so far, and observe that $\tau(\varepsilon) = w$. Without loss of generality, one may assume that $\tau$ has the shape $c_1x_1c_2x_2\ldots c_n$, where $c_1, c_n \in \Sigma^*$ and $c_2, \ldots, c_{n-1} \in \Sigma^+$. To finish the construction of $T$, the cases (i) $z = 2$ and (ii) $3 \leq z \leq 6$ will be considered separately.

Case (i): $z = 2$. Let $\Sigma = \{a, b\}$. We will apply Claim 5.1.7 (see the proof of Proposition 5.1.6) several times in this proof.

Define $(p_1, p_2, \ldots, p_{|w|})$ to be the sequence of position numbers of $\pi$ such that for all $i \in \{1, \ldots, |w|\}$, $\pi[p_i] = w[i]$. Similarly, define $(q_1, q_2, \ldots, q_{|w|})$ to be the sequence of position numbers of $\tau$ such that for all $i \in \{1, \ldots, |w|\}$, $\tau[q_i] = w[i]$. Note that since $\pi$ and $\tau$ are assumed to have the shape $c_1x_1c_2x_2\ldots x_{n-1}c_n$, where $c_1, c_2 \in \Sigma^*$ and $c_2, \ldots, c_{n-1} \in \Sigma^+$, it holds that for all $i \in \{1, \ldots, |w|\}$, either $p_{i+1} = p_i + 1$ (resp. $q_{i+1} = q_i + 1$) (no variable of $\pi$ (resp. $\tau$) occurs between $w[i]$ and $w[i+1]$) or $p_{i+1} = p_i + 2$ (resp. $q_{i+1} = q_i + 2$) (exactly one variable of $\pi$ (resp. $\tau$) occurs between $w[i]$ and $w[i+1]$). By applying Claim 5.1.7 as often as necessary, one may assume that $\pi$ and $\tau$ possess the following property.

Property 1. Suppose that for some $\alpha \in (\Sigma \cup X)^*$, $m \geq 1$ and distinct variables $x_i$ and $x_j$, $x_i a^m \alpha b x_j$ is a substring of $\pi$ (resp. $\tau$). If $b$ does not occur in $\alpha$, then $\alpha$
contains at least one variable. A similar statement holds with any of the strings in 
\{ x_i b^m a x_j, x_i a b^m x_j, x_i a a^m b x_j \} substituted for \( x_i a^m a b x_j \).

In other words, if \( \pi \) (resp. \( \tau \)) contains a substring of the shape \( x_i a^m b x_j \), where \( m \geq 1 \) and \( x_i \) and \( x_j \) are distinct variables, then one can extend \( \pi \) (resp. \( \tau \)) by inserting a new variable between \( a^m \) and \( b \). Note that one can only add a finite number of new variables to \( \pi \) since it is assumed throughout this proof that the regular patterns are always expressed as \( c_1 x_1 c_2 x_2 \ldots c_{n-1} x_n c_n \), where \( c_1, c_n \in \Sigma^* \) and \( c_2, \ldots, c_{n-1} \in \Sigma^+ \).

The remaining elements of \( T \) are defined as follows.

1. Add two labelled examples that identify the starting and ending symbols of \( \pi \).

   Fix some \( v_1 \in \Sigma \setminus \{ w[1] \} \). If \( p_1 = 1 \), that is, \( \pi \) starts with a constant, then put \((v_1 w, -)\) into \( T \). If \( p_2 = 2 \), that is, \( \pi \) starts with a variable, then put \((v_1 w, +)\) into \( T \). If \( \tau \) were consistent with \( T \), then \( \tau \) starts with a variable iff \( \pi \) starts with a variable. Similarly, fix some \( v_2 \in \Sigma \setminus \{ w[|w|] \} \); if \( p_{|w|} = |\pi| \), then put \((w v_2, -)\) into \( T \), and if \( p_{|w|} = |\pi| - 1 \), then put \((w v_2, +)\) into \( T \). If \( \tau \) were consistent with \( T \), then \( \tau \) ends with a variable iff \( \pi \) ends with a variable.

2. Now consider any substring \( w[i] w[i+1] \) of \( w \) such that \( w[i] = w[i+1] \). Fix some \( a_i \in \Sigma \setminus \{ w[i] \} = \Sigma \setminus \{ w[i+1] \} \). Let \( w' \) be the string obtained from \( w \) by inserting \( a_i \) between \( w[i] \) and \( w[i+1] \). If \( p_{i+1} = p_i + 2 \), then put \((w', +)\) into \( T \); if \( p_{i+1} = p_i + 1 \), then put \((w', -)\) into \( T \). Suppose that \((w', +) \in T \). We argue that if \( \tau \) were consistent with \( T \), then \( q_{i+1} = q_i + 2 \). Since \( \tau(\varepsilon) = w \) and \( |w'| = |w| + 1 \), \( w' \) is derived from \( \tau \) by replacing exactly one variable \( x_j \) with a constant symbol.

   Let \( \varphi : (\Sigma \cup X)^* \mapsto \Sigma^* \) be a substitution witnessing \( w' \in L(\tau) \). Suppose \( \varphi \) maps \( x_j \) to the \((j')^{th} \) position of \( w' \) for some \( j' \leq i \). Since \( \tau(\varepsilon) = \pi(\varepsilon) = w \), it follows
that $w'[l + 1] = w[l]$ for all $l \geq j'$, contradicting the fact that $w'[i + 1] \neq w[i]$. If $\varphi$ maps $x_j$ to the $(j'')^{th}$ position of $w'$ for some $j'' \geq i + 2$, then $w'[i + 1] = w[i]$, which again yields a contradiction. Hence $x_j$ occurs between $q_i$ and $q_{i+1}$, that is, $q_{i+1} = q_i + 2$. One can argue similarly that if $(w',-) \in T$ and $\tau$ were consistent with $T$, then $q_{i+1} = q_i + 1$.

3. Next, add a labelled example to $T$ so that a variable of $\pi$ occurs between $w[1]$ and $w[2]$ iff a variable of $\tau$ occurs between $w[1]$ and $w[2]$. Suppose that $p_2 = p_1 + 2$, that is, a variable of $\pi$ occurs between $w[1]$ and $w[2]$. The case $w[1] = w[2]$ was handled in 2. By symmetry of $a$ and $b$, it may be assumed that $w[1] = a$ and $w[i] = b$ for all $2 \leq i \leq m$, where either $m = |w|$ or $w[m + 1] = a$. If $\pi$ and $\tau$ do not start with variables, then let $u_1$ be the string obtained from $w$ by inserting $a$ between $w[1]$ and $w[2]$, and put $(u_1,+)$ into $T$. The consistency of $\tau$ with $T$ would imply that $q_2 = q_1 + 2$. Suppose $\pi$ and $\tau$ both start with variables. In Step 2., we added an example to $T$ so that for any $j,j+1$ with $2 \leq j,j+1 \leq m$, a variable of $\pi$ occurs between $w[j]$ and $w[j+1]$ iff a variable of $\tau$ occurs between $w[j]$ and $w[j+1]$. If a variable of $\pi$ (resp. $\tau$) occurs between $w[j]$ and $w[j+1]$ for some $j$ such that $2 \leq j,j+1 \leq m$, then by Claim 5.1.7 a variable of $\pi$ (resp. $\tau$) occurs between $w[1]$ and $w[2]$. If no variable of $\pi$ (resp. $\tau$) occurs between $w[j]$ and $w[j+1]$ whenever $2 \leq j,j+1 \leq m$, then let $u_2$ be the string obtained from $w$ by inserting $b$ between $w[1]$ and $w[2]$, and put $(u_2,+)$ into $T$. The consistency of $\tau$ with $T$ then implies that a variable of $\tau$ occurs either between $w[1]$ and $w[2]$ or just after $w[m]$; note that the latter case also implies that a variable of $\tau$ occurs between $w[1]$ and $w[2]$. An analogous argument holds if $p_2 = p_1 + 1$. Similarly, add a labelled example to $T$ so that a variable of $\pi$ occurs between $w[|w|−1]$ and $w[|w|]$ iff a variable of $\tau$ occurs
between $w[|w| - 1]$ and $w[|w|]$.

4. Finally, consider any substring of $w$ of the shape $s = w[i]w[i + 1]w[i + 2]w[i + 3]$.

We would like to add a labelled example to $T$ so that $p_{i + 2} = p_{i + 1} + 2$ iff $q_{i + 2} = q_{i + 1} + 2$ (that is, a variable of $\pi$ occurs between $w[i + 1]$ and $w[i + 2]$ iff a variable of $\tau$ occurs between $w[i + 1]$ and $w[i + 2]$). The case $w[i + 1] = w[i + 2]$ was handled in Step 2. By symmetry of $a$ and $b$, it may be assumed that one of Subcases (1)–(4) holds; in each subcase, suppose that $p_{i + 2} = p_{i + 1} + 2$.

