Semi-analytical Modeling of Fluid Flow in and Formation Evaluation of Unconventional Reservoir Using Boundary Integration Strategies

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Chang Su

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External Examiner:  
*Dr. Rick Chalaturnyk, University of Alberta*

Co-Supervisor:  
Dr. Gang Zhao, Petroleum Systems Engineering

Co-Supervisor  
Dr. Yee-Chung Jin, Environmental Systems Engineering

Committee Member:  
Dr. Na Jia, Petroleum Systems Engineering

Committee Member:  
Dr. Hairuo Qing, Department of Geology

Committee Member:  
**Dr. Tsun Wai Kelvin Ng, Environmental Systems Engineering**

Chair of Defense:  
Dr. Ken Leyton-Brown, Department of History

*Via Zoom*  
**Not present at defense**
ABSTRACT

Tight oil/gas has been increasingly playing a more and more important role in petroleum industry around the globe for the past two decades and accounting for more share of total production of oil/gas each year. Fractured vertical or horizontal wells contribute most of the tight oil/gas production. Understanding and analyzing fluid flow in the process of fracturing, therefore, become crucial in tight formation production. However, the resulting complexity of fracture propagation from fracture injection test in tight formation, which currently is one the most heated topics for unconventional reservoirs, and heterogeneity problems of tight oil/gas reservoir with non-uniform distribution of rock/fluid properties bring great difficulties in modeling of such complex well-reservoir systems so as to evaluate reservoir characteristics and simulate a variety of reservoir flowing behaviours. Using boundary integration strategies and source and sink function methodology, this dissertation tackles the problems of modeling fracture-propagation-and-closure process along with analyzing the fracture injection test data (mini-fracturing test) in homogeneous reservoir, and modeling of fluid flow in different types of heterogeneous reservoirs. In particular, they are: (1) **Analytical Modeling of Fracture Propagation on Evaluation of Transient Pressure Behavior during Injection and after Shut-in: Minifrac Test Analysis by Model-Based Type Curves**; (2) **Modeling of Multi-stage Fractured Horizontal Well Producing in Multilayered Reservoir with Inter-layer Crossflow**; (3) **Semi-analytical Modeling of 2-Dimensional Heterogeneous Reservoir by Using Boundary Element Method**.
What is presented is that a practical fracture-propagation-and-closure process is modeled, its solution in terms of pressure and leak-off rate behaviors in type curve format is generated and the model-based solutions are applied using curve matching strategy in analyzing field mini-fracturing testing data to evaluate the leak-off rate behavior along a fracture, in order to obtain the fracture geometry, attain an instantaneous shut-in pressure (ISIP), extract reservoir flowing capacity ($kh$) and detect closure pressure $P_c$. A reasonable set of parameter solutions can be obtained using the model developed in this study due to a proper modeling of the physical process.

Robust analytical results on transient pressure behavior under constant rate and rate response under constant bottom pressure are presented in type curve format as well as inter-layer crossflow for the multilayered reservoir system. By applying boundary element method (BEM), pressure- and rate-transient behaviours of reservoir with multi-scale heterogeneities bounded by arbitrarily shaped boundaries/surfaces are also presented.
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Chapter 1  Introduction

1. 1  Research Problem Statement

In the comparatively low oil price age, the petroleum industry can only prosper when operating cost is reduced and more oil/gas is produced from invested projects. Good analysis of testing data and of production rate, evaluation of formation with satisfying results, and solid simulation are essential. Not only can successful reservoir characterization give valuable guidance on optimization of primary production operation, it also to a large extent helps determine follow-up EOR projects.

Minifrac treatment is frequently performed prior to main fracture stimulation with intent to extract parameters helping the stimulation design which normally involves a understanding of fracture propagation characteristic (fracture length and width), fluid leak-off behavior, and closure pressure (Pc), considered as the minimum principal stress usually; it has also been becoming a commonly used test technology in evaluating reservoir flowing capacity in tight formation, e.g., kh, as a matter of fact that traditional buildup or falloff test is too expensive or time consuming because formation flow is very low. Thousands of minifrac tests are performed each year globally and diagnosis of testing data is mostly based on traditional G function method (Nolte 1986), which paved the way for petroleum industry to better interpreting the testing data. The linear leak-off assumption, however, applied in G function method brings inaccuracy under some situations, for example when fracture propagation rate is small or when a filter cake is building on the fracture face, as nonlinear fluid leak-off occurs (Settari 1980; Van den Hoek 2002, Barree et al. 2015). On the aspect of field practice, production rate data of
wells are the most valuable and most easily obtained data. Analysis of production rate today, however, still largely relies on traditional empirical type curves (Arps 1944; Fetkovich 1980). In unconventional tight formation, such as Bakken in Williston basin and Eagleford of south Texas, reservoir heterogeneity is routinely encountered. For example, due to complex historical diversity of reservoir geological and depositional environment variation, tight formation usually has large vertically spanned thickness and reservoir is stratified into a number of vertical layers with different permeability and porosity values; in addition to multilayered formation, large multi-scale heterogeneities with arbitrarily shaped boundaries/surfaces are also often seen in tight formation in forms of extremely low permeable regions, natural fractures or so. More complex yet, horizontal well with multi-stage hydraulically fractured transverse fractures along wellbore trajectory recently has become a widely accepted and increasingly implemented technology in unconventional reservoir development, which further complicates the wellbore and reservoir system.

With such difficulties faced now, the challenges are: can fluid flow around fracture be accurately described based on linear leak-off assumption and can such description be applied in diagnosis of minifrac testing data for all types of reservoirs? Can traditional empirical rate type curves or some other readily available rate solutions be effectively applied to analyze rate-transient behavior under such complex well-reservoir system in tight formation? Or to what level can we rely on them? Some other challenging questions are: What can be improved for designing main fracture stimulation and for analyzing post-fracturing well productivity when testing results of minifrac treatment can be evaluated based on a model that can properly describe the nonlinear fluid flow around
fracture for all types of reservoirs? What are the effects on production behaviors caused by reservoir heterogeneity and multistage hydraulic fractures? Can we evaluate fracture and reservoir characteristics from production data (Wattenbarger et al. 1998; Nobakht et al. 2011; Zhao et al. 2016)? The most important question remains to answer: can we build a model with analytical solutions dealing with fracture-propagation-closing-process to analyze minifrac test results? Can we build up solid models with analytical or semi-analytical solutions to help improve analysis of pressure- and rate-transient data and simulate well-reservoir behaviors in heterogeneous reservoirs with solid and robust solutions? The effort of this research was motivated in believing that actual fluid leak-off should be determined by a coupled system (fracture and original reservoir) in physics, instead of pre-defining it to be of certain characteristic. We hypothesize that analytical modeling of fracture propagation with leak-off rate obtained through system coupling can improve minifrac data diagnosis with evaluation results to help design main stimulation and extract key reservoir flowing characteristics, and we also hypothesize that semi-analytical modeling of heterogeneous reservoir produced by well with complicated geometry can provide solid simulation and improve analysis of pressure- and rate-transient responses for complex heterogeneous reservoirs.

1.2 Research Objectives

The general objectives of this research are to pursue solutions to the above-mentioned problems and are detailed as follows:

- To develop an analytical strategy to model fracture-propagation-and-closure process.
- To investigate pressure-transient and leak-off rate behaviours of fracturing well when fracture is propagating or is closing.
To analyze field minifrac testing data (in particular pressure falloff period) based on generated solutions from the analytical model.

To develop 2-layer and 3-layer models to deal with multi-stage hydraulically fractured horizontal well within a heterogeneous layered reservoir system with the consideration of inter-layer fluid crossflow by using source/sink function method.

To investigate effects of reservoir heterogeneity (mobility and storativity ratio of layers), fracture vertical or horizontal penetration (fracture height and length) level based on reservoir thickness respectively in vertical or horizontal orient, and various levels of permeability anisotropy of each layer on reservoir behaviours.

To provide analytical results on transient pressure behavior under constant rate and rate response under constant bottom pressure in type curve format for the complicated 2-layer and 3-layer reservoir system.

To come up with modified rate type curves, integrating well and reservoir geometry and rock/fluid properties of each layer, which help facilitate diagnosing and evaluating field data from multilayered reservoir partially penetrated by multi-stage hydraulic fractures in a systematic manner.

To establish a semi-analytical model to solve pressure- and rate-transient behaviours of 2-dimensional reservoir with multi-scale heterogeneities bounded by arbitrarily shaped boundaries/surfaces using boundary element method.

To provide results of various types of heterogeneous reservoirs using boundary element method and results are presented in type curve format.

1.3 Research Innovation

A number of key research innovations have been achieved in this PhD study:
• Analytical solutions in terms of pressure and fluid leak-off behaviours under various scenarios when fracture is propagating or closing are achieved.

• Field minifrac tests in high-to-low permeable formation are analyzed by using generated model-based solutions along with curve matching strategy to evaluate leak-off rate behavior along fracture, to obtain fracture geometry, to attain instantaneous shut-in pressure (ISIP), to extract reservoir flowing capacity (kh) and to detect closure pressure (Pc).

• The heart of the innovations of modeling and analyzing mini-fracturing, compared to conventional G function method, is that fluid leak-off along fracture is obtained through coupling of the fracture-reservoir system in physics, instead of characterizing it to be inversely proportional to square root time or to be of some other leak-off behaviors, while those aforementioned objective parameters diagnosed from minifrac testing data are all greatly affected by fluid leak-off behavior

• A physical-based coupling strategy of multilayered reservoir partially penetrated by fractured horizontal well is achieved.

• Modified dimensionless rate and modified dimensionless time integrating reservoir and well geometries as well as rock/fluid properties of each layer are originally proposed for multilayered system in this research.

• Based on the proposed modified rate type curves, a field case of a 2-layer reservoir of Viewfield Bakken pool is studied with its monthly rate data, and average hydraulic fracture length and permeability of each layer are evaluated.

• Behaviours of reservoir with multi-scale heterogeneity bounded by arbitrarily shaped boundaries/surfaces are solved by applying boundary element method.
• Computation burden is much reduced and computation speed is fast for boundary element method used in this research.

• Solutions to examples of challenging heterogeneous reservoir geometries, including fully 2D compartmentalized reservoir, naturally fractured reservoir, and enhanced-fracture-model (Stavlgorova and Mattar 2013, Zhao 2013), that routinely confront reservoir engineers are computed.

1.4 Dissertation Outline

This dissertation consists of 5 chapters. Chapter 1 is an introduction of this PhD research topics along with the critical thinking process and major objectives. Chapter 2 deals with the modeling of fracture propagation and closure and applies the generated solutions in diagnosis of minifrac tests in various types of formation. Chapter 3 deals with the modeling process for a hydraulically multi-stage fractured horizontal well producing vertically stratified multilayer reservoir and provides the calculated results. Chapter 4 solves pressure- and rate-transient behaviours of reservoir with multi-scale heterogeneity bounded by arbitrarily shaped boundaries/surfaces using boundary element method (BEM). Chapter 5 summarizes the major findings of this PhD research and outlines recommendations for further operational development.
Chapter 2 Analytical Modeling of Fracture Propagation on Evaluation of Transient Pressure Behavior during Injection and after Shut-in: Minifrac Test Analysis by Model-based Type Curves

This chapter presents a strategy of analytically modeling pressure-transient and fluid leak-off behaviors of hydraulic fracture undergoing propagation during injection and closing during pressure falloff period (after well shut-in). Although a number of works have been studied to model fracture-propagation-and-closure process and to analyze pressure data (in particular falloff period for minifrac analysis), the modeling strategy and description of fluid leak-off around fracture proposed in this chapter provide new solid theoretical results and an innovative curve matching strategy for a model-based analysis of minifrac testing data is presented. The objective of this chapter is to analytically model fracture-propagation-and-closure process, generate solutions in terms of pressure and leak-off rate behaviors in type curve format, apply the model-based solutions along with curve matching strategy in analyzing of field minifrac tests to evaluate leak-off rate behavior along fracture, to obtain fracture geometry, to attain instantaneous shut-in pressure (ISIP), to extract reservoir flowing capacity ($kh$) and to detect closure pressure ($P_c$).
2.1 Introduction and literature related

Minifrac test is an injection fall-off test as a fracture will be created without using any proppant and is frequently performed prior to main fracture stimulation with an intention to extract parameters to help the subsequent stimulation design; it has also been becoming a commonly used test technology in evaluating reservoir flowing capacity, e.g., $kh$, for the reason that traditional buildup or falloff test is too expensive or time consuming as formation flow process is extremely slow (Cramer and Nguyen 2013). Fracturing pressure falloff data analysis was initially explored by Nolte (1986) with the invention of widely known G function method, which is based on the linear leak-off assumption that may yield appreciable inaccuracy, for example, when a filter cake is building on the fracture face or when fracture length is short (Van den Hoek 2002; Barree et al. 2015). Mayerhofer et al. (1995, 1997) modeled linear fluid leak-off under unsteady-state flow with a consideration of a rate- and time-dependent skin effect at the fracture face. Nolte et al. (1997) provided a technique to analyze pressure after fracture has closed and mainly focused on the pseudo-linear flow period after closure. During this period the pressure falloff response can give information about fracture geometry. Barree et al. (2009) presented a method to analyze G function, its derivative, and its relationship to some other diagnostic techniques to interpret fracture closure time and closure pressure as well as reservoir properties from minifrac testing data. Marongiu-Porcu et al. (2011) built a comprehensive model for an analysis of fracture falloff data that allows the quantification of parameters on both fracture and reservoir. Liu and Ehlig-Economides (2017) presented a pressure model aiming at matching fracture injection testing pressure
from both injection and falloff period with generated solution from the model and desired parameters can be obtained.

Many studies have been performed to improve modeling of pressure response of fractured well and fluid flow along fracture planes by incorporating nonlinear fluid flow within reservoir. Gringarten and Ramey (1974) applied line source solution to describe transient 2-dimensional flow in reservoir resulted from a static fractured well production/injection. Settari (1980) presented 2-dimensional fluid leak-off behavior when fracture is propagating under constant injection. Koning (1988) described pressure behavior of propagating fracture by using Gringarten and Ramey (1974) formula (line source solution), which was derived from transient elliptical flow caused by static fracture, with substitution of certain dimensionless parameters. Larsen and Bratvold (1994) built up a model to cover elliptical flow and to generate pressure solutions during fracture propagation period and during falloff period but the computation method appears to fail in linear-flow period, which causes problems to evaluate well’s behaviors when fracturing is performed in low-permeability formation as long fracture is likely to be created, and computation cost from the proposed method is high. Valko and Economides (1997) developed a radial leak-off model for high-permeability fracturing. Van den Hoek (2002) presented a numerical solution to the fully transient fluid-flow equation for a propagating hydraulic fracture. Craig and Blasingame (2005, 2006) provided a modeling methodology to simulate pressure behavior of fractured well before and after closure assuming uniform flux along fracture, and type curves were generated to analyze diagnostic fractured injection test.
In this study, a target fracture is treated as a line source/sink. Pressure change caused by an instantaneous moment of fracture conducting/emitting fluid at any point within reservoir is described by integrating point source function along fracture extent. For continuously performing static fracture, pressure solution can be obtained directly by integrating the instantaneous fracture response along time. For propagating fracture, however, its length is growing from moment to moment due to propagation so that fracture propagating velocity, a continuous function of time, must be integrated into source function that yields pressure behavior. This work is successfully accomplished in an analytical manner with approximating flux (accounting for leak-off) distribution along fracture extent to be stepwise function by dividing fracture length into a number of segments, each with uniform flux within a segment; each segment is exposed to fluid flowing at different time (based on fracture propagating velocity), and a pressure change caused by all segments is summed up by applying superposition principle. In addition to pressure description, this study has taken into account the dynamic “fracture storage” effect, which accounts for partial injecting fluid stored in fracture when fracture is propagating and accounts for fluid stored in fracture being pushed into reservoir after shut-in when fracture is closing.

Since the analysis of minifrac test data is based on type curve matching method in this study, the main and desired matching time interval is when measured pressure becomes relatively steady during falloff period; once matched, the type curve pressure at the end of injection is interpreted as ISIP (instantaneous shut-in pressure) which represents the pressure at fracture face near wellbore, instead of using the pressure at bottom hole where additional pressure is always imposed during flowing process because of friction and
wellbore tortuosity (Barree et al. 2015). In general, evaluation of reservoir flowing capacity \((kh)\) requires long shut-in time as a formation flow reaches pseudo-radial flow state based on impulse-fracture injection test (Gu et al. 1993) while in this study once curve match is achieved permeability can be determined directly. The specific strategy of analyzing minifrac testing data based on the solutions from the analytical model presented in this study is introduced in the first filed case step by step and the evaluating results of this case are presented in comparison with other results from traditional linear leak-off model, from the radial leak-off model of Valko and Economides (1997) and from the transient elliptical flow model of van den Hoek (2002).

2.2 Modeling Methodology

First of all, a number of assumptions are made as follows:

1. Reservoir is homogeneous and fluid is slightly compressible.
2. The injected fluid has the same properties as the reservoir fluid (multiphase flow effect ignored).
3. Fracturing is performed in infinite-acting reservoir and fracture fully penetrates reservoir vertical thickness.
4. Fracture propagation is symmetrical: fracture propagation velocity is the same for both wings.
5. Fracture width is very small, and pressure behavior of fracture is described by line source/sink (without further considering the effect of fracture width in pressure model).
6. Fracture propagation arrests and starts to close at shut-in, and during closing period fracture length as well as height remains constant while fracture volume diminishes.
as it is closing in width. Further propagation or recession of fracture after shut-in is beyond the scope of this study.

2. 2.1 Modeling of propagating mini-fracture

Modeling of mini-fracture propagation requires the description of fracture geometry at the very first. Fig 2.1 illustrates the general schematic of the propagating fracture with account of notation and coordinate system. Fundamentally, the propagation of mini-fracture can be characterized under three states, namely: the initial state, named as State A, the propagating state, named as State B and the fracture arrest state, named as State C. During State A well is connected to only original fracture length; during State B fracture is propagating at a velocity as a continuous function of time; during State C fracture propagation has arrested that fracture length stays unchanged.
Fig 2.1 Schematic account of fracture propagation (both wings) and of notations/coordinates
Defining the beginning of injection at well as the “mathematical” zero time, fracture half length is mathematically described by Eq. (2.1) for the 3 states as

\[
L_f(t) = \begin{cases} 
L_o & t \leq t_s \\
L_o + \int_{t_s}^{t} V(t - t_s) dt & t_s < t \leq t_e \\
L_f(t_e) & t > t_e 
\end{cases}
\]  

(2.1)

where \(L_f\) denotes the fracture half length at time \(t\), \(L_o\) denotes the original fracture half length, \(V\) denotes fracture propagating velocity, a continuous function of time, \(t_s\) represents the time before fracture initiation and \(t_e\) represents the time at which fracture propagation stops.

In this work, the behavior of the fracturing well-reservoir system will be evaluated at a number of time points of interests (the total number of time points represented by a number \(N\)). The time series is described as

\[
t_1 < t_2 < t_3, \cdots < t_{i-1} < t_i < t_{i+1} \cdots < t_N .
\]

Fig 2.2 illustrates the fracture length when evaluating well-reservoir system at time \(t_i\) for the 3 states. For hydraulic fracturing or mini-fracturing treatment, well geometry stays in State A without any propagation from the beginning of injection before fracture is created. During State A, well exhibits the behavior as a traditional fractured well with fracture spanning as far as \(2L_o\). Once fracture propagation is initiated, fracture geometry enters State B in which fracture length is becoming longer and longer as propagation goes on at a propagating speed. In fracturing pressure profile, it is easy to notice that pressure often abruptly declines with a sharp angle as fracture initiates, following pressure rise during State A, which marks the entry to State B because a high-conductive path is created.
(fracture) and pressure starts to dissipate quickly. Fracture propagation will stop at time \( t_e \) due to well shut-in, no more increment in length, which marks the beginning of State C. For minifrac test, pressure falloff response in State C is the main data to analyze and evaluate. The modeling methodology for the pressure responses causing by the fracture under different states are presented in the following sections.

2.2.1.1 Pressure response for State A

This state is before fracture propagation when \( t_i \leq t_s \). It is clear that pressure response caused by fractured well with fixed fracture length/extent can be directly described by line source solution (Gringarten and Ramey 1974), which is obtained through integration of point source solution along source length expressed by Eq. (2.2). Carslaw and Jaeger (1959) have used point source solution in solving heat conduction problem and Gringarten and Ramey (1973) firstly provided instantaneous source functions to describe transient pressure behavior within reservoir. The original fracture length \( 2L_0 \) can be evenly subdivided into \( M \) segments, and flux is assumed to be uniform along each of them. Pressure response within reservoir at \( t_i \), in State A, caused by original fracture is mathematically described as

\[
\Delta p(x, y, t_i) = \Delta p_o(x, y, t_i) = \frac{1}{\phi \mu c} \int_{t_i} f \sum_{j=1}^{M} \bar{q}_{o,j}(\tau) d\tau \int_{x_{o,j-1}}^{x_{o,j}} \exp \left[ -\frac{(x-x')^2+(y-y')^2}{4\eta(t_i-\tau)} \right] dx', \tag{2.2}
\]

where, \( \Delta p_o \) denotes pressure change caused by original fracture. Fig 2.3 gives the graphic description of original fracture extent discretization.
Fig 2.2 Three states of fracture propagation defined in this study at an evaluation time $t_i$
Fig 2.3 Schematic of original fracture length, $2L_0$, discretization during State A at time $t_i$ ($t_i \leq t_3$) with fracture segment notation of $-L_0 = x_{o,0} < x_{o,1} < x_{o,2} < \cdots < x_{o,M} = L_0$.

The spatial integration in Eq. (2.2) can be analytically evaluated, which is

\[
\int_{x_1}^{x_2} \frac{1}{4\pi\eta(t_i - \tau)} \exp \left[ -\frac{(x - x')^2 + (y)^2}{4\eta(t_i - \tau)} \right] dx'
\]

\[= \frac{1}{4\sqrt{\pi\eta(t_i - \tau)}} \left( \text{erf} \left[ \frac{x_2 - x}{2\sqrt{\eta(t_i - \tau)}} \right] - \text{erf} \left[ \frac{x_1 - x}{2\sqrt{\eta(t_i - \tau)}} \right] \right) \exp \left[ \frac{y^2}{4\eta(t_i - \tau)} \right]. \quad (2.3)
\]

With Eq. (2.3), pressure change caused by original fracture length becomes

\[
\Delta p_0(x, y, t_i) = \frac{1}{\phi h c_i} \sum_{j=1}^{M} \int_{0}^{t_i} \frac{q_{o,j}(t_i) dt}{x_{o,j} - x_{o,j-1}} \cdot \left( \text{erf} \left[ \frac{x_{o,j} - x}{2\sqrt{\eta(t_i - \tau)}} \right] - \text{erf} \left[ \frac{x_{o,j-1} - x}{2\sqrt{\eta(t_i - \tau)}} \right] \right) \exp \left[ \frac{y^2}{4\eta(t_i - \tau)} \right]. \quad (2.4)
\]

where $\bar{q}_{o,j}$ is flux term along the $j$-th segment, a continuous function of time; if the rate from that segment is denoted by $q_{o,j}$, flux can be expressed as

\[
\bar{q}_{o,j}(\tau) = \frac{q_{o,j}(\tau)}{x_{o,j} - x_{o,j-1}}. \quad (2.5)
\]

By replacing flux term in Eq. (2.4) with Eq. (2.5), Eq. (2.4) can be written as

\[
\Delta p_0(x, y, t_i) = \frac{1}{\phi h c_i} \sum_{j=1}^{M} \int_{0}^{t_i} \frac{q_{o,j}(t_i) dt}{x_{o,j} - x_{o,j-1}} \frac{1}{4\sqrt{\pi\eta(t_i - \tau)}} \cdot \left( \text{erf} \left[ \frac{x_{o,j} - x}{2\sqrt{\eta(t_i - \tau)}} \right] - \text{erf} \left[ \frac{x_{o,j-1} - x}{2\sqrt{\eta(t_i - \tau)}} \right] \right) \exp \left[ \frac{y^2}{4\eta(t_i - \tau)} \right]. \quad (2.6)
\]
In this study, solutions are pursued in Laplace domain and inverted back to real-time domain numerically. Laplace transform of Eq. (2.6) with respect to time is taken, based on convolution theorem, and pressure response within reservoir for State A in Laplace domain becomes

\[ L[\Delta p(x, y, t)] = L[\Delta p_o(x, y, t)] = \int_0^{t_\infty} \Delta p_o(x, y, t) \exp(-st) \, dt \]

\[ = \frac{1}{\phi h c_i} \sum_{j=1}^M L[q_{o,j}(t)] L[F_o(x_{o,j-1}, x_{o,j}, x, y, t)] , \quad (2.7) \]

where “L[]” denotes Laplace transform operator, \( t_\infty \) denotes time at infinity, \( T \) denotes the integration variable, \( s \) denotes Laplace variable, and \( F_o \) is

\[ F_o(x_{o,j-1}, x_{o,j}, x, y, t) = \frac{1}{x_{o,j} - x_{o,j-1}} \cdot \frac{1}{4\sqrt{\pi \eta t}} \cdot \left\{ \text{erf} \left[ \frac{x_{o,j} - x}{2\sqrt{\eta t}} \right] - \text{erf} \left[ \frac{x_{o,j-1} - x}{2\sqrt{\eta t}} \right] \right\} \exp \left[ \frac{y^2}{4\eta t} \right] . \quad (2.8) \]

### 2.2.1.2 Conceptual notation system for dynamic fracture growing and closing processes

Because the accuracy of simulating a mini-fracturing process is mainly affected by the semi-analytical treatment in the following two aspects: One is the discretization of fracture total length into smaller segments with respect to time that within this aspect a unit fracture segment length, \( L_{unit} \), is chosen to facilitate the computational process; and the other is the fracture growing status at any evaluation time \( t_i \), including \( t_i \), within either fluid injection or shut-in periods. In order to describe our semi-analytical modeling process clearly, the modeling methodology is categorized for the following two modes, fracture propagation mode during fracturing fluid injection and fracture closure mode after pump shut-in.
For general dynamic fracturing description, a scheme for growing mode with a unit length scale, $L_{\text{unit}}$, during a corresponding time interval from $t_{k-1}$ to $t_k$ is designed by using the notation below. Special attention should be paid to the definition of $t_i$ and $t_k$. At any evaluation time point $t_i$, or even $t_N$, the corresponding total fracture segment number (one side wing), $N_i(t_i)$, is

$$
N_i(t_i) = \begin{cases} 
\text{Int} \left[ \frac{L_f(t_i)}{L_{\text{unit}}} \right] + 1, & \text{for propagation mode, where } t_s < t_i \leq t_e \\
\text{Int} \left[ \frac{L_f(t_k)}{L_{\text{unit}}} \right] + 1, & \text{for shut-in mode, where } t_i \geq t_e 
\end{cases}
$$

Note that the sign of “$\geq$” in the second expression makes sense since fracture at time $t_e$ is fixed and stay stationary.

The following notation system for the dynamic fracture growing system is listed:

$t_s$: the starting time for fracture propagation

$t_e$: the ending time for fracture propagation

$t_i$: evaluation time point, $t_0 < t_1 < t_2 < \ldots < t_{i-1} < t_i < t_{i+1} < \ldots < t_N$. This is a real testing time point, or a real time point of interest

$t_k$: time intervals based on unit length, $L_{\text{unit}}$, defined fracture growing scheme at $t_i$, $t_s < t_1 < t_2 < \ldots < t_{k-1} < t_k < t_{k+1} < \ldots \leq t_N(t_i)$

$x_{f,k}$: fracture intervals based on unit length. Usually, $x_{f,k-1} < x_{f,k-1} = L_{\text{unit}}$ if $k < N_i(t_i)$, for all $k < N_i$.

The defined fracture segments at any time $t_i$, $x_{f,0} < x_{f,1} < x_{f,2} < \ldots x_{f,k} < x_{f,k+1} < \ldots < x_{f,N_i}$

Note that $N_i(t_i) = \text{Int} \left[ \frac{L_f(t_i)}{L_{\text{unit}}} \right] + 1$. 

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Therefore, following the definition of these terms, a general consideration is whether a fracture tip reaches the RHS (right hand side) of an interval at any $t_i$, i.e., there are only two fundamental fracturing propagation statuses within $t_k$ scheme: one is that a fracture tip locates within an interval of $L_{\text{unit}}$ length, another is that a fracture tip happens to land on the RHS of the interval.

For the evaluation time point $t_i$, there are also two statuses as well: one is that $t_i$ is equal to the fracturing tipping time $t_{NI}(t_i)$ during propagation mode, another is that a $t_i$ for pump shut-in period is beyond the final injection time where fracture propagation stops, i.e., $t_i$ in this status is larger than any $t_k$ at $x_k$ which is fracture discretization scheme related, where $k = 1, 2, 3, \ldots, k - 1, k, k + 1, \ldots, N(t_i)$.

2.2.1.3 Pressure response for State B

In this state fracture is in propagation mode when $t_s < t_i \leq t_e$. Pressure change caused by the propagating fracture is also accounted for by using line source solution, but, with increasing source length along with time. It is also assumed that flux along fracture is uniform per unit length that the propagated fracture length at evaluation time $t_i$ is divided into a number of segments with unit length, $L_{\text{unit}}$, described in previous section. Fig 2.4 illustrates the discretization of the propagated fracture of the right hand side wing at $t_i$.

![Fig 2.4](image_url) Schematic of propagated fracture discretization during State B (showing right hand side wing only) at time $t$ ($t_s < t \leq t_e$) with fracture segment notation of
Illustrated in Fig 2.4, the rate from the \( k \)-th segment is denoted as \( \{q_{f,k}^R\}_{k=1}^{N_i} \), where the superscript “R” stands for the right hand side wing.

It is noticeable that the flow rate of each segment illustrated in Fig 2.4 is zero before its initiation time. The reason is that each segment starts to expose and conduct fluid flow at different time. The \( k \)-th segment starts conducting fluid flow at \( t_{k-1} \) as fracture tip arrives at its heel at \( t_{k-1} \). The flow rate of the \( k \)-th segment in time profile is illustrated in Fig 2.5 (the curve does not represent real flow rate trend, simply for demonstrating purpose).

![Figure 2.5](image.png)

Fig 2.5 Account of flow rate for \( k \)-th segment in time profile

Note that in fracture propagation mode, the evaluation time \( t_i \) is right corresponding to the fracture tip location and time. This means, \( t_i = t_{N_i} \) at fracture tip of \( x_{f,N_i} \). This is why the case of evaluation time \( t_i \) right at fracture tipping time is being considered.

