How children's arithmetic concepts develop

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By
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Table of Contents

I. Introduction..................................................................................................................................1
II. Methods.......................................................................................................................................6
III. Results........................................................................................................................................8
IV. Discussion................................................................................................................................16
V. Conclusion................................................................................................................................19
VI. References................................................................................................................................21
VII. Appendixes................................................................................................................................24
List of Tables and Figures

Figure 1. Accuracy (%) by operation and concept type with standard error bars.........................8

Figure 2. Conceptually-based strategy use (%) by operation and concept type with standard error bars.................................................................................................................................................10

Table 1. Pearson correlation analysis including gender, accuracy, and conceptually-based strategy use for each combination of the three problem types on addition and subtraction problems and multiplication and division problems.................................................................11
Abstract

Arithmetic is an important skill for people of all ages, and increases education and career opportunities. In particular, arithmetic concepts focus on the relationships, principles, and properties within addition, subtraction, multiplication, and division. Three concepts are of particular interest because there is little research on individual variation and the relationship between these concepts. These concepts are: inversion, associativity, and equivalence. These concepts were measured via immediately retrospective reports of problem-solving strategies. This study focused on those individual variations in the form of a longitudinal design. Forty-three participants were asked to solve 24 mathematical problems twice a year for 3 years. Children were studied from Grades 4 to 6. Accuracy and solution data were analyzed using two mixed model ANOVAs and a Pearson correlational analysis. The children had the greatest accuracy on inversion problems, followed by associativity, and then equivalence. They displayed the highest concept usage on equivalence problems, followed by inversion, and then associativity. Overall, children had higher accuracy on the addition and subtraction problems, compared to the multiplication and division problems. Children used the concepts more often on the addition and subtraction problems, compared to the multiplication and division problems. Overall, this study provides evidence that more research needs to be conducted on children's understanding and use of these three concepts on multiplication and division problems, as well as to identify individual variations and patterns.
How children's arithmetic concepts develop

Arithmetic is an important skill for people of all ages, and increases opportunities for education and career advancements (The National Mathematics Advisory Panel, 2008). In particular, the use of arithmetic concepts can remove unnecessary mental effort from an arithmetic problem (Baroody, Ginsburg, & Waxman, 1983). Arithmetic concepts are abstract ways of processing mathematical problems that do not rely on memorization (Dubé, 2014). The use of these concepts demonstrates understanding of the relationships, principles, and properties associated with the four operations that are critical for mathematics and mathematical concepts; addition, subtraction, multiplication, and division (Robinson & Dubé, 2013; Robinson, Price, & Demyen, 2018). To date, there is very little research that focuses on the operations of multiplication and division in general, and how they relate to addition and subtraction (Crooks & Alibali, 2014; Robinson et al., 2018; Robinson, Ninowski, & Gray, 2006).

In order to understand mathematical concepts, conceptual knowledge must be measured. Conceptual knowledge is a person's understanding of mathematical concepts and their underlying structure (Crooks & Alibali, 2014). In this field of research, a common way to measure conceptual knowledge is by having participants participate in a problem-solving task and collecting data on their immediately retrospective reports of how they obtained their answers (Crooks & Alibali, 2014). Research has shown that conceptual knowledge has many benefits, including instilling richer mathematical learning and more flexible problem-solving skills (Crooks & Alibali, 2014). This particular study looked at how children's arithmetic concepts developed over time. Three arithmetic concepts were looked at: inversion, associativity, and equivalence.
Inversion

Inversion is the understanding that addition and subtraction, or multiplication and division, are inversely related mathematical concepts (Robinson & Beatch, 2016). For this the problem format of \( a + b - b \) and \( d x e ÷ e \) is used (Robinson et al., 2018). This concept involves the understanding that when a number is both added and subtracted, or multiplied and divided, the first number remains unchanged (Robinson & Ninowski, 2003; Robinson et al., 2018). Studies have shown that for addition and subtraction problems, school-aged children are able to easily solve inversion problems when they are presented consecutively, but have difficulties with this concept when the inversion problems are mixed with control problems using the format of \( a + b - c \) (Stern, 1992). Other studies have shown understanding for this concept in children as young as preschoolers (Rasmussen, Ho, & Bisanz, 2003). This shows that no formal schooling is necessary for the development of this concept. Dubé (2014) found that children begin using the inversion concept on multiplication and division problems in late childhood or adolescence, usually between Grades 6 and 8, but only a select amount of children used this concept, showing that it is still developing throughout this time period. The understanding of concepts, such as inversion, can increase accuracy and speed on arithmetic problems (Lai, Baroody, & Johnson, 2008). Understanding and using inversion can also leads to a greater understanding and use of associativity (Robinson & Dubé, 2010).

