Intersecting generalised permutations

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Abstract

For any positive integers k, r, n with $r \leq \min\{k, n\}$, let $\mathcal{P}_{k,r,n}$ be the family of all sets $\{(x_1, y_1), \ldots, (x_r, y_r)\}$ such that $x_1, \ldots, x_r$ are distinct elements of $[k] = \{1, \ldots, k\}$ and $y_1, \ldots, y_r$ are distinct elements of $[n]$. The sets $\mathcal{P}_{n,n,n}$ and $\mathcal{P}_{n,r,n}$ describe *permutations* of $[n]$ and *r-partial permutations* of $[n]$ respectively. If $k \leq n$, then $\mathcal{P}_{k,k,n}$ describes permutations of k-subsets of $[n]$. A family $\mathcal{A}$ of sets is said to be *intersecting* if every pair of sets from $\mathcal{A}$ intersect. In this note we use Katona's elegant cycle method to show that a number of important Erdős-Ko-Rado-type results by various authors generalise as follows: the size of any intersecting subfamily $\mathcal{A}$ of $\mathcal{P}_{k,r,n}$ is at most $\binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}$, and the bound is attained if and only if $\mathcal{A} = \{A \in \mathcal{P}_{k,r,n} \colon (a, b) \in A\}$ for some $a \in [k]$ and $b \in [n]$.

1 Introduction

For an integer $n \geq 1$, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The *power set* $\{A \colon A \subseteq X\}$ of a set X is denoted by 2^X, and the *uniform* subfamily $\{Y \subseteq X \colon |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$. We call a set of size n an *n-set*.

If $\mathcal{F}$ is a family of sets and x is an element in the union of all sets in $\mathcal{F}$, then we call the subfamily of all the sets in $\mathcal{F}$ that contain x the *star of $\mathcal{F}$* with *centre x*. A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$. Note that a star of a family is intersecting.

The classical Erdős-Ko-Rado (EKR) Theorem [10] says that if $r \leq n/2$, then an intersecting subfamily $\mathcal{A}$ of $\binom{[n]}{r}$ has size at most $\binom{n-1}{r-1}$, i.e. the size of a star of $\binom{[n]}{r}$. If $r < n/2$, then, by the Hilton-Milner Theorem [13], $\mathcal{A}$ attains the bound if and only if $\mathcal{A}$ is a star of $\binom{[n]}{r}$. Two alternative proofs of the EKR Theorem that are particularly short and beautiful were obtained by Katona [14] and Daykin [7]. In his proof, Katona introduced a very elegant technique called the *cycle method*. Daykin's proof is based on a fundamental result known as the Kruskal-Katona Theorem [14, 15]. The EKR Theorem inspired a wealth of results and continues to do so; see [2, 9, 11].

For positive integers k, r, n with $r \leq \min\{k, n\}$, let

$$\mathcal{P}_{k,r,n} := \{\{(x_1, y_1), \dots, (x_r, y_r)\} \colon x_1, \dots, x_r \text{ are distinct elements of } [k], \\ y_1, \dots, y_r \text{ are distinct elements of } [n]\}.$$

We shall call $\mathcal{P}_{k,r,n}$ a family of *generalised permutations*. This is due to the fact that the elements of $\mathcal{P}_{n,n,n}$ are permutations of the set $[n]$; the permutation $y_1 y_2 \dots y_n$ of $[n]$ corresponds uniquely to the set $\{(1, y_1), (2, y_2), \dots, (n, y_n)\}$ in $\mathcal{P}_{n,n,n}$.

In the more general case where $k \leq n$, the set $\mathcal{P}_{k,k,n}$ describes permutations of k-subsets of $[n]$; a permutation $y_1 y_2 \dots y_k$ of a k-subset of $[n]$ corresponds uniquely to the set $\{(1, y_1), (2, y_2), \dots, (k, y_k)\}$ in $\mathcal{P}_{k,k,n}$. The set $\mathcal{P}_{n,r,n}$ describes *r-partial permutations* of $[n]$ (see [16]). The ordered pairs formulation we are using follows [1] and also [3, 4], in which very general frameworks are considered.