**Subcase (1):** $s = abaa$. Let $t_1$ be the string obtained from $w$ by inserting $ba$ between $w[i + 1]$ and $w[i + 2]$, and put $(t_1, +)$ into $T$.

**Claim 1.** If $\tau$ were consistent with $T$, then at least one of the following would hold: $q_{i + 2} = q_{i + 1} + 2$, or variables of $\tau$ occur between $w[j]$ and $w[j + 1]$ for some $j \geq i + 2$ such that $w[j'] = a$ for all $j' \in [i + 2, j]$.

*Proof of Claim 1.* Let $\phi : (\Sigma \cup X)^* \to \Sigma^*$ be a substitution witnessing $t_1 \in L(\tau)$. Since $|t_1| = |w| + 2$, and $\tau(\varepsilon) = w$, one of the following cases holds.

**Case (a):** There is exactly one variable $x_k$ of $\tau$ such that for some $j \in [1, |t_1| - 1]$, $\phi$ maps $x_k$ to $t_1[j]t_1[j + 1]$. If $j \leq i$, then $t_1[j]t_1[j + 1] = t_1[j + 2l]t_1[j + 2l + 1]$ for all $l$ such that $j + 2 \leq j + 2l, j + 2l + 1 \leq i + 4$, which is impossible since $t_1[i]t_1[i + 1]t_1[i + 2]t_1[i + 3] = abba$. If $j = i + 1$, then $a = t_1[i + 4] = w[i + 1] = b$, a contradiction. Similarly, if $j \geq i + 3$, then $b = t_1[i + 2] = w[i + 2] = a$, a contradiction. Hence $j = i + 2$.

**Case (b):** There are distinct variables $x_k, x_l$ such that $\phi$ maps $x_k$ to $t_1[j_1]$ and $\phi
maps $x_t$ to $t_1[j_2]$ for some $j_1, j_2 \in [1, |t_1|]$ such that $j_2 > j_1 + 1$. Suppose $j_1 < i + 2$.

First, suppose that $t_1[j_1] = a$. Then either $t_1[j'] = a$ for all $j' \in [j_1, i + 1]$ (which is impossible) or $j_2 \in [j_1 + 2, i + 1]$, $t_1[j_2] = b$ and $t_1[j_2 + 2h - 1]t_1[j_2 + 2h] = ab$ for all $h \geq 1$ such that $j_2 + 1 \leq j_2 + 2h - 1, j_2 + 2h \leq i + 3$, which is impossible because $t_1[i]t_1[i + 1]t_1[i + 2]t_1[i + 3] = abba$. Second, suppose that $t_1[j_1] = b$. If $j_1 \leq i$, then either $t_1[j'] = b$ for all $j' \in [j_1, i + 1]$ or $j_2 \in [j_1 + 1, i + 1]$ and $t_1[j_2 + 2h - 1]t_1[j_2 + 2h] = ba$ for all $h \geq 1$ such that $j_2 + 1 < j_2 + 2h - 1, j_2 + 2h \leq i + 3$, a contradiction.

Furthermore, if $j_1 \geq i + 3$, then $b = t_1[i + 2] = w[i + 2] = a$, a contradiction. Consequently, $j_1 \in \{i + 1, i + 2\}$.

Now suppose $j_2 \geq i + 6$. Suppose that $t_1[j_2] = b$. Then for all $j_3 \in [i + 4, j_2 - 1]$, $t_1[j_3] = b$, which is impossible since $t_1[i + 4]t_1[i + 5] = aa$. Hence we may assume that $t_1[j_2] = a$. Then for all $j_3 \in [i + 6, j_2 - 1]$, $t_1[j_3] = a$.

It follows that either $x_k$ occurs between $w[i + 1]$ and $w[i + 2]$, that is, $q_{i+2} = q_{i+1} + 2$, or $x_k$ occurs between $w[i]$ and $w[i + 1]$ and $x_l$ occurs between $w[j]$ and $w[j + 1]$ for some $j \geq i + 2$ such that $w[j'] = a$ for all $j' \in [i + 2, j]$.

Note that if variables of $\tau$ occur between $w[i]$ and $w[i + 1]$ as well as between $w[j]$ and $w[j + 1]$ for some $j \geq i + 2$ such that $w[j'] = a$ for all $j' \in [i + 2, j]$, then Lemma 5.1.7 implies that a variable of $\tau$ must occur between $w[i + 1]$ and $w[i + 2]$.

**Subcase (2):** $s = bbab$. Let $t_2$ be the string obtained from $w$ by inserting $ba$ between $w[i + 1]$ and $w[i + 2]$, and put $(t_2, +)$ into $T$. One can argue similarly to Subcase (1) that a variable of $\tau$ must occur between $w[i + 1]$ and $w[i + 2]$.

**Subcase (3):** $s = bbbaa$. Let $t_3$ be the string obtained from $w$ by inserting $ab$
between \(w[i+1]\) and \(w[i+2]\), and put \((t_3, +)\) into \(T\). The rest of the argument proceeds analogously to Subcase (1).

**Subcase (4):** \(s = abab\). Let \(t_4\) be the string obtained from \(w\) by inserting \(ba\) between \(w[i+1]\) and \(w[i+2]\), and put \((t_4, +)\) into \(T\). The rest of the argument proceeds analogously to Subcase (1).

The case \(p_{i+2} = p_{i+1} + 1\) can be handled analogously to Subcases (1)–(4).

\(T\) now contains a total of \(2|\pi| + 1\) labelled examples, and this completes the proof of Case (i).

**Case (ii):** \(3 \leq z \leq 6\). For each pair of adjacent constants \(w[i], w[i+1]\) such that \(1 \leq i, i+1 \leq |w|\), fix some \(a_i \in \Sigma \setminus \{w[i], w[i+1]\}\) (which is possible because \(|\Sigma| \geq 3\) and let \(s_i\) be the string derived from \(w\) by inserting \(a_i\) between \(w[i]\) and \(w[i+1]\).

Put \((s_i, +)\) into \(T\) if \(p_{i+1} = p_i + 2\) and put \((s_i, -)\) into \(T\) if \(p_{i+1} = p_i + 1\). Fix some \(b_1 \in \Sigma \setminus \{w[1], w[|w|]\}\). Set \(\alpha = b_1w\) and \(\beta = wb_1\). Put \((\alpha, +)\) into \(T\) if \(\pi\) starts with a variable and put \((\alpha, -)\) into \(T\) if \(\pi\) starts with a constant. Put \((\beta, +)\) into \(T\) if \(\pi\) ends with a variable and put \((\beta, -)\) into \(T\) if \(\pi\) ends with a constant. One can argue similarly to Step 2 in the proof of Case (i) that if \(\tau\) were consistent with \(T\), then \(L(\tau) = L(\pi)\).

The class \(1\Pi^2\) of 1-variable patterns has been treated quite extensively in the literature. In particular, the corresponding class of *non-erasing* languages is efficiently learnable from queries \([14]\) while its membership problem is decidable in polynomial time \([2]\). By contrast, the class of *erasing* 1-variable pattern languages is not learnable in various models of query learning \([33]\). Theorem 5.2.2 shows that the finite distinguishability problem restricted to \(1\Pi^2\) has a simple decision procedure;
further, any 1-variable pattern \( \pi \) with finite teaching dimension w.r.t. \( 1\Pi^z \) has a teaching set of size at most cubic in \( |\pi| \).

**Theorem 5.2.2** Let \( z \in \mathbb{N} \cup \{\infty\} \) and let \( \pi \) be a 1-variable pattern over \( \Sigma \). Then \( TD(\pi, 1\Pi^z) < \infty \) iff \( \pi \) contains a variable. If \( \pi \) contains a variable, then \( TD(\pi, 1\Pi^z) = O(|\pi|^3) \) if \( z \geq 2 \) (including \( z = \infty \)).

**Proof.** Let \( \pi \in 1\Pi^z \) be given. Suppose \( \pi \) does not contain any variable. If \( T \) were a finite teaching set for \( \pi \) w.r.t. \( 1\Pi^z \), then for any \( m > \max\{ |\alpha| : \alpha \in T^+ \cup T^- \} \), the 1-variable pattern \( \pi' = \pi x^m_1 \), which is not equivalent to \( \pi \), would be consistent with \( T \), a contradiction. Thus \( TD(\pi, 1\Pi^z) = \infty \).

Now suppose \( \pi \) contains at least one variable.

