POWER FUNCTION FOR THE TEST FOR
DISTINGUISHING GAMMA AND WEIBULL
DISTRIBUTION FAMILIES

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Uchenna Anthony Ndulaka, candidate for the degree of Master of Science in Statistics, has presented a thesis titled, *Power Function for the Test For Distinguishing Gamma and Weibull Distribution Families*, in an oral examination held on December 23, 2019. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

In other to satisfy his customers, the manufacturer will like to ascertain the lifetime distribution of his products for reliability. Hence, the importance of determining the lifetime distribution of the product for the purpose of locating and removing the defect in time and also examine the product reliability.

The purpose of my research is to construct statistical tests to test two main probability models in reliability theory which consist of model of aging and deterioration (Gamma distribution) and model of weak link (Weibull distribution). Some asymptotically locally most powerful test will be constructed to distinguish between the two models.
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Chapter 1

Introduction

The reliability and warranty period are the main indicators of the quality of a manufactured product delivered to a customer. In order to calculate the warranty period with the given constraint on the reliability, it is required first of all to establish the lifetime distribution of the manufactured products, that is, to establish the probability model of the random variable that describes the observed lifetime of the product. The probability model represents some family of distributions that depends on parameters is such way that the reliability and warranty period problem is reduced to hypothesis testing on the correctness of the probability model and estimation of its parameters by the result of lifetime observations for a certain amount of manufactured items.

The goal of this chapter is to construct the locally most powerful test for the choice between two probability models: model of aging and deterioration (Gamma...
distribution) and the model of a weak link (Weibull distribution). The relevance of the choice for this particular problem to be solved is defined by its practical value. Models of aging and weak link are most commonly used in practice for determination of the warranty period. The Scientific relevance is in the fact that so far there has not been found the optimal test for distinguishing these models.

The first model of aging and deterioration is defined by the Gamma distribution. Aging is a natural process of a change in properties and quality of a material. Together with deterioration it leads to fact that the material looses its initial features and starts to wear out. Therefore, if the statistical properties confirm the reliability model of aging and deterioration type (that is, the hypothesis that lifetime data have Gamma distribution is accepted), then necessary measures should be taken to it is required to eliminate the wear out, for example, to lubricate the rubbing parts.

The second model of a weak link is defined by the Weibull distribution. Here we consider a system of links connected in a series and a failure of one (weak) link implies the failure of the whole system. This model gives an explanation of failures due to small undetected defects. In order to avoid failures caused by the wear out of weak links our old systems, it is necessary to detect the warranty period and the time when the parts of the system should be changed.

In this chapter we improve the statistical test for distinguishing the Gamma $G(\theta_1, \alpha)$ and Weibull $W(\theta_2, \beta)$ distributions, presented in the previous chapters and
originated from the article [20]. Our main goal is to raise its power up to a locally most powerful test construction.

There is a extended literature for the models of aging and weak link testing. The initial point of this investigation are the articles [23] and [?] by Dr. Igor Volodin. In these article, the test for distinguishing these models and some other was suggested. It is worthwhile to mention the article of Volodin et. al [24], where numerical investigations of the power for test for distinguishing two close type of Weibull distributions is presented. Article by Adgamov and Volodin [1] is devoted to distinguishing more general probability models defined by Weibull and Generalized Gamma distributions defined in Stacy [20].

The construction of our tests for distinguishing between families of distributions is based on the theory of local asymptotic normality [21] (Chapter 7) and connected to it methods of asymptotically most power tests construction [6].

This thesis is organized as follows. In Chapters 2 and 3 we describe the tests for Gamma vs Weibull and Weibull vs Gamma. The thesis is concluded by the analysis of the presented investigations, the computer codes are also presented in Appendix.

1.1 Gamma and Weibull distribution under review

The warranty and reliability period of a product has a particular data, whose distribution is mostly Generalized gamma, Weibull or Gamma distribution. In this
study, we will be making use of Gamma and Weibull distribution. Both distributions are special cases of Generalized Gamma distribution as shown in Chapter 2.

In 1965, Stacy and Mihran introduced the Generalized Gamma distribution in order to combine Gamma and Weibull distribution. The Generalized Gamma distribution has three parameters \((\beta > 0, \lambda > 0, \tau > 0)\) and it’s also a continuous probability distribution. The Generalized Gamma distribution is useful in many ways:

i. Construction of models in reliability theory,

ii. Flood frequency analysis. Commonly used by engineers [19]

iii. Examining regression model under health cost [16]

iv. Income distribution modality in Economics [10]

The Generalized Gamma distribution has the probability density function defined as:

\[
f(x; \beta, \tau, \lambda) = \frac{\tau}{\lambda \Gamma(\beta)} \left(\frac{x}{\lambda}\right)^{\beta \tau - 1} \left(\frac{x}{\lambda}\right)^{\tau}, \quad x > 0, \beta > 0, \lambda > 0, \tau > 0.
\]

However, Volodin [22] defined a generalized gamma distribution as such that a random variable \(x\) must be associated with the density function

\[
f(x; \lambda, \beta) = \frac{1 + \beta}{\Gamma\left(\frac{1 + \lambda}{1 + \beta}\right)} x^{\lambda} \exp\left(-x^{1+\beta}\right)
\]
If $\beta = 0$, the distribution is said to be Gamma with parameter $\lambda$, $X \sim G(\lambda)$. Ageing and deterioration pattern of a system is said to have a Gamma distribution, if the reliability of the system is time dependent.

### 1.2 Fundamental concepts

*Locally Asymptotic Normality:* Sequence of statistical models has one of the properties called Local Asymptotic Normality, which allows the sequence to be asymptotically approximated by a normal location model, after parameter rescaling. The iid sampling from the regular parametric model is an example of when local asymptotic normality occurs.

*Mutually Independent Model:* When the breakdown of any of the component elements of the system is considered as the breakdown of the whole components, then the lifetime distribution is considered to be Weibull distribution. $X \sim W(\tau, \lambda)$

*Memory Less Model:* The replacement of a system by a new one immediately after it breaks down as a lifetime distribution called Gaussian Distribution. $X \sim GG(\beta)$

*Generalized Exponential Distribution:* The Gamma distribution, Weibull Distribution and Exponential Distribution as special cases of the generalized Gamma distribution which are simplified to lack of an aftereffect model, (Exponential), $\beta = \lambda = \tau = 0$
1.3 Statistical hypothesis testing. Distinguishing the distribution

In statistical analysis, model misspecification may be very severe and consequential. We will be making use of statistical hypothesis testing to determine the distribution of the lifetime data of a given process.

An \(n\)-size random samples of independent identically distributed random variables, \(X_1, X_2, \ldots, X_n\) with a common pmf/pdf \(f(x; \theta), \theta \in \Theta\) are selected. With the competing hypothesis about possible values of \(\Theta\), we are now faced with the responsibility of how to decide between them. The Neyman and Person (1920) procedure and the extending Fisher’s earlier ideas is primarily to identify a Null hypothesis

\[ H_0 : \theta \in \Theta_0 \]

against the Alternative hypothesis

\[ H_1 : \theta \in \Theta_1 \]

with \(\Theta_0\) and \(\Theta_1\) as disjoint subsets of the parameter space \(\Theta\)

A test statistic is needed to decide between \(H_0\) and \(H_1\).