**An evaluation time \( t_i \) is equal to a fracture tipping time:** There is fundamentally only one scenario for fracture propagation if the fracture tipping time and location are always noted as \( t^{*e}(t^{*e} = t_{N_i} = t_i \text{ in propagation mode}) \) and \( x^{*e}(x^{*e} = x_{f,N_i}(t_i) \text{ in propagation mode}) \), and the last segment starting time and location are always noted as \( t^{*0}(t^{*0} = t_{N_i-1} \text{ in propagation mode}) \) and \( x^{*0}(x^{*0} = x_{f,N_i-1}(t_{N_i-1}) \text{ in propagation mode}) \).
propagation mode) and \( x^{0} = x^{0} = x_{f,Ni-1} \). The pressure change caused by the last segment can be formulated as

\[
\Delta p_{\text{seg, last}}(x_{f,Ni}(t_{i}), x_{f,Ni-1}, x, y, t_{i}) = \\
\int_{t=0}^{t_{i}=x_{f,Ni-1}} \frac{\beta}{L_{f}(t-\tau)} \cdot \frac{y^{2}}{4\sqrt{\pi \eta (t-\tau)}} \cdot \left[ \text{erf} \left( \frac{x_{f,Ni}(t)-x}{2\sqrt{\eta (t-\tau)}} \right) - \text{erf} \left( \frac{x_{f,Ni-1}-x}{2\sqrt{\eta (t-\tau)}} \right) \right] dt ,
\]

(2.10)

As a matter of fact, the current notation system allows the following expression since the coordinate system specifies the \((x,y)\) as the origin is at \((0,0)\). This means:

\[
x_{f}(t) = L_{f}(t) .
\]

(2.11)

Then, Eq. (2.10) can be expressed as

\[
\Delta p_{\text{seg, last}}(L_{f}(t_{i}), x_{f,Ni-1}, x, y, t_{i}) = \\
\int_{t=0}^{t_{i}=x_{f,Ni-1}} \frac{\beta}{L_{f}(t-\tau)} \cdot \frac{1}{4\sqrt{\pi \eta (t-\tau)}} \cdot \left[ \text{erf} \left( \frac{L_{f}(t)-x}{2\sqrt{\eta (t-\tau)}} \right) - \text{erf} \left( \frac{x_{f,Ni-1}-x}{2\sqrt{\eta (t-\tau)}} \right) \right] \frac{y^{2}}{4\eta (t-\tau)} dt ,
\]

(2.12)

Eq. (2.12) has a general integration form of

\[
\Delta p_{\text{seg, last}}(L_{f}(t_{i}), x_{f,Ni-1}, x, y, t) = \int_{t=0}^{t_{i}=x_{f,Ni-1}} \beta \cdot K_{\text{iso}}(t-\tau) .
\]

(2.13)

Eq. (2.13) applies to both the cases of (A) fracture tip locates between \(x_{Ni-1}\) and \(x_{Ni}\) scheme and (B) fracture tip just reaches \(x_{Ni}\) in completing a unit length. The only difference between case A & B is that this last fracture segment length is different. The segment length in case A is basically less than \(L_{\text{uni}}\) while in case B it equates \(L_{\text{uni}}\). The equation details for case B can be show below:

\[
\Delta p_{\text{seg, last}}(L_{f}(t_{i}), x_{f,Ni-1}, x, y, t) = \Delta p_{\text{seg, last}}(x_{f,Ni}, x_{f,Ni-1}, x, y, t) ,
\]
An evaluation time \( t_i \) is larger than a fracture tipping time: This is another core of this dynamic fracturing modeling process and the solution to this question actually help complete the whole mini-fracture modeling methodology, because (1) for fracture propagation modeling, any fracture segment before the specially aligned last \( N_i(t_i) \)-th segment under an evaluation time \( t_i \) actually has been formed before \( t_i \), (2) for fracture closure modeling, the entire fracture length, with all fracture segments included, has been generated. The nature of these two physical processes are the same in light of the fact that the evaluation time always exists after the generation of a fracture segment under discussion.

The time duration for fracture growing event within the \( k \)-th segment (with \( L_{uni} \)) is: \( (t_{k-1} \sim t_k) \) with respect to the coordinates of \((x_{k-1}, x_k)\) and \( t_i > t_k \), where \( k=1, 2, 3, ..., k-1, k, k+1, ..., N_i, f(t_i) \). The solution expression is:

\[
\Delta p_{k-th \ segment}(x_f(\tau), x_{f,k}, x_{f,k-1}, x, y, t = t_i > t_k) = \\
\int_{t_{k-1}}^{t_k} q^R_{f,k}(\tau) \cdot \frac{1}{x_f(\tau) - x_{f,k-1}} \cdot \frac{1}{4\sqrt{\pi} \eta(t-\tau)} \cdot \left[ \text{erf} \left( \frac{x_f(\tau) - x}{2\sqrt{\eta(t-\tau)}} \right) - \text{erf} \left( \frac{x_{f,k-1} - x}{2\sqrt{\eta(t-\tau)}} \right) \right] \exp \left( \frac{y^2}{4\eta(t-\tau)} \right) d\tau. 
\]

Due to growing mode from from \( t_{k-1} \) to \( t_k \), Eq. (2.15) can be written as

\[
\Delta p_{k-th \ segment}(x_f(\tau), x_{f,k}, x_{f,k-1}, x, y, t = t_i > t_k) = \\
\int_{t_{k-1}}^{t_k} q^R_{f,k}(\tau) \cdot \frac{\exp \left( y^2/[4\eta(t-\tau)] \right)}{x_f(\tau) - x_{f,N_i-1}} \cdot \frac{1}{4\sqrt{\pi} \eta(t-\tau)} \cdot \left[ \text{erf} \left( \frac{x_f(\tau) - x}{2\sqrt{\eta(t-\tau)}} \right) - \text{erf} \left( \frac{x_{f,k-1} - x}{2\sqrt{\eta(t-\tau)}} \right) \right] d\tau + \\
\int_{t_k}^{t_i} q^R_{f,k}(\tau) \cdot \frac{\exp \left( y^2/[4\eta(t-\tau)] \right)}{x_{f,k} - x_{f,k-1}} \cdot \frac{1}{4\sqrt{\pi} \eta(t-\tau)} \cdot \left[ \text{erf} \left( \frac{x_f(\tau) - x}{2\sqrt{\eta(t-\tau)}} \right) - \text{erf} \left( \frac{x_{f,k-1} - x}{2\sqrt{\eta(t-\tau)}} \right) \right] d\tau.
\]
Eq. (2.16) presents the solution for a dynamically growing fracture segment and how the induced pressure response is casting into future time $t_i$, where the overall actual pressure needs to be computed through applying superposition principle. Note that here $t_i > t_{N_i - 1}(t_i)$.

Physically, **IPS1** accounts for the contribution of growing mode with dynamic flux during $t_{k-1} > \tau > t_k$. **IPS2** accounts for the contribution of continuous injecting fluid flux along a fixed fracture segment from $x_{k-1}$ to $x_k$ during $t_{k-1} > \tau > t_k$. In details,

\[
IPS_1 = \int_{t_{k-1}}^{t_k} q_{f,k}^R(\tau) \cdot \frac{\exp[y^2/(4\eta(t-\tau))]}{x_f(\tau) - x_{f,Ni-1}} \cdot \frac{1}{4\sqrt{\pi\eta(t-\tau)}} \left[ \text{erf}\left(\frac{x_f(\tau) - x}{2\sqrt{\eta(t-\tau)}}\right) - \text{erf}\left(\frac{x_f,k-1 - x}{2\sqrt{\eta(t-\tau)}}\right) \right] d\tau
\]

\[
= \int_{t_{k-1}}^{t_k} q_{f,k}^R(\tau) \cdot K_{IPS1}(t-\tau) . \tag{2.17}
\]

\[
IPS_2 = \int_{t_k}^{t_i} q_{f,k}^R(\tau) \cdot \frac{\exp[y^2/(4\eta(t-\tau))]}{x_{f,k} - x_{f,k-1}} \cdot \frac{1}{4\sqrt{\pi\eta(t-\tau)}} \left[ \text{erf}\left(\frac{x_{f,k} - x}{2\sqrt{\eta(t-\tau)}}\right) - \text{erf}\left(\frac{x_{f,k-1} - x}{2\sqrt{\eta(t-\tau)}}\right) \right] d\tau
\]

\[
= \int_{t_k}^{t_i} q_{f,k}^R(\tau) \cdot K_{IPS2}(t-\tau) . \tag{2.18}
\]

The final solution of pressure change caused by propagated fracture of right side wing for fracture propagation mode during fracturing fluid injection is:

\[
\Delta p_f^R(x, y, t_i) = \sum_{k=1}^{N_i(t_i) - 1} \Delta p_{k-th\ segment}(x_k, x_{k-1}, x, y, t_i) + \Delta p_{seg\ last}(L_f(t_i), x_{f,Ni-1}, x, y, t_i) , \tag{2.19}
\]

where the superscript “R” denotes the right hand side wing of fracture.

The solution expression in Laplace domain is:
Since it is assumed that fracture propagation is symmetrical, the left side wing of fracture is simply an image of the right side wing and its solution takes similar form of Eq. (2.19). The final solution of pressure change caused by the entire fracture is the sum of that from original fracture extent and propagated fracture extent of both right and left side wings, and in Laplace domain it is expressed as

\[ L[\Delta p(x, y, t)] = L[\Delta p^R_f(x, y, t)] + L[\Delta p^L_f(x, y, t)] + L[\Delta p_o(x, y, t)] . \]  

(2.21)

### 2.2.1.4 Pressure response for State C

In this state the fracture propagation is arrested when \( t_i > t_e \). Fracture’s tip at any time after \( t_e \) stays at \( L_f(t_e) = L_f(t_e) \) (no more fracture length increment). For fracture discretization, the number of segments, \( N_i \), is fixed—no more segments required. The discretization of propagated fracture length of the right hand side wing at \( t_i \) is illustrated in the Fig 2.6.

**Fig 2.6** Schematic of propagated fracture discretization during **State C** (showing right hand side wing only) at time \( t_i \) (\( t_i > t_e \)) with fracture segment notation of

\[ L_o = x_{f,0} < x_{f,1} < x_{f,2} < \cdots < x_{f,k} < \cdots < x_{f,N_i-1} < x_{f,N_i} = L_f(t_e) . \]
The methodology presented above has helped build the semi-analytical system to generate the solution for pressure change profiles during fracture closure mode after fracturing fluid pumping injection is stopped and a shut-in duration initiates (Note that here for shut-in duration \( t_i > t_{NI} \)). By applying the similar strategy to develop Eq. (2.16), one can write the solution for the \( k \)-th fracture segment within shut-in duration with the same expression as Eq. (2.16). Note that the generation of this \( k \)-th segment was completed physically during propagation duration. This is actually the 2\(^{nd}\) case where the evaluation time \( t_i \) is beyond the final tipping time, \( t_{NI} \), and the entire fracture length, with all fracture segments included, has been already generated. Therefore, the solution of pressure change caused by the right side wing fracture during shut-in period is:

\[
\Delta p_i^R(x,y,t_i) = \\
\sum_{k=1}^{N_i(t_i)} \Delta P_{k-th \ segment}(x_k, x_{k-1}, x, y, t_i) = \int_{t_{k-1}}^{t_k} q_{f,k}^R(\tau) \cdot K_{IPS1}(t - \tau) + \int_{t_k}^{t_i} q_{f,k}^R(\tau) \cdot K_{IPS2}(t - \tau). \quad (2.22)
\]

Its expression in Laplace domain is:

\[
L[\Delta p_i^R(x,y,t)] = \int_0^\infty \Delta p_i^R(x,y,t) \exp(-st) \, dt. \quad (2.23)
\]

Similar to State B, the final solution of pressure change caused by the entire fracture in Laplace domain is expressed as

\[
L[\Delta p(x,y,t)] = L[\Delta p_0^R(x,y,t)] + L[\Delta p_f^R(x,y,t)] + L[\Delta p_o(x,y,t)]. \quad (2.24)
\]

In this work, Laplace transforms of function \( F_o, K_{soe}, K_{IPS1} \) and \( K_{IPS2} \) are performed numerically. Zhao and Thompson (2002) devised a fast and accurate method of
numerical Laplace transformation, and the evaluation of the functions in Laplace domain is obtained by using Zhao and Thompson’s method.

2.2.2 Flow rate/Leak-off rate along fracture for State A, B and C

2.2.2.1 Flow rate/Leak-off rate in State A

This state is before fracture propagation when \( t_i \leq t_s \). When well is connected to only original fracture, part of injecting fluid stores in wellbore through compression and the rest flows/leaks into reservoir along the original fracture extent. The material balance can be described as

\[
q_{\text{inj,A}} \rho_w - q_l(t) \rho_w = V_{wb} \frac{d\rho_w}{dt} .
\]  

In Eq. (2.25), \( q_l \) denotes the leak-off rate, \( \rho_w \) denotes the density of injecting fluid, \( V_{wb} \) denotes the wellbore volume, and \( q_{\text{inj,A}} \) denotes the injection rate of the fluid injected.

According to the chain rule, one can write

\[
\frac{d\rho_w}{dt} = \frac{d\rho_w}{dp} \frac{dp}{dt} .
\]  

Injecting fluid compressibility is defined as

\[
c_w = \frac{1}{\rho_w} \frac{d\rho_w}{dp} ,
\]  

with Eq. (2.26) and Eq. (2.27), leak-off rate from material balance described in Eq. (2.25) can be expressed as

\[
q_l(t) = q_{\text{inj,A}} - V_{wb} c_w \frac{dp_w}{dt} .
\]
In Eq. (2.28), \( V_{wb} c_w \frac{dp_{wf}}{dt} \) accounts for the wellbore storage effect defined traditionally.

In this study, the wellbore storage effect is ignored, due to the fact that for mini-fracturing test, the pressure measurement is usually conducted downhole nearby the target interval to be mini-fractured, thus the wellbore storage effect is greatly minimized. Note that if wellbore storage effect needs to be considered, it can be included in this model directly. The other reason is that water is usually selected as injecting fluid and its compressibility is very small. Also, the fracture half-length in State A, \( L_o \), is a constant. Hence with an ignorance of wellbore storage in this state, leak-off rate can be directly expressed as

\[
q_l(t) = q_{inj,A} \cdot \tag{2.29}
\]

In this model proposed, the sum of flow rates of all the segments discretized from fracture extent account for fluid leak-off rate along fracture. Therefore, in State A, the sum of flow rates of the M segments constituting the original fracture length equals injection rate at well.

\[
\sum_{j=1}^{M} q_{o,j}(t) = q_l(t) = q_{inj,A} \cdot \tag{2.30}
\]

In Laplace domain, Eq. (2.30) becomes

\[
\sum_{j=1}^{M} L[q_{o,j}(t)] = L[q_l(t)] = \frac{q_{inj,A}}{s} \cdot \tag{2.31}
\]

### 2.2.2.2 Flow rate/Leak-off rate in State B
Fracture is propagating in this state when \( t_s < t_l \leq t_e \). With ignorance of wellbore storage effect, from material balance point of view, injecting rate at well equals the sum of flow rate/leak-off rate along fracture and fracture volume growing rate, which is caused by “fracture storage effect”. In State B, the material balance can be described as

\[
q_{\text{inj}, \omega} - q_l(t) \rho_w = \frac{d(V_f \rho_w)}{dt},
\tag{2.32}
\]

where \( V_f \) denotes the fracture volume and it is

\[
V_f(t) = 2h L_f(t) w_f. \tag{2.33}
\]

According to Nolte (1986), average fracture width is related to fracture length and net pressure \( (p_{w_f} - p_c) \) as:

\[
w_f = \frac{\pi L_f(t)}{E'} \left[ p_{w_f}(t) - p_c \right], \tag{2.34}
\]

\( p_{w_f} \) denotes well bottom pressure or pressure at near-well fracture face and \( p_c \) is the fracture closure pressure and \( E' \) is the plane-strain elastic module. It is worthwhile to mention that other types of fracture models, such as the commonly used KGD (Khristianovitch-Geertsma-de Klerk) and PKN (Perkins-Kern), can be properly selected and comparatively studied as well.

Plugging Eq. (2.33) and Eq. (2.34) in Eq. (2.32) and assuming a constant density of injecting fluid, the fluid leak-off rate described in Eq. (2.32) can be written as

\[
q_l(t) = q_{\text{inj}} - 4h \frac{\pi}{E'} \left[ p_{w_f}(t) - p_c \right] L_f(t) \frac{dL_f}{dt} - 2h \frac{\pi}{E'} L_f^2(t) \frac{dp_{w_f}}{dt}. \tag{2.35}
\]
Closure pressure $p_c$ is one of the most important objective unknown to be evaluated from minifrac testing data. In order to integrate Eq. (2.35) into the devised pressure model of mini-fracture, a reasonable approximation is made and the leak-off rate is approximated as:

$$q_I(t) \approx q_{inj} - 4h \frac{\pi}{E'} \left[p_{wf}(t) - p_i\right]L_f(t) \frac{dL_f}{dt} - 2h \frac{\pi}{E'} L_f^2(t) \frac{dp_{wf}}{dt}. \quad (2.36)$$

Based on dimensionless terms defined from Eq. (2.55) to Eq. (2.62), the leak-off rate in dimensionless form can be expressed as

$$q_{ID}(t_D) = 1 - \frac{1}{\phi E' c_t} \psi(t_D)p_{wfD}(t_D) - \frac{1}{\phi E' c_t} \psi(t_D) \frac{dp_{wfD}}{dt_D}. \quad (2.37)$$

where

$$\psi(t_D) = L_f^2(t_D), \quad (2.38)$$

and

$$\psi'(t_D) = \frac{d\psi(t_D)}{dt_D} = 2L_f(t_D) \frac{dL_f}{dt_D}. \quad (2.39)$$

The leak-off rate in dimensionless form in Laplace is expressed as

$$L[q_{ID}(t_D)] = \int_0^{\infty} q_{ID} (t_D) \exp(-st_D) dt_D =$$

$$= \int_0^{\infty} \left[1 - \frac{1}{\phi E' c_t} \psi'(t_D)p_{wfD}(t_D) - \frac{1}{\phi E' c_t} \psi(t_D) \frac{dp_{wfD}}{dt_D}\right] \exp(-st_D) dt_D. \quad (2.40)$$

The integration of Eq. (2.40) is evaluated numerically. Illustrated in Eq. (2.37), under certain fracture propagating velocity, the greater the value of $\frac{1}{\phi E' c_t}$, the greater the corresponding fracture storage effect is. Koning and Koninklijke (1985) has ignored
fracture storage effect during fracture propagation period in the fracturing model, with the result generated by the assumption that leak-off rate is equal to injecting rate, and only considered fracture storage effect during shut-in period when fracture length stays constant. Larsen and Bratvold (1994) and Liu and Ehlig-Economides (2017) considered wellbore storage effect but did not account for fracture storage in their fracture propagation model. This study takes into account the fracture storage effect with increasing fracture length during propagation period but with the approximation made in Eq. (2.36).

More importantly, Nolte et al. (1993) discussed the significance of the parameter group, \((\phi c_t E')\), without a clearly definition of this group of terms. In fact, this group of parameters compares the geomechanic plane strain change per unit stress change with respect to compressible bulk volumetric change per unit pore pressure change. In this study, the importance of this group of terms is realized and this group is defined as the ratio of geomechanic strain deformation to compressible volumetric change under unit stress/pressure variation. This new term can be expressed as

\[
C_{fs} = \frac{1}{\phi E' c_t}. \tag{2.41}
\]

Theoretically the sum of flow rates of all segments from the original fracture and from the propagated fracture equals leak-off rate. In dimensionless form, this is expressed as

\[
\sum_{j=1}^M q_{oD,j}(t_D) + \sum_{k=1}^{N(t_D)} q_{f,k}(t_D) + \sum_{k=1}^{N(t_D)} q_{fD,k}(t_D) = q_{ID}(t_D)
\]

\[
= 1 - C_{fs} \psi'(t_D) p_{wfr}(t_D) - C_{fs} \psi(t_D) \frac{dp_{wfr}}{dt_D}. \tag{2.42}
\]
In Laplace domain, Eq. (2.42) becomes

\[ \sum_{j=1}^{M} L[q_{oD,j}(t_D)] + \sum_{k=1}^{N_1(t_i)} L[q_{fD,k}^L(t_D)] + \sum_{k=1}^{N_2(t_i)} L[q_{fD,k}^R(t_D)] = L[q_{ID}(t_D)] . \] (2.43)

### 2.2.2.3 Flow rate/Leak-off rate in State C

In this state the fracture propagation is arrested when \( t_i > t_e \). At shut-in, injecting rate at well becomes zero and based on the assumption made in the beginning of modeling methodology, fracture starts to close.

According to the material balance based on Eq. (2.32) and assuming constant density of injecting fluid, in State C Eq. (2.32) can be written as

\[ q_i(t) = q_{in,j,c} - \frac{dV_f}{dt} . \] (2.44)

When well is in shut-in situation in State C, the injection rate becomes zero, i.e., \( q_{in,j,c} = 0 \) and Eq. (2.44) becomes

\[ q_i(t) = -\frac{dV_f}{dt} = -2L_{fe}h \frac{dw_f}{dt} , \] (2.45)

where \( L_{fe} \) denotes fracture half length after propagation arrests since shut-in time \( t_e \).

Using the fracture width formulated by Eq. (2.34) at shut-in time \( t_e \), Eq. (2.45) becomes

\[ q_i(t) = -c_f \frac{dp_{wf}}{dt} , \] (2.46)

where

\[ c_f = \frac{2\pi h L_{fe}^2}{E'} . \] (2.47)
Koning and Koninklijke (1985) defined the $c_f$ term as “fracture storage constant” in general. In this study the $c_f$ term is clearly defined using fracture half length with the assumption of $h=h_f$, and also with the involvement of plane strain modulus $E'$, which makes the connection between fluid flow and geomechanics during mini-fracture testing process.

Pressure and time in Eq. (2.46) are expressed in their dimensionless forms respectively by Eq. (2.55) and Eq. (2.56) and dimensionless leak-off rate becomes

$$q_{ID}(t_D) = -c_{fD} \frac{dp_{wFD}}{dt_D} .$$

In Eq. (2.48), dimensionless fracture storage constant is defined as

$$c_{fD} = \frac{L_{fD}^2}{\phi E' c_t} ,$$

$$c_{fD} = c_{fS}L_{fD}^2 .$$

The dimensionless leak-off rate in Laplace domain is expressed as:

$$L[q_{ID}(t_D)] = \int_0^{t_e} q_{ID} (t_D) \exp(-s t_D) dt_D = \int_0^{t_{eD}} q_{ID} (t_D) \exp(-s t_D) dt_D + \int_{t_{eD}}^{t_e} q_I (t_D) \exp(-s t_D) dt_D$$

$$= \int_0^{t_e} \left[ 1 - c_{fS} \psi(t_D) p_{wFD}(t_D) - c_{fS} \psi(t_D) \frac{dp_{wFD}}{dt_D} \right] \exp(-s t_D) dt_D + \int_{t_{eD}}^{t_e} -c_{fD} \frac{dp_{wFD}}{dt_D} \exp(-s t_D) dt_D .$$

The integration of Eq. (2.50) is evaluated numerically. Similar to the strategy applied to leak-off rate behavior for State B, The sum of flow rates of all segments from the original
fracture and from the propagated fracture accounts for leak-off rate and is expressed in Laplace domain as

\[
\sum_{j=1}^{M} L[q_{oD,j}(t_d)] + \sum_{k=1}^{N_1(t_i)} L[q_{fD,k}(t_d)] + \sum_{k=1}^{N_2(t_i)} L[q_{fD,k}^9(t_d)] = L[q_i(t_d)].
\]  

(2.51)

2.2.3 Coupling

In this study, fluid flow inside the fracture body is ignored so that there is no pressure drop along the entire fracture extent, which implies that fracture has infinite conductivity. It is most likely the case when considering fracture propagation that no proppant or sand is injected during the treatment of mini-fracturing, thereby almost no pressure drop along fracture. Under such condition, in this mini-fracturing model no pressure drop along the fracture is achieved by equating pressures at the mid of each segment of fracture in Laplace domain; when the number of divided segments for the entire fracture length is great enough, pressure difference along fracture will closely approach zero.

The sum of flow rates of fracture segments equals leak-off rate for State A, B and C in Laplace domain, which is described by Eqs. (2.31), (2.43) and (2.51). Equating pressures at midpoint of segments of fracture together along with flow balance described by Eq. (2.31) or Eq. (2.43) or Eq. (2.51) yields \(M + 2N_i\) equations in the same number of unknown rates from discretized segments along fracture. The \(M + 2N_i\) rates are then substituted into Eq. (2.7) or Eq. (2.21) or Eq. (2.24) (\(N_i=0\) for State A), and desired pressure at any location can be obtained by inverting the Laplace solution with Stehfest inversion algorithm (1970).
Solutions from a special situation where the entire fracture is divided into a single segment, which implies that flow flux/leak-off along the entire fracture extent is uniform and the rate of the segment is equal to leak-off rate, are also calculated in some examples, but they are presented only as reference cases.

2.3 Results and Discussion

In this study, fracture length $L_f(t)$ is proposed to be characterized in the form of power law with respect to time, following Nolte (1986) with new modification in this study. The detail of $L_f(t)$ is described by Eq. (2.52) and the fracture propagating velocity $\frac{dL_f(t)}{dt}$ is described by Eq. (2.53).

$$L_f(t) = \begin{cases} L_o & t \leq t_s \\ L_o + \alpha(t - t_s)^\gamma & t > t_s \end{cases}$$  \hspace{1cm} (2.52)

$$\frac{dL_f(t)}{dt} = \alpha \gamma (t - t_s)^{\gamma - 1} \quad t > t_s \quad ,$$  \hspace{1cm} (2.53)

where $L_o$ denotes the original fracture half length, $\gamma$ is the exponent, and $\alpha$ describes how fast fracture propagates under certain exponent $\gamma$.

Some other studies modeling fracture propagation have also characterized fracture length in power law form with exponent $\gamma$ ranging between 0.5 to 1 (Nolte 1986, Nolte et al. 1993, Liu and Ehlig-Economides 2017), however, with few explanations.

For the fracture half length characterization proposed in Eq. (2.52), the term $\alpha$, with a unit of $[L/s^\gamma]$, accounts for how fast the fracture propagates under certain exponent $\gamma$. In general, the value of $\alpha$ is more related to rock geomechanical properties, such as Young’s modulus, Poisson’s ratio, porosity, compressibility, etc., and it is also related to
associated fluid properties such as fluid compressibility, viscosity, etc. The dimensionless exponent term $\gamma$, mainly accounts for the curvature variation of the fracture growth under acceleration or deceleration situations. The value of $\gamma$ normally ranges from 0.1 to 1 in this study. This is because of the fact that it is unlikely that a fracture propagation process can be accelerated with $\gamma > 1$. Thus $\gamma=1$ is set as the upper bound. For the lower bond of $\gamma$ value, it can be close to zero when fracture is propagating very slowly. For this study $\gamma=0.1$ is selected as the lower bond to indicate a slow fracture growing. Note that $\gamma=0$ means that a fixed fracture half-length of $(L_o+\alpha)$ and such a case is not of any interest in this study. Theoretically, the value of $\gamma$ as an exponent helps characterize the fracture propagating pattern, which is directly associated with the physical presentation of fracture geometry as a function of time and is strongly affected by fluid leakage (function of formation permeability and differential pressure) and rock geomechanical process (rock geomechanic properties). Due to the complexity of the technical contents related to the full definition of terms $\alpha$ and $\gamma$, further extensive research regarding the values of $\alpha$ and $\gamma$ terms are required to advance our current understanding, which is out of the scope of this research.

In literature, Nolte (1986) also suggests that leak-off along fracture becomes smaller when $\gamma$ is more closer to 1, while leak-off becomes greater when $\gamma$ becomes smaller. It can be understood that as $\gamma$ goes further down from 1 fracture propagating decelerates more.

For a clear reference purpose, the proposed fracture half length profile $L_f(t)$ vs. time $t$ and propagating velocity profile $\frac{dL_f(t)}{dt}$ vs. time $t$ are presented in Fig 2.7 (a) and Fig 2.7
(b), respectively. Note that in Fig 2.7 (b) when $\gamma = 1$ fracture propagates at a constant velocity and when $\gamma < 1$ fracture propagation decelerates. When it comes the time to analyze the relationship among injection time duration, $L_f(t)$, $\alpha$ and $\gamma$, one may prefer to rewrite Eq. (2.52) in its log-log format as

$$\lg(L_f - L_o) = \lg(\alpha) + \gamma \lg(t - t_s).$$

(2.54)

According to Eq. (2.54), if plotted in the form of $(L_f-L_o)$ vs. $(t-t_s)$ on log-log scale, the curves’ slope represents $\gamma$ value and curves’ interception at vertical coordinate represents $\alpha$ value. This type of plot may be of great practical value for field minifrac data analysis.
Fig. 2.7 (a) Fracture half length $L_f$ profile (b) fracture half length propagating velocity $\frac{dL_f}{dt}$ profile along with time characterized in the form of power law under various component $\gamma$ with $L_o = 0.1$ and $\alpha = 1$
**Dimensionless group**

Results are presented in type curves format, and dimensionless groups are defined.

Dimensionless pressure is defined as follow based on constant rate:

\[
P_D = \frac{2\pi k h_f}{q_{inj} \mu} (p - p_i).
\] (2.55)

Dimensionless time is defined as

\[
t_D = \frac{\eta t}{l^2} = \frac{kt}{\phi \mu c_l l^2},
\] (2.56)

where “l” denotes a characteristic length.

Dimensionless rate is defined as follow:

\[
q_D = \frac{q}{q_{inj}}.
\] (2.57)

Dimensionless leak-off rate is defined as

\[
q_{ID} = \frac{q_I}{q_{inj}}.
\] (2.58)

Dimensionless cumulative leak-off rate is defined as

\[
Q_{ID} = \int_0^{t_D} q_{ID}(t_D)dt_D.
\] (2.59)

Dimensionless fracture half length is defined as

\[
L_{fD}(t_D) = \frac{L_f(t)}{l} = \begin{cases} L_{oD} & \text{if } t_D \leq t_{SD} \\ L_{oD} + \alpha_D(t - t_c)' & \text{if } t_D > t_{SD} \end{cases}
\] (2.60)

where,
In this section, the modeling methodology of mini-fracturing is validated by checking theoretical solutions, generated by the model, of propagating fracture in terms of pressure-transient and leak-off rate under five scenarios. The first scenario investigates pressure and leak-off rate behaviors of propagating fracture under various exponent $\gamma$ ranging from 0.3 to 1 at fixed $\alpha$; The second scenario examines the solution difference resulted from two different fracture performing condition—uniform pressure along fracture or uniform leak-off along fracture—under various $\alpha$ value, and this scenario also tests the stability of solution under different selection of unit length for fracture segment. The third scenario discusses the effect of various levels of fracture storage on fracturing pressure as well as on leak-off rate for fracture with different propagating velocities with different $\alpha$ values. The fourth scenario presents pressure solutions from propagating fracture when initiated at different time. The fifth scenario aims at checking solutions of pressure having a transition period when fracture propagates an increment and stays static after that incremental propagation.

**Scenario 1: Effect of fracture propagation under various $\gamma$ on pressure and leak-off rate behaviors.** In Fig 2.8 (a), pressure response of fracturing well is examined under various $\gamma$ values ranging from 0.3 to 1. It is noticeable that pressure tends to be constant when $\gamma = 0.5$, which is often identified in the pressure profile of hydraulic fracturing or

\[
L_{OD} = \frac{L_o}{T}, \quad (2.61)
\]

\[
\alpha_d = \frac{\alpha}{l(\eta T)^Y}. \quad (2.62)
\]
mini-fracturing treatment. It is also clearly shown that greater $\gamma$ yields smaller dimensionless pressure before dimensionless time $t_D = 1$; almost at $t_D = 1$, all curves join together forming a knot and from then curves generated by greater $\gamma$ goes below those generated by smaller $\gamma$, which results from the fact that fracture length with greater $\gamma$, characterized in the format of power law, is actually shorter before $t_D = 1$ but is longer after that as shown in Fig 2.7 (a). The shorter a fracture length propagated, the greater its dimensionless pressure would be, because the pressure drop is larger. Fracturing pressure drop goes upward when $\gamma < 0.5$ while fracturing pressure drop goes downward when $\gamma > 0.5$.