**Case (i):** \( z = 1 \). Let \( \Sigma = \{a\} \) and \( \pi = a^m x^n \). A teaching set for \( \pi \) w.r.t. \( 1\Pi^1 \) is \( \{(a^x, -) : x < m\} \cup \{(a^m, +)\} \cup \{(a^{m+n}, +)\} \cup \{(a^{m+x}, -) : 0 < x < n\} \). Note that \( \{(a^m, +)\} \cup \{(a^x, -) : x < m\} \) uniquely identifies \( a^m \) as the constant part of \( \pi \), while \( \{(a^{m+n}, +)\} \cup \{(a^{m+x}, -) : 0 < x < n\} \) uniquely identifies the variable block of \( \pi \) among all \( \pi' \) such that \( \pi'(\varepsilon) = a^m \).

**Case (ii):** \( z \geq 2 \) (including \( z = \infty \)). Let \( \pi = c_1 X_1 c_2 X_2 \ldots X_{n-1} c_n \), where \( c_1, c_2 \in \Sigma^*, c_2, \ldots, c_{n-1} \in \Sigma^+ \) and \( X_1, \ldots, X_{n-1} \in \{x\}^+ \). Build a teaching set \( T \) as follows. First, choose any two distinct \( a, b \in \Sigma \). Put \( (\pi(a), +) \) and \( (\pi(b), +) \) into \( T \). Let \( \pi' \) be any 1-variable pattern that is consistent with \( \{(\pi(a), +), (\pi(b), +)\} \). Note that since \( |\pi(a)| = |\pi(b)| \) and \( \pi' \) contains at most one variable (with possibly more than one occurrence), any substitutions \( \varphi_1, \varphi_2 : (\Sigma \cup X)^* \rightarrow \Sigma^* \) such that \( \varphi_1(\pi') = \pi(a) \) and \( \varphi_2(\pi') = \pi(b) \) satisfy \( \varphi_1^{-1}(\pi(a)[i]) = \varphi_2^{-1}(\pi(b)[i]) \).
for all $i \in [1,|\pi(a)|]$. In particular, consider any $j \in [1,|\pi|]$ such that $\pi[j]$ is a variable; since $\pi(a)[j] = a \neq b = \pi(b)[j]$, $\varphi_1^{-1}(\pi(a)[j])$ is also a variable.

Further, let $\pi' = d_1 Y_1 d_2 Y_2 \ldots Y_{k-1} d_k$, where $d_1, d_k \in \Sigma^*, d_2, \ldots, d_{k-1} \in \Sigma^+$ and $Y_1, \ldots, Y_{k-1} \in \{x\}^+$. Consider the following decomposition of $\pi(a)$:

$$\underbrace{c_1 a^{|X_1|}}_{c_1} \underbrace{a^{|X_2|}}_{c_2} \ldots \underbrace{a^{|X_{n-1}|}}_{c_{n-1}} \underbrace{c_n}_c \quad (5.9)$$

There is a sequence $(i_1, \ldots, i_k)$ such that $1 \leq i_1 \leq \ldots \leq i_k \leq n$ and $\varphi_1$ maps $d_j$ to $c_{i_j}$ for all $j \in [1,k]$ (where the $c_i$'s are indicated by braces in the decomposition $\pi(a)$). Further, for every $j \in [1,k]$, $i_j < i_{j+1}$. To see this, assume to the contrary that there exists some $l \in [1,k]$ such that $i_l = i_{l+1} = m$ for some $m \in [1,n]$. Then $\varphi_1$ and $\varphi_2$ both map $Y_i$ to the same proper substring of $c_m$. As $Y_i = x^u$ for some $u \geq 1$, it follows that $\varphi_1(x) = \varphi_2(x)$ and therefore $\varphi_1(\pi') = \varphi_2(\pi')$, a contradiction. Thus $i_1 < \ldots < i_k$ indeed holds. Further, for every $i \in [1,n]$, there are $O(\ell^2)$ substrings of $c_i$. Consequently, since $n \leq |\pi|$, $|\{\tau(\varepsilon) : \tau \in (\Sigma \cup X)^+ \land \tau \text{ is consistent with } T\}| = O(\ell^2)$. For each $w \in \{\tau(\varepsilon) : \tau \in (\Sigma \cup X)^+ \land \tau \text{ is consistent with } T\}$ such that $w \neq \pi(\varepsilon)$, put $(w,-)$ into $T$. Hence if $\pi'$ is consistent with $T$, then $\pi'(\varepsilon) = \pi(\varepsilon)$. In addition, $\pi'$ has the shape $c_1 Y_1 \ldots Y_{n-1} c_n$, where $Y_1, \ldots, Y_{n-1} \in \{x\}^+$ and for some $\mu \geq 1$, $|X_i| = \mu |Y_i|$ for all $i \in [1,n-1]$. Fix some $a \in \Sigma$, and for each possible choice of $\mu > 1$, put the negative example $(c_1 a^{|X_1|} \ldots a^{|X_{n-1}|} c_n, -)$ into $T$. There are at most $|\pi|$ possible choices of $\mu > 1$. At this stage, $T$ contains $2 + O(\ell^3) + |\pi| = O(\ell^3)$ examples and every $\pi' \in \Pi^2$ consistent with $T$ must be equivalent to $\pi$.  

5.3 Worst-Case Teaching Complexity

We have seen that there are relatively simple patterns such as constant patterns with infinite TD w.r.t. the whole class of patterns. In the present section, we shall investigate subclasses of pattern languages with finite teaching complexity in the TD and RTD models. In particular, the TD and RTD will be determined for the class of regular pattern languages over any alphabet.

Since, by Theorem 5.1.4, for any alphabet size there are patterns with an infinite teaching dimension with respect to the class of all (erasing) pattern languages, it holds that $\text{TD}(\Pi^z) = \infty$ for all $z \in \mathbb{N} \cup \{\infty\}$. By Theorem 5.2.2, the same holds for 1-variable pattern languages.

**Theorem 5.3.1** Let $z \in \mathbb{N} \cup \{\infty\}$. Then $\text{TD}(\Pi^z) = \text{TD}(1\Pi^z) = \infty$.

By contrast, for $z \geq 7$ as well as for $z = 1$, the corresponding class of regular pattern languages has a finite teaching dimension (whose exact value depends on $z$).

**Theorem 5.3.2**

1. $\text{TD}(R\Pi^1) = 3$.

2. For all $z \geq 2$ (including $z = \infty$), $\text{TD}(R\Pi^z) \geq 5$.

3. For all $z \geq 7$ (including $z = \infty$), $\text{TD}(R\Pi^z) = 5$.

**Proof.** 1. Let $\Sigma = \{a\}$. The inequality $\text{TD}(R\Pi^1) \geq 3$ follows from analogues of Propositions 4.2.3 and 4.2.4 for the class of erasing pattern languages. To prove the upper bound, first consider any $\pi \in R\Pi^1$ of the shape $a^m x$, where $m \geq 0$ (it suffices to assume that any non-constant pattern belonging to $R\Pi^1$ has this shape).
If $m = 0$, then $\{(\varepsilon, +), (a, +)\}$ is a teaching set for $\pi$ w.r.t. $\text{RII}^1$. If $m > 0$, then $\{(a^m, +), (a^{m+1}, +), (a^{m-1}, -)\}$ is a teaching set for $\pi$ w.r.t. $\text{RII}^1$. Second, consider any constant pattern $a^n$, where $n \geq 1$. This pattern may be taught w.r.t. $\text{RII}^1$ using the teaching set $\{(a^n, +), (a^{n-1}, -), (a^{n+1}, -)\}$.

2. The proof is similar to that for Lemma 4.2.7.

3. This result is immediate from 2. and the following sequence of lemmas.

**Lemma 5.3.3** Let $z = |\Sigma| \geq 7$ and $n \geq 2$. Let $\pi$ be any regular pattern of the shape $\pi = X_1c_1X_2c_2 \ldots X_{n-1}c_{n-1}X_n$ for some $c_1, c_2, \ldots, c_{n-1} \in \Sigma^+$ and $X_1, X_2, \ldots, X_n \in X^+$. Then $TD(\pi, \text{RII}^z) \leq 3$. In particular, $\pi$ has a teaching set of size three w.r.t. $\text{RII}^z$ that contains two positively labelled examples that neither start nor end with the same letter.

**Lemma 5.3.4** Let $z = |\Sigma| \geq 2$ and $\pi$ be a regular pattern that starts and ends with a block of variables. Let $T$ be a teaching set for $\pi$ w.r.t. $\text{RII}^z$ such that $T$ contains two positively labelled examples that neither start nor end with the same letter. Let $c_1, c_2 \in \Sigma^+$. Then the following hold:

1. $TD(c_1\pi, \text{RII}^z) \leq 1 + |T|$ and $TD(\pi c_1, \text{RII}^z) \leq 1 + |T|$,

2. $TD(c_1\pi c_2, \text{RII}^z) \leq 2 + |T|$.

