Underlying Cognitive Components and Conceptual Knowledge in Arithmetic

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SUPERVISORY AND EXAMINING COMMITTEE

William Tomos Edwards, candidate for the degree of Master of Arts in Experimental & Applied Psychology**,** has presented a thesis titled, ***Underlying Cognitive Components and Conceptual Knowledge in Arithmetic,*** in an oral examination held on December 9, 2013. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

Within the field of mathematical cognition there is extensive research on conceptual knowledge of arithmetic operations. There is also extensive research on the link between mathematical ability and spatial ability. This research study seeks to build on both areas of research and identify ways in which they are interrelated. Conceptual knowledge of arithmetic operations is the subject of ongoing research. When solving a three-term problem of the form *a x b* ÷ *b*, those who understand the inversion concept do not need to perform any calculations because they know that the multiplication and division operations cancel each-other out. When solving a three-term problem of the form *a x b* ÷ *c*, those who understand the associativity concept know that they can do *b* ÷ *c* first, or *a x b* first.

Research indicates that there is a complex relationship between spatial ability and mathematical ability. In some studies spatial ability is shown to have an especially strong relationship with certain measures of mathematical performance while in other studies this is not the case. Theories have already been put forth that visual-spatial abilities are initially cardinal to learning mathematics in children, but verbal and general intelligence become more important to mathematical performance later on, after these mathematical skills and forms of knowledge have been well learned. In this study it is theorized that spatial abilities are more important than other cognitive abilities for acquiring new mathematical knowledge across the lifespan, and not just in childhood. Conversely, general intelligence is more important to mathematical performance after the relevant mathematical knowledge has been well learned. This theory is supported by past research as well as the results of this study.

This study also provides important clues about the development of conceptual knowledge of arithmetic by showing that knowledge of the inversion concept and the associativity concept are both strongly related to spatial ability. Verbal reasoning ability doesn’t relate to knowledge of these concepts but it is related to performance with mathematical skills that are more basic.

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Dedication

華子荷，這論文獻給你。你值得擁有我所得到的認同。我願意竊取主賜予他人的肯定給你。言語無法表達遇到你我有多幸運。無論際遇如何，總記得你是如此美麗，特別和唯一。這世界幸好有你。

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Underlying Cognitive Components and Conceptual Knowledge in Arithmetic

The study of mathematical cognition is a sub-discipline within the broader field of cognitive psychology. An active area of research in mathematical cognition involves the different strategies that people use to carry out operations like multiplication and division (Bisanz & LeFevre, 1990; Gilmore & Papadatou- Pastou, 2009; Robinson & Dubé, 2009a, b, c). It is widely thought that some problem solving strategies are predicated upon a better understanding of mathematical principles than others (Bisanz & LeFevre, 1990; Canobi, Reeve, & Pattison, 2003; Gilmore & Papadatou- Pastou, 2009; Rittle-Johnson, Siegler, & Alibali, 2001; Schneider, Rittle-Johnson, & Star, 2011; Schneider & Stern, 2010). In order for a person to fully grasp arithmetic, he or she must understand the principles that underlie operations like multiplication and division (Piaget, 1941).

Within the domain of mathematical cognition, and specifically arithmetical cognition, conceptual understanding of arithmetic is thought of as an understanding of the underlying principles of mathematics that apply to arithmetic, and it allows a person to understand why and how arithmetical operations and problem solving algorithms follow certain rules and produce certain results (Bisanz & LeFevre, 1990; Canobi et al., 2003; Rittle-Johnson et al., 2001; Schneider et al., 2011; Schneider & Stern, 2010). A hallmark of conceptual knowledge is flexibility in that it can be applied in a variety of situations and contexts (Boaler, 1998a, b).

The purpose of this investigation is to understand the unique ways that spatial intelligence and verbal intelligence influence performance on arithmetic problems and conceptual knowledge of arithmetic operations. Previous research indicates that spatial intelligence is more strongly related to certain mathematical skills than verbal or, by extension, general intelligence (Laski, Casey, Yu, Dulaney, Heyman, & Dearing, 2013; Wei, Yuan, Chen, & Zhou, 2012). There has not been any research yet testing whether this general pattern exists for conceptual knowledge of arithmetic operations.