**Fig 2.8 (b)** exhibits the leak-off rate behavior of propagating fracture with $\gamma$ also ranging from 0.3 to 1. It shows when $\gamma$ is below 0.5 leak-off rate increases from the beginning while it decreases when $\gamma$ is above 0.5 but less sharply at late time stage. For the case $\gamma = 0.5$, at an early time leak-off rate drops a little whereas it becomes flat afterward as does the pressure behavior. **Fig 2.8 (b)** illustrates that a small $\gamma$ results in a high fluid leak-off, on the other hand, a greater $\gamma$ results in a lower fluid leak-off.
**Fig 2.8** (a) Pressure behavior of propagating fracture. (b) Leak-off rate behavior of propagating fracture, under different $\gamma$, $\frac{1}{\phi \varepsilon' c_t} = 1$, $\alpha_D = 1$, $L_{oD} = 0.1$, $t_{SD} = 0$
**Scenario 2: Testing of solution stability.** Fig 2.9 provides two sets of solutions of fracturing pressure under the same fracture propagating condition, not only with the attempt to test solution stability but also in order to check the solution difference between two different fracture performing conditions. One set of solutions is generated under the situation where fracture is performing with no pressure drop along fracture extent, which implies that fracture has infinite conductivity (in the mini-fracturing model many segments are applied for the fracture extent based on the unit length $L_{unit}$), while the other set is generated under the situation where fracture is performing with uniform fluid flux/leak-off along the entire fracture extent (in the mini-fracturing model, a single segment is applied for the fracture extent). The effect of $\alpha_D$ is examined with solutions computed under various $\alpha_D$, at fixed $\gamma=0.5$. Fig 2.9 suggests that pressure tends to be more flat during an earlier time period when $\alpha_D$ is greater. As shown in the figure, propagating fracture under uniform flux always causes greater pressure and the difference of pressure solutions due to the 2 different fracture performing conditions—infinite conductivity and uniform flux—is more pronounced with bigger $\alpha_D$ and is less so with smaller $\alpha_D$. Since $\gamma$ is directly proportional to fracture growing speed, this means that for a fast growing fracture, the pressure drop modeling difference between the two situations under study is more pronounced, and therefore it deserves more attention in field application. The unit length $L_{unit}$ for discretized fracture segments in Fig 2.9 under all $\alpha_D$ is the same as $L_{unit} = 0.1$ (when fracture length is shorter than 0.1, only 1 segment is applied for propagated fracture).

Fig 2.10, discusses the effect of unit length on fracturing pressure solutions as $\alpha_D = 10$. The impact of unit length selection—resulting in different segment numbers discretized
from fracture—on pressure solution is important to discuss. In Fig 2.10, the solutions generated from different values of unit length under $\alpha_D = 10$ are provided. It clearly shows that the solutions are unstable and fluctuating heavily at early time period for $L_{unit} = 1$ or 0.5 and are gradually becoming stable after $t_D = 5$; for $L_{unit} = 0.2$, solution quality is much better with slightly fluctuating bumps before $t_D = 1$ while the solution is robust with shorter unit length as $L_{unit} = 0.1$ throughout all the time as indicated by the solid red curve. The results in Fig 2.10 imply that early-time calculation is important and difficult since fracture propagating velocity is fast at early time, so that the selection of unit length must be careful and reasonable as to achieve robust and desired solution accuracy.

Fig 2.11 provides dimensionless pressure and its derivative for a limiting case with $\gamma = 0.5$ and $\alpha_D = 10^{-4}$, suggesting that fracture propagation velocity is extremely slow with fracture length practically approaching a point, in comparison with a point source solution. This limiting case, to a certain extent, validates our analytical solution as it shows that before very late time the slowly propagating fracture behaves very similar to a point source with flat pressure derivative at 0.5 (radial flow), due to the fact that fracture propagates so slowly that it actually results in a very short fracture length, thus the pressure solution of the “short” line source almost overlaps that of a point source.
Fig 2.9 Pressure behavior under different fracture propagation velocity subject to $\alpha_D$ without consideration of fracture storage effect, $\gamma = 0.5$, $L_{OD} = 0.1$, $t_{SD} = 0$, $L_{unit} = 0.1$
Fig 2.10 Effect of unit length selected for discretized segment from propagated fracture on pressure solution without consideration of fracture storage, $\alpha_D = 10$, $\gamma = 0.5$, $L_{OD} = 0.1$ and $t_{sD} = 0$
Fig 2.1 Limiting case for extremely slow propagating velocity with uniform flux along entire fracture as $\alpha_D = 10^{-4}, \frac{1}{\phi e' c_t} = 5, \gamma = 0.5, L_{oD} = 0, t_{sD} = 0$
Scenario 3: Effect of fracture storage under various values of $c_{fs}$, $\left(\frac{1}{\sqrt{AEc_t}}\right)$. During injection period with fracture propagation, under certain fracture propagating condition the level of fracture storage is accounted for by $c_{fs}$ illustrated in Eq. (2.41); the greater of the value, the greater the fracture storage is. The effect of fracture storage plays a crucial role in both pressure and leak-off behaviors as part of injecting fluid is stored in the expanding fracture volume. With slow-to-fast fracture propagating velocities by different $\alpha_D$ at $\gamma = 0.5$, Fig 2.12 (a) and Fig 2.12 (b) examine behaviors of pressure and leak-off respectively under various levels of fracture storage effect represented by the value of $c_{fs}$. It can be seen that with medium-to-fast fracture propagation as $\alpha_D = 1$ and $\alpha_D = 10$, a large part of the injecting fluid is used to fill the fracture volume and the levels of fracture storage effect impose huge impacts on pressure and leak-off behaviors. For $\alpha_D = 10$, dimensionless leak-off rate ranges from 0.07 to 0.45 with $c_{fs}$ varying from 0.1 to 1.0, as does leak-off rate from 0.56 to 0.95 for $\alpha_D = 1$. Fracture storage, however, barely has any influence when fracture propagating velocity is very small as for this case with $\alpha_D = 0.1$ that solutions in terms of pressure and leak-off rate responding to different levels of fracture storage effect almost overlap each other due to the fact that fracture volume is very small because of slow propagation in length.

In general, fracturing in tight formation leads to fast-propagating fracture while in high-permeable formation slow-propagating fracture is likely to be generated. Revealed by Fig 2.12 (a) and Fig 2.12 (b), accurate computation of leak-off rate is the key to diagnosing minifrac testing data in low-to-medium permeable formation. Results from G function (Nolte 1986) suggest that leak-off rate is a constant during fracture propagation before shut-in if fracture propagation is proportional to square root time based on Carter’s linear
flow assumption. In Fig 2.12 (b), however, it shows that for $\alpha_D = 1$ with strong fracture storage effect leak-off rate does not become flat until late time, which will lead to error for minifrac data diagnosis if based on G function. When it comes to high-permeable formation, the use of G function to analyze minifrac data leads to error and might yield overestimation of fracture length (van den hoek 2002) and may yield reservoir-flow capacity ($kh$) that can be very high (Barree et al. 2015) because leak-off behavior does not conform to linear flow assumption when fracture propagation is slow.
Fig 2.12 (a) Pressure behavior (b) Leak-off rate behavior, under various level of fracture storage effect represented by $c_f$, subject to different $\alpha_D$, $\gamma = 0.5$, $L_{oD} = 0.1$. 
**Scenario 4: Effect of injecting time $t_s$ before fracture is initiated.** A period of time of injection at well is necessary prior to fracture’s creation and propagation, because well bottom pressure is required to reach a point where formation breaks down leading to initiation of fracture. Fig 2.13 shows pressure behavior under different injection period before fracture propagates with $\alpha_D = 0.75$. For the three scenarios with $t_{SD} = 0.1, 1, 10$, solution computed from each one overlaps one another after $t_D = 100$, but before that the one having later fracture propagation always yields higher pressure than does the one having earlier fracture propagation even though the difference of fracture length is quite little as time approaches 100.
Fig 2.13 Effect of injecting time before fracture initiation $t_{sd}$ on pressure behavior without consideration of fracture storage effect, $\alpha_D = 0.75$, $\gamma = 0.5$, $L_{od} = 0.1$, $L_{unit} = 0.1$
Scenario 5: Arrest of fracture propagation while injection still goes on at well.
Suppose that fracture propagates an additional increment from its original tip and then propagation arrests while well is still injecting, pressure behavior must have a smooth transition between two static fractured well behavior with constant fracture length: one has original fracture length before propagation and the other one has that of original length plus the increment.

Fig 2.14 (a) provides pressure behavior of fractured well that goes through a fracture length increment at $t_{sd} = 1$ from $L_{fd} = 1$ to $L_{fd} = 10$ under different propagating velocity $\alpha_D$. The pressure transition is presented for each $\alpha_D$, and it can be seen that pressures quickly overlap the fractured well solution with constant $L_{fd} = 10$ once fracture tip arrives at there.

Instead of examining different propagating velocity $\alpha_D$, Fig 2.14 (b) provides pressure responses with different fracture propagation initiation time $t_{sd}$ at fixed $\alpha_D = 0.75$ and fracture propagation stops when fracture half length reaches $L_{fd} = 10$. 

Fig 2.14 Pressure response for fracture propagating from $L_{FD} = 1$ to 10 without consideration of fracture storage effect (a) under different $\alpha_D, \gamma = 0.5, L_{OD} = 1, t_{sD} = 1, t_{eD} = 1258, 316, 125, 79$ for the four solutions and $L_{unit} = 0.1$ (b) under different initiation time $t_{sD}, \alpha_D = 0.75, \gamma = 0.5, L_{OD} = 0.1, t_{eD} = 145, 154, 244$ for the three solutions
2.3.2 Results after Shut-in

In this section, pressure profiles including falloff period after well has been shut-in affected by fracture propagation during injection are presented for another two scenarios. Fracture length is assumed to stay constant after shut-in, no further fracture propagation or no fracture recession considered. Scenario 6 provides fracturing pressure profiles from the beginning of injection to a long time after shut-in for fracture with different propagation pattern during injection period accounted for by a number of both \( \gamma \) and \( \alpha \) values without the consideration of fracture storage effect. Scenario 7 investigates the influence of various levels of fracture storage on pressure falloff behaviors after shut-in.

**Scenario 6: fracturing pressure profile vs. time from beginning of injection to post-shut-in.** In this scenario, Fig 2.15 shows the pressure profile of fracturing well comprising the three states (State A, B, C proposed in the mini-fracturing model) without consideration of fracture storage effect (meaning fluid leak-off rate along fracture faces turns into 0 right after shut-in): pre-fracture initiation, fracture propagation, and after shut-in. Effects of various \( \gamma \) as \( \gamma = 0.2, 0.5, 0.8 \) and various \( \alpha_D \) values as \( \alpha_D = 0.3, 0.6, 1 \) for fracture propagation period on pressure response are examined and presented in Fig 2.15. It is shown that when \( \gamma = 0.8 \) fracturing pressure under all the 3 \( \alpha_D \) values declines during the time period of fracture propagation; in contrast, fracturing pressure goes slightly upward when \( \gamma = 0.2 \) under all the 3 \( \alpha_D \) values; and fracturing pressure almost stays flat for all the 3 \( \alpha_D \) when \( \gamma = 0.5 \). With the characterization of fracture length in the form of power law in this study, the exponent \( \gamma = 0.5 \) strikes a dividing line in terms of pressure solution that well behavior exhibits increasing pressure for \( \gamma \) greater than 0.5 while well behavior exhibits declining pressure for \( \gamma \) smaller than 0.5. After shut-in,
pressure falloff response is related to fracture propagation pattern during injection period. 

**Fig. 2.15** indicates that after shut-in without consideration of fracture storage effect, pressure falls more slowly under the situations where fracture ends up with longer length by the time well is shut-in either because of $\alpha_D$ or because of $\gamma$. 
Fig 2.15 Fracturing pressure including both injection and shut-in period with different $\gamma$ and $\alpha_D$ without consideration of fracture storage effect, $t_{SD} = 100$, $t_{ED} = 500$, and $L_{unit} = 0.1$
Scenario 7: Effect of fracture storage after shut-in under various values of $c_{fD}$. This scenario considers the effect of fracture storage after shut-in on pressure behavior while this effect is ignored during injection period. Fluid leak-off continues as fluid flows into formation through fracture faces after well is shut-in for the reason that fracture volume is diminishing—fracture width is closing—so that fluid in fracture is being pushed into reservoir until fracture is completely closed. Fig 2.16 examines pressure falloff behaviors under various levels of fracture storage effect, represented by dimensionless fracture storage constant $c_{fD} = 0, 10, 100, 1000$ defined in Eq. (2.49) that greater value results in greater fracture storage effect, while fracture is closing. Fig 2.16 illustrates that pressure falls less sharply with greater fracture storage effect. What accounts for this phenomenon is that with greater fracture storage effect more fluid continues to leak into formation along fracture and the leak-off also lasts longer after shut-in, which help keep the pressure from declining dramatically and help hold the pressure at a comparatively high level for a longer time. Numerous pressure falloff behaviors are observed in field cases. As far as the author’s knowledge, pressure falloff data from minifrac test in tight reservoir usually shows slow decline while falloff data in medium-to-high permeable reservoir usually shows sharp decline, which possibly results from the fact that in tight reservoir comparatively longer fracture is often created with more fluid stored in fracture resulting in greater fracture storage effect while in medium-to-high permeable reservoir shorter fracture is often created with less fluid stored in fracture resulting in smaller fracture storage effect.
Fig 2.16 Effect of fracture storage only considered during shut-in period on pressure behavior, $\alpha_D = 0.3$, $\gamma = 0.5$, $L_{OD} = 1$, $t_{sD} = 0$, $t_{eD} = 200$, $L_{unit} = 0.1$
2.3.3 Summary of Theoretical Results

In this work, fracture propagation is characterized in the form of power law along with time, specifically described by two coefficients $\gamma$ and $\alpha$ shown in Eq. (2.52). This type of characterization of fracture length, as far as the author’s knowledge, is firstly applied in fracturing model and the physical meanings behind the two coefficients have also been explained.

Theoretical solutions in terms of leak-off rate and fracturing pressure during both injection and shut-in periods under various fracturing situations are presented above. Results indicate that fracture propagation under different exponent $\gamma$ can have great effect on both pressure and leak-off behaviors not only affecting the magnitude of pressure/leak-off but also determining the trend of fracturing pressure and leak-off rate. Results also suggest that fracture storage effect has great influence on pressure and leak-off behaviors; in particular pressure falloff response after shut-in is largely determined by fracture storage effect that pressure declines much less steeply when fracture storage effect is greater. Theoretical solutions will become the basis on which field minifrac test is analyzed in this study by curve matching between measured bottom hole pressure and pressure solution generated by this model, and detailed field case analysis will be discussed in the following section.

2.4 Field Case Analysis of Minfrac Tests

An important application of this work is in minfrac test analysis. As small injection rate without adding proppant is the performing requirement for mini-fracturing, narrow fracture with high conductivity is likely to create, which approaches the assumption made and fracture coupling condition adopted in this mini-fracturing model: fracture is
regarded as a line source in pressure description; uniform pressure along fracture (infinite conductivity) is adopted. By applying the analytical model, 3 field cases of minifrac test analysis are presented in the following part.

2. 4.1 Case 1

This field case is a minifrac test in high-permeable reservoir and has been presented by Valko and Economides (1997) originally, and by van den Hoek (2002). Table 2.1 shows the summary of reservoir and treatment information; Fig 2.17 shows the bottom hole (gauge) pressure (BHP) during treatment. In order to achieve curve matching, the strategy is presented in detail in following steps.

Table 2.1: Input Information of Case 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeable height, h_f / m</td>
<td>7.01</td>
</tr>
<tr>
<td>Reservoir fluid viscosity, u / (Pa·s)</td>
<td>0.001</td>
</tr>
<tr>
<td>Porosity, φ</td>
<td>0.25</td>
</tr>
<tr>
<td>Total compressibility, c_t / (Pa·s⁻¹)</td>
<td>7.25×10⁻¹⁹</td>
</tr>
<tr>
<td>Plane strain modulus, E' / (Pa)</td>
<td>1.378×10⁶</td>
</tr>
<tr>
<td>Pumping time, t_p / (sec)</td>
<td>354</td>
</tr>
<tr>
<td>Injected volume (2 wings), V_i / m³</td>
<td>9.46</td>
</tr>
<tr>
<td>Asymptotic pressure, p_i / (Pa)</td>
<td>7.687×10⁵</td>
</tr>
</tbody>
</table>

**Step 1:** Normalize the bottom hole pressure data by subtracting the asymptotic pressure \((BHP - p_f)\), which represents the pressure difference between well bottom pressure and reservoir pressure at far point; normalize the injecting time by subtracting the time at which fracture is initiated, \(t_s\), as pressure data subsequent to fracture’s creation is the target for analysis. Normalized pressure data and time are mathematically written as

\[
p_n = (BHP - p_f) ,
\]

\[
t_n = t - t_s .
\]
Step 2: Generate type curves $p_D$ vs $t_{nD}$, from the analytical model proposed in this study, to match normalized pressure data and time. $t_{nD}$ is defined as

$$t_{nD} = \frac{k}{\phi \pi c_l l^2} (t - t_s). \quad (2.65)$$

In this field case, the fracture’s propagating length in the analytical model is characterized as being proportional to square root time.

$$L_{FD} = a_D \sqrt{t_{nD}}. \quad (2.66)$$

Due to the fact that BHP of fracturing well is almost constant while injection in this case, that is why fracture length proportional to square root time, $\gamma = 0.5$, is selected. It is assumed that there is no existing fracture connected to well prior to injection, thereby $L_0$ being set as 0.

In order to identify the shut-in time in type curve, characteristic length $l$ is considered an unknown constant that makes $t_{nD}$ equal to 1 at shut-in, which is

$$t_{e,nD} = \frac{k}{\phi \pi c_l l^2} (t_e - t_s) = 1. \quad (2.67)$$

With this definition, pressure starts to fall off at $t_{nD} = 1$ in type curves.

Step 3: Match normalized treatment data with type curve in log-log plots. Since the formation is highly permeable in this case, it is necessary to match measured data with the generated type curve shortly after shut-in (seconds or tens of seconds after shut-in) for the fact that additional pressure at well bottom (caused by friction or due to fluid flow in wellbore) will dissipate very soon in high-permeable formation and therefore measured BHP reflects real pressure at fracture face near wellbore. In tight formation, however,
additional pressure at well bottom will dissipate in a much longer time, and in order to match measured BHP it has to be case specific with consideration of the level of tightness of rock as well as wellbore storage effect, etc. Fig 2.18 shows curve match result on log-log scale. The matched measured pressure data and theoretical pressure solution generated by the model are plotted in Cartesian coordinate together with leak-off rate from the model in Fig 2.19 (a).

**Step 4:** It is necessary to analyze pressure behavior after shut-in in the similar manner as G function method does in order to detect the closure pressure. When fracture is closing, the relationship between cumulative leak-off rate and pressure response should be linear.

Recall the dimensionless leak-off rate in State C (after shut-in) by Eq. (2.48),

\[ q_{ID}(t_D) = -c_{fD} \frac{dp_{wFD}}{dt_D}. \]  

(2.48)

Integration of both sides from shut-in time gives

\[ \int_{t_{ed}}^{t_D} q_{ID}(t_D) dt_D = -c_{fD} \int_{t_{ed}}^{t_D} \frac{dp_{wFD}}{dt_D} dt_D = -c_{fD} \int_{ISIP}^{p_{wFD}} dp_{wFD}. \]  

(2.68)

Finally, we have

\[ ISIP - p_{wFD}(t_D) = \frac{1}{c_{fD}} (Q_{ID}(t_D) - Q_{ID}(t_{ed})). \]  

(2.69)

Eq. (2.69) indicates that the difference of BHP vs. the difference of cumulative leak-off after shut-in and at shut-in will present a linear relationship and that the linear trend will deviate once fracture is closed. Fig 2.19 (b) shows \((ISIP - p_{wFD}) vs (Q_{ID} - Q_{ID}(t_{ed}))\) and it suggests that the linear trend starts to deviate somewhere between 0.17 to 0.19 of
cumulative leak-off rate difference, which corresponds to the time, from the beginning of injection, between 7.38 to 7.73 minutes. The detected fracture closure time (between 7.38 to 7.73 minutes) is consistent with time when generated pressure solution deviates from the measured pressure data illustrated in Fig 2.18, which gives more confidence to this match. The reason behind is that in reality there is no more fluid pushed out of fracture when fracture is completely closed, but the proposed mini-fracturing model only models closing period after shut-in without consideration of after closure period. That is to say, fracture is closing all the time in the model and there is always fluid pushed out of fracture with leak-off rate being proportional to pressure derivative with respect to time described by Eq. (2.48). In another word, solutions from the model only describe behaviors of fracture when it is closing, and therefore theoretical solutions must deviate from measured data after fracture closure (matched pressure solution from model should be slightly above measured pressure data).

With successful match with measured data, the formation permeability is obtained as \( k = 402.3 \, \text{md} \), characteristic length \( l = 25.27 \, \text{m} \), \( \alpha_D = 0.2 \), and thereby fracture half length at end of injection is obtained as \( L_{f,e} = 0.2 \cdot 1 \cdot 25.27 = 5.05 \, \text{m} = 16.5 \, \text{ft} \); in addition, ISIP (instantaneous shut-in pressure) is interpreted as 1580 psi, based on the match between measured BHP and pressure solution generated by the analytical model. In general, ISIP is not obvious and is difficult to determine from measured data because ISIP represents fracture-propagation pressure at fracture face instead of well bottom hole pressure at the end of injection. The difference between ISIP and bottom pressure at shut-in is caused by excess friction, either through poor perforation or near-wellbore tortuosity (Barree et al. 2015), that always exists near well, which adds an additional pressure to
bottom pressure during injection; and the frictional pressure has to dissipate after shut-in but it takes a period of time, after which bottom hole pressure reflects the real near-well pressure at fracture face. In the theoretical model proposed, friction, or other factors causing additional pressure at well bottom, is not taken into account so that pressure solution at well represents near-well pressure at fracture face. In this minifrac case it shows in Fig 2.17 that well bottom (gauge) pressure at the end of pumping is 1662 psi, and that the theoretical result generated from analytical model is around 1580 psi (in Fig 2.19 a), which is regarded as ISIP. As well, fracture closure can be approximately identified at 1.5 minute after shut-in and fracture closure pressure $P_c$ at 1375 psi.
Fig 2.17 Measured bottom hole (gauge) pressure of Case 1
Fig 2.18 Measured pressure data and generated pressure solution from model match in log-log plots
Fig 2.19 (a) Measured data and matched type curve in Cartesian coordinate together with cumulative leak-off rate (b) Cumulative leak-off rate difference vs. BHP difference subsequent to shut-in
This case has been presented by Valko and Economides (1997) and by van den Hoek (2002), and they offered results evaluated by their own methods as well as traditional Carter’s liner leakoff model (leading to G function derivation). To have a clear comparison of results analyzed by all the methods, Table 2.2 presents the evaluation results by each method.

Table 2.2: Comparison of Results Analyzed by Different Methods for Case 1

<table>
<thead>
<tr>
<th></th>
<th>linear leakoff</th>
<th>Valko and Economides</th>
<th>van den Hoek</th>
<th>This work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability</td>
<td>11.2 Darcy</td>
<td>83 md</td>
<td>1.52 Darcy</td>
<td>402.3 md</td>
</tr>
<tr>
<td>Fracture half length</td>
<td>23.2 ft (radial)</td>
<td>29.5 ft (linear)</td>
<td>29.5 ft (radial)</td>
<td>16.5 ft (linear)</td>
</tr>
<tr>
<td>ISIP</td>
<td>NA</td>
<td>1550 psi (given)</td>
<td>NA</td>
<td>1580 psi</td>
</tr>
<tr>
<td>Closure time</td>
<td>NA</td>
<td>1.7 min</td>
<td>NA</td>
<td>1.5 min</td>
</tr>
<tr>
<td>Closure pressure</td>
<td>NA</td>
<td>1350 psi</td>
<td>NA</td>
<td>1375 psi</td>
</tr>
</tbody>
</table>

2.4.2 Case 2

This minifrac test is conducted in a deep tight gas reservoir, and input information is summarized in Table 2.3.

Table 2.3: Input Information of Case 2

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeable height, h_w / m</td>
<td>16</td>
</tr>
<tr>
<td>Plane strain modulus, E / pa</td>
<td>1.378 x 10^{10}</td>
</tr>
<tr>
<td>Porosity, ( \phi )</td>
<td>0.1</td>
</tr>
<tr>
<td>Compressibility, c_1 / pa^{-1}</td>
<td>2.9 x 10^{-8}</td>
</tr>
<tr>
<td>Viscosity, ( \mu ) / pa \cdot s</td>
<td>0.005</td>
</tr>
<tr>
<td>Average injection rate after formation breakdown, ( q_{\text{inj}} ) / m^3/s</td>
<td>0.0254</td>
</tr>
<tr>
<td>Pumping time, ( t_e ) / sec</td>
<td>1440</td>
</tr>
<tr>
<td>Asymptotic pressure ( p_f ) / pa</td>
<td>5.877 x 10^6</td>
</tr>
</tbody>
</table>

The measured pressure and injecting rate are provided in Fig 2.20. The injecting rate had stepwise increment along with time before formation breakdown in case that the initial injecting rate is not great enough to effectively break apart the tight formation. It has been
about 14 minutes before fracture is initiated and injection continues for another 10 minutes.

Following the same steps discussed in the previous field case study, Fig 2.21 (a) presents the matched measured BHP and theoretical pressure solution generated by the model as well as leak-off rate from the model. The matching strategy of measured pressure and theoretical pressure solution in this case is a little different from the previous case. The minifrac test in this case is conducted in deep tight reservoir, in which additional pressure, for example caused by friction, will not dissipate shortly after shut-in because fluid flow in wellbore continues for a much longer time in tight formation. We try to match the measured data about 12 minutes later than shut-in in this case, giving enough time for additional pressure at well bottom to disappear. Note that at what time measured data and theoretical pressure solution from model should try to get matched has to be case specific, because each test has its unique conditions such as the properties of rock and reservoir fluid, afterflow effect from wellbore, friction effect, etc.

Fracture closure is not observed in this test. Fig 2.21 (b) shows the cumulative leak-off rate difference versus BHP difference and linear relation continues till the end of testing time. Table 2.4 presents the evaluation results of this test.

<table>
<thead>
<tr>
<th>Table 2.4: Evaluated Results of Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability</td>
</tr>
<tr>
<td>Fracture half length</td>
</tr>
<tr>
<td>ISIP</td>
</tr>
<tr>
<td>Closure time/pressure</td>
</tr>
</tbody>
</table>
Fig 2.20 Measured BHP and injection rate profiles for Case 2
Fig 2.21 (a) Measured data and matched type curve in Cartesian coordinate (b) Cumulative leak-off rate difference vs. BHP difference subsequent to shut-in
2.5 Summary

One of the main superiority SSFM or BEM (they become the same in homogeneous infinite reservoir) holds is that flow rate along source/boundary can be obtained once performing condition of source is specified, while approximation is made on the source that rate is evenly distributed along each small segment. It became clear that, if propagating velocity can be integrated into traditional static line source solution, fracture propagation can be modeled by a propagating line source and fracturing pressure behavior can be described by the propagating line source solution, which can be properly applied to address mini-fracturing in the fact that fracture resulting from mini-fracturing treatment has very small width and has very high conductivity, compared to original formation, that approaches infinite conductivity. This study has successfully established an analytical model, in which fracture goes through three states—before initiation of fracture, during fracture propagation, after fracture propagation stopped from shut-in—and theoretical pressure-transient and leak-off rate behaviors of fracturing well under various scenarios are examined. As well, novel evaluation strategy of minifrac tests based on model-based analytical solutions are proposed and field cases of minifrac test aiming at evaluating leak-off rate behavior along fracture, obtaining fracture geometry, extracting reservoir flowing capacity ($kh$) and closure pressure are analyzed.
2.6 Nomenclature

$c_f$ = fracture storage constant, m$^3$/pa

$c_{FD}$ = dimensionless fracture storage constant

$c_{fs} = \frac{1}{\phi E c_t}$, defined in Eq. (2.41), the comparison between geomechanic plane strain and compressible bulk volumetric change per unit stress/pressure variation

$c_w$ = compressibility of injecting fluid, pa$^{-1}$

$c_t$ = total compressibility, pa$^{-1}$

$\hat{E}$ = plane strain elastic modulus, pa

$h$ = vertical reservoir thickness, m

$h_f$ = permeable thickness, m

ISIP = well instant shut-in pressure, pa

$k$ = reservoir permeability, m$^2$

$l$ = characteristic length, m

$L_0$ = original fracture half length, m

$L_{unit}$ = unit length selected for discretized fracture segment, m

$L_f$ = fracture half length at current time, m

$L_{fe}$ = fracture half length at shut-in, m
$N_t=$ number of fracture segments for propagated fracture of one side wing

$p =$ pressure, pa

$p_c =$ fracture closure pressure, pa

$p_l =$ reservoir asymptotic pressure, pa

$p_{wf} =$ well bottom pressure, pa

$\Delta p = p - p_l$, pressure change, pa

$q =$ fluid rate, m$^3$/s

$q_{inj} =$ injecting rate at well, m$^3$/s

$\bar{q} =$ flux along fracture, m$^2$/s

$q_f =$ flow rate along fracture over a segment, m$^3$/s

$q_i =$ fluid leak-off rate from fracture to reservoir, m$^3$/s

$Q_i =$ fluid cumulative leak-off rate from fracture to reservoir, m$^3$

$s =$ Laplace variable

$t =$ time variable, s

$t_c =$ fracture closure time, s

$t_e =$ time at which fracture propagation ends, s

$t_i =$ evaluation time point, s
\( t_s \) = time before fracture initiation, s

\( t_{\infty} \) = time at infinity, s

\( U \) = unit step function

\( V \) = fracture propagating velocity, \( \text{m}^3/\text{s} \)

\( V_f \) = fracture volume, \( \text{m}^3 \)

\( V_{wb} \) = wellbore volume, \( \text{m}^3 \)

\( \rho_w \) = injecting fluid density, \( \text{kg/m}^3 \)

\( \mu \) = viscosity, \( \text{pa.s} \)

\( \tau \) = a time moment, s

\( \phi \) = porosity, fraction

\( \eta \) = diffusivity, \( \text{m}^2/\text{s} \)

**Subscripts**

\( D \) = dimensionless term

\( k \) = the \( k-th \) line source segment

**Superscript**

\( R \) = right side wing of fracture

\( L \) = left side wing of fracture
Although a great many works have examined the performance of fractured horizontal wells, no analytical study has been developed to explore the behavior of fractured horizontal well producing from multilayered reservoir with inter-layer crossflow. The objective of this chapter is to develop an analytical model to deal with the transient response of multi-stage hydraulically fractured horizontal well within a heterogeneous layered reservoir system that is partially penetrated by hydraulic fractures either vertically or horizontally with the consideration of inter-layer fluid crossflow. This type of reservoir depositional environment is routinely encountered in unconventional reservoirs such as Bakken in Williston basin and Eagleford of south Texas. Therefore, the need to perform the study described is imperative. Generally, fluid flow in multilayered reservoir formations is extremely complex in nature due to the historical diversity of reservoir geological and depositional environment variation. This reality requires that reservoir evaluation and characterization performed for an individual reservoir have to be case specific.