In the case $r = k$, if two sets $\{(1, y_1), (2, y_2), \dots, (k, y_k)\}$ and $\{(1, z_1), (2, z_2), \dots, (k, z_k)\}$ in $\mathcal{P}_{k,k,n}$ intersect, then $y_i = z_i$ for some $i \in [k]$, and this is exactly what we mean by saying that the permutations $y_1 y_2 \dots y_k$ and $z_1 z_2 \dots z_k$ (of two k-subsets of $[n]$) intersect. In general, two generalised permutations intersect if and only if they have at least one ordered pair in common.

In this note we are concerned with the EKR problem for generalised permutations. We need only to consider the problem with $k \leq n$. To see this, define $\lambda \colon [k] \times [n] \to [n] \times [k]$ by $\lambda(x, y) := (y, x)$, then $\Lambda \colon \mathcal{P}_{k,r,n} \to \mathcal{P}_{n,r,k}$ by

$$\Lambda(\{(x_1, y_1), \dots, (x_r, y_r)\}) := \{\lambda(x_1, y_1), \dots, \lambda(x_r, y_r)\} = \{(y_1, x_1), \dots, (y_r, x_r)\}.$$

The functions λ and Λ are clearly both bijections. Moreover, any $P, Q \in \mathcal{P}_{k,r,n}$ are intersecting if and only if $\Lambda(P), \Lambda(Q) \in \mathcal{P}_{r,k,n}$ are intersecting. Therefore, throughout the rest of the paper it is to be assumed that $k \leq n$.

The origins of our problem lie in [8], in which Deza and Frankl prove that the size of an intersecting family of permutations of $[n]$ is at most $(n-1)!$, i.e. the size of a star of $\mathcal{P}_{n,n,n}$. Cameron and Ku [6] extended this result by establishing that only the stars of $\mathcal{P}_{n,n,n}$ attain the bound (other proofs of this result are found in [5, 12, 17, 20]). This result was also done independently by Larose and Malvenuto [18], who actually showed that the stars of $\mathcal{P}_{k,k,n}$ are the largest intersecting subfamilies of $\mathcal{P}_{k,k,n}$ (see [18, Theorem 5.1]). These results summarize as follows.

Theorem 1.1 ([6, 8, 18]) *The size of any intersecting subfamily of $\mathcal{P}_{k,k,n}$ is at most $\frac{(n-1)!}{(n-k)!}$, and the bound is attained only by the stars of $\mathcal{P}_{k,k,n}$.*

Ku and Leader [16] solved the EKR problem for r-partial permutations of $[n]$ using Katona's cycle method. Moreover, they showed that for $8 \leq r \leq n - 3$, the largest intersecting subfamilies of $\mathcal{P}_{n,r,n}$ are the stars. They conjectured that only the stars are extremal for the few remaining values of r too. A proof of this conjecture, also based on the cycle method, was obtained by Li and Wang [19].

Theorem 1.2 ([16, 19]) *For $r \in [n-1]$, the size of any intersecting subfamily of $\mathcal{P}_{n,r,n}$ is at most $\binom{n-1}{r-1}\frac{(n-1)!}{(n-r)!}$, and the bound is attained only by the stars of $\mathcal{P}_{n,r,n}$.*

The scope of this note is to show that the methods used in [16, 19] allow us to generalise Theorems 1.1 and 1.2 as follows.

Theorem 1.3 *Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{P}_{k,r,n}$. Then*

(a) $|\mathcal{A}| \leq \binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!} = \binom{n-1}{r-1}\frac{(k-1)!}{(k-r)!}$*,*

(b) the bound in (a) is attained if and only if $\mathcal{A}$ is a star of $\mathcal{P}_{k,r,n}$.

2 Proof of the result

We will prove Theorem 1.3 by extending the arguments in [16, 19] to our more general setting. Recall that we are assuming $k \leq n$ and that Theorem 1.1 settles our problem for the case $r = k$, so we will only consider $r \leq k - 1$. We will abbreviate $\mathcal{P}_{k,r,n}$ to $\mathcal{P}$.

For convenience, we shall use 'mod^*' to represent the usual *modulo operation* with the exception that for any non-zero integers a and b the value of $ba \bmod^* a$ will be a, rather than 0.

Let X be a set, and let $m = |X|$. A bijection $\sigma : X \to [m]$ is called a *cyclic ordering of* X; all the elements in X are arranged in a cycle, and $x \in X$ is the $\sigma(x)$-th element in the cycle. If σ is a cyclic ordering of X and the elements of a subset A of X are numbered consecutively, in the cyclic sense, by σ, then we say that A *meets* σ.