The inversion concept can be considered a particularly good window into people's conceptual understanding of arithmetic (Bisanz, Watchorn, Piatt, & Sherman, 2009; Robinson & Dubé, 2009a, b, c). People who understand the inversion concept know that when solving a problem with the structure $a \times b \div b$, the answer will always be a regardless of the value of b, because multiplication and division are inverse operations that cancel each other out. This same principle applies to addition and subtraction (i.e., $a + b - b$).

A consistent pattern of variability has been found across people with respect to their tendency to use the inversion shortcut (Robinson & Dubé, 2009a, b, c; Robinson, Ninowski, & Gray, 2006). People can be classified into one of several groups. The first group is *inversion shortcut users,* who consistently pay attention to the inverse nature of the last two terms in an inversion problem (Robinson & Dubé, 2009a, b, c; Robinson et al., 2006). The second group is *negation strategy users*, who perform the operation between the first two terms in the equation using computation, and then realize that the operation with the third term cancels that previous operation out, so they simply provide the first term as the answer (Bisanz & LeFevre, 1990; Robinson & Dubé, 2009a, b, c; Robinson et al., 2006; Siegler, & Stern, 1998). The third group is *no strategy* (or *left-to-right*) *users*, who simply solve equations from left-to-right in a step-by-step fashion (Robinson & Dubé, 2009a, b, c; Robinson et al., 2006). Robinson and colleagues' classification scheme is based on participants' verbal reports of strategy use, corroborated by other measures to validate their verbal reports (Robinson & Dubé, 2009a, b). As would be expected, the inversion shortcut results in greater speed and accuracy than the left-to-right strategy (Bisanz & LeFevre, 1990; Robinson & Dubé, 2009a, b, c). This underscores the advantage gained by using this shortcut, and the importance of this shortcut. The inversion shortcut is also used more frequently on problems made up of larger numbers where more difficult calculations are required (Robinson & Dubé, 2009b, c). This finding demonstrates that people use the inversion shortcut in an adaptive way to make it easier to solve math problems.

Studies have shown that children's speed and accuracy on inversion problems is not related to their speed and accuracy on arithmetic problems where they cannot use a shortcut (Bryant, Christie & Rendu, 1999; Sherman & Bisanz, 2007). Gilmore and Bryant (2006) found that children can be classified into one of three groups with respect to their understanding of inversion and their calculation skills. The first group includes children with good conceptual understanding but poor calculation skills. The second group includes children with good conceptual understanding and good calculation skills. The third group includes children with poor conceptual understanding and poor calculation skills. Gilmore and Papadatou-Pastou (2009) determined through an extensive literature review and meta-analysis that this classification system is robust and applies to children across studies and across age groups. Gilmore and Papadatou-Pastou (2009) classified participants based on their performance on arithmetic problems in which an inversion shortcut can be used (e.g., $a + b – b$), and their performance on problems in which the shortcut cannot be used (e.g., $a + b – c$). It is reasonable to theorize that conceptual knowledge of mathematics is independent from indicators of mathematical performance, yet positively related. These findings are therefore consistent with the inversion shortcut being indicative of conceptual knowledge. These findings also imply that conceptual knowledge of inversion is necessary for having good calculation skills, but good calculation skills are not necessary for understanding inversion.

Another important window into conceptual knowledge is the related associativity concept (Robinson & LeFevre, 2012). There is considerably less research on associativity than inversion (Robinson & Dubé, 2009a). The associativity concept applies to three-term problems in which each term is a distinct quantity (e.g., 10 + 23 – 7 or 40 x 20 ÷ 4). People who understand the associativity concept know that the answer is the same whether they perform the operation between the second and last term first, or the operation between the first and second term first (e.g., $(a \times b) \div c = a \times (b \div c)$). The associativity concept can be thought of as an extension of the inversion concept, and research shows that although not everyone who uses inversion uses associativity, there are very few people who use associativity who do not use inversion. This suggests that associativity may be a harder concept to grasp, and as such, may be acquired after inversion, later in development (Robinson & LeFevre, 2012). A similar feature of both problem solving strategies is that theoretically they both involve shifting attention towards the right side of an equation in order to start working with the last two terms before the first term (Dubé & Robinson, 2010b; Robinson & Dubé, 2009a; Robinson & Dubé, 2013). Another possibility is that people are paying attention to the problem as a whole before selecting a solution strategy. The associativity shortcut results in greater speed and accuracy than the left-to-right strategy (Robinson, Ninowski & Gray, 2006). This speed and accuracy advantage is, however, contingent upon the structure of the arithmetic problems (Robinson & LeFevre, 2012). When the arithmetic operation between the second and third term results in a value that is difficult to work with (e.g., fraction, large number) in comparison to doing the operation between the first and second term first, then associativity will not present an advantage (Robinson & LeFevre, 2012).