3.1 Introduction and literature related

Lee et al. (1994) pointed out long time ago that one of the main challenges reservoir engineers are facing is producing oil and gas from multiple vertical layers with different reservoir properties, concluding that simplistic reservoir description for layered reservoir
always results in an overestimated production forecast. Frantz et al. (1992) also concluded that a multi-layers reservoir description results in better evaluation of post-fracture performance in comparison with conventional single-layer reservoir description for a complex tight formation. Recently, horizontal well with multi-stage hydraulically fractured transverse fractures along wellbore trajectory has become a widely accepted and increasingly implemented technology in unconventional reservoir development, which further complicated the wellbore and reservoir system. Tight reservoirs with complicated stratified layers and relatively large vertical spanned thickness, but poor reservoir permeability and low porosity, become the central focus of reservoir engineers. For example, the tight oil reservoirs of Bakken in Williston basin across US-Canada, the Eagle Ford shale oil reserves of south Texas and the other reservoirs with geological and depositional settings under the above mentioned conditions need to address the well responses reliably and accurately. Fig 3.1 shows a typical gamma ray log profile from Viewfield Pool in Southeast Bakken Formation, illustrating simple stratigraphic division.

Analytical models for reservoirs under different types of initial and boundary conditions have been established by many researchers to describe fluid flow in reservoir. Gringarten and Ramey (1973) firstly provided instantaneous source functions corresponding to various boundary conditions in depicting transient pressure behavior over reservoir domain. Gringarten et al. (1974) extended the point source solution to transient pressure behavior of a producing or injecting fractured well with a single vertical fracture having infinite conductivity by integrating point source solution along fracture plane and time. Raghavan et al. (1978) explored the pressure behavior of a vertical fractured well with fracture partially penetrating single-layer reservoir vertical thickness.
Fig 3.1 A typical gamma ray log profile from Viewfield Pool in Bakken Formation

(Cronkwright et al. 2014)
Camocho-v et al. (1987) built an analytical model to examine the response of fractured well producing non-communicating layered reservoir. Olarewaju and Lee (1990) provided an analytical solution for rate behavior of a vertical well intercepting a two-layer reservoir. Ozkan and Raghavan (1991) derived point source solution in Laplace domain with various well configurations. Chen and Raghavan (1997) presented an algorithm to compute pressure distribution caused by a multi-stage fractured horizontal well within a rectangular shape reservoir. Zhao (2009) applied the source/sink function method in developing the technology of modeling reservoir heterogeneity with complicated wellbore and fracture system by enhancing the application of the concept of source/sink and imaging source/sink extensively. The patented technology greatly improved the technical capacity of reservoir engineers in dealing with reservoirs with complicated heterogeneity and geometry. Medeiros, Ozkan and Kazemi (2010) presents a semi-analytical approach, by using Green function method, to model transient pressure behavior of a horizontal well in composite or layered reservoir with its wellbore intercepting a number of blocks. Lin and Zhu (2010) applied slab source concept to describe pressure behavior of a horizontal well with multiple fractures. Rbeawi and Tiab (2012) formulated an analytical model describing pressure behavior of fractured horizontal well with partially-penetrating hydraulic fractures. In addition to pressure behavior, many studies have examined the production rate response of fractured horizontal well in recent years. Mederios, Ozkan and Kazemi (2008) developed a semi-analytical model incorporating key features of hydraulic fractures, naturally fractures (dual-porosity system) and wellbore flow to explore production-decline characteristics of fractured well in terms of transient-productivity index. Nobakht et al. (2012) provided a
simple method of forecasting production in tight/shale gas reservoirs, which can be applied to multi-stage fractured horizontal wells based on single fracture linear flow solution. The solution is simple and easy to generate, but may not be able to handle the interference among multiple hydraulic fractures reliably. Sureshjani and Clarkson (2015) built an enhanced-fracture-region model for analyzing and forecasting multi-stage fractured horizontal well with complex fracture geometry. Zhao et al. (2012) developed a stimulated reservoir volume (SRV) model, based on the patented modeling methodology (Zhao 2009), which confines a horizontal well with multiple stages of hydraulic fractures within SRV region, by using source/sink function with solution acquired in Laplace domain, and came up with type curves of pressure and newly modified type curves of production rate to help diagnose fracture length in a systematic manner.

This chapter presents modeling strategy of fractured horizontal well producing multilayer reservoir by applying source and sink function method, and shows computed results under various scenarios.

3.2 Methodology

Source/sink function method has been widely used in petroleum industry in analyzing transient pressure, production rate, and heat conduction behavior. Analytical solutions describing heat conduction in homogeneous reservoir, by using source/sink function, proposed by Carslaw and Jaeger (1959) become the foundation of modeling pressure- and rate-transient behavior. Pioneer researchers have extended these analytical solutions to complicated well configurations including horizontal and fractured well (Gringarten et al. 1974; Ozkan and Raghavan 1991), and to complex reservoir such as composite reservoir, commingled reservoir, complex well-reservoir system (Kuchuk and Wilkinson 1991;
Basquet et al. 1999; Medeiros, Ozkan, and Kazemi 2008) in modeling transient pressure or production rate problem in reservoir engineering.

### 3. 2.1 Point source and plane source solutions in homogeneous reservoir

Point source solution describing fluid flow behavior in porous media is the starting point in this study. Carslaw and Jaeger (1959) have used point source solution in solving heat conduction problem in a wide range of source geometry and boundary conditions. Considering an infinite one-dimension slab reservoir with source domain located at \( x' \), pressure drop, evaluated at time \( t \), of an instantaneous removal of a certain quantity \( Q \) of fluid at \( \tau \) from reservoir can be expressed as follows by Zhao and Thompson (2012)

\[
\Delta p(x', x, \tau, t) = S \cdot p_{sx}(x', x, \tau, t)dx',
\]

where \( p_{sx} \) denotes the source function with infinite reservoir boundary and has the form of

\[
p_{sx}(x', x, \tau, t) = \frac{1}{2\sqrt{\pi \eta_s(t - \tau)}} \exp \left[-\frac{(x - x')^2}{4\eta_s(t - \tau)}\right],
\]

with diffusivity defined as

\[
\eta_s = \frac{k_s}{\phi \mu c_t},
\]

and \( S \) denotes the strength of the source as

\[
S = \frac{Q}{\phi c_t Adx'}.
\]

Pressure drop in three-dimensional space of fluid removal from a point source can be obtained by applying Newman’s product method, which is the product of corresponding
source strength and source function responses in x, y and z directions (Newman 1936). Provided that source is removing fluid continuously, integrating pressure drop from an instantaneous source along time gives pressure behavior of a source with continuous removal of fluid. A continuous point source solution in an infinite reservoir, therefore, can be expressed as

\[
\Delta p(x', x, y', y, z', z, t) = \int_0^t \frac{q(\tau) d\tau}{\phi \sigma} \cdot p_{sx}(x', x, \tau, t) dx' \cdot p_{sy}(y', y, \tau, t) dy' \cdot p_{sz}(z', z, \tau, t) dz'.
\] (3.5)

The source strength S becomes

\[
S = \frac{q(\tau) d\tau}{\phi \sigma dx' dy' dz'},
\]

where q is fluid rate per unit time.

The final form of Eq. (3.5) can be written as

\[
\Delta p(x', x, y', y, z', z, t) = \int_0^t \frac{q(\tau) d\tau}{\phi \sigma} \cdot \frac{1}{8\pi(t - \tau)^{3/2}} \frac{1}{\eta_x \eta_y \eta_z} \exp \left[ - \frac{(x - x')^2}{4\eta_x(t - \tau)} \right] \exp \left[ - \frac{(y - y')^2}{4\eta_y(t - \tau)} \right] \exp \left[ - \frac{(z - z')^2}{4\eta_z(t - \tau)} \right].
\] (3.6)

When the source is between two parallel sealed or closed boundaries (as shown in Fig 3.2), the source function has been derived based on image principle by creating unlimited “imaged sources” to achieve equivalent no-flow boundary effect at original boundary locations. This is equivalent to effectively removing the original no-flow boundary in an infinite reservoir by using imaging wells. Alternatively, another derivation is achieved by solving flow equation with closed boundary condition applied (Carslaw and Jaeger 1959). Denoted by \( \overline{p}_{sx} \), source function under parallel closed boundaries (in x-direction) has
been provided by Carslaw and Jaeger (1959) and Gringarten and Ramey (1973) as the following,

\[
\bar{p}_{xx}(x', x, x_e, \tau, t) = \frac{1}{2\sqrt{\pi \eta_x (t - \tau)}} \left\{ \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{(x - x' + 2nx_e)^2}{4\eta_x (t - \tau)} \right] + \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{(x + x' + 2nx_e)^2}{4\eta_x (t - \tau)} \right] \right\}.
\]

(3.7)

Or

\[
\bar{p}_{xx}(x', x, x_e, \tau, t) = \frac{1}{x_e} \left[ 1 + 2 \sum_{n=1}^{\infty} \exp \left( -\frac{n^2 \pi^2 \eta_x t}{x_e^2} \right) \cos \left(\frac{n\pi x'}{x_e}\right) \cos \left(\frac{n\pi x}{x_e}\right) \right].
\]

(3.8)

Eq. (3.7) and Eq. (3.8) represent the two types of expressions of source function between sealed or closed parallel boundaries and they apply to early and late time computation respectively in this work. In detail, Carslaw and Jaeger (1959) explains their equivalence in mathematical derivation, and in this work we have adopted the numerical Laplace transformation algorithm established by Zhao and Thompson (2002) to generate computation results with near analytical accuracy.

\[x = 0 \quad x = x' \quad x = x_e\]

**Fig 3.2** No-flow parallel boundaries perpendicular to x-axis
The plane source solution in a rectilinear reservoir with no-flow outer boundary is derived in Appendix A and is expressed as

\[
\Delta p(x, x_e, y, y_e, z, z_e, t) =
\]

\[
\frac{1}{\phi c_t} \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{q_{i,j}(r)dr}{\Delta x \Delta z} \cdot I_{p_{3x}}(x_{i-1}, x_i, x, x_e, \tau, t) \cdot I_{p_{3z}}(z_{j-1}, z_j, z, z_e, \tau, t) \cdot \bar{p}_{3y}(y_0, y, y_e, \tau, t), \quad (A.7)
\]

and written in full form as

\[
\Delta p(x, x_e, y, y_e, z, z_e, t) =
\]

\[
\frac{1}{\phi c_t} \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{q_{i,j}(r)dr}{\Delta x \Delta z} \frac{1}{2L_f h_f/mn} \frac{1}{2\pi \eta_y(\tau - t)} \exp \left[-\frac{(y - y_0)^2}{4\eta_y(\tau - t)}\right] \]

\[
\frac{1}{2} \sum_{n=-\infty}^{\infty} \left\{ \text{erf} \left( \frac{x_i - x + 2nx_e}{2\sqrt{\eta_x(\tau - t)}} \right) - \text{erf} \left( \frac{x_{i-1} - x + 2nx_e}{2\sqrt{\eta_x(\tau - t)}} \right) \right. \\
+ \left. \text{erf} \left( \frac{x_i + x + 2nx_e}{2\sqrt{\eta_x(\tau - t)}} \right) - \text{erf} \left( \frac{x_{i-1} + x + 2nx_e}{2\sqrt{\eta_x(\tau - t)}} \right) \right\}
\]

\[
\frac{1}{2} \sum_{n=-\infty}^{\infty} \left\{ \text{erf} \left( \frac{z_i - z + 2nz_e}{2\sqrt{\eta_z(\tau - t)}} \right) - \text{erf} \left( \frac{z_{i-1} - z + 2nz_e}{2\sqrt{\eta_z(\tau - t)}} \right) \right. \\
+ \left. \text{erf} \left( \frac{z_i + z + 2nz_e}{2\sqrt{\eta_z(\tau - t)}} \right) - \text{erf} \left( \frac{z_{i-1} + z + 2nz_e}{2\sqrt{\eta_z(\tau - t)}} \right) \right\}, \quad (A.8)
\]

3.2.2 Modeling of a multi-layer reservoir partially penetrated by fractured horizontal well

Reservoirs with large-spanned vertical thickness that consists of a number of layers are commonly seen in tight formation, such as Bakken Formation and Eagle Ford Shale, and are produced by fractured horizontal well with multi-stage fractures. In heterogeneous reservoir, flowing behaviors of reservoir and well are usually modeled numerically (finite difference
method or finite element method) by using fine grids within reservoir and short time steps. For layered reservoir, however, analytical solution proposed in this study rather holds many advantages compared to numerical solution, as reservoir in the analytical model is divided into only as many locally homogeneous sub-reservoirs as the number of layers of the physical reservoir model, and within each sub-reservoir gridding is not required. In addition, solutions are pursued in Laplace domain in this study by taking numerical Laplace transforms and thereby are freed from the effect of time step. To simplify the modeling demonstration process, a reservoir consisting of only two vertically stacked layers, shown in Fig 3.3, is assumed. Layered reservoir with more than two layers can also be modeled by applying the same methodology.
Fig 3.3 Schematic of a fractured horizontal well with $N_{\text{stage}}$ transverse fractures producing from a two-layer reservoir
3. 2.3 Modeling philosophy

The two layers are in hydraulic communication with each other along their interface. The two layers are uniform and homogeneous porous mediums with constant vertical thickness $h_{e1}$ and $h_{e2}$ and have rock/fluid property $(k\phi\mu c_l)_1$ and $(k\phi\mu c_l)_2$ respectively. A horizontal well is drilled within one of the layers with $N_{stage}$ number of hydraulically fractures transversely along the wellbore; the fractures with infinite conductivity are assumed to penetrate into and produce from both layers. These fractures are regarded as plane sources in the study. Each fracture is assumed to have a fracture length of $2L_{fi} = 2L_f, 1 \leq i \leq N_{stage}$; non-uniform fracture length can be assumed if needed. The fracture heights penetrating into each layer are denoted as $h_{f1}$ in Layer 1 and $h_{f2}$ in Layer 2. In this study, the production contribution of horizontal wellbore trajectory is neglected because of the fact that fluid production from fractures dominates.

It is necessary to identify simple homogeneous reservoir components that make up this layered heterogeneous reservoir. First, considering Layer 1 only, for the $N_{stage}$ transverse fractures, it behaves as if there were two wells: the original fractured well with hydraulic fractures that reach the top of this layer and comprise only a portion of the original fracture vertical penetration, that is, $h_{f1}$; and an injecting planar fractured well at the area of communication interface of the two layers (Fig 3.4 a). Similarly, considering Layer 2 only, it behaves as if there were two wells also: original fractured well with hydraulic fractures with vertical penetration $h_{f2}$ starting from the bottom of layer 2; and a producing planar fractured well at the interface of the two layers (Fig 3.4 b)
Fig 3.4 Reservoir (a) from the perspective of Layer 1 and (b) from the perspective of Layer 2
In fact, inter-layer crossflow $q_{cf}(t)$ between the two layers is accounted for by the “planar fractured well” with its rate being equal to the crossflow rate. Because the fluid leaving Layer 2 is entering Layer 1, the production and injection rate of planar “fractured” wells in their each layer must be the same.

The following step is to decompose the reservoir into two homogeneous sub-reservoirs at their contact interface. Discretizing the reservoir components from one another at their junction is achieved by applying “no-flow boundary” concept. For simple reservoir systems that contain wells, the no-flow boundary is achieved by applying image well principle to. For the situation being considered, Layer 1 and Layer 2 are each individually discretized using this strategy and the processes are graphically presented in Fig 3.5 (a) and Fig 3.5 (b).

Planar injection or production wells are added along the plane of symmetry to account for fluid transfer. The discretized Layer 1 and Layer 2 systems are coupled by pressure and rate continuity at their contact interface. This original methodology modeling heterogeneous reservoir (decomposing complex reservoir into homogeneous ones, then coupling together) was devised by Zhao and Thompson (2002).

**Solution to Layer 1.** In Layer 1, pressure drop at any location, $\Delta p_1$, is added up based on superposition principle as containing three wells (Fig 3.5 a), namely: the original fractured well but only contains vertical fracture penetration as $h_{f1}$, and its image well, as well as the injecting planar fractured well. With the same discretizing process of plane source in Fig A.2, each original hydraulic fracture plane source is divided into $m \times n$ sub-plane sources, imaged hydraulic fracture plane source is simply a reflection of its
Fig 3.5 (a) Layer 1 discretized from reservoir by applying no-flow boundary and (b) Layer 2 discretized from reservoir by applying no-flow boundary
counterpart (original fractures), and injecting planar fractured well is divided into $M \times N$ sub-plane sources respectively (Fig 3.6); and based on plane source solution given by Eq. (A.7), pressure drop in Layer 1 can be written as

\[
\Delta p_1(x, x_e, y, y_e, z_1, 2h_{e1}, t) = 
\]

\[
\frac{1}{(\phi c_t)_1} \sum_{k=1}^{N_{\text{stage}}} \int_0^1 \sum_{i=1}^n \sum_{j=1}^m \frac{q_{k,i,j}}{2L_f h_{f1}/mn} \left[ l \vec{p}_{sx}(x_{f,i-1}, x_{f,i}, x, x_e, \tau, t) \cdot l \vec{p}_{sz}(z_{f,i-1}, z_{f,i}, z_1, 2h_{e1}, \tau, t) \cdot \vec{p}_{sy}(y_k, y, y_e, \tau, t) \right]
\]

\[
+ \frac{1}{(\phi c_t)_1} \sum_{k=1}^{N_{\text{stage}}} \int_0^1 \sum_{i=1}^n \sum_{j=1}^m \frac{q_{k,i,j}}{2L_f h_{f1}/mn} \left[ l \vec{p}_{sx}(x_{f,i-1}, x_{f,i}, x, x_e, \tau, t) \cdot l \vec{p}_{sz}(-z_{f,i}, -z_{f,i-1}, z_1, 2h_{e1}, \tau, t) \cdot \vec{p}_{sy}(y_k, y, y_e, \tau, t) \right]
\]

\[
- \frac{1}{(\phi c_t)_1} \int_0^1 \sum_{i=1}^n \sum_{j=1}^m \frac{2q_{i,j}^{f1}(\tau) d\tau}{x_{i,j} f/MN} l \vec{p}_{sx}(x_{i,j-1}, x_{i,j}, x, x_e, \tau, t) \cdot l \vec{p}_{sy}(y_{i,j-1}, y_{i,j}, y, y_e, \tau, t) \cdot \vec{p}_{sz}(0, z_1, 2h_{e1}, \tau, t) .
\]

(3.9)

Note that in this study, producing is recognized as “+” while injecting is recognized as “-”. In Eq. (3.9), $q_{k,i,j}$ represents production rate of a sub-plane from the $k$-th hydraulic transverse fracture along horizontal wellbore in Layer 1, $q_{i,j}^{f1}$ denotes the production rate of a sub-plane from injecting planar fractured well. $xyz_1$ is defined as the sub-coordinate, shown in Fig 3.5 (a), within Layer 1; the local coordinate with Layer 1 is defined as having shifted upward vertically $h_{e1}$ from global coordinate $xyz$ and having been tipped upside down, mathematically described by

\[
z_1 = h_{e1} - z .
\]

(3.10)
Fig 3.6 Discretization of the $k$-th original fracture plane penetrated in Layer 1 and of planar fractured well
Solution to Layer 2. Similar to Layer 1, pressure drop in Layer 2 is also added up as containing 3 wells (Fig 3.5 b), namely: the original fractured well but only contains vertical fracture penetration as \( h_{f2} \), and its image well, as well as the producing planar fractured well. Each original hydraulic fracture plane source is also divided into \( m \times n \) sub-plane sources (Fig 3.7) and the planar fractured well is the same as in Layer 1. It is important to notice that the planar well, in Layer 2, acts as if it was injecting while it acts as if, in Layer 1, it was producing. Pressure drop in Layer 2, \( \Delta p_2 \), can be written as

\[
\Delta p_2(x, x_o, y, y_e, z_2, 2h_{e2}, t) =
\]

\[
\frac{1}{(\phi c_l)_2} \sum_{k=1}^{N_{stage}} \int_0^t \sum_{i=1}^n \sum_{j=1}^m \frac{q_{k,i,j}^{f2}(\tau)dt}{2L_f h_{f2/mn}} \left[ l\bar{p}_{sx}\left(x_{f,i-1}, x_f, x, y, y_e, z_2, 2h_{e2}, t\right) \right] 
\cdot l\bar{p}_{sz}\left(z_{f,j-1}, z_{f,j}, z_2, 2h_{e2}, t\right)
\cdot \bar{p}_{sy}(y_e, y, y_e, t)
\]

\[
+ \frac{1}{(\phi c_l)_2} \sum_{k=1}^{N_{stage}} \int_0^t \sum_{i=1}^n \sum_{j=1}^m \frac{q_{k,i,j}^{f2}(\tau)dt}{2L_f h_{f2/mn}} \left[ l\bar{p}_{sx}\left(x_{f,i-1}, x_f, x, y, y_e, t\right) \right] 
\cdot l\bar{p}_{sz}\left(-z_{f,j-1}, z_{f,j}, z_2, 2h_{e2}, t\right)
\cdot \bar{p}_{sy}(y_e, y, y_e, t)
\]

\[
+ \frac{1}{(\phi c_l)_2} \sum_{l=1}^{N} \sum_{j=1}^{M} \frac{2q_{k,i,j}^{f2}(\tau)dt}{x_{e} x_{e} / M N} l\bar{p}_{sx}\left(x_{Li-1}, x_l, x, y, y_e, t\right) \cdot l\bar{p}_{sy}\left(y_{Li-1}, y_l, y, y_e, t\right) \cdot \bar{p}_{sz}(0, z_2, 2h_{e2}, t, t) .
\]

(3.11)

In Eq. (3.11), \( q_{k,i,j}^{f2} \) represents production rate of a sub-plane from the \( k-th \) hydraulic fracture in Layer 2. \( xyz_2 \) is defined as the sub-coordinate, shown in Fig 3.7, within Layer 2; the local coordinate with Layer 2 is defined as having shifted upward vertically \( h_{e1} \) from global coordinate \( xyz \), mathematically described by

\[
z_2 = z - h_{e1} .
\]

(3.12)
Fig 3.7 Discretization of the $k$-th original fracture plane penetrated in Layer 2
Laplace transforms with respect to time are taken for Eq. (3.9) and Eq. (3.11). Letting $L[\cdot]$ denote Laplace operator and $s$ denotes Laplace variable, the pressure response in Layer 1 and Layer 2 in Laplace domain becomes

$$L[\Delta p_1(x, x_e, y, y_e, z_1, 2h_{e1}, t)] =$$

$$+ \frac{1}{(\phi c_t)_1} \sum_{k=1}^{N_{stage}} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{m}{2L_f h_f} L[q_{k,i,j}^1(t)] \begin{bmatrix}
L[\tilde{p}_{sx}(x_{f,i-1}, x_{f,i}, x, x_e, \tau, t)] \cdot L[\tilde{p}_{sy}(y, y_e, \tau, t)] \\
L[\tilde{p}_{sz}(z_{1,i-1}, z_{1,i}, z_1, 2h_{e1}, \tau, t)] \\
L[\tilde{p}_{sz}(y, y_e, \tau, t)]
\end{bmatrix}$$

$$- \frac{1}{(\phi c_t)_1} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{M}{2L_f h_f} L[2q_{l,i}^1(t)] \begin{bmatrix}
L[\tilde{p}_{sx}(x_{l,i-1}, x_{l,i}, x, x_e, \tau, t)] \cdot L[\tilde{p}_{sy}(y, y_e, \tau, t)] \\
L[\tilde{p}_{sz}(z, z_1, 2h_{e1}, \tau, t)]
\end{bmatrix}$$

(3.13)

and

$$L[\Delta p_2(x, x_e, y, y_e, z_2, 2h_{e2}, t)] =$$

$$+ \frac{1}{(\phi c_t)_2} \sum_{k=1}^{N_{stage}} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{m}{2L_f h_f} L[q_{k,i,j}^2(t)] \begin{bmatrix}
L[\tilde{p}_{sx}(x_{f,j-1}, x_{f,i}, x, x_e, \tau, t)] \cdot L[\tilde{p}_{sy}(y, y_e, \tau, t)] \\
L[\tilde{p}_{sz}(z_{f,j-1}, z_{f,j}, z_2, 2h_{e2}, \tau, t)] \\
L[\tilde{p}_{sz}(y, y_e, \tau, t)]
\end{bmatrix}$$

$$+ \frac{1}{(\phi c_t)_2} \sum_{k=1}^{N_{stage}} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{m}{2L_f h_f} L[q_{k,i,j}^2(t)] \begin{bmatrix}
L[\tilde{p}_{sx}(x_{f,j-1}, x_{f,i}, x, x_e, \tau, t)] \cdot L[\tilde{p}_{sy}(y, y_e, \tau, t)] \\
L[\tilde{p}_{sz}(z_{f,j-1}, z_{f,j}, z_2, 2h_{e2}, \tau, t)] \\
L[\tilde{p}_{sz}(y, y_e, \tau, t)]
\end{bmatrix}$$

$$+ \frac{1}{(\phi c_t)_2} \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{N}{2L_f h_f} L[2q_{l,j}^2(t)] \begin{bmatrix}
L[\tilde{p}_{sx}(x_{l,j}, x_{l,i}, x, x_e, \tau, t)] \cdot L[\tilde{p}_{sy}(y, y_e, \tau, t)] \\
L[\tilde{p}_{sz}(z_{1,i-1}, z_{1,i}, z_1, 2h_{e1}, \tau, t)] \\
L[\tilde{p}_{sz}(y, y_e, \tau, t)]
\end{bmatrix}$$

(3.14)
Zhao and Thompson (2002) has devised a fast and highly accurate method of obtaining numerical Laplace transforms, and it is implemented in this work. The complicated Laplace transform of Eq. (3.13) or Eq. (3.14), thereby, can be generated.

**Coupling.** Eq. (3.13) and Eq. (3.14) are evaluated at midpoint of each sub-plane of hydraulic fractures, which gives $(2 \cdot m \cdot n \cdot N_{stage})$ pressure expressions in Laplace domain. Hydraulic fractures with infinite conductivity are adopted in this study, that is, pressure evaluated at each sub-plane is equated to each other. Since production from horizontal wellbore is ignored in this study, pressure along wellbore is uniform and thereby pressures at interceptions of wellbore and fractures are equal to one another. By equating pressures at midpoint of each sub-plane source of hydraulic fractures and equating pressures at interceptions of fractures and wellbore, it yields $(2 \cdot m \cdot n \cdot N_{stage} - 1)$ equations. Layer 1 and Layer 2 are interacting with each other by fluid transfer (inter-layer crossflow) and pressure continuity; pressure continuity is achieved by equating pressure at midpoint of each sub-plane of the planar injection/production well from Eq. (3.13) and Eq. (3.14), which yields $(M \cdot N)$ equations in Laplace domain. In addition to pressure continuity, material balance must be satisfied as well. Rates from every sub-plane of hydraulic fractures are summed up, amounting to the production rate at well.

The above coupling conditions of such 2-layer reservoir-well system yield $(2 \cdot m \cdot n \cdot N_{stage} + M \cdot N)$ linear equations in the unknown sub-plane rates of hydraulic fractures and of planar injection/production well. Each sub-plane rate can be obtained in Laplace domain and substitute them into Eq. (3.13) or Eq. (3.14) so that pressure at desired
location can be obtained. Rates or pressure in Laplace domain, then, can be inverted to real-time domain by using Stephest inversion algorithm (Stephest 1970).

3. 3 Two-layer Reservoir Results

In tight formation, reservoir normally has large vertical spanned thickness and therefore constitutes more than one layer where each layer has different mobility or storativity. Each layer is influenced by inter-layer crossflow from upper and lower layers, which depends on the flowing capacity and storativity of all layers, and also on fracture penetration level in each layer if hydraulic fractures are applied to horizontal well. In order to examine how reservoir heterogeneity and hydraulic fracture penetration level in either horizontal or vertical direction affect well behavior, 2-layer reservoir is the simplest but most representative candidate. Under such simple circumstance, the effect of reservoir heterogeneity and hydraulic fracture penetration ratio can be demonstrated clearly based on well response in terms of pressure or production rate behavior.

3. 3.1 Dimensionless groups

The dimensions for the two-layer model are defined in Fig 3.8. Results are presented in dimensionless form documented as type curves, which provides many advantages as dimensionless solution is universal solution.

Based on a reference length \( L_r \) (random), dimensionless reservoir and fracture geometries are defined as

\[
x_{eD} = \frac{x_e}{L_r},
\]

\[
y_{eD} = \frac{y_e}{L_r},
\] (3.15) (3.16)
Dimensionless pressure $p_D$ under constant production rate of well is defined based on properties of Layer 1:

$$ p_D = \frac{2\pi k_v x_e}{q_w \mu_1} \Delta p . \quad (3.22) $$

Dimensionless rate response of well under constant bottom pressure is defined as

$$ q_{wd} = \frac{q_w \mu_1}{2\pi k_v x_e \Delta p} . \quad (3.23) $$

Note that in this study reservoir horizontal thickness, $x_e$, is used in dimensionless pressure and rate instead of reservoir vertical thickness as traditional form.

Dimensionless rate from hydraulic fractures in each layer and inter-layer crossflow are defined based on a constant rate as

$$ q^f_D = \frac{q^f}{q_e} , \quad i = 1, 2 . \quad (3.24) $$

$$ q^{cf}_D = \frac{q^{cf}}{q_e} . \quad (3.25) $$
Dimensionless time is defined based on reservoir properties of Layer 1 as

$$t_D = \frac{k_i t}{\phi_1 \mu_i c_{ti} L_i^2}.$$  \hfill (3.26)

Dimensionless material balance time is defined as

$$t_{MBD} = \frac{\int_0^{t_D} q_{WB}(t_D) dt_D}{q_{WB}(t_D)} = \frac{k_i t_{MB}}{\phi_1 \mu_i c_{ti} L_r^2},$$  \hfill (3.27a)

where material balance time $t_{MB}$ is defined as

$$t_{MB} = \frac{\int_0^t q_w(t) dt}{q_w(t)}.$$  \hfill (3.27b)

The mobility and the storativity for each layer are defined as follows:

$$M_i = \frac{k_i}{\mu_i}, \quad i = 1, 2. \hfill (3.28)$$

$$C_{si} = \phi_i c_{ti}, \quad i = 1, 2. \hfill (3.29)$$

Based on Zhao et al. (2016) and the general understanding of Zhao and Thompson (2001), a new set of terms for the definitions of modified dimensionless rate, modified dimensionless time, and “Scaler” in this study are extended for 3D multiple layers with the following expressions.

$$q_{DM} = \frac{q_{WD} \cdot Scaler}{N_{stage}}, \hfill (3.30)$$

$$t_{DM} = \frac{t_{MBD}}{\left( h_{FD} + \sqrt{\frac{M_i C_{si}}{M_{i} C_{s1}}} \cdot Scaler^2 \right) \cdot \left( \frac{x_{eD}}{2L_{FD}} \right)^2}, \hfill (3.31)$$

where “Scaler” in this case under study is accordingly defined as
The three unique terms are proposed by Zhao et al. (2016) to help analyze rate-transient responses of fractured well in 2-dimensional homogeneous reservoir. This study extends these definitions into 3-dimensional reservoir produced by fractured well and, based on Zhao and Thompson (2001)’s semi-analytical model of communicating reservoir with linear heterogeneities, this study also integrates mobility and storativity of each layer into the 3 terms. Appendix B exemplifies the behaviors of \( q_{DM} vs t_{DM} \) under early linear flow and late time pseudo-steady state flow based on Wattenbarger et al. (1998) and Blasingame and Lee (1991) production equations respectively.