Associativity

Associativity is the understanding that the answer will remain unchanged regardless of which pair of numbers is solved first (Robinson et al., 2006; Robinson et al., 2018). For this the problem format of \( a + b - c \) and \( d x e ÷ f \) is used (Robinson et al., 2018). For addition and subtraction problems, the associativity concept has been shown as earlier as Grade 1, but was
only used by approximately 41% of the children on roughly 20% of the problems (Canobi, Reeve, & Pattison, 1998). Associativity is used more frequently by children in Grades 3 to 5, but is still only found in a small amount of participants (Robinson & Dubé, 2013). For multiplication and division problems, the associativity concept has been shown in Grades 5 to 7, but most children had difficulties with these problems (Robinson et al., 2018). The frequency of associativity use has been shown to increase significantly between Grade 7 and Grade 11 (Dubé, 2014). There is much individual variation in the usage of associativity, and research has shown that even in adults this concept was used on just under half of the associativity problems (Dubé & Robinson, 2010). These findings were similar for children in Grades 2 to 4, who also used the associativity concept on less than half of the associativity problems (Robinson & Dubé, 2009).

Understanding associativity is extremely beneficial on more difficult problems. Two examples that Robinson et al. (2018) provide for difficult problems in which associativity would be beneficial are 389 + 4438 - 4428 and 389 x 4438 ÷ 2219. Understanding and use of associativity also simplifies problem-solving, and increases speed and accuracy on these problems (Dubé, 2014; Robinson & Dubé, 2009; Robinson et al., 2016).

**Equivalence**

Equivalence is the understanding that the numbers on both sides of the equation equal the same amount (Robinson et al., 2018). For this the format of \( a + b + c = a + \_\_ \) and \( d \times e \times f = d \times \_\_ \) is used (Robinson et al., 2018). Mathematical equivalence is a precursor to algebraic thinking and typically develops later on in elementary school (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2010). This concept has been displayed, albeit weakly, for addition in Grades 2 to 5 (Robinson et al., 2018). The concept of equivalence is less understood in research in regards to children's use of the concept on multiplication and division problems. However, some children
have displayed the use of this concept between Grades 6 and 8 (Robinson et al., 2018). There is no research to date that focuses on equivalence on multiplication and division problems prior to Grade 6 (Robinson et al., 2018). For this study, equivalence will be assessed using addition and multiplication problems. Equivalence is a significant predictor of mathematical achievement in elementary school, and therefore understanding of this concept is critical (Hornburg, Riever, & McNeil, 2017).

There are many individual differences when looking at the development and usage of these concepts. Some people begin developing these concepts in early childhood, while others do not develop these concepts until middle childhood, adolescence, or even adulthood (Robinson & Dubé, 2009; Robinson et al., 2006). Regardless of the concepts examined, or the ages of the participants, people tend to have a weaker understanding of multiplication and division problems, compared to addition and subtraction problems (Robinson et al., 2018). So even if one grasps a concept, they may not necessarily transfer that concept over to a different mathematical operation.

**Individual variation**

In a study on children's understanding of inversion and associativity concepts on addition and subtraction problems, Robinson and Dubé (2013) found that if they gave participants a 5 minute demonstration of the inversion and associativity concepts before the experiment, participants' use of these shortcuts on the experimental problems increased, compared to the non-demonstration group. A second study investigated children's understanding of the inversion and associativity concepts on multiplication and division problems. In this study, Robinson et al. (2016) used the same demonstration task mentioned above, but adapted it to be used on multiplication and division problems. They found that shortcut usage for inversion and
associativity problems increased significantly in the demonstration group. This shows that these concepts can be learned, or increased in individuals, with the right tools.

There is still more information to be discovered about how these concepts develop, and no one has done a longitudinal study to look at these three concepts in more detail and identify individual differences (Robinson et al., 2018). For example, Robinson et al. (2018) found evidence of weaker understanding for certain concepts and operations in a select amount of individuals, showing that further research is needed to investigate these differences in understanding and concept usage. A longitudinal study can also help detect developmental changes that may occur in an individual participant's understanding of mathematical concepts, understanding of the four mathematical operations, and changes in their accuracy on the mathematical problems (Caruana, Roman, Hernández-Sánchez, & Solli, 2015; Robinson et al., 2018). There are risks associated with a longitudinal study, such as attrition rates and practice effects (Caruana et al., 2015). Although attrition rates are difficult to control for, this study has attempted to control for practice effects. The same mathematical problems were given to participants twice a year for 3 years, however, the interference from the participants' daily curriculum should counteract any practice effects. For these reasons, a longitudinal design was chosen for this study. This study began in the fall of 2016, and I have participated in the 5th out of 6 data collection periods. This study has been following the same students from Grades 4 to 6. These grades were chosen for multiple reasons. First, they were chosen due to the lack of research on multiplication and division problems during this time period (Robinson et al., 2018). Second, understanding of the inversion concept for multiplication and division problems has been shown to be extremely weak across Grades 5 to 8 (Robinson et al., 2016; Robinson et al., 2018). Third, understanding of the associativity concept for multiplication and division problems
has also shown to be weak for children in these grades (Robinson and Dubé, 2013; Robinson et al., 2006; Robinson et al., 2018). Finally, understanding of the equivalence concept has been found to be weak for children in Grades 2 to 5 (Robinson et al., 2018). More research needs to be conducted on children in these grades to investigate their lack of understanding (Robinson et al., 2018).