**Lemma 5.3.5** Let $z = |\Sigma| \geq 2$. Let $c \in \Sigma^+$ and $X_1 \in X^+$ be regular patterns. Then $TD(c, \text{RII}^z) = TD(X_1, \text{RII}^z) = 2$.

The proofs of these lemmas are very similar (but with a few important differences) to the corresponding proofs for the non-erasing regular pattern languages. (See
Lemmas 4.2.24, 4.2.25, 4.2.11 and 4.2.12.) First, note that any regular pattern of the shape \(X_1d_1X_2 \ldots d_{h-1}X_h\), where \(X_1, X_2, \ldots, X_h \in X^+\), is equivalent to a regular pattern in which any two distinct variables are separated by a constant block. Every regular pattern can thus be expressed in a canonical form \(c_1x_1c_2x_2 \ldots x_{n-1}c_n\), where \(c_1, c_{n-1} \in \Sigma^*\) and \(c_2, \ldots, c_{n-2} \in \Sigma^+\). Throughout this proof, it is assumed that every regular pattern is expressed in its canonical form. We recall the following notation from Chapter 4 (see Notation 4.2.1). Let \(c \in \Sigma^+\). If \(|\Sigma| \geq 3\) and \(a\) is a letter that differs from \(c[1]\) and \(c[n]\), then we define

\[
\hat{c} = c^+ac^- \quad \text{for} \quad c^+ = c[1] \ldots c[|c| - 1] \quad \text{and} \quad c^- = c[2] \ldots c[|c|].
\] (5.10)

The notation \(\hat{c}\) does not make the choice of \(a\) explicit but this choice will always be clear from the context.

Proof of Lemma 5.3.3 Let \(\prec\) be a linear order on \(\Sigma\), where \(z = |\Sigma| \geq 7\). Let \(m = |\pi|\). For each \(i \in [1, n - 1]\), let \(i'\) be the maximum index less than \(i\) such that \(c_{i'} \neq c_i\) (if no such index exists then set \(i' = i\)) and let \(i''\) be the minimum index greater than \(i\) such that \(c_{i''} \neq c_i\) (if no such index exists then set \(i'' = i\)). Let \(a_i\) be the least (w.r.t. \(\prec\)) letter in \(\Sigma\) such that \(a_i\) is different from the first and last symbols of any member of \(\{c_{i'}, c_i, c_{i''}\}\). Define the strings \(\alpha, \beta\) and \(\gamma\) as follows.
\( \alpha = \pi(\varepsilon) = c_1 c_2 \ldots c_{n-1}, \)
\( \beta = a_1^m c_1 a_1^m a_2^m c_2 a_2^m \ldots a_{n-1}^m c_{n-1} a_{n-1}^m, \)
\( \gamma = a_1^m c_1 a_1^m a_2^m c_2 a_2^m a_1^m w_1 a_1^m a_2^m c_2 a_2^m a_3^m c_3 a_3^m a_2^m w_2 a_2^m \ldots \)
\( a_1^m c_1 a_1^m a_i^m a_i+1 a_i^m w_i a_i^m \ldots \)
\( a_{n-2}^m \hat{c}_n a_{n-1}^m a_{n-1}^m a_{n-1}^m a_{n-1}^m a_{n-2}^m w_{n-2} a_{n-2}^m a_{n-1}^m \hat{c}_{n-1} a_{n-1}^m, \)

where, for each \( i \in [1, n-2], \)
\[
    w_i = \begin{cases}
        c_i & \text{if } c_i \neq c_{i+1}; \\
        \varepsilon & \text{if } c_i = c_{i+1}.
    \end{cases}
\]

Note that \( \alpha \) and \( \beta \) neither start nor end with the same letter. We shall show that
\( T = \{(\alpha,+),(\beta,+),(\gamma,-)\} \) is a teaching set for \( \pi \) w.r.t. RII\(^z\) by establishing the following claims.

**Claim 5.3.6** \( \alpha, \beta \in L(\pi) \) and \( \gamma \notin L(\pi). \)

**Claim 5.3.7** For any \( \pi' \in \text{RII}^z \) such that \( \{\alpha, \beta\} \subset L(\pi') \) and \( L(\pi') \neq L(\pi), \gamma \in L(\pi'). \)

It is immediate from Claims 5.3.6 and 5.3.7 that for any \( \pi' \in \text{RII}^z \) such that \( L(\pi') \neq L(\pi), \pi' \) is inconsistent with \( T \). This would show that \( T \) is indeed a teaching set for \( \pi \) w.r.t. RII\(^z\).
Proof of Claim 5.3.6. \( \alpha \) is obtained from \( \pi \) by substituting the empty string for every variable of \( \pi \), and \( \beta \) is obtained from \( \pi \) by substituting \( a_i^m \) for \( X_1 \), \( a_i^{m-1} \) for \( X_2 \), and \( a_{i-1}^m \) for \( X_i \) whenever \( i \in [2, n-2] \). Thus \( \{ \alpha, \beta \} \subset L(\pi) \). Now it is shown by induction that \( \gamma \notin L(\pi) \). First, note that by construction \( c \) is not a substring of \( \hat{c} \) for all \( c \in \Sigma^+ \). In particular, \( c_i \) is not a substring of \( \hat{c}_i \) for all \( i \in [1, n-1] \). Furthermore, suppose \( c_i \) were a proper substring of \( c_{i+1} \). Then \( w_i = c_i \) and \( c_{i+1} \) cannot be a substring of \( c_i \). Combining the last two facts with the requirements on \( a_{i+1} \) and \( a_{i+2} \), it follows that \( a_i^m \hat{c}_i a_i^m a_i^m a_i^m \) does not contain a substring of the shape \( s_1 c_1 s_2 c_2 s_3 \) for any \( s_1, s_2, s_3 \in \Sigma^+ \). Similarly, if \( c_i \) is not a proper substring of \( c_{i+1} \), then the definitions of \( w_i, a_{i+1} \) and \( a_{i+2} \) again imply that \( a_i^m \hat{c}_i a_i^m a_i^m a_i^m a_i^m w_i a_i^m \) does not contain a substring of the shape \( s_1 c_1 s_2 c_2 s_3 \) for any \( s_1, s_2, s_3 \in \Sigma^+ \). Assume inductively that \( a_i^m \hat{c}_i a_i^m a_i^m c_2a_1^m \) does not contain a substring of the shape \( s_1 c_1 s_2 c_2 \) for any \( s_1, s_2 \). By the definition of \( a_i^m \), \( a_i^m \hat{c}_i a_i^m a_i^m a_i^m a_i^m a_i^m \) cannot contain a proper substring of \( \hat{c}_i \). Consequently, as \( c_{i+1} \) is not a substring of \( a_i^m \hat{c}_i a_i^m a_i^m a_i^m a_i^m a_i^m \) does not contain a substring of the shape \( s_1 c_1 s_2 c_2 s_3 \) for any \( s_1, s_2, s_3 \in \Sigma^+ \), one has that \( a_i^m \hat{c}_i a_i^m a_i^m c_2a_1^m \) does not contain a substring of the shape \( s_1 c_1 s_2 c_2 s_3 \) for any \( s_1, s_2, s_3 \). It follows by induction that \( \gamma \notin L(\pi) \). \( \blacksquare \) (Claim 5.3.6)

Proof of Claim 5.3.7. Consider any \( \pi' \in RI^\sharp \) such that \( L(\pi') \neq L(\pi) \) and \( \{ \alpha, \beta \} \subset L(\pi') \). Since \( \alpha \) and \( \beta \) start (as well as end) with different symbols, \( \pi' \) is of the shape
Suppose otherwise. Since \( \alpha \in L(\pi') \) and \( \pi' \) contains at least one variable, \( |\pi'(\varepsilon)| < m \). Thus, for each of the strings \( a_1^m, a_1^m a_2^m, \ldots, a_{n-2}^m a_{n-1}^m, a_{n-1}^m \) indicated by braces in (5.11), \( h \) maps some variable of \( \pi' \) to at least one position in each of these strings. As \( \pi' \neq \pi \), \( \pi' \) is in canonical form, and no variable of \( \pi' \) occurs between \( h^{-1}(c_i[j]) \) and \( h^{-1}(c_i[j + 1]) \) for all \( i \in [1, n - 1] \) and \( j \in [1, |c_i|] \). \( \pi'(\varepsilon) \) must be of the shape \( s_1 c_1 s_2 c_2 \ldots s_{n-1} c_{n-1} s_n \), where \( s_1, s_2, \ldots, s_n \in \Sigma^* \) and at least one \( s_i \) is nonempty. This contradicts the fact that \( \alpha \in L(\pi') \) and \( |\alpha| < |\pi'(\varepsilon)| \). Now let \( i \in [1, n - 1] \) be the least number that satisfies (*), and let \( j \in [1, |c_i|] \) be the least number for which either \( h^{-1}(c_i[j]) \) is a variable or \( h^{-1}(c_i[k]) \) is a constant for all \( l \in [1, n - 1] \) and \( k \in [1, |c_i|] \) and a variable of \( \pi' \) occurs between \( h^{-1}(c_i[j]) \) and \( h^{-1}(c_i[j + 1]) \). (Note that we are referring to the specific occurrence of \( c_i \) in \( \beta \) indicated by the sequence of braces in the decomposition [5.11].) We shall define a substitution \( \varphi : (\Sigma \cup X)^* \mapsto \Sigma^* \) such that \( \varphi(\pi') = \gamma \). In order to define \( \varphi \), we will use the decomposition (5.11) of \( \beta \); for each prefix \( a_1^m c_1 a_1^m \ldots a_k^m c_k \) of \( \beta \), \( \varphi \) will map \( h^{-1}(a_1^m c_1 a_1^m \ldots a_k^m c_k) \) to a prefix
\(\omega\) of \(\gamma\). (In what follows, the specific occurrence of \(\omega\) in \(\gamma\) will be given w.r.t. the decomposition (5.12) of \(\gamma\) below.)