Not only can modified dimensionless time and “Scaler” be defined in the form of Eq. (3.31) and Eq. (3.32), they can also alternately be defined in the following format as

\[
Scaler = \frac{x_{eD} y_{eD} \left( h_{e1D} + \frac{C_{S2}}{C_{S1}} h_{e2D} \right)}{(2L_{fD})^2 \cdot N_{stage}} \cdot \frac{x_{eD}}{2L_{fD}} \cdot \left( h_{f1D} + \frac{M_2 C_{S2}}{M_1 C_{S1}} h_{f2D} \right),
\]

(3.32)

\[
t_{DM} = \frac{t_{MBD}}{(2L_{fD})^2 \cdot Scaler^2} \cdot \left( h_{eD} \right)^2 \cdot \left( h_{f1D} + \frac{M_2 C_{S2}}{M_1 C_{S1}} h_{f2D} \right),
\]

(3.33)

\[
Scaler = \frac{x_{eD} y_{eD} \left( h_{e1D} + \frac{C_{S2}}{C_{S1}} h_{e2D} \right)}{(2L_{fD})^2 \cdot N_{stage}} \cdot \left( h_{eD} \right)^2 \cdot \left( h_{f1D} + \frac{M_2 C_{S2}}{M_1 C_{S1}} h_{f2D} \right),
\]

(3.34)

with the above definitions, dimensionless rate of well under constant pressure needs to be defined as

\[
q_{WD} = \frac{q_{w1} \mu_1}{2\pi k_1 h_e \Delta p}.
\]

(3.35)
3.3.2 Results and Discussions

To validate the 2-layer heterogeneous reservoir model and to illustrate the system response from well behavior, solutions are checked against various scenarios. Five scenarios are designed to meet this purpose and they are tabulated in Table 3.1.

Table 3.1: Various Well-Reservoir Systems for Each Scenario in the 2-Layer Reservoir Model

<table>
<thead>
<tr>
<th>Scenario 1-1: Testing case</th>
<th>Parameters</th>
<th>Sensitivity Analysis</th>
<th>Brief Description</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( M_2/M_1 = 1 )</td>
<td>( 2L_{FD} = 0.001, 0.1, 0.5, 1 )</td>
<td>To test if solution of well behavior can approach that of a horizontal well when ( L_{FD} ) is extremely short</td>
</tr>
<tr>
<td></td>
<td>( h_{f1D} = h_{f2D} = 25 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( h_{e1D} = h_{e2D} = 50 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x_e = 5 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( C_{S2}/C_{S1} = 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario 1-2: Effect of mobility ratio</th>
<th>Parameters</th>
<th>Sensitivity Analysis</th>
<th>Brief Description</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( h_{f1}/h_{e1} = 0.1 )</td>
<td>( 2L_f/x_e = 1 )</td>
<td>To check how inter-layer crossflow from upper layer, without penetrated by fractures, affect well response</td>
</tr>
<tr>
<td></td>
<td>( h_{e1} = h_{e2} )</td>
<td>( M_2/M_1 = 0.01,0.1, 1,10, 100 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 2L_f/x_e = 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( C_{S1} = C_{S2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario 1-3: Effect of mobility ratio</td>
<td>( \frac{h_{f1}}{h_{e1}} = 0.1 )  ( \frac{h_{f2}}{h_{e2}} = 0.1 )  ( h_{e1} = h_{e2} )  ( 2L_f/x_e = 1 )  ( C_{s1} = C_{s2} )</td>
<td>( \frac{M_2}{M_1} = 0.01, 0.1, 1, 10, 100 )</td>
<td>To present complex flow regimes and show differences between ( q_{DM} vs t_{DM} ) and ( q_{WD} vs t_D ) while both layers are penetrated by fractures.</td>
</tr>
</tbody>
</table>

| Scenario 1-4: Effect of fracture length | \( \frac{M_2}{M_1} = 10 \)  \( \frac{h_{f1}}{h_{e1}} = 0.02 \)  \( \frac{h_{f2}}{h_{e2}} = 0.02 \)  \( h_{e1} = h_{e2} \)  \( C_{s1} = C_{s2} \) | \( 2L_f/x_e = 0.05, 0.2, 0.4, 0.6, 0.8, 1 \) | To present \( q_{DM} vs t_{DM} \) and inter-layer crossflow flux profile while fractures partially penetrate reservoir in both horizontal and vertical directions |

| Scenario 1-5: Effect of vertical-to-horizontal permeability ratio | \( \frac{M_2}{M_1} = 20 \)  \( \frac{h_{f1}}{h_{e1}} = 0.1 \)  \( \frac{h_{f2}}{h_{e2}} = 0.1 \)  \( h_{e1} = h_{e2} \)  \( 2L_f/x_e = 1 \)  \( C_{s1} = C_{s2} \) | \( (k_v/k_r)_1 = 0.001 \) while \( (k_v/k_r)_1 = 1 \) \( (k_v/k_r)_2 = 1 \) \( (k_v/k_r)_2 = 0.001, 0.01, 0.1, 1 \) while \( (k_v/k_r)_1 = 1 \) | To show the well response considering reservoir vertical-to-horizontal permeability ratio |
Fig 3.8 A 2-layer reservoir model for scenarios under study
Scenario 1-1: A validating and testing case. When the schematic of the 2-layer model is designed to stand for some scenarios with already existing standard solutions, the results generated by this model must closely approach the standard solutions. For example, suppose that there is only one transverse fracture along horizontal wellbore with very short fracture length while fracture height is comparatively much greater, and that Layer 1 has identical properties to those of Layer 2; the result generated by the model must approach the behavior of a horizontal well or a partially penetrating vertical well producing in homogeneous reservoir because the fracture geometry is closely approaching a line due to extremely short horizontal penetration; and if fracture length becomes greater to a certain point, early linear flow of the fractured well should be exhibited.

Fig 3.9 shows the transient pressure and its derivative of a one-stage fractured well with various dimensionless fracture lengths of $2L_{FD} = 0.001, 0.1, 0.5, 1$, and presents a standard horizontal well solution as reference. When $2L_{FD} = 0.001$ and 0.1, the one-stage fractured well behavior almost overlaps that of a horizontal well and flow regimes are clearly exhibited: at early time, pseudo-radial flow is presented with flat pressure derivative; after pressure-transient reaches horizontal boundaries (x-axis), linear flow with pressure derivative 1/2 slope is evolved and finally pseudo-steady state flow is shown after all boundaries are felt. When $2L_f = 1$, early linear flow is successfully exhibited but lasts only a short period of time and pressure derivative shows pseudo-radial flow behavior—after all fracture length is still much shorter than height—overlapping that of standard horizontal well solution.
Fig 3.9 Comparison of horizontal well behavior and results from the 2-layer model with $M_2/M_1=1$, $C_{s2}/C_{s1}=1$, $N_{stage}=1$; $2L_{FD} = 0.001, 0.1, 0.5$ and 1 while $x_{eD} = 5$ and $(h_{f1D} + h_{f2D}) = 50$
Scenario 1-2: Effect of reservoir heterogeneity under various $M_2/M_1$ with fractures penetrating only the lower Layer. Fig 3.10 (a) presents pressure derivative response of well under various $M_2/M_1$. Early formation linear flow, with half slope pressure derivative, marks the first flow regime explaining that each fracture is producing within its own individual linear-expanding drainage area, and pressure derivative is followed by a sharp rise with slope almost reaching unit (Chen and Raghavan 2013), which is caused by the intense interference among fractures and the unit slope might account for the SRV (Stimulated Reservoir Volume) boundary. The effect of inter-layer crossflow from upper layer takes place around $t_D = 1$; in general, when mobility of upper layer is greater than the lower layer, pressure derivative goes below 0.5 as flow regime enters pseudo-radial flow, and vice versa. It is notable that the well sensitivity to the level of mobility ratio is quite different when the upper layer has greater or smaller mobility. It can be seen that, in Fig 3.10 (a), well behavior is much better with $M_2/M_1 = 100$ compared to that with $M_2/M_1 = 10$; in contrast, well behavior does not show a big difference between $M_2/M_1 = 0.1$ and 0.01. Fig 3.10 (b) presents the corresponding rate response of well producing at constant bottom pressure, and in this figure material balance time is used. In terms of production rate decline, Fig 3.10 (b) shows that it declines with 1/2 slope during early linear flow and is followed by a sharp drop with almost unit slope during interference period among fractures. Following the almost unit slope drop of rate is a less steep decline period, in which greater upper layer mobility leads to better production rate. Due to the use of material balance time, rates under all mobility ratios overlap each other exhibiting a unit slope decline during boundary dominant flow, which indicates that
production during boundary dominant flow is independent of reservoir mobility when material balance time is applied. Inter-layer crossflow rate profiles along with time is presented in Fig. 3.11, they all end up with half production rate of well regardless of mobility ratio, meaning that each layer provides half well production during boundary dominant flow, for the reason that each layer has the same volume in the scenario and storativity is homogeneous throughout the reservoir and that well production rate is provided simply by fluid expansion within the reservoir.
Fig 3.10 (a) Dimensionless pressure derivative $dp_D/ln t_D$ vs. dimensionless time $t_D$ and (b) dimensionless rate $q_{wd}$ vs. dimensionless material balance time $t_{MBD}$, under various $M_2/M_1$ ratios while only Layer 1 is penetrated by fractures, $\frac{2L_f}{xe}=1$, $\frac{h_{f1}}{h_{e1}}=0.1$, $h_{f2}=0$, $h_{e1}=h_{e2}$, $N_{stage}=10$ and $C_{s2}/C_{s1}=1$
Fig 3.11 Inter-layer crossflow rate $q_D^{cf}$ under various $M_2/M_1$ ratios while only Layer 1 is penetrated by fractures, $\frac{2L_f}{x_e} = 1$, $\frac{h_{f1}}{h_{e1}} = 0.1$, $h_f^2 = 0$, $h_{e1} = h_{e2}$, $N_{stage} = 10$ and $C_{s2}/C_{s1} = 1$
Scenario 1-3: Effect of reservoir heterogeneity under various $M_2/M_1$ with fractures penetrating both layers. The influence of distinct mobility of the two layers is examined in this scenario as fractures penetrate both layers. Fig 3.12 presents pressure derivative behavior of a well producing at constant rate under various levels of heterogeneities of the two layers as $M_2/M_1 = 100, 10, 1, 0.1, 0.01$. Similar to the previous scenario, pressure behavior of well first shows early formation linear flow but the time marking the end of linear flow (deviating from 1/2 slope) is different, depending on the mobility ratio. It is because fractures in both layers start producing from the beginning and interference among fractures takes place earlier if mobility in the upper layer is greater or interference takes place later if mobility in the upper layer is smaller.

In this scenario, fracture vertical penetration is comparatively much smaller than the whole reservoir vertical thickness, such that stabilization of reservoir flowing mobility (pseudo-radial flow) might be evolved as pressure-transient travels far out into reservoir before hits boundaries and such flowing mobility must be approaching a type of average of the two layers as if well was producing in a single-layer homogeneous reservoir.

Fig 3.12 exhibits the flat pressure derivative values. Zhao and Thompson (2002) and Spivey and Lee (2013) have shown that the stabilizing pressure derivative during pseudo-radial flow in such two-region heterogeneous reservoir should approach the value described by the following formula.

$$\frac{dP_D}{d\ln P_D} \rightarrow \frac{1}{1 + \frac{M_2}{M_1}}$$  \hspace{1cm} (3.36)
Fig 3.12 Dimensionless pressure derivative \( dp_D / lnt_D \) vs. dimensionless time \( t_D \) under various \( M_2/M_1 \) ratios, with \( \frac{2lr}{xe} = 1 \), \( h_{f1} = h_{f2} = h_f \), \( h_{e1} = h_{e2} = h_e \), \( \frac{h_f}{h_e} = 0.1 \), \( N_{stage}=10 \) and \( C_{s2}/C_{s1}=1 \).
The rate response of fractured well producing at constant bottom pressure is presented in Fig 3.13 (a). It can be seen that comparison and understanding of the rate outcomes are hard to reach. Zhao et al. (2016) devised method for analyzing sequences of flow regimes from well rate response by defining new terms of modified dimensionless production rate $q_{DM}$ and modified dimensionless time $t_{DM}$, which, in a general physical sense, unifies the geometrical relationship between fracture and reservoir system in a systematic manner. In this work, further improvement has been made by integrating heterogeneous reservoir characteristics including mobility $(k/\mu)$ and storativity $(\phi c_r)$ of each layer into the definition of $q_{DM}$, in Eq. (3.30), and $t_{DM}$, in Eq. (3.31), so that effect of various mobility and storativity in multiple layers can be evaluated systematically along with fracture and reservoir geometries. Curves in Fig 3.13 (b) represent the production rate responses from Fig 3.13 (a) again, but in the new manner of $q_{DM}$ vs $t_{DM}$. It can be seen clearly now in Fig 3.13 (b) that all curves merge together in early linear flow and late pseudo-steady state periods, which reduces complexity and difficulty greatly in analyzing production rate from such an intricate reservoir-well system. Appendix B gives the mathematical description of $q_{DM}$ vs $t_{DM}$ under early linear and late pseudo-steady state flow conditions; the line 1 and the line 2 in Fig 3.13 (b) are generated by Eq. (B.15) and Eq. (B.25) respectively that are the derived mathematical expressions of $q_{DM}$ vs $t_{DM}$ for early and late time periods.
Fig 3.13 (a) Dimensionless rate $q_D$ vs. dimensionless time $t_D$ and (b) modified dimensionless rate $q_{DM}$ vs. modified dimensionless time $t_{DM}$, under various $M_2/M_1$ ratios with $\frac{2L_f}{x_e} = 1$, $h_{f1} = h_{f2} = h_f$, $h_{e1} = h_{e2} = h_e$, $\frac{h_f}{h_e} = 0.1$, $N_{stage} = 10$ and $C_{s2}/C_{s1} = 1$
**Fig 3.14** presents production rate from each layer as well as inter-layer crossflow rate (from less permeable layer to more permeable layer) when well is producing at constant rate. General observation indicates that the overall production rate from the more permeable layer is always higher than that of the less permeable layer. However, further careful scrutinization of **Fig 3.14** tells that the production rate profiles can be categorized into three phases.

**Phase 1:** The production rate from the more permeable layer, $q_{f1}^{p}$, decreases a bit while the rate from the less permeable layer, $q_{f2}^{p}$, increases a bit after the linear flow regime while interference in the more permeable layer begins;

**Phase 2:** During the periods of elliptical and pseudo-radial flow regimes, on the other hand, the production rate from the more permeable layer, $q_{f2}^{p}$, increases continuously while the rate from the less permeable layer, $q_{f1}^{p}$, reduces at the meantime until the initiation of the pseudo-steady state flow regime;

**Phase 3:** After the start of the pseudo-steady state (PSS) flow regime, the production rate from the more permeable layer, $q_{f2}^{p}$, decreases gradually while the rate from the less permeable layer, $q_{f1}^{p}$, increases slightly. These rates are finally stabilized when full pseudo-steady state is reached for the entire reservoir system.

During this production process, it is clear to observe how much the proceeding evolvement of flow regimes around the fractures affects the inter-layer crossflow over time. Despite of the variation involved in the three-phase process described above, the rate of inter-layer crossflow, $q_{c}^{f}$, is steadily increasing until it is stabilized at full pseudo-steady state flow regime condition. More confidence in solution accuracy can be reached
based on the obvious fact shown in Fig 3.14 that, for the homogeneous reservoir case ($M_2/M_1=1$), the solution of the dimensionless production rate in each layer is strictly 0.5 without any deviation and the rate of inter-layer crossflow $q_D^{cf} = 0$, which coordinates consistently with physics because now the geometry of well and reservoir system is symmetrical as $M_2/M_1=1$. 
Fig 3.14 production rates of fractures in layer 1, $q_{D}^{f1}$, and layer 2, $q_{D}^{f2}$, as well as the rate of inter-layer crossflow (from less permeable layer 1 to more permeable layer 2) $q_{D}^{cf}$ under various $M_2/M_1$ ratios, with $\frac{2l_f}{xe} = 1$, $h_{f1} = h_{f2} = h_f$, $h_{e1} = h_{e2} = h_e$, $\frac{h_f}{h_e} = 0.1$, $N_{stage} = 10$ and $C_{s2}/C_{s1} = 1$
Scenario 1-4: Effect of various levels of fracture horizontal penetration (fracture length $2L_f$) to reservoir horizontal thickness (reservoir width $x_w$) ratio. This scenario explores pressure and rate responses of well corresponding to different fracture length and reservoir horizontal thickness ratio, with fixed fracture height partially penetrating both layers under $M_2/M_1 = 10$.

**Fig 3.15 (a)** provides pressure derivative behavior under various horizontal fracture penetration ratios, namely: 0.05, 0.2, 0.4, 0.6, 0.8 and 1. For the one with shortest fracture length ($2L_f/x_w = 0.05$), it shows that spherical flow is clearly evolved with -$1/2$ slope during infinite-acting flow period, as fracture vertical penetration and reservoir vertical thickness ratio is very small at 0.02.

**Fig 3.15 (b)** shows the modified dimensionless rate versus modified dimensionless time. One of the powerful advantages of production rate in the manner of $q_{DM}$ vs $t_{DM}$ is that by plotting it this way, the early and late-time rate responses corresponding to various reservoir and fracture geometry hold together, which is very useful and meaningful when matching field data; it will be discussed in field case application section.

Inter-layer crossflow flux profile (from the less permeable layer to the more permeable layer) over the interface, with various fracture length and reservoir horizontal thickness ratio as $2L_f/x_w = 0.1, 0.4, 0.7$, at comparatively early time $t_D = 10$ and late time $t_D = 10000$ are provided in **Fig 3.16 (a)** and **Fig 3.16 (b)** respectively. At early time, we can see from the flux distribution, in **Fig 3.16 (a)**, that inter-layer flux around well (hydraulic fractures) is greater than flux far out at the sides; while, at late time showing in **Fig 3.16 (b)**, inter-layer flux is smaller around well than flux far out at the sides.
Fig 3.15 (a) Dimensionless pressure derivative $dp_D/d\ln t_D$ vs. dimensionless time $t_D$ and (b) modified dimensionless rate $q_{DM}$ versus modified dimensionless time $t_{DM}$, under various fracture length penetration ratios with $h_f^1 = h_f^2 = h_f$, $h_{e1} = h_{e2} = h_e$, $\frac{h_f}{h_e} = 0.02$; $M_2/M_1 = 10$ and $C_{S2}/C_{S1} = 1$; $N_{stage}=10$
Fig 3.16 (a) Inter-layer crossflow flux distribution at early time $t_D = 10$, under three

\[ \frac{2L_f}{x_e} = 0.1, 0.4, 0.7 \] with $h_{f1} = h_{f2} = h_f$, $h_{e1} = h_{e2} = h_e$, $\frac{h_f}{h_e} = 0.05$; $M_2/M_1 = 10$ and

$C_{S2}/C_{S1} = 1$; $N_{stage}=10$
Fig 3.16 (b) Inter-layer crossflow flux distribution at late time $t_D = 10000$, under three

$$\frac{2L_I}{x_e} = 0.1, 0.4, 0.7$$
with $h_f = h_{f2} = h_f$, $h_{e1} = h_{e2} = h_e$, $\frac{h_f}{h_e} = 0.05$; $M_2/M_1 = 10$ and $C_{S2}/C_{S1} = 1$; $N_{stage}=10$
Scenario 1-5: Effect of various level of reservoir permeability anisotropy. When reservoir anisotropy of permeability in vertical direction is considered, there is huge difference for well behavior depending on which layer has the anisotropy, the less or the more permeable layer. Fig 3.17 (a) and Fig 3.17 (b) provide dimensionless pressure derivative and modified dimensionless rate solution to investigate various levels of reservoir anisotropy effect, with vertical/horizontal permeability ratio \( (k_v/k_r)_2 = 0.001, 0.01, 0.1 \) and 1 in the more permeable layer while the less permeable layer is isotropic, on well performance as \( M_2/M_1 = 20 \). It shows that greater flowing resistance in vertical direction causes obvious rate reduction or extra pressure drop from well, which is proportional to the level of anisotropy. The presence of anisotropy distorts characteristic flow regimes associated with layered reservoir, and, it also postpones the time arriving at pseudo-steady state.

Fig 3.18 (a) and Fig 3.18 (b) show the effect of reservoir vertical anisotropy in the less permeable layer on well pressure and rate behavior. In this case, the upper or more permeable layer is isotropic. In contrast to the previous case, solutions indicate that anisotropy in the less permeable layer barely has an effect on well performance.
Fig 3.17 (a) Pressure derivative response $dp_D/d\ln t_D$ vs. dimensionless time $t_D$ and (b) modified dimensionless rate $q_{DM}$ versus modified dimensionless time $t_{DM}$, subject to various level of anisotropy effect in the more permeable layer, with $\frac{2L_f}{xe} = 1; h_{f1} = h_{f2} = h_f, h_{e1} = h_{e2} = h_e, \frac{h_{f}}{h_{e}} = 0.1; M_2/M_1 = 20$ and $C_{S2}/C_{S1} = 1; N_{stage} = 10$
Fig 3.18 (a) Pressure derivative response $dp_D/dln^{t_D}$ vs. dimensionless time $t_D$ and (b) modified dimensionless rate $q_{DM}$ versus modified dimensionless time $t_{DM}$, subject to anisotropy effect in the less permeable layer, with $\frac{2L_f}{x_e} = 1$; $h_{f1} = h_{f2} = h_f$, $h_{e1} = h_{e2} = h_e$, $\frac{h_f}{h_e} = 0.1$; $M_2/M_1 = 20$ and $C_{S2}/C_{S1} = 1$; $N_{stage} = 10$
3.4 Three-Layer Reservoir Results

3-layer reservoir system is also modeled in this study with similar strategy as in the proposed model of 2-layer reservoir with multistage fractured horizontal well, and results generated from the 3-layer model are presented and discussed in following section.

3.4.1 Dimensionless groups

Dimensionless vertical thickness of each layer is defined as

\[ h_{e1D} = \frac{h_{e1}}{L_r}, \]  \hspace{1cm} (3.17)

\[ h_{e2D} = \frac{h_{e2}}{L_r}, \]  \hspace{1cm} (3.18)

\[ h_{e3D} = \frac{h_{e3}}{L_r}, \]  \hspace{1cm} (3.37)

Dimensionless fracture height in each layer is defined as

\[ h_{f1D} = \frac{h_{f1}}{L_r}, \]  \hspace{1cm} (3.19)

\[ h_{f2D} = \frac{h_{f2}}{L_r}, \]  \hspace{1cm} (3.20)

\[ h_{f3D} = \frac{h_{f3}}{L_r}, \]  \hspace{1cm} (3.38)

Inter-layer crossflow rate \( q_{D}^{cf1} \) refers to the crossflow rate between Layer 1 and Layer 2

\[ q_{D}^{cf1} = \frac{q^{cf1}}{q_c}, \]  \hspace{1cm} (3.39)

and \( q_{D}^{cf2} \) refers to the crossflow rate between Layer 2 and Layer 3
\[ q^e_{f2} = \frac{q_{ef2}}{q_e}. \]  
(3.40)

The mobility \( M \) and the storativity \( C_S \) of each layer are defined as follows

\[ M_i = \frac{k_i}{\mu_i}, \quad i = 1, 2, 3. \]  
(3.41)

\[ C_i = \phi_i c_i, \quad i = 1, 2, 3. \]  
(3.42)

Dimensionless modified rate and dimensionless modified time in the 3-layer reservoir model are defined as

\[ q^{DM} = \frac{q_{WD} \cdot \text{Scaler}}{N_{stage}}, \]  
(3.32)

\[ t^{DM} = \frac{t_{MBD}}{\left[ (h_{f1D} + \frac{M_2C_{S2}}{M_1C_{S1}}h_{f2D} + \frac{M_3C_{S2}}{M_1C_{S1}}h_{f3D}) \right]^2 \cdot \text{Scaler}^2 \cdot \left( \frac{x_{eD}}{2L_{fD}} \right)^2}, \]  
(3.43)

where “Scaler” is defined as

\[ \text{Scaler} = \frac{x_{eD}y_{eD} \left( h_{e1D} + \frac{C_{S2}}{C_{S1}} h_{e2D} + \frac{C_{S3}}{C_{S1}} h_{e3D} \right)}{(2L_{fD}) \left[ (h_{f1D} + \frac{M_2C_{S2}}{M_1C_{S1}}h_{f2D} + \frac{M_3C_{S2}}{M_1C_{S1}}h_{f3D}) \right]^2 \cdot N_{stage}} \cdot \frac{x_{eD}}{2L_{fD}}. \]  
(3.44)

The rest dimensionless terms are defined in the same way as in the 2-layer model.

### 3. 4.2 Results and discussions

A simple 3-layer reservoir penetrated by fractured horizontal well is shown schematically in Fig 3.19. The validation of the outcomes for three-layer reservoir model is also conducted by analyzing the well responses of the 2 scenarios listed in Table 3.2.
Table 3.2: Various Well-Reservoir Systems for Each Scenario in the 3-Layer Reservoir Model

<table>
<thead>
<tr>
<th>Scenario 2-1: Middle layer is sandwiched between two comparatively thick layers</th>
<th>Schematic of Well-Reservoir System (front view)</th>
<th>Sensitivity Analysis</th>
<th>purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 2-2: Middle layer is sandwiched between two comparatively thick layers</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>parameters</th>
<th>$M_1 = M_2 = M_3 = 1$</th>
<th>$h_{e1D} = h_{e3D} = 47.5$</th>
<th>$h_{e2D} = 5$</th>
<th>$h_{f1D} = h_{f2D} = h_{f3D} = 5$</th>
<th>$2L_f/x_e = 1$</th>
<th>$C_{S2}/C_{S1} = 5$</th>
<th>$C_{S3}/C_{S1} = 10$</th>
<th>$C_{S2}/C_{S1} = 1$</th>
<th>$C_{S3}/C_{S1} = 0.2$</th>
<th>$C_{S3}/C_{S1} = 0.1$</th>
</tr>
</thead>
</table>

To present how different storativity ratios among 3 layers affect pressure and production rate behavior of fractured well

To present how different mobility ratios among 3 layers affect pressure and production rate behavior of fractured well

Scenario 2-1:
- Middle layer is sandwiched between two comparatively thick layers
- Schematic of Well-Reservoir System (front view)
- $M_1 = M_2 = M_3 = 1$
- $h_{e1D} = h_{e3D} = 47.5$
- $h_{e2D} = 5$
- $h_{f1D} = h_{f2D} = h_{f3D} = 5$
- $2L_f/x_e = 1$
- $C_{S2}/C_{S1} = 5$
- $C_{S3}/C_{S1} = 10$
- $C_{S2}/C_{S1} = 1$
- $C_{S3}/C_{S1} = 0.2$
- $C_{S3}/C_{S1} = 0.1$

Scenario 2-2:
- Middle layer is sandwiched between two comparatively thick layers
- Schematic of Well-Reservoir System (front view)
- $C_{S1} = C_{S2} = C_{S3}$
- $h_{e1D} = h_{e3D} = 47.5$
- $h_{e2D} = 5$
- $h_{f1D} = h_{f2D} = h_{f3D} = 5$
- $2L_f/x_e = 1$
- $M_2/M_1 = 5$
- $M_3/M_1 = 10$
- $M_2/M_1 = 1$
- $M_3/M_1 = 0.2$
- $M_3/M_1 = 0.1$
Fig 3.19 A three-layer reservoir model for scenarios under study
Scenario 2-1: Effect of storativity ($\phi c_l$) on well behavior and inter-layer crossflow.

**Fig 3.20** presents pressure derivative behavior of fractured well under three situations with different storativity ratios of the three layers, while mobility is uniform throughout all 3 layers. It shows the fact that smaller storativity ratios in the upper two layers result in greater pressure drop along the entire life of well than big storativity ratios. **Fig 3.20** also shows that early linear flow ends later for reservoir with greater storativity simply because greater storativity leads to smaller diffusivity that pressure-transient travels more slowly, which also leads to the later arrival of pseudo-steady state flow. As described by Spivey and Lee (2013), pressure derivative should approach 0.5 once flow regime enters pseudo-radial flow regardless of storativity distribution as long as mobility throughout reservoir is homogeneous; shown in **Fig. 3.20**, pressure derivative of the three situations in the scenario all approach 0.5 at different time.

**Fig 3.21 (a)** presents the corresponding rate of well producing at constant pressure and it illustrates that greater storativity boosts production rate along the entire life of well. By the use of material balance time, production rate from reservoir with different storativity in each layer also declines in a unit slope during boundary dominant flow. Replotting the production behavior of well from **Fig 3.21 (a)**, in the manner of $q_{DM}$ vs $t_{DM}$ on **Fig 3.21 (b)**, curves generated from different storativity ratios of the 3 layers merge together during the early and late time periods and they even closely lie next to each other between early and late time.
Fig 3.20 Dimensionless pressure derivative behavior of well $dp_D/dln^{t_D}$ vs. dimensionless time $t_D$ under various storativity ratio among layers, with $h_{e1D} = h_{e3D} = 47.5$, $h_{e2D} = 5$; $h_{f1D} = h_{f2D} = h_{f3D} = 5$; $\frac{2L_f}{x_e} = 1$; $M_1 = M_2 = M_3$; $N_{skge} = 6$
Fig 3.21 (a) Dimensionless production rate \( q_{WD} \) versus material balance time \( t_{MBD} \) and (b) modified dimensionless production rate \( q_{DM} \) versus modified dimensionless time \( t_{DM} \), under various storativity ratio among layers, with \( h_{e1D} = h_{e3D} = 47.5 \), \( h_{e2D} = 5 \); \( h_{f1D} = h_{f2D} = h_{f3D} = 5; \frac{2L}{x_e} = 1; M_1 = M_2 = M_3; N_{stage}=6 \)
It is notable that inter-crossflow from **upper layer to lower layer** is defined as “**positive**” and the opposite as “**negative**”. In **Fig 3.22**, it illustrates that crossflow between Layer 1 and Layer 2 ($q_{D}^{cf1}$) and between Layer 2 and Layer 3 ($q_{D}^{cf2}$) flow towards the middle layer when reservoir is homogeneous, but when reservoir is heterogeneous in storativity among the 3 layers fluid in the layer with greater storativity crosses inter-plane into the layer with smaller storativity, simply because the layer with greater storativity emits more fluid through expansion per unit pressure drop and therefore fluid flows to the layer with smaller storativity to provide fluid.
Fig 3.22 Inter-layer crossflow rate between adjacent layers along time under various storativity ratio among layers, with $h_{e1D} = h_{e3D} = 47.5$, $h_{e2D} = 5$; $h_{f1D} = h_{f2D} = h_{f3D} = 5$; $\frac{2L_f}{x_e} = 1$; $M_1 = M_2 = M_3$; $N_{stage} = 6$
Scenario 2-2: Effect of mobility (k/μ) on well behavior and inter-layer crossflow. In Fig 3.23, pressure derivative behavior of well under different mobility ratios of 3 layers are provided, while storativity is uniform through all layers. The pressure behavior in this scenario is similar to that as in the 2-layer model provided in Fig 3.12 and discussed then.

Production rate of well under constant bottom pressure \( q_{WB} \) vs \( t_{MBD} \), and modified production rate \( q_{DM} \) vs \( t_{DM} \) are provided in Fig 3.24 (a) and Fig 3.24 (b) respectively. The heterogeneity of mobility of the 3 layers affects production rate all the time (greater mobility boosts production rate) before pseudo-steady flow is fully evolved as shown in Fig 3.24 (a), while all production rates generated from different mobility ratios of 3 layers merge together at early time and late time—even almost overlap each other through all the production history—by plotted in the manner of \( q_{DM} \) vs \( t_{DM} \) (Fig 3.24 b).