Let a simple hypothesis be of the form, \(\theta = \theta_0\), a constant and composite otherwise.

Choosing between the null and alternative hypothesis requires a test statistic \(T = h(\overline{x})\) values in a space partitioned into disjoint subsets \(A\) and \(R\), known as acceptance and rejection regions which is similar to \(\Theta_0\) and \(\Theta_1\), respectively. We
reject $H_0$ in favor of $H_1$, if an observed value of $T, t = h(\bar{x}) \in R$ and accept $H_0$ if $t \in A$. $T$ is governed by the unknown value of $\theta$. An important role is played by the law of $T$.

Specifically speaking, $A$ and $R$ are chosen in such a way that if $H_0$ is true, the event $t \in R$ occurs with a small probability.

In general, we choose a very small number of $\alpha$ and then $R$ such that $P(T \in R|\theta) \leq \alpha$ for all $\theta \in \Theta_0$.

If $t \in R$, we reject $H_0$ at 100$\alpha$% level of significance. $\alpha$ is usually called the size of the text. So if $H_0$ is false, we expect that $P(T \in R|\theta) > \alpha$ if $\theta \in \Theta_1$; that is the probability of rejection is greater than the chosen level of significance if $H_0$ is false.

A test with this type of characteristics is said to be unbiased. For an unbiased test the following conditions applies for making decisions whether to accept or reject the hypothesis. Hence, if $t \in R$, then

i. Either $H_0$ is true and an event of small $\{\leq \alpha\}$ probability has taken place; or

ii. $H_0$ is false, i.e an event of which its probability is greater than $\alpha$ has occurred.

This has a better characteristic of the maximum likelihood concept. Two types of error occur under this procedure:

*Type 1 error:* Reject $H_0$ when it is true, and

*Type 11 error:* Accept $H_0$ when it is false.
The type 1 error is considered to be a serious error and that explain why we design the test to control its probability of occurrence: 

\[ P(\text{Type 1 error}) = P(T \in R|H_0) \leq \alpha. \]

For the Type 11 error, the determination of its probability is only possible if \( H_1 \) is simple.

In general, the power function is given by \( \beta_r(\theta) = P(T \in R|\theta) \) for all \( \theta \in \Theta \). Hence, the test is planned in such a way that the power function \( \beta_r(\theta) \leq \alpha \) if \( \theta \in \Theta_0 \). Generally, the power function is close to \( \alpha \) if \( \theta \in \Theta_1 \) but nearer to its boundary and increasing as \( \theta \) tends towards the boundary. The sensitivity of a test can be detected by how fast \( \beta_r(\theta) \) increases above the significance level, \( \alpha \) as \( \theta \in \Theta_1 \) moves away from the boundary.

The Neyman-Pearson testing procedure is the most commonly used and the meaning of the confirmation that \( H_0 \) is rejected at 100\( \alpha \)% level of significance is that if the experiment is independently and randomly repeated many times with the same population, then a Type I error occurs in a proportion \( \leq \alpha \) such replications.

However \( H_0 \) may be in form of model assumptions, such that errors are normally distributed. For instance in quality control, \( H_0 \) may be: “the process is in control”. i.e the probability of having a faulty product is minimal. Hence \( t \in A \) gives the opportunity of accepting \( H_0 \) as a practical assumption, a well-deserved outcome.

An important question now remain how to choose the test statistics? It is obvious
we want an unbiased test, which has the property of $\beta_\tau(\theta)$ that is as large as possible for all $\theta \in \Theta_1$, that is to say maximum power under $H_1$.

Uniformly most powerful (UMP) test statistic is such that for any other test statistic $\tau'$, we obtain $\beta_\tau(\theta) \geq \beta_{\tau'}(\theta)$ for all $\theta \in \Theta_1$.

In this study, the test statistic $T = g(T_1, T_2, T_3)$ is a function of three main statistics. Delta method is used to obtain the asymptotic distribution of $T$. Function $g$ is simplified by the use of Taylor series expansion by the powers of $T_i - \mu_i$, $i = 1, 2, 3$, and $\mu_i = ET_i$ and the mathematical expectation is obtained with the assumption that one the hypotheses is true ($H_0$ or $H_1$). In general, we are taking the expectation of the Generalized Gamma distribution.

Now both Gamma and Weibull distribution are reduced to the ordinary Exponential distribution $E^0$ with $\beta = \lambda = \tau = 0$. Hence, the Exponential distribution $E^0$ can be interpreted as a boundary that separates the null and the alternative hypotheses.

In a similar circumstance, Volodin [23] came up with another method of testing statistical hypotheses based on the type of distribution by small samples. But the smallness of each sample makes it hard to conclude anything about the overall population from which the samples were taken. Nevertheless, it is reasonable to have the hypothesis that all these samples obey same the distribution law which depends on certain parameters, and each may be different for different samples. Hence, the issue of testing the statistical hypothesis that all samples belong a particular class of an
overall population, all of which obey the same distribution law and separated by each other only by different distribution which no hypothesis is made. He also proposed a different method for constructing the critical set on the basis of estimating the nuisance parameter. This invariably reduces the calculation necessary for hypothesis testing for the type of distribution for any arbitrary distributional hypotheses.
Chapter 2

The test construction and its power

evaluation

Gamma and Weibull distributions are particular cases of the Generalized Gamma distribution, introduced in Stacy [20]. Its density function is

$$f\left(\frac{x}{\theta}; \lambda, \beta\right) = \frac{1 + \beta}{\Gamma\left(\frac{1+\lambda}{1+\beta}\right)\theta} \left(\frac{x}{\theta}\right)^{\lambda} \exp\left(-\left(\frac{x}{\theta}\right)^{1+\beta}\right).$$

If we take $\beta = 0$, then we obtain usual Gamma distribution. If we take $\lambda = \beta = \tau$, then we obtain Weibull distribution. Therefore, in terms of the parameters of the Generalized Gamma distribution, the main problem considered of this chapter is in distinguishably hypotheses $\beta = 0$ and $\lambda = \beta$ with the nuisance scale parameter $\theta$. 

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2.1 Derivation of test statistic, mean variance and covariance of the distributions

Let $X$ have a Generalized Gamma distribution, with the density function:

$$f(x|\lambda, \beta) = \frac{1 + \beta}{\Gamma\left(\frac{1+\beta}{1+\lambda}\right)}x^\lambda \exp(-x^{1+\beta}), x, \lambda, \beta > 0,$$

then the random variable $X$ has a Gamma distribution with $\beta = 0$ and parameter $\lambda$, i.e. $X \sim G(\lambda)$.

Similarly, if $X$ comes from the family of Generalized Gamma distribution with $\lambda = 0$, then the random variable $X$ has a generalized exponential distribution with parameter $\beta$, hence, $X \sim E(\beta)$.

In a similar situation, if $X$ has the Generalized Gamma distribution, with $\beta = \lambda = \tau$, we say that the random variable $X$ has a Weibull distribution with parameter $\tau$, i.e., $X \sim W(\tau)$.

Note that $G(\lambda)$, is the model for aging and deterioration while $W(\tau)$, is the model for weak link and $E(\beta)$ is that of manufacture defect.