The current research questions are, for children at the beginning of Grade 6: Do children understand the concepts of inversion, associativity, and/or equivalence? Are there any gender differences? Do they find a certain concept easier? Do they find a certain mathematical operation easier? And how accurate are they?

There are three hypotheses. The first one is that inversion will have the strongest understanding, followed by associativity, and then equivalence (Robinson et al., 2018). The second is that accuracy and understanding of the concepts will be stronger for the addition and subtraction problems, compared to the multiplication and division problems (Robinson et al., 2018). The third one is that there will be no gender differences on accuracy or understanding of the concepts (Robinson et al., 2018).

**Methods**

**Participants**

Participants were 43 Grade 6 children (20 boys and 23 girls) from two schools in a large Canadian city. The students were predominantly Caucasian and from middle socioeconomic status families.

**Materials**

Participants were asked to answer 24 mathematical problems on a laptop using E-prime software. There was a set of 12 addition and subtraction problems, followed by a set of 12
multiplication and division problems (see Appendixes A and B). Each set included 4 inversion problems, 4 associativity problems, and 4 equivalence problems. The questions were designed so that all answers would be whole numbers, and a particular concept would never appear more than once in a row. Participants were individually videotaped to capture their answers and explanations, as well as for reliability coding.

**Procedure**

Participants were asked to solve the 24 mathematical problems on a laptop in an approximately 15 minute individually recorded session. The study took place during the first 2 weeks of November, 2018, at the participants' school, during school hours. Parents and guardians received a renewal of consent form approximately 1 week prior to data collection. Participants were also asked to sign an assent form prior to data collection.

Participants were given a maximum of 30 seconds to answer each of the 24 mathematical problems. After they answered, or after this time period, they were asked some probing questions to gain understanding of their concept usage. These included questions such as "How did you get that answer?" and "How were you trying to get that answer?", no accuracy feedback was given. Once a participant had completed the study they were given an incentive in the form of a coloured gel pen.

**Measures**

Data was collected on participants' age, gender, accuracy, and their problem-solving strategy. Age and gender were immediately entered into the software, while the other two variables were coded during the study using a keyboard. Whether participants used conceptual knowledge during problem-solving was assessed by asking participants to give immediately
Results

Boys and girls were collapsed together because no significant results involving gender were found.

Accuracy

A 2 (Operation: addition and subtraction or multiplication and division) x 3 (Concept: inversion, associativity, or equivalence) (ANOVA) was conducted on correct responses. There were main effects for operation and concept type. Performance on addition and subtraction problems was more accurate than on multiplication and division problems (78.3% vs. 59.7%), $F(1, 42) = 48.48, p < .001, \eta_p^2 = .54$. Performance was highest on inversion problems (87.8%), followed by associativity problems (77.0%), and ending with the lowest accuracy on equivalence problems (42.2%), $F(2, 84) = 83.63, p < .001, \eta_p^2 = .67$ (see Figure 1).

Figure 1. Accuracy (%) by operation and concept type with standard error bars.
Conceptually-based problem-solving strategy use

Conceptual knowledge was assessed through children's immediately retrospective verbal reports of their problem-solving strategies. Strategies were coded as either conceptually-based or not conceptually-based. Participants were credited with using conceptually-based problem-solving strategies if (a) on inversion problems (e.g., \(2 + 5 - 5 = 2\), \(7 \times 4 \div 4 = 7\)) they reported that the second and third numbers cancel each other out, (b) on associativity problems (e.g., \(3 + 9 - 6 = 6\), \(5 \times 6 \div 2 = 15\)) they reported that they first subtracted the third number from the second number and then added the solution to the first number, or that they first divided the second number by the third number and then multiplied that number by the first number, and (c) on equivalence problems (e.g., \(5 + 22 + 3 = 5 + ?\), \(5 \times 2 \times 3 = 5 \times ?\)) they reported that the answer was the sum of the second and third numbers on the left side of the equation, or that the answer was the product of the second and third numbers on the left side of the equation (Robinson et al., 2018).