\[
\begin{align*}
& a^m_1 c^m_1 a_1^m 2^m_1 a_2^m c^m_2 a_2^m w_1 a^m_1 a_2^m c^m_2 a_2^m 3^m_3 c^m_3 a_2^m w_2 a^m_2 \ldots a^m_1 c^m_1 a^m_2 a^m_{i+1} c^m_{i+1} a^m_{i+1} a^m_i w^m_i a^m_i \ldots \\
& a^m_{n-2} c^m_{n-2} a^m_{n-2} c^m_{n-1} a^m_{n-1} a^m_{n-2} w^m_{n-2} a^m_{n-2} \ldots a^m_{n-1} c^m_{n-1} a^m_{n-1}.
\end{align*}
\]

(5.12)

Assume that \(i \in [2, n-2]\). (The cases \(i = 1\) and \(i = n-1\) can be handled in a very similar way.) Consider the decomposition (5.11) of \(\beta\). We first map \(h^{-1}(a^m_1 c^m_1)\) to \(a^m_1 c^m_1 a^m_2 c^m_2 a^m_2 w_1 a^m_1\) if \(c_1 \neq c_2\), and to \(a^m_1 c^m_1 a^m_2 c^m_2 a^m_2\) if \(c_1 = c_2\). To construct such a map, note that since \(|\pi'| \leq m\) and \(\pi'\) is not a constant pattern, there is a least position \(p_1\) of \(\pi'\) occupied by a variable \(x_1\) such that \(h\) maps \(x_1\) to some substring of \(a^m_1\) (the first occurrence of \(a^m_1\) in the decomposition (5.11)). If \(c_1 \neq c_2\), so that \(w_1 = c_1\), then one can define \(\varphi(x_1)\) to be an extension of \(h(x_1)\) so that \(\varphi(x_1)\) covers the substring \(v^m_1 c^m_1 a^m_2 c^m_2 a^m_2 v'\) for some suffix \(v\) of \(a^m_1\) starting at the first position in \(a^m_1\) that \(h\) maps \(x_1\) to and some prefix \(v'\) of \(a^m_1\) ending at the last position in \(a^m_1\) that \(h\) maps \(x_1\) to. Letting \(a^m_1 = v' v''\) and \(a^m_1 = v'' v\) for some \(v', v'' \in \Sigma^*\), one can then define \(\varphi(h^{-1}(v'' w_1)) = h(h^{-1}(v'' w_1)) = v'' c_1\) and \(\varphi(h^{-1}(v'')) = h(h^{-1}(v'')) = v''\). If \(c_1 = c_2\), so that \(a_1 = a_2\), then \(\varphi\) can be defined so that it extends \(h(x_1)\) to cover the substring \(v^m_1 c^m_1 a^m_2 a^m_2 u\), where \(v\) is defined as above and \(u\) is the prefix of \(a^m_2\) ending at the last position in \(a^m_1\) (\(= a^m_2\)) that \(h\) maps \(x_1\) to. Letting \(a^m_2 = u u'\) for some \(u' \in \Sigma^*\), one then defines \(\varphi(h^{-1}(u' c_2)) = h(h^{-1}(u' c_2)) = u' c_1\) and \(\varphi(h^{-1}(v'')) = h(h^{-1}(v'')) = v''\).

Inductively, assume that for all \(k < j\), where \(j < i\), \(\varphi\) maps \(h^{-1}(a^m_1 c^m_1 a^m_1 \ldots\).
Chapter 5. The Teaching Complexity of Erasing Pattern Languages

\(a_k^{m}c_k\) to

\[a_1^{m}\hat{c}_1a_1^{2m}a_2^{m}c_2a_2^{m}a_1^{m}a_1w_1a_1^{m}\hat{c}_2a_2^{2m}a_3^{m}c_3a_3^{m}a_2^{m}w_2a_2^{m}\ldots a_k^{m}\hat{c}_ka_k^{2m}a_{k+1}^{m}c_{k+1}^{m}a_{k+1}^{m}w_k\]

if \(c_k \neq c_{k+1}\), or to

\[a_1^{m}\hat{c}_1a_1^{2m}a_2^{m}c_2a_2^{m}a_1^{m}a_1w_1a_1^{m}\hat{c}_2a_2^{2m}a_3^{m}c_3a_3^{m}a_2^{m}w_2a_2^{m}\ldots a_k^{m}\hat{c}_ka_k^{2m}a_{k+1}^{m}c_{k+1}^{m}\]

if \(c_k = c_{k+1}\). We now define the image of \(h^{-1}(a_1^{m}c_1a_1^{m}\ldots a_j^{m}c_{j-1}a_{j-1}^{m}a_j^{m}c_j)\) under \(\varphi\).

**Case (i):** \(c_{j-1} \neq c_j\). Again, since \(|\pi'| \leq m\) and \(\pi'\) contains at least one variable, there is a least position \(p'\) such that \(\pi'[p']\) is a variable \(x'\) and \(h\) maps \(x'\) to some substring of \(a_j^{m}\) (where the specific occurrence of \(a_j^{m}\) in \(\beta\) being referred to is indicated by braces below).

\[a_1^{m}c_1a_1^{m}\ldots a_{j-1}^{m}c_{j-1}a_{j-1}^{m}a_j^{m}c_j\]

Let \(p''\) be the position of \(\pi'\) that \(h\) maps to the first position of the substring \(a_{j-1}^{m}\) whose occurrence in \(\beta\) is indicated by braces below.

\[a_1^{m}c_1a_1^{m}\ldots a_{j-1}^{m}c_{j-1}a_{j-1}^{m}a_j^{m}c_j\]

For every symbol \(s\) of \(\pi'\) between the \((p'')_{th}\) position and the \((p' - 1)_{st}\) position inclusive, define \(\varphi(s) = h(s)\). If \(c_j \neq c_{j+1}\), then \(\varphi(x')\) can be defined as an
extension of $h(x')$ so that $\varphi(x')$ covers the substring $v_1 \hat{c}_j a_j^{2m} a_{j+1}^{m} c_{j+1} a_{j+1}^{m+1} v_2$ for some suffix $v_1$ of $a_j^{m}$ starting at the first position in $a_j^{m}$ that $h$ maps $x'$ to and some prefix $v_2$ of $a_j^{m}$ ending at the last position in $a_j^{m}$ that $h$ maps $x'$ to. If $c_j = c_{j+1}$ (so that $a_{j+1} = a_j$), then $\varphi(x')$ can be defined as an extension of $h(x')$ so that $\varphi(x')$ covers the substring $w_1 \hat{c}_j a_j^{2m} w_2$, where $w_1$ is the suffix of $a_j^{m}$ starting at the first position in $a_j^{m}$ that $h$ maps $x'$ to and $w_2$ is the prefix of $a_{j+1}^{m}$ ending at the last position in $a_j^{m}$ ($= a_{j+1}^{m}$) that $h$ maps $x'$ to. Proceeding as in the case $j = 1$, one can then extend the definition of $\varphi$ so that $\varphi$ maps $h^{-1}(a_1^{m} c_1 a_1^{m} \ldots a_j^{m} c_j)$ to

$$\overbrace{a_1^m \hat{c}_1 a_1^{2m} a_2^m a_2 a_2^m a_1^m w_1 a_1^{m+1} a_2^m \hat{c}_2 a_2^m a_3^m a_3 a_2^m w_2 a_2^m} \ldots$$

$$\overbrace{a_j^m \hat{c}_j a_j^{2m} a_{j+1}^m c_{j+1} a_{j+1}^{m+1} a_j^{m} w_j}$$

if $c_j \neq c_{j+1}$, and to

$$\overbrace{a_1^m \hat{c}_1 a_1^{2m} a_2^m a_2 a_2^m a_1^m w_1 a_1^{m+1} a_2^m \hat{c}_2 a_2^m a_3^m a_3 a_2^m w_2 a_2^m} \ldots \overbrace{a_j^m \hat{c}_j a_j^{2m} a_{j+1}^m c_{j+1} a_{j+1}^{m+1} c_{j+1}}$$

if $c_j = c_{j+1}$.