The inter-layer crossflow affected by different mobility ratios among layers is different to that affected by different storativity ratios. In Fig 3.25, results in this scenario display that fluid in the layer with smaller mobility crosses the inter-plane towards the layer with greater mobility, which is exactly the opposite to the scenario when reservoir is heterogeneous in storativity (results shown Fig 3.22).
Fig 3.23 Dimensionless pressure derivative behavior of well $dp_D/dln t_D$ vs. dimensionless time $t_D$ under various mobility ratio among layers, with $h_{e1D} = h_{e3D} = 47.5$, $h_{e2D} = 5$; $h_{f1D} = h_{f2D} = h_{f3D} = 5$; 

$$\frac{2L_f}{x_o} = 1; \; C_{S1} = C_{S2} = C_{S3}; \; N_{stage} = 6$$
Fig 3.24 (a) Dimensionless production rate $q_{WD}$ versus material balance time $t_{MBD}$ and (b) modified dimensionless production rate $q_{DM}$ versus modified dimensionless time $t_{DM}$, under various mobility ratio among layers, with $h_{e1D} = h_{e3D} = 47.5, h_{e2D} = 5; h_{f1D} = h_{f2D} = h_{f3D} = 5; \frac{2L_f}{x_e} = 1; C_{S1} = C_{S2} = C_{S3}; N_{stage}=6$
Fig 3.25 Inter-layer crossflow rate between adjacent layers along time under various mobility ratio among layers, with $h_{e1D} = h_{e3D} = 47.5$, $h_{e2D} = 5$; $h_{f1D} = h_{f2D} = h_{f3D} = 5$; $\frac{2L_{f}}{x_e} = 1$;

$C_{S1} = C_{S2} = C_{S3}; N_{stage}=6$
3. 5  Field Case Application

The Viewfield Bakken oil pool is located between township road 6 and township road 12, and range road 12 and range road 5 (west of the second meridian) in the southeast section of Saskatchewan. The Bakken formation consist of three main members, and the top and the bottom members contain rich organic shales, which sandwiches the middle member characterized from dolomitic siltstone to silty sandstone. With respect to Viewfield oil pool, the middle member has proven to be the major source of oil production since horizontal drilling and multistage sand fractures have been the favored method of developing Canadian tight oil Bakken formations (Cronkwright et al. 2014).

The field data is for a multi-staged horizontal well producing the middle member of Bakken formation from a reservoir that consists of 2 vertical layers. The production dataset is shown in **Fig 3.26** and basic information of reservoir and well is provided in **Table 3.3.**
Fig 3.26 Production data (Viewfield Bakken well producing a 2-layer reservoir)
### Table 3.3: Input Parameters for Multi-Staged Fractured Horizontal Well in 2-Layer Reservoir in Bakken Formation (Data is Collected from Accumap)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_e$, m</td>
<td>155.44</td>
</tr>
<tr>
<td>$y_e$, m</td>
<td>1828.8</td>
</tr>
<tr>
<td>$h_{e1}$, m</td>
<td>10.66</td>
</tr>
<tr>
<td>$h_{e2}$, m</td>
<td>15.85</td>
</tr>
<tr>
<td>Porosity, %</td>
<td>0.1</td>
</tr>
<tr>
<td>Compressibility, $p_a^{-1}$</td>
<td>$6.45 \times 10^{-9}$</td>
</tr>
<tr>
<td>Viscosity, $p_a \cdot s$</td>
<td>0.00171</td>
</tr>
<tr>
<td>Oil FVF</td>
<td>1.25</td>
</tr>
<tr>
<td>$P_i$, pa</td>
<td>$2.337 \times 10^7$</td>
</tr>
<tr>
<td>Estimated $P_{wf}$, pa</td>
<td>$5.515 \times 10^8$</td>
</tr>
<tr>
<td>Horizontal well length, m</td>
<td>1361</td>
</tr>
<tr>
<td>$N_{stage}$</td>
<td>16</td>
</tr>
</tbody>
</table>
The desired parameters to evaluate in this field case are reservoir permeability of each layer and fracture length (on the assumption that fractures fully penetrate reservoir’s vertical thickness); to evaluate the production data of the multi-staged fractured well, type curve match is applied. With 2 variables (fracture length and reservoir permeability ratio of the two layers), numerous type curves need to be generated in our analytical model to achieve a good match, which is too costly, tedious and inconvenient. By plotting theoretical production behavior in the form $q_{DM} \text{ vs } t_{DM}$, we can see from the two scenarios—effect of mobility ratio of layers (in both 2-layer and 3-layer results shown in Fig 3.13 b and Fig 3.24 b)—that permeability heterogeneity of layers barely displays difference as all curves almost overlap each other. A strategy consisting of 2 steps, therefore, is proposed here: using type curves $q_{DM} \text{ vs } t_{DM}$ generated from model with homogeneous reservoir permeability (ignoring the permeability heterogeneity in this step) to match measured data and evaluate fracture length; as fracture length is obtained, type curves generated by two-layer model with different permeability ratios of the two layers can be plotted in the form of $q_{wD} \text{ vs } t_{MBD}$ to match measured data.

*Step 1:* Based on the input information provide in Table 3.3, a set of type curves are generated in the form of $q_{DM}$ and $t_{DM}$ by 2-layer model with homogeneous reservoir characteristics, and curve match process is followed to make a reasonable match. In this case, the early time production data analysis is not the target of this work in the fact that production at early time is enormously affected, distorted from theoretical solution, by many factors including fluid flow back, fluid multiphase effect, nature of fracture finite conductivity, etc. Nevertheless, a fairly satisfied match is successfully achieved as shown
in Fig 3.27 (a) that fracture length/horizontal reservoir thickness ratio is roughly $\frac{2L_f}{x_e} = 0.4$.

**Step 2:** type curves are generated based on $2L_f/x_e = 0.4$ with various mobility ratio of the two layers and are plotted in the form of $q_{wD}$ vs $t_{MBD}$. Curve match process match is followed to achieve a fairly good match. A reasonable match is made in Fig 3.27 (b) that the mobility ratio is analyzed as 0.08.

The analyzed results are summarized as follows:

Fracture length is

$$L_f = 102.00 \text{ ft}$$

Permeability of the lower layer is

$$k_1 = 0.101md$$

Permeability of the upper layer is

$$k_2 = 0.00808md$$
Fig 3.27  (a) Production rate response match by $q_{DM}$ vs $t_{DM}$ to evaluate fracture length/reservoir horizontal thickness ratio (type curves are generated with homogenous reservoir characteristics) and (b) Production rate response match by $q_{WD}$ vs $t_{MBD}$ with various mobility ratio under constant fracture length/reservoir horizontal thickness ratio $2L_fx_e = 0.4$
3.6 Summary

Multistage hydraulically fractured horizontal wells while producing vertically stratified multilayered reservoir are found to have very complicated pressure- and rate-transient behaviors as well as inter-layer crossflow profiles. A physical-based model with analytical solution dealing with such complex reservoir-well is successfully built and validated, and type curves generated by the model are able to effectively analyze well behaviors. In addition to normal type curve format with dimensionless rate versus dimensionless time, the proposed modified dimensionless rate and time to a large degree unified reservoir and well geometry and properties of each layer so that the rate behavior of horizontal well can be analyzed in a systematic manner, which helped to facilitate evaluation of field data and is discussed in the field case application section.
3.7 Nomenclature

c_t = total compressibility of reservoir, pa\(^{-1}\)

C_s = storativity, pa\(^{-1}\)

\(h_{ej}\) = reservoir vertical thickness of the j-th layer, m

\(h_{fj}\) = fracture height in the j-th layer, m

\(k\) = reservoir permeability, m\(^2\)

\(k_v\) = reservoir vertical permeability, m\(^2\)

\(k_r\) = reservoir horizontal permeability, m\(^2\)

\(L_f\) = fracture half length, m

\(M\) = mobility defined in, m\(^2\)/pa \cdot s

\(P\) = pressure, pa

\(p_i\) = initial pressure, pa

\(q\) = rate, m\(^3\)/s

\(q_c\) = a constant rate, m\(^3\)/s

\(q^{fj}\) = production rate of fractures in the j-th layer, m\(^3\)/s

\(q_w\) = producing rate of well, m\(^3\)/s

\(q^{cf}\) = inter-layer crossflow rate, m\(^3\)/s
\( q_{DM} \) = modified dimensionless rate

\( \tilde{q} \) = flux of fracture planes or of planar fractured well per unit area, \( m^2/s \)

\( S \) = strength of source, \( m \cdot Pa/s \)

\( t \) = time, \( s \)

\( t_{MB} \) = material balance time, \( s \)

\( t_{DM} \) = modified dimensionless time

\( x \) = horizontal coordinate, \( m \)

\( x' \) = horizontal coordinate of the source/sink point, \( m \)

\( y \) = horizontal coordinate, \( m \)

\( y' \) = horizontal coordinate of the source/sink point, \( m \)

\( z \) = vertical coordinate, \( m \)

\( z' \) = vertical coordinate of the source/sink point, \( m \)

\( \Delta p \) = \( p_i - p \) = pressure drop, \( Pa \)

\( \eta \) = reservoir diffusivity, \( m^2/s \)

\( \mu \) = fluid viscosity, \( Pa \cdot s \)

\( \tau \) = time variable, \( s \)

\( \phi \) = porosity of the reservoir, fraction
\( N_{\text{stage}} = \) Number of fracture stages along horizontal wellbore

**Superscripts**

\( f_j \) = flow rate of fracture in the \( j\)-th layer

\( c_f \) = inter-layer crossflow

**Subscripts**

\( D \) = dimensionless term

\( i, j \) = i-th and j-th fracture segment

\( 1 \) = parameter of Layer 1

\( 2 \) = parameter of Layer 2
Chapter 4 Semi-Analytical Modeling of 2-Dimensional Heterogeneous Reservoir by Using Boundary Element Method

Fluid flow in reservoir with multi-scale heterogeneities that often exist with arbitrarily shaped boundaries is extremely complex in nature due to the historical diversity of reservoir geological and depositional environment variation. This reality requires that reservoir evaluation and characterization performed for an individual reservoir have to be case specific. Existence of reservoir heterogeneity generally affects production throughout well’s entire life time so that detection of such characteristics from testing data is of great importance for evaluating or predicting well’s performance. For example, natural fractures distributed randomly over reservoir, whether highly conductive or unconductive acting as sealing faults, can have huge impact on well production. A comprehensive technique, therefore, that is able to generate solutions, including pressure or production rate of well under such complex heterogeneous reservoir, with high level of accuracy is significantly demanded. It is reasonable to say that only analytical solution or solution close to analytical solution accuracy can satisfy the demand.
4.1 Introduction and literature related

Analytical solutions describing heat conduction in homogeneous reservoir, by using source/sink function, proposed by Carslaw and Jaeger (1959) become the foundation of modeling pressure and production-transient behavior. Gringarten and Ramey (1973) firstly provided instantaneous source functions corresponding to various boundary conditions in depicting pressure-transient of fluid flow over reservoir domain. Pioneer researchers have extended these analytical solutions to complicated well configurations including horizontal and fractured well (Gringarten et al. 1974; Ozkan and Raghavan 1991), and to complex reservoir such as composite reservoir, commingled reservoir, complex well-reservoir system (Kuchuk and Wilkinson 1991; Basquet et al. 1999; Medeiros, Ozkan, and Kazemi 2008) in modeling transient pressure or production rate problem in reservoir engineering. Medeiros, Ozkan and Kazemi (2010) presents a semi-analytical approach, by using Green function method, to model transient pressure behavior of a horizontal well in composite or layered reservoir with its wellbore intercepting a number of blocks. Zhao and Thompson (2001, 2002) came up with semi-analytical solutions, by using source/sink function, describing pressure response and flow characteristics in complex geometry reservoirs comprising T-shape, splay and linear composite reservoir. A limitation of the two works is that adjacent reservoir compartments have to be orthogonal. Zhao and Thompson (2001), then, built up a model to account for reservoir consisting of two channels that connect to each other at an angle but it has to be some special angles such as 30°, 45°, 60° and 90°. Zhao (2009, 2012) has successfully extended source/sink function method to its application in heterogeneous reservoir system, where reservoir heterogeneity generally encountered in field practice.
can be modeled effective in two-dimensional and three-dimensional domain with analytical accuracy using semi-analytical strategy. The previously cited Zhao’s works account for boundary condition by creating image wells to fill the entire space, which is very tedious and demands too much computing efforts. In two-dimension flow field, Zhao’s model requires the entire reservoir to be discretized whether different compartments have the same properties or not, and is limited to rectangular-gridding system instead of irregular-shape gridding. To overcome such limitations and come up with solutions that apply to reservoir composed of locally homogeneous parts that have arbitrarily shaped boundaries, boundary element method became the best candidate.

Boundary element method uses free-space Green’s function as global weighting function, or referred as fundamental solution, in integral equation only along boundary (Cheng and Cheng 2015). Computation burden is much relieved as no image well is created and pressure response of reservoir mapped with input information from geological model can be generated as boundary element method allows boundaries/surfaces of reservoir compartments to be any shape. Kikani and Horne (1992, 1993) derived the fundamental solution as weighting function, which represents pressure response caused by an instantaneous point source in infinite reservoir, for boundary element method in reservoir engineering and applied the method to solve pressure-transient behavior in arbitrarily bounded homogeneous reservoir, and extended the solution to composite reservoir with multiple heterogeneous regions. Pecher and Stanislav (1996) applied boundary element method to calculate pressure behavior of vertical fractured well with uniform flux in linear composite reservoir with contact interfaces between adjacent two compartments and outer boundary being arbitrarily shaped. Xiao, Zhao and Qing (2016) made a
breakthrough by using boundary element method to solve transient flow problem in application to heterogeneous reservoir with a partition method to divide locally homogeneous region into a number of subsystems and form a fully compartmentalized 2-dimensional reservoir model with irregular gridding system. Comparisons to semi-analytical solutions by Zhao’s 2-dimensional heterogeneous reservoir model and other existed analytical solutions, to a large degree, validated their work.

In this study, grids are only used to capture the physically defined boundary of each locally homogeneous region with average rock/fluid properties; sub-dividing of a region into subsystems is not needed because solution in a region with multiply connected surfaces can be formulated and solved, which means that no “waste computation” would be conducted since solutions are only solved on the physically defined boundary of each region.

4.2 Methodology

4.2.1 Complex heterogeneous reservoir definition

A heterogeneous reservoir, consisting of 8 regions, is shown in Fig 4.1 (note that the reservoir schematic is only for demonstration purpose of proposing a heterogeneous reservoir example). Any number of source/sinks can be included in each region. In this study, heterogeneous systems are considered to consist of a number of domains that are locally homogeneous with average rock/fluid property. The defined reservoir generally stands for a typical problem involving multi-scale heterogeneities with complex regional boundary and outer reservoir boundary: Region 1 is only connected to another flow domain (Region 8), and Region 2 is partially intercepted by the physical outer boundary
of reservoir, and Region 3-7 forms a cross-like area that four regions are respectively collected to one side of the Region 4. In Fig 4.1, \( \Gamma_j (\{j\}) \) represents the closed contour of boundary enclosing the \( j \)-th region, and \( \Gamma_{j,i} \) represents the boundary/interface between Region \( i \) and Region \( j \). That is to say, \( \Gamma_{j,i} \subseteq \Gamma_i \text{ or } \Gamma_j \). The outer boundary, \( \Gamma_{\text{outer}} \), \( (\Gamma_{2,\text{outer}} \text{ is part of } \Gamma_{\text{outer}} \text{ as interface between } \Gamma_2 \text{ and } \Gamma_{\text{outer}}) \) can be prescribed to Dirichlet condition, Neuman condition, and mixed boundary condition. To simplify, and for the purpose of demonstrating the proposed methodology, we assume that the outer boundary is prescribed to no-flow condition.
Fig 4.1 Schematic of a heterogeneous reservoir with 8 regions
4.2.2 Mathematical description

From Region 1 to Region 7, each one of them alone is a simply connected region as if it was a spatially continuous homogeneous flow field/reservoir; Region 8, however, is a multiply connected region with complex multiple surfaces composed of interfaces between the 7 interior regions and Region 8, and the outer boundary surface. Modeling of pressure- and rate-transient responses within reservoir is achieved by solving unsteady fluid flow problem in such complex heterogeneous system. To achieve this, a number of assumptions are made as follows:

1. Fluid is slightly compressible and fluid flow in each region is single phase.
2. Rock and fluid properties in each region are uniform and static.
3. Formation has uniform thickness and is fully penetrated by sources/sinks.
4. Any two regions with interface are in hydraulic contact (no flow resistivity across) and interfaces/boundaries between the two regions are static.
5. Initial pressure of reservoir is uniform everywhere.

In this study, dimensionless terms are defined as follows:

Dimensionless pressure and dimensionless time are both defined in terms of reference rock/fluid properties

\[
p_D = \frac{2 \pi k_{ref} h}{(q\mu)_{ref}} \Delta p,
\]  
\[
t_D = \frac{k_{ref} t}{(\phi \mu c_{\tau})_{ref} l^2},
\]

where reference properties of rock/fluid can be random and \( l \) denotes characteristic length that can be random as well.
Dimensionless rate of source/sink is defined in terms of a constant reference rate

\[ q_D = \frac{q}{q_{ref}}. \]  \hspace{1cm} (4.3)

Dimensionless x- and y-coordinates are defined as

\[ x_D = \frac{x}{l}, \]  \hspace{1cm} (4.4)

\[ y_D = \frac{y}{l}. \]  \hspace{1cm} (4.5)

Dimensionless mobility ratio of each region is based on reference mobility

\[ \frac{M_{ij}}{M_{ref}} = \frac{(k)}{(\mu_{ij})_{ref}}, \quad j \in [1,8]. \]  \hspace{1cm} (4.6)

Dimensionless storativity of each region is also based on reference storativity

\[ \frac{C_{sj}}{C_{sref}} = \frac{(\phi c_t)_j}{(\phi c_t)_{ref}}, \quad j \in [1,8]. \]  \hspace{1cm} (4.7)

Dimensionless diffusivity ratio is

\[ \frac{C_{sj}}{C_{sref}} = \frac{(M)}{(M)_{ref}}, \quad j \in [1,8]. \]  \hspace{1cm} (4.8)

**Pressure in Region 1 to 7.** With assumptions made above, flow equation in locally homogeneous Region 1 to 7 can be mathematically described as

\[ \frac{k_j}{\mu_j} \left( \frac{\partial^2 p_j}{\partial x^2} + \frac{\partial^2 p_j}{\partial y^2} \right) = \phi_j c_t \frac{\partial p_j}{\partial t} + \sum_{k=1}^{N_{wj}} q_{j,k} \delta(x - x'_{j,k}) \delta(y - y'_{j,k}) \quad \{ j \leq 7, j \in I \}, \]  \hspace{1cm} (4.9)
where $\mu_j, k_j, c_tj$ are the reservoir and fluid properties in the $j$-th region. $\delta$ is the Dirac delta function, $(x'_{jk}, y'_{jk})$ is location of the $k$-th source/sink of the number $N_{wj}$ in the $j$-th region and $q_{lj,k}$ is the corresponding source/sink flow rate.

For simply connected region, pressure response corresponding to Eq. (4.9) is an integral equation and is given by Kikani and Horne (1992). With dimensionless terms defined through Eq. (4.1) to Eq. (4.8), the integral equation is written as

$$\theta_j = 2\pi \quad \text{if } (x, y) \in \Omega_j$$
$$\theta_j = \theta_j \quad \text{if } (x, y) \in f_j, \quad \{j|1 \leq j \leq 7, j \in I\}$$

where $\theta_j$ is the internal angle between two boundary elements in Region $j$. In Eq. (4.10), $G$ is the 2D free-space Green’s function and is used as a global weighting function in integral equation, which is given by Eq. (4.12). The physical meaning of $G$ accounts for pressure response at $(x_D, y_D)$ at evaluating time $t_D$ resulted from an instantaneous disturbance of a source/sink occurred in an infinite reservoir at location $(x'_D, y'_D)$ at time $t_D$.

**Fig 4.2** shows the path integral schemes for the 7 regions.
Fig 4.2 Reservoir from the perspective of Region 1 to Region 7 respectively and their integration path for integral equation
It is very important to note that the $\frac{\partial p_i}{\partial n}$ or $\frac{\partial g_i}{\partial n}$ is considered to be “positive” when it is pointing to the outward normal direction as shown in Fig 4.2 and is considered to be “negative” when it is pointing to the inward normal direction; $n$ is the outward-pointing normal on a boundary element.

$$G_i(x_D,y_D,x'_D,y'_D,t_D,\tau_D) = \frac{1}{4\pi \frac{c_{Rij}}{c_{Ref}} (t - \tau)_D} U(t - \tau)_D \exp \left[ -\frac{(x - x')^2_D + (y - y')^2_D}{4 \frac{c_{Rij}}{c_{Ref}} (t - \tau)_D} \right], \quad (4.12)$$

where $U(t - \tau)_D$ is the unit step function.

**Pressure in Region 8.** First of all, it is necessary to identify the total 8 locally homogenous reservoir compartments that make up the heterogeneous reservoir. The first 7 compartments are straight and simple that Fig 4.2 describes each of their flow domain and boundary. It is schematically shown in Fig 4.3 that Region 8 is a multiply connected domain with complex internal surfaces, comprising interfaces between itself and the rest of regions, and with the portion of the outer no-flow boundary that directly surrounds this region. From the perspective of Region 8, the other 7 regions behave as if they were “holes” extracted out of the reservoir with their pressures and pressure gradients along boundaries exerting on this region. It is, therefore, complicated to formulate pressure behavior within this region for the fact that the integral Eq. (4.10) describes pressure within a simply connected domain as integration path is only a simple surface encircling that flow domain (Fig 4.2).
Fig 4.3 Reservoir from the perspective of Region 8
The next step is to fictitiously divide the multiply connected domain into a simply connected domain by introducing necessarily auxiliary curves. It is shown in Fig 4.4 that, by auxiliary lines AB and CD as well as MN and PQ with negligible distance between as \( \varepsilon \to 0 \), Region 8 becomes a simply connected domain with surface/boundary expressed \( \Gamma_8 \) as

\[
\Gamma_8 = \Gamma_{AB} + (\Gamma_{outer} - \Gamma_{2,outer}) + \Gamma_{MN} + \Gamma_{NRSP} + \Gamma_{PQ} + \Gamma_{CD} + \Gamma_1. \tag{4.13}
\]

The flow equation in this domain is the same as in the other 7 regions expressed in Eq. (4.9); and with boundaries given in Eq. (4.13), pressure in this region, similar to Eq. (4.10), now can be written as

\[
\frac{\theta_B}{2\pi} p_{DB}(x_D, y_D, t_D) = \int_0^{t_D} d\tau \left\{ \int_{\Gamma_{AB}} \vartheta \, d\Gamma + \int_{\Gamma_{outer}-\Gamma_{2,outer}} \vartheta \, d\Gamma + \int_{\Gamma_{2}-\Gamma_{2,outer}} \vartheta \, d\Gamma + \int_{\Gamma_{MN}} \vartheta \, d\Gamma + \int_{\Gamma_{NRSP}} \vartheta \, d\Gamma \\
+ \int_{\Gamma_{PQ}} \vartheta \, d\Gamma + \int_{\Gamma_{CD}} \vartheta \, d\Gamma + \int_{\Gamma_1} \vartheta \, d\Gamma + \sum_{k=1}^{N_{wb}} q_{DB,k} \vartheta_B \right\},
\tag{4.14}
\]

where

\[
\vartheta = G_B \frac{\partial p_{DB}}{\partial n} \frac{M_B}{M_{ref}} \frac{C_{sref}}{C_{sB}} \left( 1 - \frac{p_{DB}}{p_{DB}} \frac{M_B}{M_{ref}} \frac{C_{sref}}{C_{sB}} \right). \tag{4.15}
\]

The \( \theta_B \) has the same definition as Eq. (4.11) as

\[
\begin{align*}
\theta_B &= 2\pi & \text{if } (x, y) \in \Omega_8 \\
\theta_B &= \theta_B & \text{if } (x, y) \in \Gamma_8.
\end{align*}
\tag{4.16}
\]
Fig 4.4 Region 8 fictitiously divided into a simply connected region
Note that in Fig 4.4 the arrows do not account for the direction of integration path, instead they are there simply for the purpose of helping demonstrate or show the integration contour for Region 8 as is fictitiously divided into a simply connected domain by auxiliary lines AB, CD, MN, and PQ. In mathematics, the integration path for line integral, or for curve integral or for sometimes also referred as contour integral is a scalar instead of a vector.

The pressure expression in Eq. (4.14) can be further simplified as the sum of the 4 integrals—integration along line AB, CD, MN and PQ—equals to zero. The derivatives $\frac{\partial P_D}{\partial n}$ and $\frac{\partial g}{\partial n}$ in Eq. (4.10) or Eq. (4.14) are assumed pointing to the outward normal direction along the boundary of respective each region that $n$ is the outward-pointing normal on a boundary element. For the Region 8, the outward-pointing normals on line AB and CD, on MN and PQ, are opposite towards each other and are shown in Fig 4.5; since the distance between both lines are assumed to be negligible, the pressure gradient (fluid flowing rate) normal to them in fact must be consistent towards one direction and the fluid leaving AB (CD) is entering CD (AB) so that the rate (pressure gradient) must be the same, which makes

$$\int_{AB} \varphi d\Gamma = - \int_{CD} \varphi d\Gamma , \quad (4.17)$$

$$\int_{MN} \varphi d\Gamma = - \int_{PQ} \varphi d\Gamma . \quad (4.18)$$

Also, we assumed that pressure gradient, from the perspective of each of the first 7 regions, on the interface connected to Region 8 is pointing consistently alongside the outward normal direction.
Fig 4.5 Outward-pointing normals for the 4 auxiliary lines pointing opposite towards each other
In contrast, from the perspective of Region 8 pressure gradient or weighting function gradient on the same interface is pointing alongside the inward normal direction; there should be a “negative” sign to account for it for curve integrals integrated along \( \Gamma_2 - \Gamma_{2,outer} \), \( \Gamma_1 \), \( \Gamma_{NRSP} \). Now, in addition to Eq. (4.17) and Eq. (4.18), pressure response within Region 8 (Eq. 4.14) can be written as

\[
\frac{\theta_k}{2\pi} p_{D8}(x_D, y_D, t_D) = \int_0^{\tau_D} d\tau \left\{ \int_{\Gamma_{outer} - \Gamma_{2,outer}} \varphi d\Gamma - \int_{\Gamma_2 - \Gamma_{2,outer}} \varphi d\Gamma - \int_{\Gamma_{NRSP}} \varphi d\Gamma - \int_{\Gamma_1} \varphi d\Gamma + \sum_{k=1}^{N_{w_{in}}} q_{D8,k} \beta_k \right\}. \quad (4.19)
\]

As auxiliary lines are removed, the multiply connected surfaces for Region 8 becomes

\[
\Gamma_8 = (\Gamma_{outer} - \Gamma_{2,outer}) + (\Gamma_2 - \Gamma_{2,outer}) + \Gamma_{NRSP} + \Gamma_1. \quad (4.20)
\]

**Boundary discretization.** The integral equations of Eq. (4.10) and Eq. (4.19) are directly solved on the boundaries of respective problem domains. The integrals involve integration of pressure and pressure gradient that are unknown, continuous functions of position and time. In order to achieve numerical solution to Eq. (4.10) and Eq. (4.19), in this study we discretize a boundary into a desiring number of segments, or more often named as boundary elements, by assigning nodes on the boundary and each boundary element is formed with linear interpolation between two adjacent nodes, and pressure as well as pressure gradient on each boundary element is assumed to be uniform. Boundary element resulted from this type of discretization is referred as “constant element”, which has two advantages as Xiao, Zhao and Qing (2016) points out that “first, integrals of the Green’s function along boundary can be obtained analytically, resulting in improvement
of calculating speed and accuracy; second, singularity at sharp corners of adjacent elements is naturally avoided”.

As shown in Fig 4.6, $[I_j]^7$ is respectively discretized into $[Nj]^7$ boundary elements and $(I_{outer} - I_{2 OUTER})$ is discretized into $Ne$ boundary elements. With the discretization, pressure response within each region by Eq. (4.10) and Eq. (4.19) can be expressed as discrete forms consisting of contributions of all boundary elements discretized from corresponding boundary.

**Pressure in discrete form.** For pressure in Region 1 to 7, the contour integral of Eq. (4.10) is now written in discrete form, which gives

$$\frac{\theta_j}{2\pi} p_{Dj}(x_D, y_D, t_D) =$$

$$\int_0^{t_D} \left\{ \sum_{k=1}^{N_j} \frac{\partial p_{Dj}^k}{\partial n} M_j \frac{C_{sref}}{C_{s1}} \int_{r_j^k} G_j d\Gamma - \sum_{k=1}^{N_j} \frac{p_{Dj}^k}{M_{ref}} \frac{C_{sref}}{C_{s1}} \int_{r_j^k} \frac{\partial G_j}{\partial n} d\Gamma + \sum_{i=1}^{N_{w_j}} q_{Dj,i} G_j \right\} d\tau, \quad (4.21)$$

where

$$\theta_j = 1 \text{ if } (x, y) \in \Omega_j$$

$$\theta_j = 0.5 \text{ if } (x, y) \in I_j \quad j \in [1, 7]. \quad (4.22)$$

In Eq. (4.21), the superscript $k$ for pressure, integration path, stands for the $k$-th boundary element of $I_j$. As spatial pressure and pressure gradient along each element are assumed to be constant approximately, we have them on the $k$-th boundary element as

$$p_{Dj}(x_D, y_D, t_D) \approx p_{Dj}^k(t_D), \text{ for } (x_D, y_D) \in I_j^k, \quad (4.23)$$

$$\frac{\partial p_{Dj}(x_D, y_D, t_D)}{\partial n} \approx \frac{\partial p_{Dj}^k(t_D)}{\partial n}, \text{ for } (x_D, y_D) \in I_j^k. \quad (4.24)$$
**Fig 4.6** Boundary discretization strategy in order to approximately evaluate line integral along each boundary element
The evaluation of line integrals, which are free-space Green’s function and its derivative integrated along each boundary element, is tedious and inconvenient as the integration path spans in two dimensions and the slope of each element is different. To avoid this, a local coordinate system is defined based on the element on which the spatial integration is performed for line integral.