A 2 (Operation: addition and subtraction or multiplication and division) x 3 (Concept: inversion, associativity, or equivalence) (ANOVA) was conducted on the verbally reported use of conceptually-based problem-solving strategies. There were main effects for operation and concept type. Conceptually-based problem-solving strategies were used more frequently on addition and subtraction problems, as compared to multiplication and division problems (61.4% vs. 39.5%), \(F(1, 42) = 42.71, p < .001, \eta^2_p = .50\). Conceptually-based strategies were used most frequently on equivalence problems (78.2%), followed by inversion problems (49.1%), and least frequently on associativity problems (24.1%), \(F(2, 84) = 39.83, p < .001, \eta^2_p = .49\). These main effects of operation and concept type were qualified by a two-way interaction between them, \(F(2, 84) = 24.99, p < .001, \eta^2_p = .37\). This interaction stemmed from the equivalence problems
that were solved equally frequently via a conceptually-based strategy on addition and subtraction problems, as well as on multiplication and division problems, whereas conceptually-based strategies were used more frequently on addition and subtraction problems for inversion and associativity problems (see Figure 2).

![Figure 2](image-url)

**Figure 2.** Conceptually-based strategy use (%) by operation and concept type with standard error bars.

**Correlations**

A multiple Pearson correlational analysis was conducted. The variables were gender, accuracy, and conceptually-based strategy use for each combination of the three problem types on addition and subtraction problems, and multiplication and division problems.

**Accuracy**

Results of the Pearson correlational analysis indicate that for addition and subtraction problems, there was a positive association between accuracy on associativity problems and accuracy on equivalence problems, \( r(41) = .36, p < .02 \). Those who obtained correct answers on
associativity problems were more likely to obtain correct answers on equivalence problems for addition and subtraction (see Table 1).

Table 1. *Pearson correlation analysis including gender, accuracy, and conceptually-based strategy use for each combination of the three problem types on addition and subtraction problems and multiplication and division problems.*

<table>
<thead>
<tr>
<th>Column</th>
<th>Sex</th>
<th>AccA SInv</th>
<th>AccA SAss</th>
<th>AccA SEQ</th>
<th>AccM DInv</th>
<th>AccM DAss</th>
<th>AccM DEQ</th>
<th>Strat ASInv</th>
<th>Strat ASAss</th>
<th>Strat ASEQ</th>
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*Note.* Acc, accuracy; AS, addition and subtraction; Inv, inversion; Ass, associativity; EQ, equivalence; MD, multiplication and division; Strat, strategy.

** Correlation is significant at the .001 level (2-tailed). * Correlation is significant at the .05 level (2 tailed).

For accuracy on multiplication and division problems, there was a positive association between inversion and associativity, \( r(41) = .69, p < .001 \), between inversion and equivalence, \( r(41) = .45, p < .002 \), and between associativity and equivalence, \( r(41) = .56, p < .001 \). Those who obtained correct answers on the inversion problems were more likely to obtained correct
answers on the associativity and equivalence problems. Those who obtained correct answers on the associativity problems were more likely to obtained correct answers on the equivalence problems.

For accuracy, there was a positive correlation between associativity on the addition and subtraction problems and inversion on the multiplication and division problems, \( r(41) = .41, p < .007 \), between equivalence on the addition and subtraction problems and inversion on the multiplication and division problems, \( r(41) = .60, p < .001 \), and between associativity on the addition and subtraction problems and associativity on the multiplication and division problems, \( r(41) = .58, p < .001 \). There was also a positive correlation between equivalence on the addition and subtraction problems and associativity on the multiplication and division problems, \( r(41) = .56, p < .001 \), between associativity on the addition and subtraction problems and equivalence on the multiplication and division problems, \( r(41) = .35, p < .02 \), and between equivalence on the addition and subtraction problems and equivalence on the multiplication and division problems, \( r(41) = .62, p < .001 \). Those who had higher accuracy on the associativity and equivalence problems using addition and subtraction were more likely to have higher accuracy on the inversion problems using multiplication and division. Those who had higher accuracy on the associativity problems using addition and subtraction were more likely to have higher accuracy on the associativity problems using multiplication and division. Those who had higher accuracy on the equivalence problems using addition and subtraction were more likely to have higher accuracy on the associativity problems using multiplication and division. And those who had higher accuracy on the associativity and equivalence problems using addition and subtraction were more likely to have higher accuracy on the equivalence problems using multiplication and division.
Conceptually-based strategy use

For strategy on the addition and subtraction problems, there was a positive correlation between inversion and associativity, $r(41) = .52$, $p < .001$, and between associativity and equivalence, $r(41) = .37$, $p < .01$. For the addition and subtraction problems, those who used inversion were more likely to use associativity, and those who used associativity were more likely to use equivalence. For strategy on the multiplication and division problems, there was a positive correlation between inversion and associativity, $r(41) = .56$, $p < .001$. For multiplication and division problems, those who used inversion were more likely to use associativity.