**Case (ii):** $c_{j-1} = c_j$. Then $w_{j-1} = \varepsilon$. Define $p', p'' \in \mathbb{N}$ and the variable $x'$ as in Case (i). If $c_j \neq c_{j+1}$, then $\varphi(x')$ can be defined as an extension of $h(x')$ so that $\varphi(x')$ covers the substring $v_1 a_j^{2m} a_j^{m} \hat{c}_j a_j^{2m} a_{j+1}^{m} c_{j+1} a_{j+1}^{m+1} v_2$ for some suffix $v_1$ of $a_j^{m}$ starting at the first position in $a_j^{m}$ that $h$ maps $x'$ to and some prefix $v_2$ of $a_j^{m}$ that ends at the last position in $a_j^{m}$ that $h$ maps $x'$ to. If $c_j = c_{j+1}$ (so that $a_{j+1} = a_j$), then $\varphi(x')$ can be defined as an extension of $h(x')$ so that
\( \varphi(x') \) covers the substring \( u_1a_j^m a_{j-1}^m a_j^m c_j a_j^m u_2 \), where \( u_1 \) is the suffix of \( a_j^m \) starting at the first position in \( a_j^m \) that \( h \) maps \( x' \) to and \( u_2 \) is the prefix of \( a_{j+1}^m \) ending at the last position in \( a_j^m \) (= \( a_{j+1}^m \)) that \( h \) maps to. Proceeding as in the case \( j = 1 \), one can then extend the definition of \( \varphi \) so that \( \varphi \) maps \( h^{-1}(a_1^m a_1^m \ldots a_j^m c_j) \) to

\[
\begin{align*}
&\underbrace{a_1^m \hat{c}_1 a_1^m a_2^m c_2 a_2^m a_1^m w_1 a_1^m a_2^m c_2 a_2^m a_3^m c_3 a_3^m a_2^m w_2 a_2^m \ldots} \\
&\underbrace{a_j^m \hat{c}_j a_j^m a_{j+1}^m c_{j+1} a_{j+1}^m w_j}
\end{align*}
\]

if \( c_j \neq c_{j+1} \), and to

\[
\begin{align*}
&\underbrace{a_1^m \hat{c}_1 a_1^m a_2^m c_2 a_2^m a_1^m w_1 a_1^m a_2^m c_2 a_2^m a_3^m c_3 a_3^m a_2^m w_2 a_2^m \ldots} \\
&\underbrace{a_j^m \hat{c}_j a_j^m a_{j+1}^m c_{j+1}}
\end{align*}
\]

if \( c_j = c_{j+1} \).

For \( j = i \), \( \varphi \) maps the string

\[
a_1^m c_1 a_1^m \ldots a_{i-1}^m c_{i-1} a_{i-1}^m a_i^m c_i
\]

to the substring

\[
\underbrace{a_1^m \hat{c}_1 a_1^m a_2^m c_2 a_2^m a_1^m w_1 a_1^m a_2^m c_2 a_2^m a_3^m c_3 a_3^m a_2^m w_2 a_2^m \ldots} \\
\underbrace{a_i^m \hat{c}_i}
\]

note that such a mapping can be defined because either \( h^{-1}(c_i[j]) \) (w.r.t. the decomposition \((5.11)\)) is a variable for at least one \( j \in [1, |c_i|] \), or \( h^{-1}(c_i[k]) \) is a constant.
for all \( l \in [1, n - 1] \) and \( k \in [1, |c_i|] \) and \( \pi' \) contains a variable between \( h^{-1}(c_i[j]) \) and \( h^{-1}(c_i[j + 1]) \) for some \( j \in [1, |c_i|] \). To see this, first suppose there exists some \( q' \) such that \( q' \) is the least position of \( \pi' \) for which \( \pi'[q'] \) is a variable \( y \) and \( h \) maps \( y \) to some substring of \( c_i \); now choose the least \( j \) such that \( h \) maps \( y \) to the \( j^{th} \) position of \( c_i \). (The specific occurrence of \( c_i \) in \( \beta \) being referred to is indicated by braces below.)

\[
a_i^m c_i a_i^m \ldots a_{i-1}^m c_i \underbrace{a_i^m}_{\text{as a prefix of } a_{i-1}^m a_i^m c_i}\underbrace{a_i^m}_{\text{as a suffix of } a_{i-1}^m a_i^m c_i}
\] (5.13)

Let \( \theta \) and \( \eta \) be strings such that \( \hat{c}_i = c_i[1] \ldots c_i[j - 1] \theta c_i[j] \alpha c_i[j + 1] \ldots c_i[|c_i|] \). One can define \( \varphi(y) \) so that \( \varphi(y) \) covers the substring \( \theta c_i[j] \alpha \) of \( \hat{c}_i \). Now consider the following case distinction.

**Case (i):** \( c_{i-1} \neq c_i \). Define \( \varphi(h^{-1}(a_{i-1}^m a_i^m c_i[1] \ldots c_i[j - 1])) = a_{i-1}^m a_i^m c_i[1] \ldots c_i[j - 1] \) (as a prefix of \( a_{i-1}^m a_i^m \hat{c}_i \)) and \( \varphi(h^{-1}(c_i[j + 1] \ldots c_i[|c_i|])) = c_i[j + 1] \ldots c_i[|c_i|] \) (as a suffix of \( a_{i-1}^m a_i^m \hat{c}_i \)).

**Case (ii):** \( c_{i-1} = c_i \). Then \( w_{i-1} = \varepsilon \) and \( a_{i-1} = a_i \). There is a least position \( r \) of \( a_{i-1}^m \) (where \( a_{i-1}^m \) is indicated by braces in (5.13)) such that for some variable \( z \) of \( \pi' \), \( h \) maps \( z \) to the \( r^{th} \) position of \( a_{i-1}^m \). \( \varphi(z) \) can be defined as an extension of \( h(z) \) so that \( \varphi(z) \) covers \( u_1 a_{i-1}^m u_2 \), where \( u_1 \) is the suffix of \( a_i^m \) that starts at the \( r^{th} \) position of \( a_i^m \) and \( u_2 \) is the prefix of \( a_{i-1}^m \) (as \( a_{i-1}^m \)) that ends at the last position of \( a_{i-1}^m \) that \( h \) maps \( z \) to. Letting \( a_i^m = u_3 u_1 = u_2 u_4 \) for some \( u_3, u_4 \in \Sigma^* \), define \( \varphi(h^{-1}(u_3)) = u_3 \) and \( \varphi(h^{-1}(u_4 a_i^m c_i[1] \ldots c_i[j - 1])) = u_4 a_i^m c_i[1] \ldots c_i[j - 1] \) (as a prefix of \( u_4 a_i^m \hat{c}_i \)) and \( \varphi(h^{-1}(c_i[j + 1] \ldots c_i[|c_i|])) = c_i[j + 1] \ldots c_i[|c_i|] \) (as a suffix of \( u_4 a_i^m \hat{c}_i \)).
Now suppose that \( h^{-1}(c_1[k]) \) is a constant for all \( l \in [1, n - 1] \) and \( k \in [1, |c_1|] \) and \( \pi' \) contains a variable \( z \) between \( h^{-1}(c_1[j]) \) and \( h^{-1}(c_1[j + 1]) \) for some \( j \in [1, |c_1|] \). The definition of \( \varphi(h^{-1}(a_1^ma_1^m \ldots a_{i-1}^m c_{i-1}^m c_i c_1 a_1 \ldots a_i^m)) \) here is very similar to that in the previous case. Let \( \theta' \) be the string such that \( \hat{c}_i = c_1[i] \ldots c_1[j] \theta' c_1[j + 1] \ldots c_1[|c_1|] \). One can define \( \varphi(z) \) so that \( \varphi(z) \) covers the substring \( \theta' \) of \( \hat{c}_i \). Further, one defines \( \varphi(h^{-1}(a_1^m c_1 a_1^m \ldots a_{j-1}^m c_{j-1}^m a_j a_j^m c_j a_j^m)) \) and \( \varphi(h^{-1}(c_1[j + 1] \ldots c_1[|c_1|])) \) according to a case distinction similar to that in the previous case.

By applying an argument similar to that in the preceding paragraph, one can extend the definition of \( \varphi \) to \( h^{-1}(a_1^ma_1^m \ldots a_{j-1}^m c_{j-1}^m a_j a_j^m c_j a_j^m) \) for all \( j \in [1, n - 1] \).