All are exceptional cases of the Generalized Gamma distribution, which will be decomposed to the general model of lack of an aftereffect called Exponential distribution, $E^0$, with $\beta = \lambda = \tau = 0$.

Volodin [22], stated that there is locally most powerful test distinguishing these Gamma and Weibull distributions in the Generalized Gamma Distribution, which
depends on the test statistic:

\[ J = c_1 J_1 + c_2 J_2 \]

and,

\[ J_1 = \frac{1}{n} \sum \ln X_k - \ln \left( \frac{1}{n} \sum X_k \right), \]

\[ J_2 = \ln \left( \frac{1}{n} \sum X_k \right) - \frac{1}{n} \sum \frac{X_k \ln X_k}{\frac{1}{n} \sum X_k} \]

where \( X_1, \ldots, X_n \) is the random sample, and \( n \) is the sample size.

Our choice of the constants \( c_1 \) and \( c_2 \) determines uniformly most power and locally most powerful test for hypothesis that the model of an aftereffect is correct with alternatives \( G, W, \) or \( E^3 \).

Our interest will be to construct asymptotically locally most powerful tests to distinguish these two models- \( G \) and \( W \).

The Rejection Region will be \( J > c \).

Critical value \( c = c(\alpha) \).

The statistics \( J_1 \) and \( J_2 \) are dependent on three pivot statistics:

\[ T_1 = \frac{1}{n} \sum_{1}^{n} X_k, \]

\[ T_2 = \frac{1}{n} \sum_{1}^{n} \ln(X_k), \]

\[ T_3 = \frac{1}{n} \sum_{1}^{n} X_k \ln(X_k). \]

We can rewrite \( J_1 \) and \( J_2 \) in terms of \( T_1, T_2, \) and \( T_3 \).
Hence, $I_1 = T_2 - \ln T_1,$

$$ I_2 = \ln T_1 - \frac{T_3}{T_1}, $$

and

$$ I = c_1 (T_2 - \ln T_1) + c_2 = \ln T_1 - \frac{T_3}{T_1}. $$

The following conditions are true for the derivation of $T_1, T_2,$ and $T_3.$

**Lemma 1.** Let $X$ be a random variable with Generalized Gamma distribution, the mean values of the statistics are:

$$ \mu_{m,n} = E(X^m \ln^n X), m, n = 0, 1, 2, \ldots $$

$$ \mu_1 = ET_1 = EX = \mu_{10}, $$

$$ \mu_2 = ET_2 = E \ln(X) = \mu_{01}, $$

$$ \mu_3 = ET_3 = EX \ln(X) = \mu_{11}. $$

The Variances:

$$ n\delta^2_1 = n \text{Var}T_1 = EX^2 - \mu_{10}^2 = \mu_{20} - \mu_{10}^2, $$

$$ n\delta^2_2 = n \text{Var}T_2 = E \ln^2 X - \mu_{01}^2 = \mu_{02} - \mu_{01}^2, $$

$$ n\delta^2_3 = n \text{Var}T_3 = EX^2 \ln^2 X - \mu_{11}^2 = \mu_{22} - \mu_{11}^2. $$

Covariances:

$$ n\lambda_{12} = n \text{cov}(T_1, T_2) = EX \ln X - \mu_{10}\mu_{01} = \mu_{11} - \mu_{10}\mu_{01}, $$
\[ n \lambda_{13} = ncov(T_1, T_3) = EX^2 \ln X - \mu_{10} \mu_{11}, \]
\[ = \mu_{21} - \mu_{10} \mu_{11}, \]

\[ n \lambda_{23} = ncov(T_2, T_3) = EX \ln^2 X - \mu_{01} \mu_{11} \]
\[ = \mu_{12} - \mu_{01} \mu_{11} \]

**Proof.**

The variance is giving by

\[ EX^2 - E^2 X = \mu_{20} - \mu_{10}^2. \]

\[ n \delta_2^2 = n \text{Var}T_2 = n \left( \frac{1}{n} \sum_{k=1}^{n} \text{Var}(\ln X_k) \right) \]
\[ = \frac{1}{n} \sum_{k=1}^{n} E(\ln X_k)^2 - E^2(\ln X_k) \]
\[ = E(\ln X)^2 - E^2(\ln X) = \mu_{02} - \mu_{01}^2. \]

\[ n \delta_3^2 = n \text{Var}T_3 = n \text{Var} \left( \frac{1}{n} \sum_{k=1}^{n} (X_k \ln X_k) \right) \]
\[ = \frac{1}{n} \sum_{k=1}^{n} \text{Var}(X_k \ln X_k) \]
\[ = \frac{1}{n} \sum_{k=1}^{n} E(X_k \ln X_k)^2 - E^2(X_k \ln X_k) = \mu_{22} - \mu_{11}^2. \]

The Covariances of the three statistic: The Proof: Given,

\[ n \lambda_{12} = ncov(T_1, T_2) = EX \ln X - \mu_{10} \mu_{01} = \mu_{11} - \mu_{10} \mu_{01}, \]
then,

\[
ncov\left(\frac{1}{n} \sum_{k=1}^{n} X_k, \frac{1}{n} \sum_{k=1}^{n} \ln X_k\right) = \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \text{cov}(X_k, \ln X_l)
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \text{cov}(X_k, \ln X_l) + \sum_{k \neq l} \text{cov}(X_k, \ln X_l)
\]

and \(X_k \sim \text{iid.}\)

\[
= \text{cov}(X_k, \ln X_k) = E(X_k, \ln X_k) - E(X_k)E(\ln X_k)
\]

\[
= \mu_{11} - \mu_{10}\mu_{01},
\]

and

\[
n\lambda_{13} = n\text{cov}(T_1, T_3) = n\text{cov}\left(\frac{1}{n} \sum_{k=0}^{n} X_k, \frac{1}{n} \sum_{l} X_l \ln X_l\right)
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \text{cov}(X_k, X_l \ln X_l)
\]

\[
= \frac{1}{n} \sum_{k} \text{cov}(X_k, X_k \ln X_k) + \sum_{k \neq l} \text{cov}(X_k, X_l \ln X_l),
\]

\(X_k \sim \text{iid};\)

\[
= \text{cov}(X_k, X_k \ln X_k) = E(X_k^2 \ln X_k) - E(X_k)(E(X_k \ln X_k))
\]

\[
= \mu_{21} - \mu_{10}\mu_{11}.
\]

Similarly,

\[
n\lambda_{23} = n\text{cov}(T_2, T_3) = n\text{cov}\left(\frac{1}{n} \sum_{k=0}^{n} \ln X_k, \frac{1}{n} \sum_{l} X_l \ln X_l\right)
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \text{cov}(\ln X_k, X_l \ln X_l)
\]

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\[
\sum_{k=1}^{n} cov(X_k, X_k \ln X_k) + \sum_{k \neq l} cov(X_k, X_l \ln X_l),
\]
since \(X_k \sim \text{iid};\)

\[
cov(X_k, X_k \ln X_k)
= E(X_k \ln^2 X_k) - E(\ln X_k)E(X_k \ln X_k)
= \mu_{12} - \mu_{01}\mu_{11}.
\]

From Lemma 1, the mean, variances, and the covariances of the pivot statistics \(T_k, k = 1, 2, 3\) for our Generalized Gamma distribution, can be calculated if we know the mean values of the random variables \(X^i \ln^j X, i, j = 0, 1, 2\). Hence, we start by deriving the moments of the random variables.