For strategy, there was a positive correlation between inversion on the addition and subtraction problems and inversion on the multiplication and division problems, $r(41) = .38$, $p < .01$, between associativity on the addition and subtraction problems and inversion on the multiplication and division problems, $r(41) = .47$, $p < .001$, between associativity on the addition and subtraction problems and associativity on the multiplication and division problems, $r(41) = .56$, $p < .001$, and between equivalence on the addition and subtraction problems and equivalence on the multiplication and division problems, $r(41) = .88$, $p < .001$. Those who used inversion on the addition and subtraction problems were more likely to use inversion on the multiplication and division problems. Those who used associativity on the addition and subtraction problems were more likely to use both inversion and associativity on the multiplication and division problems. And those who used equivalence on the addition and subtraction problems were more likely to use equivalence on the multiplication and division problems.

Accuracy and conceptually-based strategy use

For accuracy and strategy use on the addition and subtraction problems, there was a positive correlation between accuracy on the inversion problems and strategy use on the
inversion problems, $r(41) = .53, p < .001$, between accuracy on the equivalence problems and strategy use on the inversion problems, $r(41) = .44, p < .003$, and between accuracy on the inversion problems and strategy use on the associativity problems, $r(41) = .33, p < .03$. There was also a positive correlation between accuracy on the associativity problems and strategy use on the associativity problems, $r(41) = .33, p < .03$, between accuracy on the associativity problems and strategy use on the equivalence problems, $r(41) = .32, p < .04$, between accuracy on the equivalence problems and strategy use on the associativity problems, $r(41) = .36, p < .02$, and between accuracy on the equivalence problems and strategy use on the equivalence problems, $r(41) = .73, p < .001$. For addition and subtraction problems, those who obtained correct answers on the inversion problems were more likely to have used the inversion concept. Those who obtained correct answers on the equivalence problems were more likely to have used the inversion concept. Those who obtained correct answers on the inversion problems were more likely to have used the associativity concept. Those who obtained correct answers on the associativity problems were more likely to have used the associativity concept. Those who obtained correct answers on the associativity problems were more likely to have used the equivalence concept. Those who obtained correct answers on the equivalence problems were more likely to have used the equivalence concept. And those who obtained correct answers on the equivalence problems were more likely to have used the equivalence concept.

For accuracy and strategy use on the multiplication and division problems, there was a positive correlation between accuracy on the inversion problems and strategy use on the inversion problems, $r(41) = .41, p < .007$, between accuracy on the inversion problems and strategy use on the equivalence problems, $r(41) = .43, p < .004$, between accuracy on the associativity problems and strategy use on the equivalence problems, $r(41) = .43, p < .005$, and
between accuracy on the equivalence problems and strategy use on the equivalence problems, \( r(41) = .47, p < .002 \). Those who had higher accuracy on the inversion problems were more likely to have used the inversion concept. Those who had higher accuracy on the inversion problems were more likely to have used the equivalence concept. Those who had higher accuracy on the associativity problems were more likely to have used the equivalence concept. And those who had higher accuracy on the equivalence problems were more likely to have used the equivalence concept.

For strategy and accuracy, there was a positive correlation between accuracy on addition and subtraction equivalence problems and strategy use on multiplication and division inversion problems, \( r(41) = .46, p < .002 \), between strategy use on addition and subtraction equivalence and accuracy on multiplication and division inversion problems, \( r(41) = .36, p < .02 \), and between strategy use on addition and subtraction equivalence problems and accuracy on multiplication and division associativity problems, \( r(41) = .40, p < .009 \). There was also a positive correlation between strategy use on addition and subtraction equivalence problems and accuracy on multiplication and division equivalence problems, \( r(41) = .46, p < .002 \), between accuracy on addition and subtraction associativity problems and strategy use on multiplication and division equivalence problems, \( r(41) = .34, p < .03 \), and between accuracy on addition and subtraction equivalence problems and strategy use on multiplication and division equivalence problems, \( r(41) = .70, p < .001 \). Those who obtained correct answers on the addition and subtraction equivalence problems were more likely to have used the inversion concept on the multiplication and division problems. Those who used the equivalence concept on the addition and subtraction problems were more likely to have obtained correct answers on the multiplication and division inversion problems. Those who used the equivalence concept on the
addition and subtraction problems were more likely to have obtained correct answers on the multiplication and division associativity problems. Those who used the equivalence concept on the addition and subtraction problems were more likely to obtain correct answers on the multiplication and division equivalence problems. Those who obtained correct answers on the addition and subtraction associativity problems were more likely to have used the equivalence concept on the multiplication and division problems. And those who obtained correct answers on the addition and subtraction equivalence problems were more likely to have used the equivalence concept on the multiplication and division problems.