This establishes that \( T \) is a teaching set for \( \pi \) w.r.t. RII\(^{z} \).

**Proof of Lemma 5.3.4** We prove that \( TD(c_1 \pi c_2, \text{RII}^z) \leq 2 + |T| \); the remaining cases can be proved similarly. We follow the proof of Lemma 4.2.25 (the analogue of Lemma 5.3.4 for the class of non-erasing pattern languages). Suppose \( T \) is a teaching set for \( \pi \) w.r.t. RII\(^{z} \) containing at least two positively labelled examples \((w_1, +), (w_2, +)\) that neither start nor end with the same letter. Let \( T' = \{(c_1w_2, +) : (w, +) \in T\} \cup \{(c_1v_2, -) : (v, -) \in T\} \cup \{(\hat{c}_1w_1c_2, -), (c_1w_1\hat{c}_2, -)\} \). Let \( \pi' = d_1\rho d_2 \) be a regular pattern that is consistent with \( T' \), where \( \rho \) starts and ends with variables and \( d_1, d_2 \in \Sigma^* \). Since \((c_1w_1c_2, +), (c_1w_2c_2, +) \in L(\pi') \) and \( w_1, w_2 \) both start as well as end with different symbols, \( d_1 \) is a prefix of \( c_1 \) and \( d_2 \) is a suffix of \( c_2 \). We argue that \( d_1 \) is in fact equal to \( c_1 \). Let \( \varphi : (\Sigma \cup X)^* \mapsto \Sigma^* \) be a substitution witnessing \( c_1w_1c_2 \in L(\pi') \). If \( d_1 = c_1[1] \ldots c_1[k] \) for some \( k < |c_1| \), then \( v = \hat{c}_1w_1c_2 \in L(\pi') \): one can map the variable \( x_1 \) in \( \pi' \) occurring just after \( d_1 \)
to \(c_1[k + 1] \ldots c_1[|c_1| - 1]ac[2] \ldots c_1[2] \ldots c_1[|c_1|]\) (where \(a \notin \{c_1[1], c_1[|c_1|]\}\)), and for each position \(j\) of \(v\) after \(c_1\), one maps \(\varphi^{-1}(v[j])\) (which may be equal to \(x_1\)) to \(v[j]\).

This contradicts the fact that \(\pi'\) is consistent with \(T'\). A similar argument shows that if \(d_2\) were a proper suffix of \(c_2\), then \(v' = c_1w_1c_2 \in L(\pi')\), a contradiction. Thus \(\pi' = c_1\rho c_2\). Furthermore, note that for all \(u \in \Sigma^*\) and \(l \in \{+, -\}\), \(\pi' = c_1\rho c_2\) is consistent with \((c_1uc_2, l)\) iff \(\rho\) is consistent with \((u, l)\). Hence if \(T\) is a teaching set for \(\pi\) w.r.t. \(\mathbb{RII}^z\), then \(T'\) is a teaching set for \(c_1\pi c_2\) w.r.t. \(\mathbb{RII}^z\). (Lemma 5.3.4)

**Proof of Lemma 5.3.5.** Let \(c \in \Sigma^+\) and \(X_1 \in \mathbb{X}^+\) for some regular pattern \(X_1\). Fix distinct \(a, b \in \Sigma\). One may directly verify that \(\{(c, +), (c^2, -)\}\) is a teaching set for \(c\) w.r.t. \(\mathbb{RII}^z\) while \(\{(a, +), (b, +)\}\) is a teaching set for \(X_1\) w.r.t. \(\mathbb{RII}^z\). Furthermore, \(TD(c, \mathbb{RII}^z) \geq 2\) because a single positive example is consistent with \(X_1\) while a single negative example \((v, -)\) for some \(v \in \Sigma^*\) is consistent with \(c'\) for any \(c' \in \Sigma^* \setminus \{c, v\}\). Also, \(TD(X_1, \mathbb{RII}^z) \geq 2\) because a single positive example \((w, +)\) is consistent with \(w\) while every teaching set for \(X_1\) contains only positive examples. (Lemma 5.3.5)

This concludes the proof of Theorem 5.3.2.3.

In view of the fact that the erasing pattern languages are often hard to teach in the TD model (an exception being the class of regular erasing pattern languages), it may be asked whether the recursive teaching model improves on the sample efficiency of teaching these languages. Unfortunately, the next result shows that no recursive teaching sequence exists even for the class of 1-variable pattern languages.

**Theorem 5.3.8** If \(z \in \mathbb{N} \cup \{\infty\}\), then no recursive teaching sequence for \(\mathbb{RII}^z\) exists.
Proof. The proof is very similar to that of Proposition 3.2.12. Assume for the sake of a contradiction that \( R = ((\mathcal{L}_0, d_0), (\mathcal{L}_1, d_1), \ldots) \) were a teaching sequence for \( 1\Pi^2 \). Fix any constant pattern \( c \). Suppose \( c \in \mathcal{L}_i \), and let \( T \) be a minimal teaching set for \( c \) w.r.t. \( \bigcup_{j \geq i} \mathcal{L}_j \). Choose \( N > \max\{d_j : j \leq i\} \) such that \( N > |w| \) for every \( w \in X(T) \). Further, let \( p_0, \ldots, p_{N+i-1} \) be a strictly increasing sequence of \( N+i \) primes. Observe that by the choice of \( N \), \( \pi_0 = cx \prod_{j=0}^{N+i-1} p_j \) is consistent with \( T \) and so it must belong to \( \mathcal{L}_{j_0} \) for some \( j_0 < i \). For any two distinct \((N+i-1)\)-subsets \( S, S' \) of \( \{p_0, \ldots, p_{N+i-1}\} \), \( L(\pi_0) \subseteq L\left(cx \prod_{y \in S} y\right) \) and \( L(\prod_{y \in S} y) \cap L\left(cx \prod_{y \in S'} y\right) \subseteq L(\pi_0) \). Thus by the choice of \( N \), there must exist some \((N+i-1)\)-subset \( S_1 \) of \( \{p_0, \ldots, p_{N+i-1}\} \) such that \( \pi_1 = cx \prod_{y \in S_1} y \in \mathcal{L}_{j_1} \) for some \( j_1 < j_0 \). By iterating a similar argument, one can show that for some \( N \)-subset \( S'' \) of \( \{p_0, \ldots, p_{N+i-1}\} \), \( cx \prod_{y \in S''} y \in \mathcal{L}_0 \). But the teaching dimension of \( cx \prod_{y \in S''} y \) w.r.t. \( 1\Pi^2 \) is at least \( N > d_0 \), a contradiction.

For regular pattern languages, recursive teaching is provably more efficient than teaching according to the TD model over non-binary alphabets, as the next theorem shows. Determining \( \text{RTD}(R\Pi^2) \) remains an open problem.

**Theorem 5.3.9** Let \( z \in \mathbb{N} \cup \{\infty\} \). If \( z \neq 2 \), then \( \text{RTD}(R\Pi^2) = 2 \).

Proof. \( \text{RTD}(R\Pi^2) \geq 2 \) follows from the fact that every pattern other than \( x \) has teaching dimension at least 2 w.r.t. \( R\Pi^2 \). We next focus on proving the upper bound.

We first consider the case \( z \geq 3 \). Let \( \pi = B_1 x_1 \ldots B_2 x_2 B_3 \ldots B_{n-1} x_{n-1} B_n \in \text{RII}^2 \) be given, where \( B_1, \ldots, B_n \) are blocks of constants over \( \Sigma \) and \( B_2, \ldots, B_{n-1} \) are nonempty. It suffices to show that the teaching dimension of \( \pi \) w.r.t. the subclass
of all regular patterns $\tau$ such that either (i) $|\tau(\varepsilon)| > |\pi(\varepsilon)|$ or (ii) $|\tau(\varepsilon)| = |\pi(\varepsilon)|$ and $|\text{Var}(\tau)| \leq |\text{Var}(\pi)|$ is 2. If the latter were true, then one could construct a recursive teaching sequence $((\mathcal{L}_0, d_0), (\mathcal{L}_1, d_1), \ldots)$ for RII such that for any $\eta, \eta' \in \text{RII}$ with $\eta \in \mathcal{L}_i$ and $\eta' \in \mathcal{L}_{i'}$, $i < i'$ holds if $|\eta(\varepsilon)| < |\eta'(\varepsilon)|$ or $|\eta(\varepsilon)| = |\eta'(\varepsilon)| \wedge |\text{Var}(\eta')| \leq |\text{Var}(\eta)|$.