We will need the following Lemma in these calculations:

**Lemma 2.** Let \(X\) be a random variable with the Generalized Gamma distribution, then

\[
\mu_{ij} = E(X^i \ln^j X) = \frac{\Gamma(j) \left(\frac{\lambda+i+1}{1+\beta}\right)}{(1 + \beta)^j \Gamma\left(\frac{1 + \lambda}{1 + \beta}\right)}.
\]

Note that \(\Gamma^j(b) = \frac{d^j}{dx^j} \Gamma(x)|_{x=b}\).

**Proof:**

Given \(\mu_{ij} = \frac{1+\beta}{\Gamma} \left(\frac{1+\lambda}{1+\beta}\right) \int_0^\infty x^{\lambda+i} \ln^j x e^{-x^{1+\mu}} dx\).

Let \(x^{1+\beta} = t\). Hence, \(x = t^{\frac{1}{1+\beta}}, dx = \frac{1}{1+\beta} t^{-\frac{\beta}{1+\beta}} dt\).
So,

\[ \mu_{ij} = \frac{1}{\Gamma} \left( \frac{1 + \lambda}{1 + \beta} \right) \int_0^\infty t^{\frac{\lambda+i}{1+\beta} - \frac{\beta}{1+\beta}} \ln^j t^\frac{1}{1+\beta} e^{-t} dt \]

\[ = \frac{(1 + \beta)^{-i}}{\Gamma} \left( \frac{1 + \lambda}{1 + \beta} \right) \int_0^\infty t^{\frac{\lambda+i-\beta}{1+\beta}} \ln^j t e^{-t} dt, \]

because \( \int_0^\infty t^b \ln^j t e^{-t} dt = \frac{d^j}{db^j} \Gamma(b + 1) = \Gamma(j)(b + 1). \)

So we will obtain that

\[ \mu_{ij} = \frac{\Gamma(j) \left( \frac{\lambda+i+1}{1+\beta} \right)}{(1 + \beta)^j} \Gamma \left( \frac{1 + \lambda}{1 + \beta} \right). \]

The following corollary will be concluded based on the above Lemma.

**Corollary:** \( X \sim G(\lambda), \ i.e \ Generalized\ Gamma\ distribution\ with\ \beta = 0, \) then

\[ \mu_{ij} = \Gamma(j) \frac{\lambda + i + 1}{\Gamma(1 + \lambda)}. \]

In a similar condition, if \( X \sim E(\beta), \ (\ with\ \lambda = 0), \) then

\[ \mu_{ij} = \frac{\Gamma(j) \left( \frac{i+1}{1+\beta} \right)}{(1 + \beta)^j \Gamma \left( \frac{1}{1+\beta} \right)}. \]

The values of \( \mu_{ij}, i, j = 0, 1, 2 \) will be used for an application in successive calculations in Lemma 3.

Note:

\( \ln(\Gamma(x)) \) is known as Digamma Euler function, denoted as

\[ \varphi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \]

\[ \varphi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left( \frac{\Gamma'(x)}{\Gamma(x)} \right)^2. \]
known as Trigamma Euler function.

Therefore, $\Gamma'(x) = \Gamma(x)\varphi(x)$ and $\Gamma''(x) = \Gamma(x)\varphi'(x) + \varphi^2(x)$.

**Lemma 3:** Under this Lemma, these formulas are true:

\[
\mu_{01} = \frac{\Gamma'\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta + 1)\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\varphi\left(\frac{\lambda+1}{\beta+1}\right)}{\beta + 1},
\]

\[
\mu_{02} = \frac{\Gamma^2\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta + 1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta + 1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}\varphi'\left(\frac{\lambda + 1}{\beta + 1}\right) + \varphi^2\left(\frac{\lambda + 1}{\beta + 1}\right)
\]

\[
= \frac{1}{(\beta + 1)^2} \varphi'\left(\frac{\lambda + 1}{\beta + 1}\right) + \varphi^2\left(\frac{\lambda + 1}{\beta + 1}\right)
\]

\[
\mu_{10} = \frac{\Gamma\left(\frac{\lambda+2}{\beta+1}\right)}{\Gamma\left(\frac{\lambda+1}{\beta+1}\right)};
\]

\[
\mu_{11} = \frac{\Gamma'\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta + 1)\Gamma\left(\frac{\lambda+2}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+2}{\beta+1}\right)\varphi\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta + 1)\Gamma\left(\frac{\lambda+1}{\beta+1}\right)},
\]

\[
\mu_{12} = \frac{\Gamma^2\left(\frac{\lambda+2}{\beta+1}\right)}{(\beta + 1)^2\Gamma\left(\frac{\lambda+2}{\beta+1}\right)}
\]

\[
= \frac{\Gamma^2\left(\frac{\lambda+1}{\beta+1}\right)}{(\beta + 1)^2\Gamma\left(\frac{\lambda+1}{\beta+1}\right)}\left\{\varphi'\left(\frac{\lambda + 2}{\beta + 1}\right) + \varphi^2\left(\frac{\lambda + 2}{\beta + 1}\right)\right\}.
\]

\[
\mu_{20} = \frac{\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}{\Gamma\left(\frac{\lambda+2}{\beta+1}\right)};
\]

\[
\mu_{21} = \frac{\Gamma'\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta + 1)\Gamma\left(\frac{\lambda+3}{\beta+1}\right)} = \frac{\Gamma\left(\frac{\lambda+3}{\beta+1}\right)\varphi\left(\frac{\lambda + 3}{\beta + 1}\right)}{(\beta + 1)\Gamma\left(\frac{\lambda+2}{\beta+1}\right)},
\]

\[
\mu_{22} = \frac{\Gamma^2\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta + 1)^2\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}
\]

\[
= \frac{\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}{(\beta + 1)^2\Gamma\left(\frac{\lambda+3}{\beta+1}\right)}\left\{\varphi'\left(\frac{\lambda + 3}{\beta + 1}\right) + \varphi^2\left(\frac{\lambda + 3}{\beta + 1}\right)\right\}.
\]

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Now let us consider the Generalized Gamma distribution with $\lambda = 0, X \sim E(\beta)$.