Discussion

The first hypothesis - that inversion will have the strongest understanding, followed by associativity, and then equivalence - was partially supported by the accuracy data and children's verbal reports of conceptually-based problem-solving strategies. Participants in this study showed the strongest understanding for, but lowest accuracy on, equivalence problems. Understanding of equivalence is necessary for participants to obtain the correct answer (Robinson et al., 2018), therefore, it makes sense that those who obtained the correct answer on equivalence problems were more likely to have used the equivalence concept. Inversion had the highest accuracy, and second highest understanding. The accuracy and understanding were stronger for the addition and subtraction problems, as compared to the multiplication and division problems. This shows that although this concept is well developed by Grade 6 for addition and subtraction problems, more effort needs to be made to strengthen understanding for this concept on multiplication and division problems (Robinson et al., 2018). Finally, associativity had the second highest accuracy, but the lowest understanding. This could be due to children using basic mathematical skills without applying concepts on these problems. It could
also be due to children automatically retrieving the answer for the first half of the problem from memory, without realizing that it would be quicker to do the second half of the equation first (Robinson et al., 2018). These results are consistent with previous research, and show children's lack of understanding for the associativity concept (Robinson et al., 2018).

The second hypothesis - that accuracy and understanding of the concepts will be stronger for the addition and subtraction problems, as compared to the multiplication and division problems - was partially supported. Addition and subtraction problems had higher rates of accuracy than multiplication and division problems. For conceptually-based strategy use, the inversion and associativity concept were used more on addition and subtraction problems, as compared to multiplication and division problems. However, the equivalence concept was used approximately the same amount on both addition and subtraction, as well as multiplication and division problems. These findings are consistent with past research (Robinson et al., 2006; Robinson et al., 2018), and support the assertion that children have a weaker understanding for multiplication and division problems.

The third hypothesis - that there will be no gender differences on accuracy or understanding of the concepts - was supported by the accuracy data and children's verbal reports of conceptually-based problem-solving strategies. Accuracy and conceptually-based problem-solving strategies did not differ based on sex.

A Pearson correlational analysis revealed mixed results in comparison with the two ANOVAs, as well as some unexpected results. The correlational analysis showed that for strategy, the results were inconsistent with the strategy ANOVA results. The correlational analysis showed that for conceptually-based problem-solving strategy use, those who used the associativity concept on addition and subtraction problems were more likely to use the inversion
concept on multiplication and division problems. It also showed that those who used the equivalence concept on addition and subtraction problems were more likely to use the equivalence concept on multiplication and division problems. Both of these results are consistent with the strategy ANOVA, however, the other strategy correlation results are inconsistent.

For the correlational analysis for accuracy and strategy, the results are also mixed. For addition and subtraction problems, those who obtained the correct answer on inversion problems were more likely to have used the inversion strategy. Those who obtained the correct answer on associativity problems were more likely to have used the equivalence strategy. And those who obtained the correct answer on equivalence problems were more likely to have used the associativity strategy. These results are consistent with the conceptually-based problem-solving strategy ANOVA. For multiplication and division problems, those who had higher accuracy on the associativity problems were more likely to have used the equivalence strategy. These results are also consistent with the ANOVA. Finally, those who obtained the correct answer on addition and subtraction associativity problems were more likely to have used the equivalence strategy on multiplication and division problems. And those who used the equivalence strategy on addition and subtraction problems were more likely to have had higher accuracy on the inversion and associativity multiplication and division problems. These results are also consistent with the ANOVA, however, the other correlation results are inconsistent.

Surprisingly, the correlational analysis showed that those who had higher accuracy on the associativity problems for both addition and subtraction were more likely to have had higher accuracy on the equivalence problems for addition and subtraction, and they were also more likely to have had higher accuracy on all concept types for multiplication and division problems. Seeing as accuracy was so low for associativity concept on the accuracy ANOVA, these results
were unexpected. These surprising results were even more pronounced on the equivalence problems, where participants who obtained the correct answer on addition and subtraction equivalence problems were more likely to have obtained the correct answer on all three concepts for multiplication and division problems. Another unexpected finding from the correlation analysis is that those who used one of the three concepts on the addition and subtraction problems were more likely to have used that same concept on the multiplication and division problems. This finding is surprising because according to the conceptually-based problem-solving strategy ANOVA, strategy use for the multiplication and division problems is significantly lower than for the addition and subtraction problems for both inversion and associativity problems.