We claim that $\{(w_1, +), (w_2, +)\}$ is a teaching set for $L(\pi)$ w.r.t. the required class, where $w_1 = B_1 \ldots B_n$ ($= \pi(\varepsilon)$) and $w_2$ is obtained from $\pi$ by replacing every variable occurrence $x_i$ with a letter that is different from the last symbol of $B_i$ and the first symbol of $B_{i+1}$ (which is possible because $|\Sigma| \geq 3$). For each $i \in \{1, \ldots, n - 1\}$, let $c_i$ denote the letter that is substituted for $x_i$.

Consider any regular pattern $\tau$ that is consistent with $\{(w_1, +), (w_2, +)\}$. Since $w_1 \in L(\tau)$ and $|\tau(\varepsilon)| \geq |w_1|$, it follows that the constant part of $\tau$ is equal to $w_1$, that is, $\tau(\varepsilon) = B_1 \ldots B_n$. Let $h$ be a homomorphism $(\Sigma \cup X)^* \mapsto \Sigma^*$ such that $h(\tau) = w_2 = B_1 c_1 \ldots B_{n-1} c_{n-1} B_n$. For each $i \in \{1, \ldots, |w_2|\}$, let $p(i)$ denote the position of the symbol in $\tau$ such that $h$ maps $\tau[p(i)]$ to the $i^{th}$ symbol of $w_2$. We shall argue that for each $i$, if $\pi[i] = x_{j_i}$ for some $j_i \in \{1, \ldots, n - 1\}$, then $\tau[p(i)]$ is some variable $x_{j'}$, and if $\pi[i]$ is some letter $a \in \Sigma$, then $\tau[p(i)]$ is also $a$. Suppose for the sake of a contradiction that this were false for some least position $i_0$ of $w_2$. It cannot happen that $\tau[p(i_0)]$ is some constant $d$ and $\pi[i_0]$ is some variable: otherwise, since $d \neq \pi[i_0 + 1]$ and the constant parts of $\tau$ and of $\pi$ before the $i_0^{th}$ position of $\pi$ coincide, one would have $\pi(\varepsilon) \neq \tau(\varepsilon)$, which is impossible. So the only possible case is that $\tau[p(i_0)]$ were some variable and $\pi[i_0]$ were some constant. Since $\tau(\varepsilon) = \pi(\varepsilon)$, there must be some $i_1 > i_0$ such that $\pi[i_0] = \tau[p(i_1)] = w_2[i_1]$. Note that if $\pi[i_1]$ were some variable, then $i_1 = i_0 + 1$ would imply that $w_2[i_0] \neq w_2[i_0 + 1]$; thus
$i_1 > i_0 + 1$. In this case, there must be some $i_2 > i_1$ with $\tau[p(i_2)] = \pi[i_0 + k']$ for the least $k' > 0$ such that $\pi[i_0 + k']$ is a constant and $i_0 + k' < i_2$. If $\pi[i_2]$ were again a variable, then the above argument may be repeated, giving some $i_3 > i_2$ for which $\tau[p(i_3)] = \pi[i_0 + k'']$, where $k''$ is the least positive number such that $\pi[i_0 + k'']$ is a constant and $i_0 + k' < i_0 + k'' < i_3$. On the other hand, if $\pi[i_1]$ were a constant, then there is some $i_3' > i_1$ such that $\tau[p(i_3')] = \pi[i_1]$. One can then apply the same argument for $i_2$ to $i_3'$. Thus by repeatedly applying the preceding line of argument, one can construct an indefinitely long sequence $i_0 < i_1 < i_2 < \ldots$ such that each $\tau[p(i_j)]$ is a constant, which is impossible as $\tau$ is finite. Since $\pi$ has at least as many distinct variables as $\tau$, $\pi = h^{-1}(w_2) = \tau$.

We now consider the case $z = 1$ and $\Sigma = \{a\}$. Any regular pattern can be normalised to either $a^n$ or $a^{n-1}x_1$ for some $n \geq 1$. We construct a teaching sequence $R'$ for $\Pi^z$ that lists patterns in increasing order of the number of constant symbols. The pattern $a^{n-1}x_1$ uses $\{(a^{n-1}, +), (a^n, +)\}$ as a recursive teaching set w.r.t. $R'$, while $a^n$ uses $\{(a^n, +), (a^{n+1}, -)\}$. \hfill \qed
Chapter 6

Conclusions and Open Problems

In this thesis, we studied two main teaching parameters, the TD and RTD (as well as their variants, TD\(^+\) and RTD\(^+\)), for classes of linear sets with a fixed dimension and classes of non-erasing as well as erasing pattern languages. It may be observed that even though many of the concept classes considered here have an infinite TD, there are finer notions of teachability that occasionally yield different finite sample complexity measures. In particular, there are families of linear sets that have an infinite TD and RTD\(^+\) and yet have a finite RTD. We broadly interpret a class that has an infinite RTD as being “unteachable” in a stronger sense than merely having an infinite TD. Quite interestingly, the fact that many infinite concept classes we considered have an infinite TD (see, for example, Table 3.1) contrasts with positive results on the learnability of the same classes; it has been proven, for example, that the family of linear subsets of \(\mathbb{N}_0^m\) and the family of non-erasing pattern languages over any alphabet are learnable in the limit from just positive examples \([3, 45]\). One
possible interpretation of this contrast is that it may be generally harder to teach than to learn a given concept class.

We conclude with a number of problems left open in the present work.

**Open Problem 1.** Is $\text{RTD}(\text{LINSET}_k)$ finite for each $k \geq 4$?

We have shown that $\text{RTD}(\text{LINSET}_k)$ is finite whenever $1 \leq k \leq 3$. Further, we constructed for every $k \geq 1$ a class of $\mathcal{L}_k$ of cofinite linear sets, each of which has exactly $k$ periods, such that $\text{TD}_{\text{min}}(\mathcal{L}_k) \geq k - 1$. Problem 1 is equivalent to the question: for each $k \geq 4$, does there exist a family $\{F_{k,0}, F_{k,1}, F_{k,2}, \ldots\}$ of classes of linear sets with a fixed number $k$ of periods such that $\text{TD}_{\text{min}}(F_{k,i})$ grows to infinity with $i$?

**Open Problem 2.** What is the RTD of the class of arbitrary non-erasing pattern languages over finite alphabets of size at least 2?

A partial result was obtained by Hans Simon [18, Theorem 11]: over any alphabet of size $z$, $\text{RTD}(\Pi^z(\leq n))$ (where $\Pi^z(\leq n)$ is the class of patterns of length at most $n$) is at most $1 + \lceil \log_z(n) \rceil$, and this bound is tight if all recursive teaching sequences for $\Pi^z(\leq n)$ are restricted to examples of length $n$.

**Open Problem 3.** What is the TD of the class of regular non-erasing pattern languages over alphabets of size between 2 and 7?

It is worth noting that pattern languages tend to behave more pathologically when the alphabet size is small. For example, it was recently shown that over any alphabet with at least four distinct letters, a non-erasing pattern language $L(\pi)$ is regular only
if \( \pi \) is block-regular; yet this result fails when the alphabet size is exactly two: there are non-erasing pattern languages that are regular and cannot be generated by any block-regular pattern \[26\]. Thus it would not be unreasonable to expect the TD of the class of regular non-erasing pattern languages to be more than 5 when the alphabet size is less than eight. Generally speaking, a smaller alphabet means that it is more difficult to construct a negative example \((w, -)\) such that \(w\) contains certain subsequences of strings while “avoiding” a particular pattern – in other words, ensuring that \(w\) cannot be expressed in the form \(X_1c_1X_2c_2 \ldots X_{n-1}c_{n-1}X_n\) for some \(X_1, X_n \in X^+, X_2, \ldots, X_{n-1} \in X^*\) and \(c_1, \ldots, c_{n-1} \in \Sigma^*\).

It follows from the proof of Lemma 4.2.24 that if the alphabet size is at least four, then three labelled examples (two positive and one negative) are sufficient for distinguishing any regular pattern \(\pi = X_1c_1X_2c_2 \ldots X_{n-1}c_{n-1}X_n\) from all regular patterns of the shape \(Y_1d_1Y_2d_2 \ldots Y_{n-1}d_{n-1}X_n\), where \(c_i\) is a substring of \(d_i\) for all \(i = 1, \ldots, n - 1\). It is less clear whether or not a single additional negative example can distinguish \(\pi\) from the class of all regular patterns obtained from \(\pi\) by replacing at least one constant in \(\pi\) with a variable not occurring in \(\pi\) when the alphabet has fewer than eight letters.

**Open Problem 4.** What is the TD of the class of regular erasing pattern languages over alphabets of size between 2 and 6?

**Open Problem 5.** Is the finite distinguishability problem decidable for the class of erasing pattern languages over binary and ternary alphabets?

It seems plausible that a positive answer to Problem 5 would be found, though we do not know for sure. Owing to time constraints, we have left this problem for
future work.
References


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