THE IMPORTANCE OF STUDYING CONCEPTUAL UNDERSTANDING

There are a number of compelling reasons to study the inversion and associativity shortcuts as measures of conceptual knowledge. Gilmore and Papadatou-Pastou's (2009) meta-analysis suggests that children who use the inversion shortcut are more likely to have good calculation skills in general. Studying the development of the inversion concept in children allows researchers to understand the development of mathematical knowledge more broadly as well (Bisanz et al., 2009), especially given that understanding and performance can both be readily assessed through inversion problems. By collecting data on individuals' speed, accuracy and solution strategies for inversion problems ($a \times b \div b$), inferences can be made about the dynamic relationship between performance and conceptual knowledge across a variety of domains (e.g., algebra, calculus, personal finance) in mathematical knowledge (Bisanz et al., 2009; Bryant et al., 1999; Gilmore & Papadatou-Pastou, 2009; Rasmussen, Ho, & Bisanz, 2003; Sherman & Bisanz, 2007). Similarly, children's conceptual understanding of the inversion concept can inform mathematical education more broadly (Bisanz et al., 2009). Based on how students assimilate mathematical procedures and concepts into their mathematical inventory (Bisanz et al., 2009), teaching can be designed to accommodate the needs of the students so that they are able to develop strong conceptual knowledge that they can use to solve a variety of different problems in different contexts (Bisanz et al., 2009; Boaler, 1998a, b).

Many of the same advantages associated with studying inversion also apply to studying associativity. Because the inversion shortcut appears to be a prerequisite to using the associativity shortcut (Robinson & LeFevre, 2012), it can be inferred that associativity reflects even greater conceptual knowledge than inversion. Therefore, the associativity shortcut could be used as a further gauge of an individual's level of conceptual knowledge. Having said that, failure to use the associativity shortcut may not necessarily be indicative of lower conceptual understanding in all circumstances. Depending on the structure of a given three term arithmetic problem, the relative advantage gained by using the associativity concept may not be that great, and thus people's incentive to use it will not be that great either (Dubé & Robinson, 2010a; Robinson & Dubé, 2009b). Technically, no calculations are required for inversion problems but they are for associativity problems, thus the inversion shortcut increases speed and accuracy to a far greater degree than does the associativity shortcut (Robinson & Dubé, 2009a). In order to ensure that participants do have a reasonable incentive to use associativity, problems can be structured to facilitate use of the associativity concept (Dubé & Robinson, 2010b). Some standard three-term arithmetic problems are not as conducive to associativity shortcut use as others. If solving the rightmost operation first is much easier than solving the leftmost operation first (i.e., 3 x 14 ÷ 2), people will have a greater incentive to use the associativity shortcut than when the difficulty of both operations is somewhat similar (i.e., 3 x 4 ÷ 2; Dubé & Robinson, 2010b; Robinson & Dubé, 2009a). People will also have less of an incentive to use the associativity shortcut when solving the rightmost operation first results in a fraction (i.e., 3 x 2 ÷ 4; Dubé & Robinson, 2010b; Robinson & Dubé, 2009a). The degree to which standard three-term problems facilitate associativity use is an important dimension of these problems that researchers should control. In order to consider associativity use as a valid measure of conceptual knowledge, researchers may need to ensure that the associativity shortcut can offer a clear advantage on some arithmetic problems.

Research from mathematics education literature indicates the importance of conceptual understanding (Boaler, 1998a, b; Fuson, Carroll, & Drueck, 2000; Wood & Sellers, 1997). When mathematics curricula focus on teaching students various procedures that they can memorize through rote learning, the results can be problematic (Masingila, 1993; Schoenfeld, 1988). Students that have rote-learned mathematics in such curricula often develop inflexible knowledge that can only be applied to very specific and clearly defined mathematical problems and cannot be generalized to other related areas of mathematics, or to real world situations (Masingila, 1993; Schoenfeld, 1988). What is more problematic is that students who have memorized mathematical procedures may be able to perform well on standardized tests, but not be able to extrapolate the learned information to other important contexts (Schoenfeld, 1998). Students that are taught in curricula that foster conceptual knowledge perform better on a variety of mathematical measures than equivalent groups of students that have memorized procedures (Boaler, 1998a, b; Fuson, Carroll, & Drueck, 2000; Wood & Sellers, 1997), and this can include standardized tests (Wood & Sellers, 1997). Furthermore, students in these curricula are better able to retain knowledge of what they have learned, and they are able to extrapolate their mathematical knowledge to multiple contexts including “real world” situations (Boaler, 1998a, b).