Katona's cycle method is based on the following fundamental result.

Lemma 2.1 ([14]) *Let X be a set of size at least $2r$, and let σ be a cyclic ordering of X. Let $\mathcal{B} := \{B \in \binom{X}{r} \colon B$ meets $\sigma\}$, and let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{B}$. Then $|\mathcal{A}| \leq r$. Moreover, if $|X| > 2r$, then $|\mathcal{A}| = r$ if and only if $\mathcal{A}$ is a star of $\mathcal{B}$.*

The union of all sets in $\mathcal{P}$ is the Cartesian product $[k] \times [n]$. We say that a cyclic ordering σ of $[k] \times [n]$ is *r-good* if every set of r elements $(x_1, y_1), \ldots, (x_r, y_r)$ of $[k] \times [n]$ that are numbered consecutively, in the cyclic sense, by σ are such that $x_1, \ldots, x_r$ are distinct and $y_1, \ldots, y_r$ are distinct. In an r-good cyclic ordering any r consecutive elements form a generalized permutation in $\mathcal{P}$.

We will define a cyclic ordering of $[k] \times [n]$ that is r-good for all $r \in [k-1]$. (It is interesting to note that no such cyclic ordering exists if $r = k = n$.) Let $\tau \colon [k] \times [n] \to [kn]$ be defined by

$$\tau(x, y) := k(y - x \bmod^* n) + x.$$

As one can immediately see from the following example with $k = 5$ and $n = 7$, where each element (x, y) of $[k] \times [n]$ is given the label $\tau(x, y)$ shown in bold superscript, τ is $(k-1)$-good, and hence τ is r-good for all $r \in [k-1]$.

$(1,7)^{\mathbf{31}}$	$(2,7)^{\mathbf{27}}$	$(3,7)^{\mathbf{23}}$	$(4,7)^{\mathbf{19}}$	$(5,7)^{\mathbf{15}}$
$(1,6)^{\mathbf{26}}$	$(2,6)^{\mathbf{22}}$	$(3,6)^{\mathbf{18}}$	$(4,6)^{\mathbf{14}}$	$(5,6)^{\mathbf{10}}$
$(1,5)^{\mathbf{21}}$	$(2,5)^{\mathbf{17}}$	$(3,5)^{\mathbf{13}}$	$(4,5)^{\mathbf{9}}$	$(5,5)^{\mathbf{5}}$
$(1,4)^{\mathbf{16}}$	$(2,4)^{\mathbf{12}}$	$(3,4)^{\mathbf{8}}$	$(4,4)^{\mathbf{4}}$	$(5,4)^{\mathbf{35}}$
$(1,3)^{\mathbf{11}}$	$(2,3)^{\mathbf{7}}$	$(3,3)^{\mathbf{3}}$	$(4,3)^{\mathbf{34}}$	$(5,3)^{\mathbf{30}}$
$(1,2)^{\mathbf{6}}$	$(2,2)^{\mathbf{2}}$	$(3,2)^{\mathbf{33}}$	$(4,2)^{\mathbf{29}}$	$(5,2)^{\mathbf{25}}$
$(1,1)^{\mathbf{1}}$	$(2,1)^{\mathbf{32}}$	$(3,1)^{\mathbf{28}}$	$(4,1)^{\mathbf{24}}$	$(5,1)^{\mathbf{20}}$

Let S_n denote the set of all bijections from $[n]$ to $[n]$. For any $(\phi, \psi) \in S_k \times S_n$, define $\tau_{\phi,\psi} : [k] \times [n] \to [kn]$ by

$$\tau_{\phi,\psi}(x, y) := \tau(\phi^{-1}(x), \psi^{-1}(y))$$

(i.e. $\tau_{\phi,\psi}(\phi(i), \psi(j)) := \tau(i, j)$). Note that $\tau_{\phi,\psi}$ is a cyclic ordering of $[k] \times [n]$ and let

$$T_{k,n} := \{\tau_{\phi,\psi} \colon (\phi, \psi) \in S_k \times S_n\}.$$

Further, for any $(\phi, \psi) \in S_k \times S_n$, define $f_{\phi,\psi} : [k] \times [n] \to [k] \times [n]$ by

$$f_{\phi,\psi}(x, y) := (\phi(x), \psi(y)).$$

Lemma 2.2 *For all $(\phi, \psi) \in S_k \times S_n$ the ordering $\tau_{\phi,\psi}$ is an r-good cyclic ordering of $[k] \times [n]$.*