Fig 4.7 shows the global Cartesian coordinate system, $xOy$, and the local Cartesian coordinate system for the $k$-th element of $I_j$, $\xi_j^k O_j^k \zeta_j^k$. The locations of the starting node and ending node of the $k$-th element are marked as $(x_{nod,j}^k, y_{nod,j}^k)$ and $(x_{nod,j}^{k+1}, y_{nod,j}^{k+1})$ in the global coordinate system. The local coordinate has the element to lie on its $\xi_j^k$-axis, has the included angle $\partial_j^k$ between $x$-axis and $\xi_j^k$-axis, and has its origin $O_j^k$ in the global coordinate system at $(g, h)$. From the perspective of this local coordinate system, any point with $x$- and $y$-coordinate in the global coordinate system now has $\xi_j^k$- and $\zeta_j^k$-coordinates as

$$\xi_j^k = (x - g) \cos \partial_j^k + (y - h) \sin \partial_j^k,$$  \hspace{1cm} (4.25) \\
$$\zeta_j^k = (y - h) \cos \partial_j^k - (x - g) \sin \partial_j^k.$$

With the defined local coordinate system and Eq. (4.25) and Eq. (4.26), the line integrals in Eq. (4.21) now can be evaluated analytically, and we have
Fig 4.7 Local coordinate system defined for $k$-th element on $\Gamma_j$
\[ \hat{\theta}^k_{\frac{1}{2}}(x_D, y_D, t_D, \tau_D) = \]

\[ \int_{\Gamma_j} G_j(x, y, x', y', t, \tau) \, d\Gamma = \int_{\Gamma_j} \frac{1}{4\pi \frac{C_{\refl}}{\sqrt{\frac{C_{\refl}}{t}}} (t - \tau)_D} \exp \left[ - \frac{(x - x')^2_D + (y - y')^2_D}{4 \frac{C_{\refl}}{C_{\refl}} (t - \tau)_D} \right] \, d\Gamma \]

\[ = \int_{\Gamma_{\text{nod},j}^k} \frac{1}{4\pi \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \exp \left[ - \frac{\left( \xi^k_{\text{nod},j} - \xi^0 \right)^2_D + \left( \zeta^k \right)^2_D}{4 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \right] d\xi' \]

\[ = -\frac{1}{4 \sqrt{\frac{C_{\refl}}{C_{\text{refl}}} \pi (t - \tau)_D}} \exp \left[ \frac{\left( \xi^k_{\text{nod},j} - \xi^0 \right)^2_D}{2 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \right] - \exp \left[ \frac{\left( \xi^k_{\text{nod},j} - \xi^0 \right)^2_D}{2 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \right] \]

In Eq. (4.27), \( \xi^k_{\text{nod},j} \) and \( \zeta^k \) are given by Eq. (4.25) and Eq. (4.26). Similarly, the \( k \)-th and \( (k+1) \)-th nodal coordinates from the local coordinate system is

\[ \xi^k_{\text{nod},j} = \left( x^k_{\text{nod},j} - g \right) \cos \vartheta^k_j + \left( y^k_{\text{nod},j} - h \right) \sin \vartheta^k_j, \quad (4.28) \]

\[ \zeta^k_{\text{nod},j} = \left( y^k_{\text{nod},j} - h \right) \cos \vartheta^k_j - \left( x^k_{\text{nod},j} - g \right) \sin \vartheta^k_j. \quad (4.29) \]

By applying the same procedure, the other integral, which is derivative of free-space Green’s function integrated along the element, can be written as

\[ \hat{\theta}^k_{\frac{1}{2}}(x_D, y_D, t_D, \tau_D) = \int_{\Gamma_j} \frac{\partial G_j(x, y, x', y', t, \tau)}{\partial n} \, d\Gamma \]

\[ = -\frac{\xi^k}{2 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \exp \left[ \frac{\left( \xi^k \right)^2_D}{4 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \right] \exp \left[ \frac{\left( \xi^k_{\text{nod},j} - \xi^0 \right)^2_D}{2 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \right] - \exp \left[ \frac{\left( \xi^k_{\text{nod},j} - \xi^0 \right)^2_D}{2 \frac{C_{\refl}}{C_{\text{refl}}} (t - \tau)_D} \right]. \quad (4.30) \]
With analytical evaluation of line integrals, the pressure behavior within Region \( j \) \((j \in [1,7])\) can be expressed as

\[
\frac{\theta_j}{2\pi} p_{Dj}(x_D, y_D, t_D) = \int_0^{t_D} \left\{ \sum_{k=1}^{N_j} \frac{\partial p_{Dj}^k(\tau_D)}{\partial \psi_j} \frac{M_j}{M_{ref}} C_{sref} \phi_j^k(x_D, y_D, t_D, \tau_D) - \sum_{k=1}^{Nw_j} p_{Dj}^k(\tau_D) \frac{M_j}{M_{ref}} C_{s1} \phi_j^k(x_D, y_D, t_D, \tau_D) + \sum_{i=1}^{Nw_j} q_{Dj,i} G_j \right\} d\tau.
\]

(4.31)

As for **Region 8** with multiply connected surfaces, pressure is also needed to be expressed in discrete form. With similar procedures as we take for pressure in the first 7 regions, pressure response within Region 8, Eq. (4.19), can be expressed in discrete form, which gives

\[
\frac{\theta_8}{2\pi} p_{D8}(x_D, y_D, t_D) = \int_0^{t_D} \sum_{k=1}^{N_8} \left\{ \frac{\partial p_{D8}^{outer,k}(\tau_D)}{\partial \tau_{\theta_b}^{outer,k}} \frac{M_8}{M_{ref}} C_{sref} \phi^{outer,k}_8(x_D, y_D, t_D, \tau_D) \right\} d\tau
\]

\[
- \int_0^{t_D} \sum_{k=1}^{N_{2,8}} \left\{ \frac{\partial p_{D8}^{2,k}(\tau_D)}{\partial \theta_b^{2,k}} \frac{M_8}{M_{ref}} C_{s8} \phi_b^{2,k}(x_D, y_D, t_D, \tau_D) \right\} d\tau
\]

\[
- \int_0^{t_D} \sum_{k=1}^{N_{NRSP}} \left\{ \frac{\partial p_{D8}^{NRSP,k}(\tau_D)}{\partial \theta_b^{NRSP,k}} \frac{M_8}{M_{ref}} C_{s8} \phi_b^{NRSP,k}(x_D, y_D, t_D, \tau_D) \right\} d\tau
\]

\[
- \sum_{k=1}^{N_{NRSP}} \left\{ \frac{\partial p_{D8}^{NRSP,k}(\tau_D)}{\partial \theta_b^{NRSP,k}} \frac{M_8}{M_{ref}} C_{s8} \phi_b^{NRSP,k}(x_D, y_D, t_D, \tau_D) \right\} d\tau
\]

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\[- \int_{0}^{T_D} \sum_{k=1}^{N_1} \left( \frac{\partial p_{DB}^{1,k}(\tau_D)}{\partial \theta_{DB}^{1,k}} \frac{M_B}{M_{ref}} \frac{C_{ref}}{C_{ref}} \theta_B^{1,k}(x_D, y_D, t_D, \tau_D) \right) \, d\tau + \int_{0}^{T_D} \sum_{l=1}^{N_{WS}} q_{DB,l} G_{B} \, d\tau, \quad (4.32)\]

with

\[
\theta_B = 1 \text{ if } (x, y) \in \Omega_B \\
\theta_B = 0.5 \text{ if } (x, y) \in \Gamma_B.
\]  \quad (4.33)

On the right hand side of Eq. (4.32), the first term accounts for line integrals along \((\Gamma_{outer} - \Gamma_{2,outer})\), the second along \((\Gamma_2 - \Gamma_{2,outer})\), the third along boundary of the cross-like shape area \(\Gamma_{NSP}\), and the fourth along \(\Gamma_1\); the superscript \(k\) in Eq. (4.32) for terms of \(p_{DB}, \theta_B, \theta_B\) stands for the \(k\)-th element on the above mentioned multiply connected surfaces of Region 8. The number of boundary elements for \((\Gamma_2 - \Gamma_{2,outer})\) is denoted by \(N_{2,B}\) and for \(\Gamma_{NSP}\) is denoted by \(N_{NSP}\), which equals to

\[
N_{NSP} = N3 + N5 + N6 + N7 - N4. \quad (4.34)
\]

**Laplace transforms.** Laplace transforms with respect to time are taken for Eq. (4.31) and Eq. (4.32). As both equations apply to convolution theorem, they are expressed in Laplace domain as

\[
\frac{\theta_j}{2\pi} L\left[ p_{DJ}(x_D, y_D, t_D) \right] = \sum_{k=1}^{N_j} \left\{ \frac{M_j}{M_{ref}} \frac{C_{ref}}{C_{ref}} L\left[ \frac{\partial p_{DB}^{j,k}(\tau_D)}{\partial \theta_{DB}^{j,k}} \right] \right\} L\left[ \theta_B^{j,k}(x_D, y_D, t_D, \tau_D) \right] \right\} + \sum_{l=1}^{N_{WS}} L[q_{D,j,l}] L[G_{j}], \quad j \in [1,7] \quad (4.35)
\]

and
\[
\frac{\theta_0}{2\pi} L[P_{DB}(x_D, y_D, t_D)] = \sum_{k=1}^{N_e} \left\{ \frac{M_B}{M_{ref}} \frac{C_{sref}}{C_{sb}} L\left[ \frac{\partial p_{DB}^{outer,k}(t_D)}{\partial \xi_{sB}^{outer,k}} \right] L[\theta_0^{outer,k}(x_D, y_D, t_D)] \right\} \\
- \sum_{k=1}^{N_{SB}} \left\{ \frac{M_B}{M_{ref}} \frac{C_{sref}}{C_{sb}} L[p_{DB}^{2,k}(t_D)] L[\theta_0^{2,k}(x_D, y_D, t_D)] \right\} \\
- \sum_{k=1}^{N_{NSP}} \left\{ \frac{M_B}{M_{ref}} \frac{C_{sref}}{C_{sb}} L[p_{DB}^{NSP,k}(t_D)] L[\theta_0^{NSP,k}(x_D, y_D, t_D)] \right\} \\
+ \sum_{i=1}^{N_{WB}} L[q_{DB,i}] L[G_0].
\]

(4.36)

where “L[ ]” denotes the Laplace transform operator.

Zhao and Thompson (2002) has devised a fast and highly accurate method of obtaining numerical Laplace transforms, and it is implemented in this work. The complicated Laplace transforms of \( \tilde{\Theta} \) and \( \hat{\Theta} \) in Eq. (4.35) or Eq. (4.36), thereby, can be obtained.

**Linear matrix equations in Laplace domain.** For each of the 7 simply connected regions, \([N_j]^7\) number of linear equations can be obtained in Laplace domain by evaluating pressure, based on Eq. (4.35), at the midpoint of every element on respective boundaries \([I_j]^7\); for the Region 8, \(Ne\) number of linear equations can be obtained by
evaluating pressure, based on Eq. (4.36), at the midpoint of element along the boundary \((I_{\text{outer}} - I_{2,\text{outer}})\). By evaluating pressure at midpoint of every boundary element discretized, thereby, gives a set of \((N1 + N2 + N3 + N4 + N5 + N7 + Ne)\) linear equations.

The 8 locally homogeneous regions are interacting with each other along their contact interfaces by fluid transfer. Each region is coupled with its adjacent one(s) along interface(s) based on pressure and rate continuity. Taking the coupling between Region 1 and Region 8 for example, we have

\[
\frac{M_i}{M_{\text{ref}}} L \left[ \frac{\partial P_{D1}^k(t_D)}{\partial \xi_1^k} \right] = \frac{M_8}{M_{\text{ref}}} L \left[ \frac{\partial P_{D8}^{L_k}(t_D)}{\partial \xi_8^{L_k}} \right], \quad k \in [1, N1].
\] (4.37)

Similar to Eq. (4.37), coupling of the 8 regions gives \(N1 + N2 + N3 + N4 + N5 + N6 + N7\) number of linear equations in total. Note that \(N_{2,8}\) is the number of boundary elements on \((I_2' - I_{2,\text{outer}}')\). In addition, the prescribed outer boundary condition, no-flow boundary assumed in the beginning of this work, gives \((Ne + N2 - N_{2,8})\) number of linear equations as

\[
\frac{\partial P_{D8}^{\text{outer},k}(t_D)}{\partial \xi_{8,\text{outer},k}} = 0 \quad k \in [1, Ne + N2 - N_{2,8}].
\] (4.38)

Hence, another set of \((N1 + N2 + N3 + N4 + N5 + N7 + Ne)\) linear equations are given. In total, \(2(N1 + N2 + N3 + N4 + N5 + N7 + Ne)\) linear equations are obtained in the exactly same number of unknowns—\((N1 + N2 + N3 + N4 + N5 + N7 + Ne)\) pressures plus \((N1 + N2 + N3 + N4 + N5 + N7 + Ne)\) pressure gradients of all boundary elements.
By solving these linear equations, pressure and pressure gradient of each boundary element can be obtained in Laplace domain, and substitute them into Eq. (4.35) or Eq. (4.36) so that pressure at desired location within the reservoir can be obtained. Pressure and pressure gradient in Laplace domain, then, can be inverted to real-time domain by using Stephest inversion algorithm (Stephest 1970).
4.3 Model Validity and Results

To validate the BEM-based model and to investigate the response of heterogeneous reservoir, solutions in terms of pressure- and rate- transient generated by the proposed method are checked against 3 different well-reservoir system examples where wells—including vertical well and fractured horizontal well—produce reservoirs with multi-scale heterogeneities bounded by complex boundary shapes. In each example, sensitivity analysis of well behavior is conducted under a number of scenarios with differing well or reservoir parameters. Pressure-transient response of well—producing at constant rate—and rate-transient response of well—producing at constant downhole pressure—presented in this section are documented in type curve format. By applying Duhamel principle, the rate solution under constant pressure can be obtained from pressure solution under constant rate in Laplace domain as

\[ L[q_{WD}] = \frac{1}{s^2 L[p_D]}, \]  \hspace{1cm} (4.39)

\[ q_{WD} = \frac{q_w \mu_{ref}}{2\pi k_{ref} h \Delta p}, \]  \hspace{1cm} (4.40)

where \( q_{WD} \) is the dimensionless rate behavior of well at constant producing pressure defined in Eq. (4.40) while \( p_D \) is the dimensionless pressure under constant producing rate, which is obtained in this study by Eq. (4.35) or Eq. (4.36), and “s” denotes the Laplace variable.
4. 3.1 Example 1: Fully compartmentalized reservoir

In this example, Fig 4.8 (a) shows that the reservoir consists of 25 compartments/regions with box-shaped no-flow outer boundary and each compartment is numbered consecutively. Reservoir is produced by a vertical well placed in the Region 13, which is described by a point source.

Scenario 1: effect of grid shape. One of the greatest advantages offered by BEM is that this method is able to handle fluid flow in a locally homogeneous domain with irregular-shaped boundary in a comparatively easy and convenient manner as proposed in the methodology section. In contrast, traditional analytical method such as source/sink function method can only generate solutions in rectangular-bounded flow domain by superposing particular types of image wells corresponding to the boundary conditions. It is necessary to examine how grid shape can affect solution accuracy by checking results generated with different gridding systems against an already known analytical solution.

Suppose that reservoir is homogeneous with uniform mobility and storativity through all the 25 compartments; pressure behavior of the well can be described by the widely known standard solution given by Gringarten et al (1973), Carslaw and Jaeger (1959). Fig 4.8 (b) shows the schematic of reservoir with the same size to that in Fig 4.8 (a) but with rectangular gridding system. Solutions from the two reservoir schematics with homogeneity for all 25 compartments generated from the BEM model, if accurate enough, and the standard solution must overlap each other. Fig 4.9 provides solutions of pressure and pressure derivative in response to reservoir schematic in Fig 4.8 (a) and Fig 4.8 (b) respectively with uniform properties through all 25 compartments and standard solution of Eq. (4.41) is also provided.
Fig 4.8 Schematic of fully compartmentalized reservoir (a) with irregular-gridding system and (b) with rectangular-gridding system, \( X_{eD} = 50 \) and \( Y_{eD} = 50 \)
\[
\Delta p_D(x_D,y_D,t_D) = \int_0^{t_D} \frac{1}{X_{ed}} \left[ 1 + 2 \sum_{n=1}^{\infty} \exp\left( -\frac{n^2\pi^2 t_D}{X_{ed}^2} \right) \cos\left( \frac{n\pi x_D}{X_{ed}} \right) \cos\left( \frac{n\pi y_D}{X_{ed}} \right) \right] 
\cdot \frac{1}{Y_{ed}} \left[ 1 + 2 \sum_{n=1}^{\infty} \exp\left( -\frac{n^2\pi^2 t_D}{Y_{ed}^2} \right) \cos\left( \frac{n\pi x_D}{Y_{ed}} \right) \cos\left( \frac{n\pi y_D}{Y_{ed}} \right) \right] dt_D.
\]

(4.41)

It clearly shows in Fig 4.9 that the 3 solutions almost overlap each other with barely noticeable difference for both pressure and pressure derivative responses, which illustrates that the boundary shape of locally homogeneous compartment barely affect result accuracy by using this BEM model. It is of great importance in reservoir modeling for BEM to have the two advantages: 1. flow can be modeled in arbitrarily shaped domain; 2. boundary shape does not affect or comprise solution accuracy, which is close to analytical solution accuracy.
Fig 4.9 Pressure response to reservoir schematic in Fig 4.8 (a) and Fig 4.8 (b) and to the standard solution
**Scenario 2: linear composite reservoir.** To validate the accuracy of solution for heterogeneous reservoir generated by the BEM proposed, results generated by this model are checked against the semi-analytical solution provided by Zhao and Thompson (2001), in which reservoir consists of two compartments linearly lined up. To model linear composite reservoir with the reservoir schematic in Fig 4.8 (a), the 1st to 15-th compartments are having uniform mobility and the 16-th to 25-th regions are also having uniform mobility but its value is different to that for the first 15 compartments.

Solutions from boundary element method and solutions from semi-analytical method are in high agreement for this scenario. Zhao and Thompson (2001) has applied source/sink function method to deal with linear composite reservoir with complex outer boundary geometry (each individual compartment has to be rectangular). It is shown in Fig 4.10 that solutions from the two methods have high agreement. Boundary element method preserves the analytical nature of solution as it retains the free-space Green’s function that governs differential flow equation as global weighting function and this method has largely eliminated problems such as numerical dispersion, grid shape, grid orientation (Kikani and Horne 1992, 1993).
Fig 4.10 Comparison between solutions generated by boundary element method and semi-analytical method.
4.3.2 Example 2: Naturally fractured reservoir

This example is going to discuss how natural fractures (not connected to well), whether highly conductive, less conductive or extremely unconductive serving the role as sealing faults, can affect well behavior and to provide flux profiles along fracture faces. By using boundary element method in this study where natural fractures are treated as porous media conducting fluid that its geometry (including length and aperture), permeability, and storativity can be assigned, natural fractures thereby are modeled in a more realistic and more physical-driven manner. Two scenarios are considered in this example. In the first one, well is producing between two parallel natural fractures; in the second one, one fracture intercepts the other, forming a crossing conjunction area.

Scenario 1: two parallel fractures. The schematic of naturally fractured reservoir bounded by an arbitrarily shaped no-flow boundary with two parallel natural fractures is shown in Fig 4.11. Pressure- and rate- transient behaviors of well are examined under various level of fracture conductivity that is reflected on permeability ratio between fracture and formation matrix while fracture width and length are fixed. Note that the two fractures are assumed to have the same properties. Pressure behavior with constant production rate is shown in Fig 4.12 (a) while production rate decline with constant downhole producing pressure is shown in Fig 4.12 (b).
Fig 4.11 Reservoir with two parallel natural fractures bounded by arbitrarily shaped outer boundary. \( H_D = 50 \), \( w_{FD} = 0.01 \)
Fig 4.12 (a) Pressure behavior of well and (b) Production rate behavior of well, with respect to various level of natural fracture and formation permeability ratio as $2L_{fD} = 10$, $X_{VD} = 2$
As illustrated in **Fig 4.12 (a)** and **Fig 4.12 (b)**, it is difficult for well to detect the existence of natural fractures with such close distance to well even if fracture permeability is 100 times greater or smaller than original formation permeability. Effect of natural fracture clearly affects pressure- or rate-transient response when fracture is 10000 times more permeable than formation, while natural fracture’s impact on well behavior is even more pronounced when fracture is 10000 times less permeable than formation, acting as non-conductive faults, as a channel flow with 1/2 unit slope on pressure derivative and -1/2 unit slope on rate is evolved.

Flux profile exhibits large difference along different fracture planes—including the one closer to well (left hand side of fracture) and the one further to well (right hand side of fracture) illustrated in **Fig 4.13**—when fracture is much more permeable than formation with great conductivity; flux profile along fracture planes (flux for only one fracture is provided) under the situation with $k_{frac}/k_{matrix} = 10000$ is presented in **Fig 4.14 (a)**. In contrast, flux profile is almost the same on the two fracture planes when fracture is less permeable than is formation; **Fig 4.14 (b)** provides flux along fracture under the situation with $k_{frac}/k_{matrix} = 0.01$.

Illustrated in **Fig 4.14 (a)** where fracture is significantly conductive, under $k_{frac}/k_{matrix} = 10000$, for the LHS of fracture the portion of fracture near two tips (roughly 30% and 20% of half length for respective early and late time) extracts fluid from reservoir and injects it back into reservoir from mid part, which helps the production as fracture serves as the main fluid passing path in reservoir. Note that fluid flowing towards the well represents positive flux. Fluid quite uniformly flows into the RHS of fracture except a bit more around tips. On the other hand, **Fig 4.14 (b)** shows that fluid flow rate
along the RHS and LHS barely has difference overlapping each other when $k_{frac}/k_{matrix} = 0.01$, that fracture only passes as much fluid it receives from RHS into reservoir from LHS, and that low-conductive fracture makes it difficult for fluid to pass because flow rate near two tips are so small (nearly zero), which indicates that fracture serves as a “barrier” as fluid flow is impeded and it has to pass around tips in order to flow towards well. Zhao (2013) came up with semi-analytical solutions dealing with natural fracture network based on source/sink function method, and flux profiles along fracture planes are also discussed.
**Fig 4.13** Definition of right hand side and left hand side of fracture and of positive flow direction for flux along fracture plane
Fig 4.14 Flux profile along both RHS and LHS of fracture (a) with $k_{frac}/k_{matrix} = 10000$ and (b) with $k_{frac}/k_{matrix} = 0.01$
Fig 4.15 (a) and Fig 4.15 (b) examined pressure and rate behavior when well is producing between two parallel sealing faults, which are described by setting natural fracture permeability extremely low as $k_{frac}/k_{matrix} = 0.0001$, under various spatial spanning extent. Channel flow has been clearly exhibited on pressure derivative with $1/2$ slope, shown in Fig 4.15 (a), for sealing faults spans as far as or further than $2L_{fD}/H_{D} = 0.2$ when pressure-transient hits faults. Pressure derivative starts to go downward as fluid, beyond fault tips, flows into the channel before outer no-flow boundary is felt.
Fig 4.15 (a) Pressure behavior of well and (b) rate behavior of well, producing between two sealing faults as $k_{frac}/k_{matrix} = 0.0001$, $X_{vD} = 5$
Scenario 2: two crossing natural fractures. Effect of two crossing natural fractures on well behavior is investigated in this scenario as Fig 4.16 presents the schematic of the naturally fractured reservoir. Two crossing fractures, intercepting each other with an included angle and forming a parallelogram of conjunction area, are modeled simply by consisting of 5 regions shown in Fig 4.16 filled with different colors (using different colors is for the purpose of demonstrating compartments of modeling crossing natural fractures and they are assumed to have same properties in following computed results), and details are discussed in the methodology section as dealing with the cross-like area.

Pressure and rate behaviors of well under various permeability ratio between fracture and formation matrix are presented in Fig 4.17 (a) and Fig 4.17 (b) with fractures’ included angle fixed at 60° and the distance from left tip to their intersecting point fixed at 3/4 of fracture length.
Fig 4.16 Reservoir with two crossing natural fractures as $H_D = 50$, $2L_{FD} = 20$, $L_D = 15$, $d_D = 1$ and $w_{FD} = 0.1$
Fig 4.17 (a) Pressure behavior of well and (b) rate behavior of well, under various level of permeability ratio between fracture and formation with included angle=60 degrees
The effect of included angle between two sealing faults for $k_{frac}/k_{matrix} = 0.0001$ is also examined under a number of values $\alpha = 90^0, 75^0, 60^0$ and $45^0$. It shows in Fig 4.18 that wider included angle causes slightly greater pressure drop as $L_D/2L_{f_D} = 3/4$ is fixed as well as distance from the well to faults $d_D = 1$, which responds to the fact that wider included angle results in wider extent that sealing faults impede fluid from the other side to flow towards the well and fluid has to bypass sealing faults. It is noticeable when fractures are orthogonal (angle=90 degrees) pressure derivative goes up and stays at $dp_D/\ln t_D = 2$ for a period of time before it drops as a result that fluid beyond faults start to flow towards well. By superposing image well theory, it is known that pressure derivative for well producing in a quarter of an infinite reservoir—reservoir intercepted by two orthogonal sealing faults—would become 4 times (which is 2 caused by original well and three more image wells) as much as that in a free-space flow domain (which is 0.5). Before pressure-transient travels beyond faults’ tip, the image well theory applies to the case with $\alpha = 90^0$ and the generated pressure derivative does meet the theoretical value $dp_D/\ln t_D = 2$.

Although this simple example is far from enough to comprehensively examine the effect caused by all types of natural fracture networks, the main purpose of this example is to show the ability of modeling natural fractures and the high level of solution accuracy. Tight oil/gas has been increasingly playing a more and more important role in oil/gas production, with multi-stage fractures being implemented on horizontal wells, and there exists a great need to understand and analyze complex fracture network as formed by connection of hydraulic fractures and natural fractures. The BEM modeling technology proposed gives a powerful tool to simulate complex natural network, which is able to
handle as many as fractures and is only limited to the storage of computer. Using the same strategy as for natural fractures, fluvial wandering reservoir can also be modeled by the proposed BEM technology as fluvial channels or splays are treated as heterogeneities of porous media.
Fig 4.18 Pressure behavior of well under different included angle as $k_{frac}/k_{matrix} = 0.0001$
4.3.3 Example 3: Enhanced-fracture-region model

A physical model, proposed by Stavlgorova and Mattar (2013), has recently been often applied to deal with multi-stage fractured horizontal well. This model provides an excellent attempt to capture the effect of an “enhanced” permeability region in the vicinity of each main fracture by assigning a local stimulated region volume (SRV) with higher permeability than formation matrix permeability around each stage of fracture, and the local SRV can be considered to represent fracture branching/complex networks or densely distributed natural fractures. Many other researchers, such as Sureshjani and Clarkson (2015), Lougheed et al. (2013), have applied this model to simulate or analyze testing data of multi-stage fractured well performance. In fact a region of “reduced” permeability may be created in the vicinity of fracture as well due to poor flowback performance that only a small portion of fracturing fluid flowed back to well, which results in comparatively high saturation of fracturing fluid near fracture and thereby decreases relative permeability of oil/gas. In the following results, we will examine well behaviors under various levels of permeability “enhanced” or “reduced” for local SRV, under various fracture length, and under various local SRV width.

The schematic of the EFR model in this example is shown in Fig. 4.19 that 6-stage transverse fractures are placed along the horizontal wellbore (it is assumed that each stage has only one fracture). With the flexibility of the boundary element method we proposed, it is of no problem at all to compute pressure or rate behavior when fractures have different length for each stage or when each local SRV around fracture has different permeability. In order to capture and understand behaviors of this EFR model in a general
manner, we assume that all fractures have the same length, and that the permeability of each local SRV is the same.
Fig 4.19 Enhanced-fracture-region model with 6-stage fractures, $X_{eD} = 200$ and $Y_{eD} = 100$
Scenario 1: Effect of various level of near-fracture region permeability. Pressure- and rate- transient behaviors of fractured horizontal well are provided in Fig 4.20 (a) and Fig 4.20 (b) respectively. It can be seen, in Fig 4.20 (a), that early formation linear flow with 1/2 slope is exhibited for all permeability ratio $k_1/k_2$ and it lasts shorter when $k_1$ is greater because under which situation pressure-transient travels faster and hits the boundary of local SRV region at a comparatively earlier time, which marks the end of early linear flow with fluid influx from original formation and with pressure derivative picking up ($k_1/k_2 > 1$). For multi-stage fractured well producing in homogeneous reservoir, it is known that pressure derivative, following 1/2 slope, picks up with slope greater than 1/2, sometimes approaching unit slope, because of interference among fractures (as shown in this scenario with the blue curve). With $k_1/k_2 = 20$ by EFR model, however, pressure derivative brings up right after linear flow because of enhanced-fracture region instead of interference among fractures, which may lead to misinterpretation when analyzing testing data by using homogeneous reservoir model.

In this scenario, Other than the case when permeability of near-fracture region is much smaller than formation permeability as $k_1/k_2 = 0.05$, pseudo-radial flow has not been successfully evolved. Pseudo-radial flow with $dp_D/lnr_D = 0.5$ is evolved only for $k_1/k_2 = 0.05$ indicating that flowing mobility behavior would move from the inner region to the outer region if the mobility difference between the two regions is big enough (Spivey and Lee 2013).
Fig 4.20 (a) Pressure response of fractured horizontal well and (b) rate response of fractured horizontal well, under different permeability ratio $k_1/k_2$ as $2L_{FD}/Y_{eD} = 0.1$, $L_{SRVD}/d_{fSD} = 0.5$
**Scenario 2: Effect of fracture length.** In this scenario, the effect of fracture length on pressure and production rate is investigated for both situations when permeability of the region around fracture is enhanced or reduced. **Fig 4.21 (a)** provides the pressure response and **Fig 4.21 (b)** provides the production rate response. As shown in **Fig 4.21 (b)**, production rate based on the EFR model is dominated by the permeability of the region in vicinity of fracture that well with the longest fractures $2L_{FD}/Y_{eD} = 0.7$ under $k_1/k_2 = 0.05$ produces less than does the well with shortest fractures $2L_{FD}/Y_{eD} = 0.1$ under $k_1/k_2 = 20$ until $t_D = 2$ and even after that the difference of production rate is quite small. It suggests that there exists big difference between results generated by EFR model and by homogeneous reservoir model for multi-stage fractured well, and it implies that near-fracture region imposes huge influence on well production performance.
Fig 4.21 (a) Pressure response of fractured horizontal well and (b) rate response of fractured horizontal well, under different fracture length for both permeability around fracture “enhanced” or “reduced” as $k_1/k_2 = 20$ and 0.05, $L_{SRV}/d_{fSD} = 0.5$.
**Scenario 3: Effect of width of the local SRV in vicinity of fracture.** In this scenario, we examine how the width of local SRV region can affect well behavior when the permeability of the region is enhanced. If the enhanced region accounts for natural fractures around hydraulic fracture, the wider of the region, more natural fractures are distributed between hydraulic fractures.

Neither pressure nor rate shows much of a difference among the three curves ([Fig 4.22 a](#) and [Fig 4.22 b](#)). The difference in time profile lies mainly between $t_D = 0.1$ to 5, during which production rate is roughly as much as 2 times, when local SRV regions between hydraulic fractures are connected as $L_{srvd}/d_{fsd} = 1$, the rate when $L_{srvd}/d_{fsd} = 0.5$.

For the case $L_{srvd}/d_{fsd} = 1$, the EFR model becomes a two-region composite reservoir as local SRV regions between hydraulic fractures are connected. Zhao (2012) proposed a model integrated SRV for fractured horizontal well to capture the relation between SRV and original formation. When $L_{srvd}/d_{fsd} = 1$ in this example, it represents similar reservoir model schematic described in Zhao (2012)’s.

Pressure profiles all over the reservoir with such complex reservoir-well system at different time $t_D = 10, 100$ and 1000 are provided in respective [Fig 4.23 (a)](#), [Fig 4.23 (b)](#) and [Fig 4.23 (c)](#) under $L_{srvd}/d_{fsd} = 0.5$. 
Fig 4.22 (a) Pressure response of fractured horizontal well and (b) rate response of fractured horizontal well, under different ratio of local SRV width and space between fractures as $k_1/k_2 = 20$ and $2L_{fD}/Y_{eD} = 0.1$
Fig 4.23 Pressure profile over reservoir (a) at $t_D = 10$ and (b) at $t_D = 100$ and (c) at $t_D = 1000$, with $L_{SRvD}/d_{fS} = 0.5$, $2L_{fD}/Y_{eD} = 0.2$ and $k_1/k_2 = 20$
Results from Example 3 indicate that solutions generated by the proposed BEM can accurately capture flowing characteristics in near-wellbore regions, even with complex well geometry, which provides solid basis in analyzing test data with confidence as test data mainly reflects the flowing ability of reservoir in near-wellbore region and which can also help with operation of improving well’s productivity such as removing near-wellbore damage.

Theoretical results of various reservoir-well systems from the 3 examples suggest that the modeling strategy by using proposed BEM technology can deal with complex reservoir with multi-scale heterogeneities in an efficient and convenient manner and solutions generated have good accuracy. In addition, this technology has quite fast computation. 