COGNITIVE COMPONENTS THAT FACILITATE CONCEPTUAL UNDERSTANDING

An important emerging area of research is the association between underlying cognitive abilities and the use of the inversion shortcut (Robinson & Dubé, 2013, Dubé & Robinson, 2010a). Recently Dubé and Robinson (2010a) demonstrated a link between analogical reasoning ability and use of the inversion shortcut on multiplication and division problems, as well as between working memory capacity and inversion shortcut use on those problems. Previous research did not find a link between performance on memory tasks and inversion shortcut use for addition and subtraction problems (Rasmussen, Ho, & Bisanz, 2003). Robinson and Dubé (2013) demonstrated that children who frequently use the negation strategy (i.e., performing the operation between the first two terms in an inversion problem using computation, and then realizing that the operation with the third term cancels that previous operation out, and then simply providing the first term as the answer) have weaker inhibition than children who use inversion or associativity. It is important to know how these and other cognitive components relate to shortcut use because this knowledge can benefit educators and students alike by informing educational programs.

One of the most important reasons for studying the relationship between cognitive components and conceptually-based strategy use is that it allows researchers to move beyond a nomothetic understanding of conceptual shortcuts that does not account for individual differences in various psychological factors, towards a comprehensive understanding that accounts for how those factors will influence an individual's probability of choosing a conceptual shortcut (Robinson & LeFevre, 2012). Based on a large body of empirical evidence, Robinson and LeFevre (2012) demonstrate that a range of individual factors often dictate whether or not someone will choose to use the inversion or associativity shortcut. Importantly, they argue it is not just the problem itself or the context of the problem that determines whether someone will use a conceptually-based shortcut, but factors that are unique to that individual. Studying underlying cognitive components will shed light on the factors that influence a person's probability of choosing a conceptually-based strategy.

DIRECTION OF THIS STUDY

In this study, the impact of verbal deductive reasoning, as well as the impact of general spatial processing abilities on the usage of the inversion and associativity shortcuts was assessed in adults. This was accomplished by assessing people's performance on multiplication and division arithmetic problems as well as their performance on measures of the relevant cognitive components. As discussed above, the impact of several cognitive components on inversion shortcut use has already been studied in children (Dubé & Robinson, 2010a; Robinson and Dubé, 2013). There is a gap in the literature with respect to adults, and many cognitive components that remain to be explored. A measure of verbal deductive reasoning was used, since mathematics uses deductive reasoning. The use of this measure will ensure that any associations or dissociations between arithmetic measures and the measure of verbal deductive reasoning can be attributed to the fundamental properties of verbally stored and processed information and not to a particular kind of reasoning process.

SPATIAL ABILITY AND MATHEMATICS

There is a considerable body of research demonstrating an intimate link between spatial intelligence and mathematical ability. A link has been established between numerical processing and spatial processing (Hubbard, Piazza, Pinel, & Dehaene, 2005). The SNARC effect (spatial numerical association of response codes) demonstrates an association between numerical and spatial processing (Dehaene, Bossini, & Giraux, 1993; Hubbard et al., 2005). It has been observed that numbers presented further left in space can be classified in terms of relative magnitude more quickly if they are smaller, and numbers presented on the right side of space can be classified more quickly if they are larger.

There is evidence from cognitive neuroscience that spatial and numerical relationships are processed in the same areas of the brain. Using repetitive transcranial magnetic stimulation to suppress activity in the angular gyrus of the parietal lobe has been shown to decrease performance on a visual- spatial search task, as well as the judgement of numerical magnitudes. Stimulation of control areas did not generate the same effect (Gobel, Walsh, & Rushworth, 2001). The relationship between numerical and spatial processing is complex; performing exact calculations activates brain areas normally associated with verbal processing, while mathematical operations that involve approximate estimates activate brain areas normally associated with spatial processing (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999).