**Conclusion**

Examining the inversion, associativity, and equivalence concepts on addition and subtraction, as well as multiplication and division problems, highlighted children's difficulties with the inversion concept on the multiplication and division problems. It also highlighted children's difficulty with the associativity concept regardless of the operation. Children also displayed extremely low accuracy on equivalence problems. One potential limitation is that conceptual knowledge is difficult to measure through the use of conceptually-based strategies (Robinson et al., 2018). The 30 second time limit may not have been enough time for a child to apply conceptually-based strategies. Children may also be relying on memorization and automatic retrieval, instead of looking for other ways to solve the problem (Dubé, 2014). A possible option for assessing conceptually-based understanding of the associativity problems is to allot more time for solving these problems. A possible option for increasing accuracy on equivalence problems is to discuss the function of the equal sign with the participants prior to
them solving the problems. Having experimenters demonstrate specific concepts before the problem-solving task has also proven to be beneficial (Robinson and Dubé, 2013; Robinson et al., 2016). This study adds to the limited research on these concepts, as well as children's use of these concepts on multiplication and division problems. While it is clear that most children of this age group have a fairly strong understanding of the inversion concept on addition and subtraction problems, they struggle with this concept on multiplication and division problems. Associativity is quite low, regardless of operation, but is significantly lower for multiplication and division problems. However, the results from the correlational analysis showed that those who obtained the correct answer on addition and subtraction associativity problems were more likely to obtain the correct answer on the multiplication and division problems for all concept types. Further research should investigate this relationship. The equivalence concept was used approximately the same amount, regardless of operation. However, those who had higher accuracy on the addition and subtraction problems were more likely to have higher accuracy on the multiplication and division problems for all three concepts. More research needs to be conducted to determine why accuracy is so low on equivalence problems across all operations, and to explore the relationship between equivalence on addition and subtraction problems and all three concepts on multiplication and division problems. Overall, this study provides evidence that more research needs to be conducted on children's understanding and use of these three concepts on multiplication and division problems, as well as to identify individual variations and patterns.
References


Appendix A

F2016 Study Time 1/6          Add/Sub          Date: ____________

Subject #: _____          Gender  M  F          Age: _____          Birthday: ________

Expt’er: _____          School: _____

Answer          VR

1. 2 + 5 - 5 = ___
2. 5 + 22 + 3 = 5 + ___
3. 3 + 9 - 6 = ___
4. 4 + 2 + 7 = 4 + ___
5. 8 + 26 - 26 = ___
6. 2 + 29 - 27 = ___
7. 6 + 3 + 5 = 6 + ___
8. 3 + 24 - 24 = ___
9. 7 + 25 - 22 = ___
10. 6 + 3 - 3 = ___
11. 8 + 21 + 6 = 8 + ___
12. 5 + 4 - 2 = ___
Appendix B

Fall 2016 Time 1/6  Mult/Div  Date: __________

Subject #:  _____  Gender   M     F Age: _____  Birthday: ________

Expt’er:  _____  School: ______

Answer    VR

1. 5 x 2 x 3 = 5 x ___
2. 7 x 4 ÷ 4 = ___
3. 5 x 6 ÷ 2 = ___
4. 3 x 5 ÷ 5 = ___
5. 4 x 6 ÷ 3 = ___
6. 6 x 5 x 3 = 6 x ___
7. 7 x 9 ÷ 3 = ___
8. 6 x 2 ÷ 2 = ___
9. 4 x 8 x 2 = 4 x ___
10. 8 x 9 ÷ 9 = ___
11. 3 x 4 x 7 = 3 x ___
12. 2 x 8 ÷ 4 = ___
Participant Assent Form

Research Study Title: How children’s arithmetic concepts develop: A three-year study

Researchers: Dr. Katherine Robinson and TBA

Reason for the Research: Remember the last time you did the math study? We talked about how we might be talking to you again during the next three years? This is the 2nd/3rd/4th/5th/6th time we will be talking to you as only by talking to you several times will we understand more about how children your age think about and solve math problems and how this changes over time.

Once again you will be solving some math problems on a laptop and then telling me about how you got the answer or you were trying to get the answer. I will be videotaping the math problems so we can watch the videos back later on and listen to what you said. Only the other researchers might look at the videos. The videos will be kept in a safe, locked place and then permanently deleted once we are done with the data.