The following are the values of $\mu_{ij}; i, j = 0, 1, 2$, under this particular case:

$$
\mu_{01} = \frac{\varphi\left(\frac{1}{\beta+1}\right)}{\beta+1},
$$

$$
\mu_{02} = \frac{\varphi'(\frac{1}{\beta+1}) + \varphi'(1)}{(\beta+1)^2},
$$

$$
\mu_{10} = \frac{\Gamma\left(\frac{2}{\beta+1}\right)}{\Gamma\left(\frac{1}{\beta+1}\right)},
$$

$$
\mu_{11} = \frac{\Gamma\left(\frac{2}{\beta+1}\right)\varphi\left(\frac{2}{\beta+1}\right)}{(1+\beta)\Gamma\left(\frac{1}{\beta+1}\right)},
$$

$$
\mu_{12} = \frac{\Gamma\left(\frac{2}{\beta+1}\right)}{(1+\beta)^2\Gamma\left(\frac{1}{\beta+1}\right)} \left\{ \varphi'\left(\frac{2}{\beta+1}\right) + \varphi^2\left(\frac{2}{\beta+1}\right) \right\},
$$

$$
\mu_{20} = \frac{\Gamma\left(\frac{3}{1+\beta}\right)}{\Gamma\left(\frac{1}{1+\beta}\right)},
$$

$$
\mu_{21} = \frac{\Gamma\left(\frac{3}{1+\beta}\right)}{(1+\beta)\Gamma\left(\frac{1}{1+\beta}\right)} \varphi\left(\frac{3}{1+\beta}\right),
$$

$$
\mu_{22} = \frac{\Gamma\left(\frac{3}{1+\beta}\right)}{(1+\beta)^2\Gamma\left(\frac{1}{1+\beta}\right)} \left\{ \varphi'\left(\frac{3}{1+\beta}\right) + \varphi^2\left(\frac{3}{1+\beta}\right) \right\},
$$

hence, we can calculate the mean values, variances and covariances for statistics $T_1, T_2, T_3$ for the case $X \sim G(\lambda)$, which is the generalized gamma distribution with $\beta = 0$. So in this case all the values of $\mu_{ij}$ and $i, j = 0, 1, 2$ are written as follows:

$$
\mu_{01} = \varphi(\lambda+1),
$$

$$
\mu_{02} = \varphi'(\lambda+1) + \varphi^2(\lambda+1),
$$
\[ \mu_{10} = \lambda + 1, \]
\[ \mu_{11} = (\lambda + 1) \varphi (\lambda + 2), \]
\[ \mu_{12} = (\lambda + 1) \varphi' (\lambda + 2) + \varphi^2 (\lambda + 2), \]
\[ \mu_{20} = \frac{\Gamma(\lambda + 3)}{\Gamma(\lambda + 1)} = (\lambda + 1) (\lambda + 2), \]
\[ \mu_{21} = (\lambda + 2) (\lambda + 1) \varphi(\lambda + 3), \]
\[ \mu_{22} = (\lambda + 2) (\lambda + 1) \varphi'(\lambda + 3 + \varphi^2 (\lambda + 3)). \]

So the mean values, variances and the covariance of \( T_1, T_2, T_3 \) are:

\[ \mu_1 = \mu_{10} = \lambda + 1, \]
\[ \mu_2 = \mu_{01} = \varphi (\lambda + 1), \]
\[ \mu_3 = \mu_{11} = (\lambda + 1) \varphi(\lambda + 2), \]
\[ \delta_1^2 = \frac{1}{n} (\mu_{20} - \mu_{10}^2) = \frac{1}{n} (\lambda + 1), \]
\[ \delta_2^2 = \frac{1}{n} (\mu_{02} - \mu_{01}^2) = \frac{1}{n} \varphi'(\lambda + 1) \]
\[ \delta_3^2 = \frac{1}{n} (\mu_{22} - \mu_{11}^2) \]
\[ = \frac{1}{n} (\lambda + 1) (\lambda + 2) (\varphi' (\lambda + 3) + \varphi^2 (\lambda + 3)) - \]
\[ (\lambda + 1) \varphi^2(\lambda + 1). \]
\[ \lambda_{12} = \frac{1}{n} (\mu_{11} - \mu_{10} \mu_{01}) = \frac{1}{n}, \]
\[
\lambda_{13} = \frac{1}{n}(\mu_{21} - \mu_{10}\mu_{11})
\]
\[
= \frac{1}{n} (\lambda + 1) (\varphi (\lambda + 2) + 1),
\]
\[
\lambda_{23} = \frac{1}{n} (\mu_{12} - \mu_{01}\mu_{11}) = \frac{1}{n} (\lambda + 1)(\varphi'(\lambda + 2) + \varphi^2(\lambda + 2) - \varphi(\lambda + 1)\varphi(\lambda + 2)).
\]

In all, we obtain the following results for the case of \( X \sim G(\lambda) \)

**Mean Values**

i. \( \mu_1 = \lambda + 1 \),

ii. \( \mu_2 = \varphi(\lambda + 1) \),

iii. \( \mu_3 = (\lambda + 1)\varphi(\lambda + 2) \).

**Variances**

i. \( \delta_1^2 = \frac{1}{n}(\lambda + 1) \),

ii. \( \delta_2^2 = \frac{1}{n}\varphi'(\lambda + 1) \),

iii. \( \delta_3^2 = \frac{1}{n} (\lambda + 1) (\lambda + 2) (\varphi'(\lambda + 3) + \varphi^2(\lambda + 3)) - (\lambda + 1)\varphi^2(\lambda + 1) \)

**Covariances:**

i. \( \lambda_{12} = \frac{1}{n} \),

ii. \( \lambda_{13} = \frac{1}{n} (\lambda + 1) (\varphi (\lambda + 2) + 1) \),

iii. \( \lambda_{23} = \lambda_{n}^1(\lambda + 1)(\varphi'(\lambda + 2) + \varphi^2(\lambda + 2) - \varphi(\lambda + 1)\varphi(\lambda + 2)). \)
Similarly, the mean values, variances and covariances of the statistics $T_1, T_2, T_3$ will be derived for the case of $X \sim W(\tau)$, (that is generalized gamma distribution with $\lambda = \beta = \tau = 0$) and the values of $\mu_{ij}, i, j = 0, 1, 2$ in this case are as follows:

$$
\mu_{01} = \frac{\varphi(1)}{\tau + 1},
$$

$$
\mu_{02} = \frac{1}{(\tau + 1)^2} \left\{ \varphi'(1) + \varphi^2(1) \right\},
$$

$$
\mu_{10} = \frac{\Gamma(1/(\tau + 1))}{(\tau + 1)},
$$

$$
\mu_{11} = \frac{\Gamma(1/2)\varphi(1/2)}{(\tau + 1)^2},
$$

$$
\mu_{12} = \frac{\Gamma(\tau/\tau + 1)}{(\tau + 1)^2} \left\{ \varphi'(\tau/\tau + 1) + \varphi^2(\tau/\tau + 1) \right\},
$$

$$
\mu_{20} = \frac{\Gamma(\tau + 3)}{\tau + 1},
$$

$$
\mu_{21} = \frac{\Gamma(\tau + 3/\tau + 1)}{\tau + 1} \varphi(\tau + 3/\tau + 1),
$$

$$
\mu_{22} = \frac{\Gamma(\tau + 3/\tau + 1)}{(\tau + 1)^2} \left\{ \varphi'(\tau + 3/\tau + 1) + \varphi^2(\tau + 3/\tau + 1) \right\},
$$

Hence, we can now calculate all the mean values, Variances and Covariances of the statistics $T_1, T_2, T_3$.