Proof. Suppose $\tau_{\phi,\psi}$ is not an r-good cyclic ordering. Then there exist two distinct elements (a_1, b_1) and (a_2, b_2) of $[k] \times [n]$ such that

$$\tau_{\phi,\psi}(a_2, b_2) = (\tau_{\phi,\psi}(a_1, b_1) + p) \operatorname{mod}^* kn$$

for some $p \in [r - 1]$, with either $a_1 = a_2$ or $b_1 = b_2$. If $a_1 = a_2$, then

$$\tau(\phi^{-1}(a_1), \psi^{-1}(b_2)) = \big(\tau(\phi^{-1}(a_1), \psi^{-1}(b_1)) + p\big) \operatorname{mod}^* kn,$$

but this contradicts the definition of τ. Similarly, we cannot have $b_1 = b_2$. □

Let Z be a set and σ be a cyclic ordering of Z. Let m be an integer with $2 \leq m \leq |Z|$ and suppose that $z_1, \dots, z_m$ are distinct elements of Z. If $\sigma(z_{i+1}) = \sigma(z_i) + 1 \operatorname{mod}^* |Z|$ for each $i \in [m - 1]$, then we say that the tuple $(z_1, \dots, z_m)$ is an *m-interval of σ*, and we call $\{z_1, \dots, z_m\}$ the *set corresponding to* $(z_1, \dots, z_m)$. If $1 \leq m_1 \leq m_2 \leq m$ and $\ell = m_2 - m_1 + 1$, then we call the ℓ-interval $(z_{m_1}, \dots, z_{m_2})$ of σ an *ℓ-subinterval of* $(z_1, \dots, z_m)$.

Lemma 2.3 *Each member of $\mathcal{P}$ meets exactly $r!(k - r)!(n - r)!kn$ members of $T_{k,n}$.*

Proof. Let $P, Q \in \mathcal{P}$. Clearly, $Q = \{f_{\pi,\rho}(x, y) \colon (x, y) \in P\}$ for some $(\pi, \rho) \in S_k \times S_n$.

Let $\tau_{\phi,\psi} \in T_{k,n}$ such that P meets $\tau_{\phi,\psi}$. Then Q meets $\tau_{\phi,\psi} \circ (f_{\pi^{-1},\rho^{-1}})$. For any $(x, y) \in S_k \times S_n$

$$\begin{aligned} \tau_{\phi,\psi} \circ (f_{\pi^{-1},\rho^{-1}})(x, y) &= \tau_{\phi,\psi}(\pi^{-1}(x), \rho^{-1}(y)) \\ &= \tau(\phi^{-1} \circ \pi^{-1}(x), \psi^{-1} \circ \rho^{-1}(y)) \\ &= \tau((\pi \circ \phi)^{-1}(x), (\rho \circ \psi)^{-1}(y)). \end{aligned}$$

Thus, since $(\pi \circ \phi)^{-1} \in S_k$ and $(\rho \circ \psi)^{-1} \in S_n$, we have

$$\tau_{\phi,\psi} \circ (f_{\pi^{-1},\rho^{-1}}) = \tau_{(\pi\circ\phi)^{-1},(\rho\circ\psi)^{-1}} \in T_{k,n}.$$

So Q meets at least as many members of $T_{k,n}$ as P does. Conversely, we can do this for every ordering that Q meets, thus P and Q meet the same number of members of $T_{k,n}$.

Each of the $k!n!$ members of $T_{k,n}$ contains exactly kn r-intervals, and, by Lemma 2.2, the sets corresponding to these r-intervals are members of $\mathcal{P}$. Thus, for each $\tau_{\phi,\psi} \in T_{k,n}$, the number of members of $\mathcal{P}$ that meet $\tau_{\phi,\psi}$ is kn. Since $|\mathcal{P}| = \binom{k}{r}\frac{n!}{(n-r)!}$, each member of $\mathcal{P}$ meets exactly