In (EDIT FOR EACH SESSION) Fall 2017, Spring 2018, Fall 2018, and Spring 2019 we will be asking you only to solve some math problems again and to talk about how you solved them. Each time we will ask you if you still want to be part of our math study.

Please feel free to ask any questions you might have whenever you want.

If you want to stop or withdraw at any time, for any reason, that's perfectly OK.

Instead of using your name, I am going to use the same number you got the first time so that if someone saw your answers or what you told me about how you solved the problems, they would not know who gave the answers because they will not know what your number is. The computer file with your answers will be kept for many years using only your number and not your name.

I am not going to tell your teacher or your parents/guardians or anyone else about how you did today but you can tell them anything you want about the math study and what you did.

Do you have any questions for me right now? You can ask me questions whenever you want. I will give you a copy of this form that has contact information on it in case you have any questions later on.

This study has been approved by the University of Regina Research Ethics Board and the Regina Board of Education.

Assent

If you write and sign your name below it means that you understand:

(1) what we are going to be doing in this study (solving some math problems and talking about how you solved the problems and then solving more math problems in (EDIT AS NECESSARY) Fall 2017, Spring, 2018, Fall 2018, and Spring 2019),

(2) that you understand that you can stop or withdraw whenever you want,

(3) that you will be videotaped during the math problems

(4) that I won't be talking to anyone about how you did, and

(5) that I might be asking you EDIT AS NECESSARY 4 more times in the next three years to solve some more math problems.

Signing your name means that you agree to be part of this math study. A copy of this form will be kept by the researchers and I will give you a copy too.
Contact:
Katherine Robinson, Professor
Department of Psychology
University of Regina
email: katherine.robinson@uregina.ca
phone: (306) 359-1248
Research Ethics Board:
email: research.ethics@uregina.ca
phone: (306) 585-4775 (out of town residents can call collect)
Appendix D

How children’s arithmetic concepts develop: A three-year study

Dear Parent or Guardian:

In Fall 2016 your child started participating in our three-year study on how children’s arithmetic concepts and problem solving strategies change over time. This letter is to let you know that the 2nd/3rd/4th/5th/6th data collection period will begin in approximately 1-2 weeks. Once again, we will be asking your child to solve some arithmetic problems and to talk about the problem solving strategies that they use in one approximately 15 minute videotaped session.

All responses provided by the students will be completely confidential and will only be viewed by the researchers. The videos will be kept on a password-protected external hard drive (using participant numbers only as identifiers) and stored in a secure location. The video files will be destroyed after the results of the study have been published.

The students will be completing the arithmetic problem solving task on a laptop and their responses will be coded by participant number only. All data will be entered into a computer data file which will be kept indefinitely as no names can be associated with the data.

This study has been approved by XXXX, principal of XXXX School, by the Regina Board of Education, and by the University of Regina Research Ethics Board. General results will be made available to teachers, school administrators, and others who are interested.

If you no longer wish your child to participate, please call or email K. Robinson at (306) 359-1248 or katherine.robinson@uregina.ca or return this letter with your child’s full name, school, and class clearly indicated on the back by DATE.

Before we collect data, we will always ask your child if he/she still wishes to participate and your child is free to withdraw from the study at any time.

This project has been approved on ethical grounds by the University of Regina Research Ethics Board. Any questions about your child's rights as a participant may be addressed to the board at (306) 585-4775 or e-mail: research.ethics@uregina.ca. Out of town participants may call collect.

Sincerely,

Katherine M. Robinson, Ph.D.
Professor
Department of Psychology
Appendix E

Participant Debriefing Form

Research Study Title: How children’s arithmetic concepts develop: A three-year study

Researchers: Dr. Katherine Robinson and TBA

Thank you for participating in this study and helping us learn more about how children think about arithmetic problems. You solved a lot of different math problems in the last three years! By solving the math problems, telling me about the ways in which you solved some of the those problems, and then playing the memory games, you are helping us understand how kids your age think and learn about math.

If you or your parent(s) or guardian(s) have any questions that you didn't get a chance to ask me today, any concerns, or you want to find out what we have learned about how kids your age think about and solve math problems between Grades 4 and 6, please feel free to contact the principal researcher.

Contact Information:
Katherine Robinson, Ph.D.
Professor
Psychology
University of Regina
(306) 359-1248
katherine.robinson@uregina.ca

This project has been approved on ethical grounds by the University of Regina Research Ethics Board. Any questions regarding your rights as a participant may be addressed to the board at (306) 585-4775 or email: research.ethics@uregina.ca. Out of town participants can call collect.

This study was approved by the University of Regina Research Ethics Board (REB) and by the Regina Board of Education.

Thank you again for helping us out with our math study and helping us learn more about how you learn and think about math!