The mean values:

$$
\mu_1 = \mu_{10} = \frac{\Gamma(1/2)}{\tau + 1},
$$

$$
\mu_2 = \mu_{01} = \frac{\varphi(1)}{\tau + 1},
$$

$$
\mu_3 = \mu_{11} = \frac{\Gamma(1/2)\varphi(1/2)}{(\tau + 1)^2},
$$
Variances:

\[ \delta_1^2 = \frac{1}{n} (\mu_{20} - \mu_{10})^2 = \frac{1}{n} \{ \Gamma\left(\frac{\tau + 3}{\tau + 1}\right) - \left(\frac{1}{\tau + 1}\right)^2 \} \]

\[ = \frac{1}{n} \left\{ \frac{2\Gamma\left(\frac{2}{\tau + 1}\right)}{\tau + 1} - \left(\frac{\Gamma\left(\frac{1}{\tau + 1}\right)}{\tau + 1}\right)^2 \right\}, \]

\[ \delta_2^2 = \frac{1}{n} (\mu_{02} - \mu_{01})^2 = \frac{1}{n} \left\{ \frac{1}{(\tau + 1)^2} \left( \varphi'(1) + \varphi^2(1) \right) - \frac{\varphi^2}{(\tau + 1)^2} \right\} \]

\[ = \frac{1}{n} \frac{\varphi'(1)}{(\tau + 1)^2}, \]

Covariance:

\[ \lambda_{12} = \frac{1}{n} (\mu_{11} - \mu_{10}\mu_{01}) = \frac{1}{n} \frac{\Gamma\left(\frac{1}{\tau + 1}\right)}{\tau + 1} \frac{\varphi(1)}{(\tau + 1)} - \frac{\varphi(1)}{\tau + 1} \frac{\tau}{\tau + 1}, \]

\[ = \frac{1}{n} \left\{ \frac{\Gamma\left(\frac{1}{\tau + 1}\right)}{(\tau + 1)^2} \left( \varphi\left(\frac{\tau + 2}{\tau + 1}\right) - \varphi(1) \right) \right\}, \]

\[ \lambda_{13} = \frac{1}{n} (\mu_{21} - \mu_{10}\mu_{11}) = \frac{1}{n} \frac{\Gamma\left(\frac{\tau + 3}{\tau + 1}\right)}{\tau + 1} \varphi\left(\frac{\tau + 3}{\tau + 1}\right) - \frac{\Gamma\left(\frac{\tau + 3}{\tau + 1}\right)}{\tau + 1} \frac{\varphi\left(\frac{\tau + 2}{\tau + 1}\right)}{(\tau + 1)^2} \}

\[ = \frac{1}{n} \left\{ \frac{2\Gamma\left(\frac{2}{\tau + 1}\right) \varphi\left(\frac{\tau + 3}{\tau + 1}\right)}{(\tau + 1)^2} - \left(\frac{\Gamma\left(\frac{1}{\tau + 1}\right)\varphi\left(\frac{\tau + 2}{\tau + 1}\right)}{(\tau + 1)^3} \right)' \right\} \]

\[ \lambda_{23} = \frac{1}{n} (\mu_{12} - \mu_{01}\mu_{11}) \]

\[ = \frac{1}{n} \frac{\Gamma\left(\frac{\tau + 2}{\tau + 1}\right)}{(\tau + 1)^2} \left( \varphi'\left(\frac{\tau + 2}{\tau + 1}\right) + \varphi^2\left(\frac{\tau + 2}{\tau + 1}\right) \right) - \frac{\varphi(1)}{\tau + 1} \frac{\Gamma\left(\frac{1}{\tau + 1}\right)\varphi\left(\frac{\tau + 2}{\tau + 1}\right)}{(\tau + 1)^2}, \]

\[ = \frac{1}{n} \left\{ \frac{\Gamma\left(\frac{1}{\tau + 1}\right)}{(\tau + 1)^3} \left[ \varphi'\left(\frac{\tau + 2}{\tau + 1}\right) + \varphi^2\left(\frac{\tau + 2}{\tau + 1}\right) \right] - \varphi(1) \Gamma\left(\frac{1}{\tau + 1}\right) \varphi\left(\frac{\tau + 2}{\tau + 1}\right) \right\} \]

Therefore, in this particular case \( X \sim W(\tau) \), we obtain the following mean values, variances and covariances:
Mean Values

\[ \mu_1 = \frac{\Gamma\left(\frac{1}{\tau+1}\right)}{\tau + 1}, \]
\[ \mu_2 = \frac{\varphi(1)}{\tau + 1}, \]
\[ \mu_3 = \frac{\Gamma\left(\frac{1}{\tau+1}\right)\varphi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau + 1)^2}. \]

Variances:

\[ \delta_1^2 = \frac{1}{n} \left\{ \frac{2\Gamma\left(\frac{2}{\tau+1}\right)}{\tau + 1} - \left(\frac{\Gamma\left(\frac{1}{\tau+1}\right)}{\tau + 1}\right)^2 \right\}, \]
\[ \delta_2^2 = \frac{1}{n} \frac{\varphi'(1)}{(\tau + 1)^2}, \]
\[ \delta_3^2 = \frac{1}{n} \frac{2\Gamma\left(\frac{2}{\tau+1}\right)\left(\varphi\left(\frac{\tau+3}{\tau+1}\right) + \varphi^2\left(\frac{\tau+3}{\tau+1}\right)\right) - \left(\frac{\Gamma\left(\frac{1}{\tau+1}\right)\varphi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau + 1)^2}\right)}{(\tau + 1)^3}. \]

Covariances:

\[ \lambda_{12} = \frac{1}{n} \left\{ \frac{\Gamma\left(\frac{1}{\tau+1}\right)}{\tau + 1}^2 \left(\varphi\left(\frac{\tau + 2}{\tau + 1}\right) - \varphi(1)\right) \right\}, \]
\[ \lambda_{13} = \frac{1}{n} \left\{ \frac{2\Gamma\left(\frac{2}{\tau+1}\right)\varphi\left(\frac{\tau+3}{\tau+1}\right)}{(\tau + 1)^2} - \left(\frac{\Gamma\left(\frac{1}{\tau+1}\right)}{\tau + 1}\right)^2\varphi\left(\frac{\tau+2}{\tau+1}\right) \right\}', \]
\[ \lambda_{23} = \frac{1}{n} \left\{ \frac{\Gamma\left(\frac{1}{\tau+1}\right)\left\{\varphi\left(\frac{\tau+2}{\tau+1}\right) + \varphi^2\left(\frac{\tau+2}{\tau+1}\right)\right\} - \varphi(1)\Gamma\left(\frac{1}{\tau+1}\right)\varphi\left(\frac{\tau+2}{\tau+1}\right)}{(\tau + 1)^3} \right\}. \]

2.2 Hypothesis testing

Local most powerful invariant tests for distinguishing of special types connected with the Generalized Gamma distribution depend on three basic statistics:

\[ T_1 = \frac{1}{n} \sum_{k=1}^{n} X_k, \quad T_2 = \frac{1}{n} \sum_{k=1}^{n} \ln X_k, \quad T_3 = \frac{1}{n} \sum_{k=1}^{n} X_k \ln X_k. \]
Following the ideas of the article [?], we will search for a consistent test with good power properties in the form of the linear combination of these test statistics:

\[ T = c_1(T_2 - \ln T_1) + c_2 \left( \ln T_1 - \frac{T_3}{T_1} \right), \]

where constants \(c_1, c_2\) are chosen from the condition that the test is consistent. From the previous material we know that the statistic \(T\), is asymptotically normal, hence the rejection region takes the form \(T > C\), where the critical constant \(C\) is defined by the significance level as

\[ C = C(\alpha) = \frac{\sigma_0}{\sqrt{n}} \Phi^{-1}(1 - \alpha) + \mu_0. \]

Here \(\Phi^{-1}(1 - \alpha)\) is the quantile function of the standard normal distribution, \(\mu_0 = C(c_2 - c_1) - c_2\) and \(\sigma_0^2 = c_1^2(\pi^2/6 - 1) + 2c_1c_2(2 - \pi^2/6) + c_2^2(\pi^2/3 - 3)\) are the mean and variance of the statistic \(T\) when the sample is taken from the exponential distribution \((\lambda = \beta = 0), C = 0.577216\ldots\) is the Euler’s constant.