$$\frac{k!n!kn}{\binom{k}{r}\frac{n!}{(n-r)!}} = r!(k-r)!(n-r)!kn$$

members of $T_{k,n}$. □

For each $\tau_{\phi,\psi} \in T_{k,n}$ the *characteristic vector of* $\tau_{\phi,\psi}$ is the length-$\left(\binom{k}{r}\frac{(n)!}{(n-r)!}\right)$ vector in which each position corresponds to a member P of $\mathcal{P}$, and the entry is 1 if P meets $\tau_{\phi,\psi}$, and 0 otherwise. Similarly, for any $\mathcal{A} \subseteq \mathcal{P}$, the *characteristic vector* $\chi_{\mathcal{A}}$ *of* $\mathcal{A}$ is the length-$\left(\binom{k}{r}\frac{(n)!}{(n-r)!}\right)$ vector in which each position corresponds to a member P of $\mathcal{P}$, and the entry is 1 if $P \in \mathcal{A}$, and 0 otherwise. We now have the tools to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{P}$ of maximum size. Define a matrix M in which the rows are indexed by the members of $\mathcal{P}$, the columns are indexed by the members $\tau_{\phi,\psi}$ of $T_{k,n}$, and the column for $\tau_{\phi,\psi}$ is the characteristic vector of $\tau_{\phi,\psi}$. For any $\ell \in \mathbb{N}$, let $\mathbf{1}_\ell$ denote the all ones vector of length ℓ. By Lemma 2.3,

$$M\mathbf{1}_{|T_{k,n}|}{}^T = r!(k-r)!(n-r)!kn\mathbf{1}_{|\mathcal{P}|}{}^T.$$

Define $\mathcal{A}_{\phi,\psi}$ to be the set of all the members of $\mathcal{A}$ that meet $\tau_{\phi,\psi}$. Then the $\tau_{\phi,\psi}$-entry of $\chi_{\mathcal{A}}M$ is equal to $|\mathcal{A}_{\phi,\psi}|$; by Lemma 2.1, this value is no more than r. So

$$(k!)(n!)r \geq \chi_{\mathcal{A}}M\mathbf{1}_{|T_{k,n}|}{}^T = r!(k-r)!(n-r)!kn\chi_{\mathcal{A}}\mathbf{1}_{|\mathcal{P}|}{}^T = r!(k-r)!(n-r)!kn|\mathcal{A}|, \quad (1)$$

which implies that

$$|\mathcal{A}| \leq \frac{(k!)(n!)r}{r!(k-r)!(n-r)!kn} = \binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}.$$

This gives the first statement of Theorem 1.3.

The intersecting family $\{P \in \mathcal{P} \colon (1,1) \in P\}$ meets this bound, so the size of $\mathcal{A}$ is $\binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}$. Thus, equality holds in (1), and $|\mathcal{A}_{\phi,\psi}| = r$ for each $\tau_{\phi,\psi} \in T_{k,n}$. So Lemma 2.1 tells us that for each $\tau_{\phi,\psi} \in T_{k,n}$ the set $\mathcal{A}_{\phi,\psi}$ consists of those r sets that meet $\tau_{\phi,\psi}$ and contain a fixed element $(x_{\phi,\psi}, y_{\phi,\psi})$. Thus, for each $\tau_{\phi,\psi} \in T_{k,n}$,

$$\mathcal{A}_{\phi,\psi} = \{A \colon A \text{ corresponds to an } r\text{-subinterval of } L_{\phi,\psi}\}, \quad (2)$$

where $L_{\phi,\psi}$ is the $(2r-1)$-interval of $\tau_{\phi,\psi}$ with middle entry $(x_{\phi,\psi}, y_{\phi,\psi})$.

Let β be the identity function from $[k]$ to $[k]$, and let γ be the identity function from $[n]$ to $[n]$. So $\tau = \tau_{\beta,\gamma}$. We may assume that $(x_{\beta,\gamma}, y_{\beta,\gamma}) = (k,k)$. So $\mathcal{A}_{\beta,\gamma}$ consists of the r sets corresponding to all the r-subintervals of the $(2r-1)$-interval

$$L_{\beta,\gamma} = ((k-r+1, k-r+1), \ldots, (k,k), (1,2), \ldots, (r-1,r)).$$

Define

$$I := \{(i,i)\colon i \in [k-1]\}, \qquad \bar{I} := ([k]\times[n])\backslash(I \cup \{(k,k)\}).$$