**Table 4.1** summarizes the computing speed for a complete curve of each example.

**Table 4.1**: Summary of CPU run time (on the platform of 2.3 GHz Intel i5 core CPU) of the proposed BEM model for each of the 3 examples for a complete curve

<table>
<thead>
<tr>
<th></th>
<th>Example 1 (Scenario 1)</th>
<th>Example 2 (Scenario 1)</th>
<th>Example 3 (Scenario 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Boundary Element Number for Reservoir</strong></td>
<td>400</td>
<td>96</td>
<td>200</td>
</tr>
<tr>
<td><strong>Total CUP Run Time (min)</strong></td>
<td>41</td>
<td>37</td>
<td>153</td>
</tr>
<tr>
<td><strong>Percentage of Numerical Laplace Transforms Computation</strong></td>
<td>57.2%</td>
<td>94.7%</td>
<td>93.4%</td>
</tr>
</tbody>
</table>

### 4.4 Summary

A comprehensive modeling strategy by using boundary element (BEM) is presented to solve transient fluid flows in multi-scale and arbitrarily shaped heterogeneous systems. The proposed BEM is fully aware of realistic geology and only needs to discretize
boundaries of each locally homogeneous region and reservoir physical outer boundary. The modeling results are independent of gridding size and time steps. Transient pressure and production responses of wells producing in various reservoir-well systems are shown in 3 examples and robust as well as solid solutions are achieved because BEM preserves the analytical nature of solution as it retains the free-space Green’s function that governs differential flow equation as global weighting function. This work has shown promise of developing BEM-based reservoir simulators, which will be able to model transient fluid flows and compatible with realistic geology.
4.5 Nomenclature

c_t = total compressibility of reservoir, \( \text{pa}^{-1} \)

\( C_s = \text{storativity, pa}^{-1} \)

\( C_R = \text{diffusivity, m}^2/\text{s} \)

\( G = \text{free-space Green's function} \)

\( h = \text{reservoir vertical thickness, m} \)

\( k = \text{permeability, m}^2 \)

\( L_f = \text{fracture half length, m} \)

\( M = \text{mobility, m}^2/\text{pa} \cdot \text{s} \)

\( n = \text{the outward-pointing normal on a boundary element} \)

\( N_j = \text{discretized number of boundary element, } j = 1 \text{ to } 7 \)

\( N_e = \text{discretized number of boundary element for } (\Gamma_{\text{outer}} - \Gamma_{2,\text{outer}}) \)

\( p = \text{pressure, pa} \)

\( p_i = \text{initial pressure, pa} \)

\( \Delta p = p - p_i, \text{ pa} \)

\( q = \text{rate of source/sink, m}^3/\text{s} \)

\( q_{w} = \text{producing rate of well, m}^3/\text{s} \)
\( s = \text{Laplace variable} \)

\( t = \text{time, s} \)

\( w_f = \text{fracture width, m} \)

\( x = \text{horizontal coordinate, m} \)

\( x' = \text{horizontal coordinate of the source/sink point, m} \)

\( y = \text{horizontal coordinate, m} \)

\( y' = \text{horizontal coordinate of the source/sink point, m} \)

\( \xi = \text{local coordinate for boundary element, m} \)

\( \zeta = \text{local coordinate for boundary element, m} \)

\( \eta = \text{reservoir diffusivity, m}^2/\text{s} \)

\( \mu = \text{fluid viscosity, pa} \cdot \text{s} \)

\( \tau = \text{time variable, s} \)

\( \phi = \text{porosity of the reservoir, fraction} \)

\( \theta = \text{included angle between two boundary elements} \)

\( \Omega = \text{locally homogeneous domain} \)

\( \Gamma = \text{enclosed boundary for domain } \Omega \)

\( \Gamma_{\text{outer}} = \text{reservoir outer boundary} \)
Superscripts

\( k \) = the \( k \)-th boundary element

Subscripts

D = dimensionless term
Chapter 5  Conclusions and Recommendations

5.1 Conclusions

During this research, boundary integration approach (including source/sink function method and boundary element method) has been proposed to solve the research problems stated in Chapter 1.

Firstly, source/sink function method is applied to model the fracture-propagation-and-closing process. Theoretical solutions in terms of pressure transient and fluid leak-off along fracture plane during both fracture propagating and closing periods are examined and provided in type curve format. In addition, model-based diagnosis of field minifrac testing data from various types of formation is conducted based on generated solutions from the analytical model.

Secondly, source/sink function method is applied to model unsteady state flow (transient flow) for vertically stratified reservoir produced by multistage hydraulically fractured horizontal well with inter-layer crossflow. Multistage fractures are integrated in the model with flexibility of either penetrating a single layer or multiple layers. Theoretical type curves are generated as well as modified rate type curves under various scenarios. Modified type curves are successfully applied to analyze field production data in estimating fracture length and permeability of each layer.

Thirdly, boundary element method (BEM) is applied to model unsteady state flow in 2-dimensional heterogeneous reservoir with large multi-scale heterogeneities bounded by arbitrarily shaped boundaries/surfaces. Discretizations are only made on the boundary, which not only conforms to the reservoir’s geological description but also saves
computation efforts because no gridding inside the flow domain is necessary. Solutions in terms of transient pressure and rate behaviours of 3 examples are presented in type curve format.

The following key conclusions are drawn from the research:

- Fracture propagation velocity, characterized as a continuous function of time, is physically and analytically integrated into line source function (fracture is regarded as a line source/sink), and this study successfully extends sessile line source solution, which accounts for traditional fractured well behavior with constant fracture length, to propagating line source solution, which accounts for fracturing well behavior with propagating fracture.
- Three minifrac field cases—a test in high-permeable reservoir, a test in deep tight gas reservoir, and a test in capping mudstones above oil sands reserve—are analyzed based on type curves generated from the analytical model and key informations, including fracture geometry, fluid leak-off behavior, closure pressure, instantaneous shut-in pressure, and reservoir flowing capacity (kh), are evaluated.
- Well behavior is greatly influenced by heterogeneity of reservoir with multiple layers through inter-layer crossflow, which has a number of flow patterns.
- Heterogeneities of layered system distort flow regimes from homogeneous reservoir, and flow regimes become much more complex.
- Effect of permeability anisotropy in the more permeable layer is much more pronounced than in the less permeable layer.
- Proposed modified type curves can effectively help analyze field data from layered reservoir with fractured horizontal well.
• By applying the boundary element method (BEM) to deal with 2D flow problem in heterogeneous reservoir, discretization or approximation is only made on the boundary enclosing the specific flow domain with uniform fluid/rock properties. No gridding or sub-partition inside the domain is needed, which saves a lot of numerical problems and computation efforts.

• Due to the fact that BEM preserves the characters of analytical solution by using free-space Green’s function as a global weighting function (or fundamental solution) in integral equation along the boundaries, which largely eliminates the numerical problems such as numerical dispersion, grid shape, grid orientation, and due to the pursuit of solution in Laplace domain in this work, which frees the effect of time step, satisfying level of solution accuracy is successfully achieved.

• With such powerful ability to handle complex arbitrarily shaped heterogeneous reservoir, light load of computation burden in comparison with numerical methods, and good solution accuracy, fast computation, good promise of building up BEM-based reservoir simulator has been shown in this work.

• Near-wellbore/fracture region largely affects production rate of well almost through well’s life time and is playing a crucial role in improving well productivity.

5.2 Recommendations

Source/sink function method and boundary element method have great capability of semi-analytical modeling fluid flow in either heterogeneous or homogeneous reservoir and the use of the two methods in future research tackling reservoir engineering problems could include the followings:
• Moving boundary problems such as modeling main fracture stimulation, wormhole propagation, etc.

• Water and gas two-phase flow problem and combining it with the proposed multilayer model to deal with bottom-water gas reservoir as the lower layer in model serving as water region and the upper one or more layers serving as gas region or gas-water transition region.

• Naturally fractured reservoir with complex natural fracture network, fluvial wandering reservoirs, etc.

• Multiple wells production analysis from pad drilling.

• Building BEM-based reservoir simulation.
Reference


Appendix A: Plane Source Solution in Homogeneous & Rectilinear Reservoir with No-Flow Outer Boundary

In this Appendix, what is discussed is the pressure solution of a plane source in a rectilinear reservoir that is homogeneous and is enclosed with no-flow boundaries. Fig A.1 illustrates the schematic of such a system, and the coordinate system is assigned accordingly.

**Fig A.1** Schematic of a plane source in a rectilinear reservoir with closed outer boundaries

Based on the source function shown in Eq. (3.7) or Eq. (3.8) and applying Newman product method (Newman, 1936), the pressure drop generated by the plane source at an arbitrary orientation within the system, \( M(x, y, z) \), showing in Fig. A. 1, has the following form (note that only the application of Eq.(3.7) is demonstrated here)

\[
\Delta p(x, x_e, y, y_e, z, z_e, t) = \int_0^\tau \frac{d\tau}{\phi c_t} \bar{p}_{xy}(y_0, y, y_e, \tau, t) \int_{x_0}^{x_0+h_f} \int_{x_0}^{x_0+2l_f} q(x', z', \tau) \bar{p}_{xz}(x', x, x_e, \tau, t) dx' \bar{p}_{zz}(z', z, z_e, \tau, t) dz'
\]
The pressure drop caused by plane source in reservoir, shown in Fig A.1 is analytically formulated by Eq. (A.1). The flux distribution over source domain, \( \tilde{q}(x', z', \tau) \), however, is a continuous function of position over the plane, \( i.e., \) along the \( x \)- and \( z \)- axes, and of time. To obtain numerical solution, the spatial flux distribution over source plane is approximated to be piecewise function by dividing plane source into sub-sources or sub-planes where individual flux is treated as a uniform quantity (uniform per unit area). Fig A.2 illustrates the division of the source plane whose horizontal and vertical extension is evenly divided into \( n \) and \( m \) segments respectively, \( i.e., \) the source plane consists of \( n \times m \) sub-sources.

Let the rate from the sub-plane, located at \( x_{i-1} \leq x' \leq x_i \ (i \in [1, n]) \) and \( z_{j-1} \leq z' \leq z_j \ (j \in [1, m]) \), be denoted as \( q_{i,j} \); the flux of this sub-plane \( \tilde{q}_{i,j} \) is approximated as

\[
\tilde{q}(x', z', \tau) \approx \tilde{q}_{i,j}(\tau) = \frac{q_{i,j}(\tau)}{\Delta x \Delta z}, \quad x_{i-1} \leq x' \leq x_i, \quad z_{j-1} \leq z' \leq z_j,
\]

where

\[
\Delta x = \frac{2l_f}{n} \text{ and } \Delta z = \frac{h_f}{m}.
\]

(A.3)
With approximation of Eq. (A.2), pressure behavior of the plane source can be written as

\[
\Delta p(x,x_e,y,y_e,z,z_e,t) =
\]

\[
\frac{1}{\phi c_e} \int_0^t \sum_{i=1}^n \sum_{j=1}^m \frac{q_{ij}(\tau)d\tau}{\Delta x \Delta z} \int_{x_{i-1}}^{x_i} \tilde{p}_x(x',x,x_e,\tau,t)dx' \int_{z_{j-1}}^{z_j} \tilde{p}_z(z',z,z_e,\tau,t)dz' \tilde{p}_{yz}(y_0,y,y_e,\tau,t). \quad (A.4)
\]

The spatial integration of point source function can be analytically evaluated, which is (e.g., x-axis)

\[
\int_{x_{i-1}}^{x_i} \tilde{p}_{yx}(x',x,x_e,\tau,t)dx' =
\]

\[
= \int_{x_{i-1}}^{x_i} \frac{1}{2\sqrt{\pi} \eta_x(t-\tau)} \left\{ \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{(x-x'+2nx_e)^2}{4\eta_x(t-\tau)} \right\} + \sum_{n=0}^{\infty} \exp \left\{ -\frac{(x+x'+2nx_e)^2}{4\eta_x(t-\tau)} \right\} \right\} dx'
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left\{ \text{erf} \left[ \frac{x_i - x + 2nx_e}{2\sqrt{\eta_x(t-\tau)}} \right] - \text{erf} \left[ \frac{x_{i-1} - x + 2nx_e}{2\sqrt{\eta_x(t-\tau)}} \right] \right\} + \left\{ \text{erf} \left[ \frac{x_i + x + 2nx_e}{2\sqrt{\eta_x(t-\tau)}} \right] - \text{erf} \left[ \frac{x_{i-1} + x + 2nx_e}{2\sqrt{\eta_x(t-\tau)}} \right] \right\}
\]

\[
(A.5)
\]

where “erf” denotes the error function.
Fig A.2 Division of plane source
The integral of point source function can be denoted by a simple notation defined as

\[
I_p(x_{i-1}, x_i, x, x_e, \tau, t) = \int_{x_{i-1}}^{x_i} p(x', x, x_e, \tau, t)dx'.
\] (A.6)

With this analytical evaluation of spatial integration of source function, the plane source solution described by Eq. (A.4) can be expressed as

\[
\Delta p(x, x_e, y, y_e, z, z_e, t) = \frac{1}{\phi c_t} \int_0^L \sum_{i=1}^n \sum_{j=1}^m \frac{q_{l,j}(\tau)dt}{\Delta x \Delta z} \cdot I_p(x_{i-1}, x_i, x, x_e, \tau, t) \cdot I_p(z_{j-1}, z_j, z, z_e, \tau, t) \cdot \tilde{p}_{xy}(y_0, y, y_e, \tau, t). \] (A.7)

Eq. (A.7) written in full detail becomes

\[
\Delta p(x, x_e, y, y_e, z, z_e, t) = \frac{1}{\phi c_t} \int_0^L \sum_{i=1}^n \sum_{j=1}^m \frac{q_{l,j}(\tau)dt}{2Lf/\eta_f/\epsilon n} \frac{1}{2 \sqrt{\pi \eta_y(t-\tau)}} \exp \left[ \frac{-(y-y_0)^2}{4 \eta_y(t-\tau)} \right]
\]

\[
+ \frac{1}{2} \sum_{n=\infty}^{\infty} \left\{ \text{erf} \left[ \frac{x_i - x + 2nx_e}{2 \sqrt{\eta_x(t-\tau)}} \right] - \text{erf} \left[ \frac{x_{i-1} - x + 2nx_e}{2 \sqrt{\eta_x(t-\tau)}} \right] \right\}
\]

\[
+ \text{erf} \left[ \frac{x_i + x + 2nx_e}{2 \sqrt{\eta_x(t-\tau)}} \right] - \text{erf} \left[ \frac{x_{i-1} + x + 2nx_e}{2 \sqrt{\eta_x(t-\tau)}} \right]
\]

\[
+ \frac{1}{2} \sum_{n=\infty}^{\infty} \left\{ \text{erf} \left[ \frac{x_i - z + 2nz_e}{2 \sqrt{\eta_z(t-\tau)}} \right] - \text{erf} \left[ \frac{x_{i-1} - z + 2nz_e}{2 \sqrt{\eta_z(t-\tau)}} \right] \right\}
\]

\[
+ \text{erf} \left[ \frac{x_i + z + 2nz_e}{2 \sqrt{\eta_z(t-\tau)}} \right] - \text{erf} \left[ \frac{x_{i-1} + z + 2nz_e}{2 \sqrt{\eta_z(t-\tau)}} \right], \] (A.8)

Similarly, if Eq. (3.8) is applied to perform the late time computation, the full detailed expression is

\[
\Delta p(x, x_e, y, y_e, z, z_e, t) =
\]
\[
\frac{1}{\phi c_t} \int_0^t \sum_{i=1}^n \sum_{j=1}^m \frac{q_{i,j}(r) \, dr}{2 \pi \eta_y(t - r)} \exp \left[ - \frac{(y - y_0)^2}{4 \eta_y(t - r)} \right] \\
\cdot \left\{ \frac{(x_i - x_{i-1})}{x_e} + 4 \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( - \frac{n^2 \pi^2 \eta_x t}{x_e^2} \right) \sin \left[ \frac{n\pi(x_i - x_{i-1})}{2x_e} \right] \cos \left[ \frac{n\pi(x_i + x_{i-1})}{2x_e} \right] \cos \left[ \frac{n\pi x}{x_e} \right] \right\} \\
\left\{ \frac{(z_j - z_{j-1})}{h_e} + 4 \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( - \frac{n^2 \pi^2 \eta_z t}{z_e^2} \right) \sin \left[ \frac{n\pi(z_j - z_{j-1})}{2z_e} \right] \cos \left[ \frac{n\pi(z_j + z_{j-1})}{2z_e} \right] \cos \left[ \frac{n\pi z}{z_e} \right] \right\}. 
\]
(A.9)
Appendix B  Exemplification of Mathematical Description of

$q_{DM}$ vs. $t_{DM}$ in Early and Late Time Periods with Fractures Penetrating Multiple Layers

The dimensionless modified rate and modified time behavior of well, $q_{DM}$ vs. $t_{DM}$, offers a great many advantages either in organizing theoretical type curves or applying them for field cases. The two unique dimensionless terms were firstly proposed by Zhao et al. (2016) to help organize and analyze production rate from multistage fractured horizontal well in homogeneous reservoir in a systematic manner by incorporating reservoir and well geometry information into the two terms. Extended from the work of Zhao et al. (2016), this research further integrated mobility and storativity of each layer into the two terms so that production rate of the multilayered system with multistage fractured well, when plotted as $q_{DM}$ vs. $t_{DM}$, presents in a more organized manner as in homogeneous reservoir and therefore can be analyzed in systematically as well.

For all of the modified dimensionless terms defined from Eqs. (3.30) to (3.35), the multilayered system with distinctive mobility and storativity of each layer, as well as fracture and reservoir geometries, merge together during early linear flow and late pseudo-steady flow when plotted as $q_{DM}$ vs. $t_{DM}$. In this Appendix, detailed mathematical derivation using other researchers published work will be given to exemplify the similar outcomes.

During early linear flow
To simplify the mathematical deriving process, a 2-layer reservoir is assumed; similar deriving strategy and process apply to multi-layered reservoir that consists of more than 2 layers.

Wattenbarger et al. (1998) gives the production rate response of a fractured well with single fracture producing at constant bottom pressure in homogeneous reservoir, which is

\[ q_w(t) = \frac{2\pi k(2L_f)\Delta p}{\mu} \cdot \frac{h_f}{\pi \sqrt{\frac{k_t}{\phi \mu c_s}}} , \quad (B.1) \]

where “\( h_f \)” denotes the vertical fracture penetration (fracture height).

Considering multiple fractures and no interference among fractures in early linear flow, it is reasonable to multiply single fracture solution, represented by Eq. (B.1), by the number of fractures transversely along multi-stage fractured horizontal well. Therefore, production rate of multi-stage fractured well can be formulated as

\[ q_w(t) = \frac{2\pi k(2L_f)\Delta p}{\mu} \cdot \frac{h_f}{\pi \sqrt{\frac{k_t}{\phi \mu c_s}}} \cdot N_{stage} . \quad (B.2) \]

While in a 2-layer reservoir, the production rate at well equals to the summation of production rates of hydraulic fractures in each layer, in which fractures are producing as if in a homogeneous reservoir. Therefore, production rate of multistage fractured well in the 2-layer reservoir becomes

\[ q_w(t) = q^{f1}(t) + q^{f2}(t) , \quad (B.3) \]
According to the dimensionless production rate of well and dimensionless time defined from Eq. (3.23) and Eq. (3.26),

$$q_{wD} = \frac{q_{w} \mu_1}{2\pi k_1 x_e \Delta p} \cdot \frac{h_{f1} \cdot N_{stage}}{\pi \left( \frac{k}{\phi \mu c_t} \right)_1 t} \cdot \frac{h_{f2} \cdot N_{stage}}{\pi \left( \frac{k}{\phi \mu c_t} \right)_2 t} \cdot \frac{1}{2} \frac{\sqrt{1 + \frac{1}{\phi_0}}} {\phi_1 \mu_1 c_{i1}} L_r^2. \tag{3.23}$$

$$t_D = \frac{k_1 t}{\phi_1 \mu_1 c_{i1} L_r^2}. \tag{3.26}$$

by replacing production rate and time with their dimensionless form, Eq. (B.4) can be written as

$$q_{wD}(t_D) = \frac{2\pi k_1 x_e \Delta p}{\mu_1} \left[ \frac{2\pi k_1 (2L_r) \Delta p}{\mu_1} \cdot \frac{h_{f1} \cdot N_{stage}}{\pi \left( \frac{k}{\phi \mu c_t} \right)_1 \left( \frac{\phi \mu c_t}{k} \right)_1 L_r^2} + \frac{2\pi k_2 (2L_r) \Delta p}{\mu_2} \cdot \frac{h_{f2} \cdot N_{stage}}{\pi \left( \frac{k}{\phi \mu c_t} \right)_2 \left( \frac{\phi \mu c_t}{k} \right)_2 L_r^2} \right]. \tag{B.5}$$

By organizing Eq. (B.5), it gives

$$q_{wD}(t_D) = \frac{2L_f D h_{f1D} \cdot N_{stage}}{x_e \pi \sqrt{\pi t_D}} + \frac{2L_f D h_{f2D} \cdot N_{stage}}{x_e \pi \sqrt{\pi t_D}} \cdot \frac{k_2 \mu_1}{k_1 \mu_2} \cdot \frac{1}{\sqrt{\frac{k_2 \mu_1 \phi_1 c_{i1}}{k_1 \mu_2 \phi_2 c_{i2}}} t_D} \cdot \frac{\sqrt{k_2 \mu_1 \phi_2 c_{i2}}}{\sqrt{k_1 \mu_2 \phi_1 c_{i1}} t_D}. \tag{B.6}$$

With mobility and storativity defined in Eq. (3.28) and Eq. (3.29), Eq. (B.6) can be expressed as
Recall the definition of $q_{DM}$ and $t_{DM}$ in Eq. (3.30) and Eq. (3.31)

$$q_{DM} = \frac{q_{WD}}{N_{stage}} \cdot \frac{y_{eD} \left( h_{e1D} + \frac{C_{S2}}{C_{S1}} h_{e2D} \right)}{\left[ \frac{2L_{FD}}{x_{eD}} \left( h_{f1D} + \frac{C_{S2} M_2}{C_{S1} M_1} h_{f2D} \right) \right]^2 \cdot N_{stage}}.$$  \hspace{1cm} (3.30)

$$t_{DM} = t_{MBD} \cdot \frac{\left[ \frac{2L_{FD}}{x_{eD}} \left( h_{f1D} + \frac{C_{S2} M_2}{C_{S1} M_1} h_{f2D} \right) \right]^2 \cdot N_{stage}}{y_{eD} \left( h_{e1D} + \frac{C_{S2}}{C_{S1}} h_{e2D} \right)}.$$  \hspace{1cm} (3.31)

The ratio of time to material balance time is a constant (the ratio is 0.5) during early linear flow. According to the definition of material balance time $t_{MB}$, the ratio can be formulated as

$$\frac{t}{t_{MB}} = \frac{t}{\int_0^t q_w(t) dt}.$$  \hspace{1cm} (B.8)

Based on the production rate of well by Eq. (B.4),

$$\int_0^t \frac{q_w(t) dt}{q_w(t)} = \frac{\int_0^t 2\pi k_1 (2L_f) \Delta p}{\mu_1} \cdot \frac{h_{f1} \cdot N_{stage}}{\pi \left( \frac{k}{\phi_\mu c_i} \right) t} + \frac{2\pi k_2 (2L_f) \Delta p}{\mu_2} \cdot \frac{h_{f2} \cdot N_{stage}}{\pi \left( \frac{k}{\phi_\mu c_i} \right) t} \cdot \frac{\int_0^t 2\pi k_1 (2L_f) \Delta p}{\mu_1} \cdot \frac{h_{f1} \cdot N_{stage}}{\pi \left( \frac{k}{\phi_\mu c_i} \right) t} + \frac{2\pi k_2 (2L_f) \Delta p}{\mu_2} \cdot \frac{h_{f2} \cdot N_{stage}}{\pi \left( \frac{k}{\phi_\mu c_i} \right) t}.$$
Therefore, based on Eq. (3.30), \( q_{WD} \) can be expressed in terms of \( q_{DM} \) as

\[
q_{WD} = q_{DM} \cdot \frac{N_{\text{Stage}}^2 \left[ \frac{2L_{FD}}{x_{eD}} \left( h_{f_{1D}} + \sqrt{\frac{C_{S2}M_1}{C_{S1}M_1}} h_{f_{2D}} \right) \right]^2}{y_{eD} \left( h_{e_{1D}} + \frac{C_{S2}}{C_{S1}} h_{e_{2D}} \right)}.
\]  

(B.11)

By equating Eq. (B.7) and Eq. (B.11), it gives

\[
q_{DM} \cdot \frac{N_{\text{Stage}}^2 \left[ \frac{2L_{FD}}{x_{eD}} \left( h_{f_{1D}} + \sqrt{\frac{C_{S2}M_1}{C_{S1}M_1}} h_{f_{2D}} \right) \right]^2}{y_{eD} \left( h_{e_{1D}} + \frac{C_{S2}}{C_{S1}} h_{e_{2D}} \right)} = \frac{2L_{FD} h_{f_{1D}} N_{\text{Stage}}}{x_{eD} \pi \sqrt{\pi t_D}} + \frac{2L_{FD} h_{f_{2D}} N_{\text{Stage}}}{x_{eD} \sqrt{\pi t_D}} \sqrt{\frac{C_{S2}M_2}{C_{S1}M_1}}.
\]  

(B.12)

Replacing dimensionless time \( t_D \) in Eq. (B.12) by Eq. (B.10), Eq. (B.12) becomes

\[
q_{DM} \cdot \frac{N_{\text{Stage}}^2 \left[ \frac{2L_{FD}}{x_{eD}} \left( h_{f_{1D}} + \sqrt{\frac{C_{S2}M_1}{C_{S1}M_1}} h_{f_{2D}} \right) \right]^2}{y_{eD} \left( h_{e_{1D}} + \frac{C_{S2}}{C_{S1}} h_{e_{2D}} \right)} = \left( \frac{2L_{FD}}{x_{eD}} \right) \cdot \frac{h_{f_{1D}} N_{\text{Stage}} + h_{f_{2D}} N_{\text{Stage}}}{\pi \sqrt{0.5 \pi t_{MBD}}} \sqrt{\frac{C_{S2}M_2}{C_{S1}M_1}}.
\]  

(B.13)

The next step is to replace \( t_{MBD} \) in Eq. (B.13) by \( t_{DM} \) described in Eq. (3.31), and Eq. (B.13) can be expressed as

\[
\text{(B.13)}.
\]
By organizing Eq. (B.14), it gives

$$
q_{DM} = \frac{1}{\pi \sqrt{0.5 \pi t_{DM}}} .
$$  \( B.15 \)

Eq. (B.15) gives the expression for \( q_{DM} \) vs \( t_{DM} \) during early linear flow as reservoir is made up of 2 vertical layers.

By applying similar strategy, the general expression of \( q_{DM} \) vs \( t_{DM} \) in a multi-layer reservoir that consists of more than 2 layers during early linear flow also results in the same expression as Eq. (B.15) but the modified dimensionless rate \( q_{DM} \) and modified dimensionless time \( t_{DM} \) are defined as

$$
q_{DM} = q_{WD} \cdot \frac{\sqrt{N_{stage}} 2 L_{FD} \sum_{k=1}^{NL} \frac{C_{SK} M_K}{C_{SK} M_K + h_{fKD}}}{\sqrt{N_{stage}} 2 L_{FD} \sum_{k=1}^{NL} \frac{C_{SK} M_K}{C_{SK} M_K + h_{fKD}}} ,
$$  \( B.16 \)

$$
t_{DM} = t_{MBD} \left[ \frac{N_{stage} 2 L_{FD} \sum_{k=1}^{NL} \frac{C_{SK} M_K}{C_{SK} M_K + h_{fKD}}}{\sqrt{N_{stage}} 2 L_{FD} \sum_{k=1}^{NL} \frac{C_{SK} M_K}{C_{SK} M_K + h_{fKD}}} \right]^{2} ,
$$  \( B.17 \)

where “NL” represents the number of vertical layers of reservoir
As is shown in Eq. (B.15), \(q_{DM} vs. t_{DM}\) exhibits a straight line in log-log plot with -1/2 slope, independent of fracture and reservoir geometries as well as mobility and storativity of each layer.

**During pseudo-steady state flow**

Blasingame and Lee (1991) gives the production rate response of a vertical well during pseudo-steady state flow in a homogeneous reservoir producing at constant bottom pressure by using material balance time as

\[
\frac{\Delta p}{q_w(t_{MB})} = mt_{MB} + b, \quad (B.18)
\]

where

\[
m = \frac{1}{c_i\phi x_e y_e h_e},
\]

\[
b = \frac{\mu}{\pi h_e} \ln \left( \frac{4A}{\pi \gamma C \mu \phi x_e y_e} \right).
\]

It is widely known that the term “b” accounts for the effect of well geometry (vertical, horizontal, fractured well, etc). As time elapses longer and longer after entering boundary dominant flow, it has minimum impact on production rate, and thereby “b” could be neglected that Eq. (B.18) should closely approach

\[
\frac{\Delta p}{q_w(t_{MB})} \approx \frac{t_{MB}}{c_i\phi x_e y_e h_e}. \quad (B.19)
\]

Since well production rate is simply provided by expansion of pore volume of the reservoir, in a multilayered reservoir it is physically straightforward to formulate the production as summation of fluid expansion from all layers. Based on Eq. (B.19), it is
Expressing production rate and material balance time of Eq. (B.20) in their respective dimensionless forms, Eq. (B.20) can be written as

\[
\frac{\Delta p}{q_w(t_{MB})} \approx \frac{t_{MB}}{\sum_{k=1}^{NL} (\phi c_i)_k x_e y_e h_{ek}}.
\]  

(B.20)

By organizing Eq. (B.21), it gives

\[
q_w(t_{MB}) \approx \frac{2\pi k_1 x_e \Delta p}{\mu_1} \approx \frac{\Delta p \cdot \sum_{k=1}^{NL} (\phi c_i)_k x_e y_e h_{ek}}{t_{MB} \cdot \phi_1 \mu_1 c_{i1} l_{i1}^2 k_1}.
\]  

(B.21)

As shown in Eq. (B.16), \( q_{WD} \) can be expressed in terms of \( q_{DM} \). By equating Eq. (B.16) and Eq. (B.22), it gives

\[
q_{DM} \approx \frac{y_{eD} \sum_{k=1}^{NL} C_{SK} h_{ekD}}{2\pi t_{MB}}.
\]  

(B.22)

Replacing \( t_{MB} \) in Eq. (B.23) by Eq. (B.17), it gives the \( q_{DM} \) vs \( t_{DM} \) in late time as

\[
q_{DM} \approx \frac{N_{stage}^2 \cdot \left[ \frac{2L_{FD}}{x_e} \sum_{k=1}^{NL} \frac{C_{SK} M_K}{C_{S1} M_1} h_{fKD} \right]^2}{y_e \sum_{k=1}^{NL} C_{SK} h_{eKD}} \quad \approx \quad \frac{y_{eD} \sum_{k=1}^{NL} C_{SK} h_{eKD}}{2\pi \cdot t_{DM}}.
\]  

(B.23)

By organizing Eq. (B.24), it gives

\[
q_{DM} \approx \frac{1}{2\pi t_{DM}}.
\]  

(B.25)
As is shown in Eq. (B.25), $q_{DM} vs. t_{DM}$ exhibits a straight line on log-log plot with -1 slope during pseudo-steady state, independent of fracture and reservoir geometry as well as mobility and storativity of each layer.