The main goal of this section is in the choice of constants \(c_1\) and \(c_2\) that provide the largest power of the test for the hypothesis

\(H_0\): sample is obtained from the Gamma distribution

with the alternative

\(H_1\): sample is obtained from the Weibull distribution

and also a similar solution of the opposite problem when \(H_1\) is the null hypothesis and \(H_0\) is the alternative.
First we derive the conditions on constants $c_1$ and $c_2$ so that there are consistent
tests for distinguishing these hypotheses. For this we use the mean values of the three
basic (derived in the previous chapters) to obtain the asymptotic of the probability
of the critical region when the sample is taken from the testing distributions. For
Gamma distribution

$$P_\lambda(T > C) \sim 1 - \Phi \left( \frac{\sigma_0}{\sigma(\lambda)} \Phi^{-1}(1 - \alpha) + \frac{\mu_0 - \mu(\lambda)}{\sigma(\lambda)} \sqrt{n} \right),$$

and for the Weibull distribution

$$P_\tau(T > C) \sim 1 - \Phi \left( \frac{\sigma_0}{\sigma(\tau)} \Phi^{-1}(1 - \alpha) + \frac{\mu_0 - \mu(\tau)}{\sigma(\tau)} \sqrt{n} \right).$$

If the difference of mean values $\mu_0 - \mu(\cdot)$ is negative, then the probabilities in the left
hand sides of the previous formulae tends to one as $n \to \infty$, which guaranties the
consistency of the test. If the opposite inequality for the difference of means is true,
then the probability of rejecting the null hypothesis tends to zero. This inequality
may be useful for controlling the size of the test of the given level $\alpha$.

Therefore constants $c_1$ and $c_2$ should be chosen from the above inequalities for the
mean values. These inequalities should be true for all possible values of $\lambda$ and $\tau$. Let’s
start with the choice of constants $c_1$ and $c_2$ which guarantee that the inequalities are
true in the neighborhood of small values of $\lambda$ and $\tau$. For this we expand the difference
of mean values by the Maclaurine series formula keeping only linear terms:

$$\mu_0 - \mu(\lambda) = -\lambda(c_1(\psi'(1) - 1) + c_2(2 - \psi'(1))) + O(\lambda^2),$$
\[ \mu_0 - \mu(\tau) = -\tau(c_2(\psi'(1) - 1) - c_1) + O(\tau^2), \]

(here the value of tri-Gamma function \( \psi'(1) = 1.64 \ldots \)).

Hence if the null hypothesis is the Gamma distribution and the alternative is Weibull distribution, then \( c_1 \) and \( c_2 \) must satisfy the system of inequalities

\[
c_1(\psi'(1) - 1) + c_2(2 - \psi'(1)) < 0, \quad c_2(\psi'(1) - 1) - c_1 > 0,
\]

which is equivalent to the two-sided inequality

\[
\frac{c_1}{\psi'(1) - 1} < c_2 < -\frac{c_1(\psi'(1) - 1)}{2 - \psi'(1)}.
\]

Since the critical region has the form \( T > C \), then one of the constants \( c_1 \) or \( c_2 \) can take value 1 or \(-1\). In order for the previous inequality to be true, choose \( c_1 = -1 \), hence \( c_2 \) must satisfy inequality

\[
\frac{-1}{\psi'(1) - 1} < c_2 < \frac{(\psi'(1) - 1)}{2 - \psi'(1)}.
\]

In numerical expressions this inequality means \(-1.56 < c_2 < 1.77\).

The choice of \( c_2 \) from this inequality guarantees the given constraint on the probability of the first type error and its consistency only in the neighbourhood of zero values of \( \lambda \) and \( \tau \). In the next section the value of \( c_2 \) is found by a numerical method in order to guarantee these type of conditions for all values of \( \lambda \) and \( \tau \).

If the null hypothesis is the Weibull distribution, then \( c_1 \) and \( c_2 \) should satisfy the system of inequalities

\[
c_1(\psi'(1) - 1) + c_2(2 - \psi'(1)) > 0, \quad c_2(\psi'(1) - 1) - c_1 < 0,
\]
which are equivalent to the two-sided inequality

\[-\frac{c_1(\psi'(1) - 1)}{2 - \psi'(1)} < c_2 < \frac{c_1}{\psi'(1) - 1}.

For the previous inequality to be true, we choose \(c_1 = 1\), so that \(c_2\) must satisfy the inequality

\[-\frac{(\psi'(1) - 1)}{2 - \psi'(1)} < c_2 < \frac{1}{\psi'(1) - 1}.

In numerical expressions this inequality means \(-1.77 < c_2 < 1.56\).
Chapter 3

Optimization of the test power

In the previous chapter we found the regions of the values for constant $c_2$, for which guarantees the local consistency of the test. With the help of the software Wolfram Alpha, we can numerically choose the particular values of $c_2$ that produce the maximum of the power.

For the case when the sample was taken from the Weibull distribution with the alternative Gamma, the value of $c_1 = 1$ and the choice of the $c_2$ value started from $c_2 = -1.77$ moving to the right (increasing) with the step 0.1. The algorithm of choosing $c_2$ stops on the step for which the type I error probability is first time became smaller that the given significance level $\alpha = 0.05$. This step corresponds to $c_2 = -1.3$. After, starting for this $c_2$, move to the right until the power function became maximum. It was noted that changing to the step as 0.01, the value of the power decreases. Therefore $c_2 = -1.3$. Figure 3.1 gives the graphs of the power
function for the value \( c_2 = -1.30 \) (solid line), \( c_2 = -1.31, c_2 = -1.32 \) (dotted lines) and also the graph of the type I error probability for the best value for the power \( c_2 = -1.3 \).

For the case when the sample is taken from the Gamma distribution with the Weibull alternative, similar computations are done. The value \( c_1 = -1 \) is fixed, the choice of \( c_2 \) stated from \( c_2 = -1.56 \) moving to the right with step 0.1 until the type I error probability is smaller than the given significance level \( \alpha = 0.05 \). Since the type I error probability becomes smaller that the significance level starting from \( c_2 = -1.56 \), we search the value of \( c_2 \) such that the region of the decrease of the power function is the minimal. It was found that with changing the step by 0.01, the minimal region of the decrease of the power is for \( c_2 = 1.7 \).

![Graph](image.png)

**Figure 3.1:** Power function and the type I error probability
From these figures it is possible to make the conclusion that the type I error probability is smaller than the given significance level $\alpha = 0.05$ and the power function achieves its maximum value of 0.125 at $c_2 = -1.3$.