If $P \subseteq I$, then P does not intersect the set $\{(k,k),(1,2),\dots,(r-1,r)\} \in \mathcal{A}_{\beta,\gamma}$; similarly, if $P \subseteq \bar{I}$, then P does not intersect the set $\{(k-r+1,k-r+1),\dots,(k,k)\} \in \mathcal{A}_{\beta,\gamma}$. Thus, for each $A \in \mathcal{A}$, it is the case that $A \not\subseteq I$ and $A \not\subseteq \bar{I}$, so

$$1 \le |A \cap I| \le r-1, \qquad 1 \le |A \cap \bar{I}| \le r-1. \tag{3}$$

Define the sets

$$T' := \{\tau_{\pi,\rho} \in T_{k,n}\colon \pi(k) = \rho(k) = k\}, \qquad T^* := \{\tau_{\pi,\rho} \in T'\colon \pi(i) = \rho(i), i = 1,\dots,k\}.$$

Note for each $\tau_{\pi,\rho} \in T^*$ that

$$\{(\pi(i),\rho(i))\colon (i,i) \in I\} = I, \qquad \{(\pi(i),\rho(j))\colon (i,j) \in \bar{I}\} = \bar{I}. \tag{4}$$

If $(x_{\pi,\rho}, y_{\pi,\rho}) \in I$, then, by (4), I has an r-subset R that corresponds to an r-subinterval of $L_{\pi,\rho}$, and hence $R \in \mathcal{A}$ by (2), but this contradicts the first inequality in (3). Similarly, $(x_{\pi,\rho}, y_{\pi,\rho}) \in \bar{I}$ contradicts the second inequality in (3). So $(x_{\pi,\rho}, y_{\pi,\rho}) = (k,k)$ for each $\tau_{\pi,\rho} \in T^*$.

Now suppose $(x_{\pi,\rho}, y_{\pi,\rho}) \neq (k,k)$ for some $\tau_{\pi,\rho} \in T'$. Then $L_{\pi,\rho}$ has an r-subinterval which does not have (k,k) as one of its entries. Let B be the set corresponding to this interval; according to (2), $B \in \mathcal{A}$. By (3), $1 \le s := |B \cap I| \le r-1$. Let $(a_1,a_1),\dots,(a_s,a_s)$ be the s distinct elements of $B \cap I$. Define $a_{s+1},\dots,a_k$ to be the $k-s$ distinct elements of $[k]\backslash\{a_1,\dots,a_s\}$. Since $(k,k) \notin B \cap I$, we may assume that $a_k = k$.

Choose $(\pi^*,\rho^*) \in S_k \times S_n$ such that $\pi^*(i) = \rho^*(i) = a_i$ for each $i \in [k]$. So $\tau_{\pi^*,\rho^*} \in T^*$ and hence $(x_{\pi^*,\rho^*}, y_{\pi^*,\rho^*}) = (k,k) = (a_k,a_k)$ (as shown above). Therefore,

$$L_{\pi^*,\rho^*} = ((a_{k-r+1},a_{k-r+1}),\dots,(a_k,a_k),(a_1,a_2),\dots,(a_{r-1},a_r)),$$

and the r-set

$$C := \{(a_{k-r+s},a_{k-r+s}),\dots,(a_k,a_k),(a_1,a_2),\dots,(a_{s-1},a_s)\}$$

corresponds to an r-subinterval of L_{π^*,ρ^*}; by (2), $C \in \mathcal{A}$. Since $k-r+s > s$, the pairs $(a_{k-r+s},a_{k-r+s}),\dots,(a_{k-1},a_{k-1}),(a_k,a_k)$ are not in B. Further, $(a_i,a_{i+1}) \notin B$ for each $i \in [s-1]$ since $(a_i,a_i) \in B$. Thus B and C are not intersecting, but this is a contradiction since $B, C \in \mathcal{A}$. We conclude that

$$(x_{\pi,\rho}, y_{\pi,\rho}) = (k,k) \text{ for every } \tau_{\pi,\rho} \in T'. \tag{5}$$

Finally, let A be a set $\{(x_1,y_1),\dots,(x_r,y_r)\}$ in $\mathcal{P}$ that contains (k,k). We may assume that $(x_r,y_r) = (k,k)$. Let $(\pi,\rho) \in S_k \times S_n$ be such that $\pi(i+k-r) = x_i$ and $\rho(i+k-r) = y_i$ for each $i \in [r]$. Then $\tau_{\pi,\rho} \in T'$ and A meets $\tau_{\pi,\rho}$. By (5) and (2), $A \in \mathcal{A}$. Hence the result. $\square$

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