Therefore the graphs of the power function (right graph of Figure 3.2) and type I error probabilities (left graph of Figure 3.2) are as follows:

![Figure 3.2: Power function and the type I error probability](image)

By this graph it is possible to see that the test is biased, the power function is smaller that the given significance level $\alpha$, but starting from the parameter value $\tau = 0.5$ the power function increases up to the maximum value of one. The values of the constants for two tests under consideration that we found in this section are presented in Table 3.1.
<table>
<thead>
<tr>
<th>$H_0$</th>
<th>$H_1$</th>
<th>$c_1$</th>
<th>$c_2$</th>
</tr>
</thead>
<tbody>
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<td>$G$</td>
<td>$W$</td>
<td>-1</td>
<td>1.7</td>
</tr>
<tr>
<td>$W$</td>
<td>$G$</td>
<td>1</td>
<td>-1.3</td>
</tr>
</tbody>
</table>

**Table 3.1:** The parameters values for the tests
Concluding remarks and Future Research

In this chapter we construct the locally most powerful test for the choice between two probability models of the reliability theory. The statistical test for the distinguishing of distribution from the increasing its power point of view is improved up to the construction of the locally most powerful test.

For the constructed tests the asymptotic power is calculated and graphical illustrations of the power function are presented. The graphs show that the test of Gamma vs Weibull possesses an unpleasant property of parametric inconsistency, namely the power function tends to zero when the parameter of the alternative distribution tends to infinity. The test Weibull vs Gamma does not obey the unbiasedness property in an neighborhood of small values of the alternative parameter, but for bigger values of this parameter the power of the test approaches one sufficiently quickly.
Bibliography


Appendix: Wolfram Mathematica Codes

Codes for the graphs of the power function and the type I error probability.

\[ \mu[\lambda, \beta, c1, c2] := c1 \frac{1}{1+\beta} \psi \left[ \frac{1+\lambda}{1+\beta} \right] + \] 

\[ (c2 - c1) \log \left( \frac{\Gamma(\frac{2+\lambda}{1+\beta})}{\Gamma(\frac{1+\lambda}{1+\beta})} \right) - c2 \frac{1}{1+\beta} \psi \left[ \frac{2+\lambda}{1+\beta} \right]; \]

\[ m01[\lambda, \beta] := \frac{1}{1+\beta} \psi \left[ \frac{1+\lambda}{1+\beta} \right]; \]

\[ m10[\lambda, \beta] := \frac{\Gamma(\frac{2+\lambda}{1+\beta})}{\Gamma(\frac{1+\lambda}{1+\beta})}; \]

\[ m11[\lambda, \beta] := \Gamma(\frac{3+\lambda}{1+\beta}) \psi \left[ \frac{3+\lambda}{1+\beta} \right]; \]

\[ m12[\lambda, \beta] := \frac{\Gamma(\frac{2+\lambda}{1+\beta}) (\psi(\frac{1+\lambda}{1+\beta}))^2}{(1+\beta)^2 \Gamma(\frac{3+\lambda}{1+\beta})}; \]

\[ m20[\lambda, \beta] := \frac{\Gamma(\frac{3+\lambda}{1+\beta})}{\Gamma(\frac{1+\lambda}{1+\beta})}; \]

\[ m02[\lambda, \beta] := \frac{\Gamma(\frac{3+\lambda}{1+\beta}) (\psi(\frac{1+\lambda}{1+\beta}))^2}{(1+\beta)^2 \Gamma(\frac{3+\lambda}{1+\beta})}; \]

\[ m21[\lambda, \beta] := \frac{\Gamma(\frac{4+\lambda}{1+\beta}) \psi \left[ \frac{4+\lambda}{1+\beta} \right]}{(1+\beta)^2 \Gamma(\frac{5+\lambda}{1+\beta})}; \]

\[ m22[\lambda, \beta] := \frac{\Gamma(\frac{4+\lambda}{1+\beta}) (\psi(\frac{1+\lambda}{1+\beta}))^2}{(1+\beta)^2 \Gamma(\frac{5+\lambda}{1+\beta})}; \]

\[ \sigma1[\lambda, \beta] := (m20[\lambda, \beta] - m10[\lambda, \beta]^2); \]

\[ \sigma2[\lambda, \beta] := (m02[\lambda, \beta] - m01[\lambda, \beta]^2); \]
\[\sigma^3[\lambda, \beta.] := (m22[\lambda, \beta] - m11[\lambda, \beta]^2);\]
\[\lambda12[\lambda, \beta.] := (m11[\lambda, \beta] - m10[\lambda, \beta]m01[\lambda, \beta]);\]
\[\lambda13[\lambda, \beta.] := (m21[\lambda, \beta] - m10[\lambda, \beta]m11[\lambda, \beta]);\]
\[\lambda23[\lambda, \beta.] := (m12[\lambda, \beta] - m01[\lambda, \beta]m11[\lambda, \beta]);\]
\[\text{DIS}[\lambda, \beta., n, c1., c2.]:= \sqrt{\left(\left(\frac{c2 - c1}{m10[\lambda, \beta]} + \frac{c2m11[\lambda, \beta]}{m10[\lambda, \beta]^2}\right)^2 + c1^2}\sigma1[\lambda, \beta, n] + c1^2}\sigma2[\lambda, \beta, n] + \frac{c2^2}{m10[\lambda, \beta]^2}\sigma3[\lambda, \beta, n] + 2c1\left(\frac{c2 - c1}{m10[\lambda, \beta]} + \frac{c2m11[\lambda, \beta]}{m10[\lambda, \beta]^2}\right)\lambda12[\lambda, \beta, n] - \frac{2c2c1}{m10[\lambda, \beta]^2}\lambda13[\lambda, \beta, n] - \frac{2c1c2}{m10[\lambda, \beta]^2}\lambda23[\lambda, \beta, n]\];

\[\text{F1}[\lambda, n, c1., c2.]:= 1 - \text{CDF}[\text{NormalDistribution}[0, 1],\]
\[-\frac{\text{DIS}[0, 0, n, c1., c2.]}{\text{DIS}[\lambda, 0, n, c1., c2.]}\text{Quantile}[\text{NormalDistribution}[0, 1], 0.95] + \frac{\mu[0, 0, c1., c2.] - \mu[0, 0, c1., c2.]}{\text{DIS}[\lambda, 0, n, c1., c2.]}\sqrt{n}\];

\[\text{F2}[\tau, n, c1., c2.]:= 1 - \text{CDF}[\text{NormalDistribution}[0, 1],\]
\[-\frac{\text{DIS}[0, 0, n, c1., c2.]}{\text{DIS}[\tau, 0, n, c1., c2.]}\text{Quantile}[\text{NormalDistribution}[0, 1], 0.95] + \frac{\mu[0, 0, c1., c2.] - \mu[0, 0, c1., c2.]}{\text{DIS}[\tau, 0, n, c1., c2.]}\sqrt{n}\];

\[\text{Plot}[\text{F1}[\lambda, 10, 1, -1.3], \{\lambda, 0, 3\}] (*W\sim G*)\]

\[\text{Plot}[\text{F2}[\tau, 10, 1, -1.3], \{\tau, 0, 1\}]\]

\[\text{Plot}[\text{F1}[\lambda, 10, -1, 1.7], \{\lambda, 0, 1\}] (*G\sim W*)\]

\[\text{Plot}[\text{F2}[\tau, 10, -1, 1.7], \{\tau, 0, 3\}]\]