There is substantial evidence from psychometric research to suggest an important link between spatial and mathematical abilities. Importantly, spatial ability has been shown to predict problem solving strategies (Laski et al., 2013). First-grade girls with better spatial abilities are more likely to use efficient problem solving strategies on simple arithmetic problems (Laski et al., 2013). Verbal abilities are not as strongly related to efficient problem solving strategies in first grade girls (Laski et al., 2013). Rohde and Thompson (2007) provided undergraduate students with several measures of cognitive ability and obtained their SAT scores and two other measures of academic achievement. It was found that measures of general cognitive ability were able to predict almost all measures of academic achievement with domain specific abilities making no unique contribution. The exception was the math portion of the SAT in which spatial ability and information processing speed uniquely predicted performance. It has been demonstrated that verbal working memory and central executive functioning do not correlate with ninth graders' performance on a national mathematics test, while visual-spatial working memory and mental rotation ability do (Reuhkala, 2001). When mathematically gifted children in kindergarten and first grade were given 15 cognitive measures assessing quantitative, visual-spatial, and verbal abilities, it was found that a quantitative factor and a visual-spatial factor were strongly related to one another while a verbal factor was less strongly related to either (Robinson, Abbott, Berninger, & Busse, 1996). Similarly, amongst gifted high-school students certain kinds of spatial abilities have been shown to predict success in advanced math courses (Stumpf, 1994).

Visser, Ashton, and Vernon (2006) provided undergraduate students with a battery of eight measures representing specific domains of cognitive ability. The only measures that correlated with performance on an arithmetic test, besides another test of mathematical ability, were performance on a visual-spatial test (map planning) and performance on a test categorizing objects based on relationships (diagraming relationships). Diagramming relationships was found to be highly inter-correlated with other measures, including map planning, leaving open the possibility that diagraming relationships predicts arithmetic performance indirectly through map planning. The data presented by Visser and colleagues were used to determine that map planning predicts arithmetic performance while controlling for diagraming relationships, *partial* $r = .20$, $p < .05$. Conversely, diagraming relationships does not correlate with arithmetic performance while controlling for map planning, *partial* $r = .076$, $p > .05$. Thus, Visser and colleagues' data provides further evidence that spatial abilities surpass other abilities in predicting mathematical performance.

Wei, Yuan, Chen, and Zhou (2012) provided a battery of cognitive measures to adult participants with no prior experience in advanced mathematics. Participants were given materials explaining advanced mathematics concepts and tested on what they had learned. A composite measure of spatial ability based on three separate measures was shown to account for over 25% of the variance in learning advanced mathematics, while controlling for other aspects of cognitive functioning (Wei et al., 2012). Conversely, basic numerical processing did not account for any unique variance in learning advanced mathematics, and out of four verbal measures only semantic processing of words accounted for unique variance in learning advanced mathematics (Wei et al., 2012). It is notable that the advanced mathematics concepts were presented with verbal descriptions so the need for semantic understanding could be a measurement artifact rather than a real effect. The spatial measures however do not seem to have any obvious relation to advanced mathematics concepts, strengthening the theory that there is an inherent relation between the two types of cognitive processing. Research demonstrating a direct link between adults' acquisition of new mathematical concepts and spatial ability (Wei et al., 2012) is consistent with longitudinal research showing that spatial ability predicts females' mathematical development throughout high school, above and beyond other aspects of intelligence (Sherman, 1979).

There is also experimental evidence demonstrating that spatial abilities are strongly related to mathematical abilities (Cheng & Mix, in press). When 6 to 8 year olds were given spatial training tasks their scores on mathematical measures improved. A control group of children were given crossword puzzles to solve and this resulted in no improvement in mathematical performance.

There is opposing research indicating that general intelligence primarily predicts mathematical performance, with spatial abilities making no appreciable independent contribution (Floyd, Evans, & McGrew, 2003; McGrew & Hessler, 1995; Parkin & Beaujean, 2012). Amongst subscales from the Woodcock-Johnson III it has been found that crystallized intelligence primarily predicts mathematical performance, in large standardized samples representative of the general population (Floyd, Evans, & McGrew, 2003; McGrew & Hessler, 1995). However, spatial ability was shown to make no independent contribution to mathematical performance (Floyd, Evans, & McGrew, 2003; McGrew & Hessler, 1995). Similarly, using data obtained from a large, representative sample of children, Parkin and Beaujean (2012) found that general intelligence (measured by the Wechsler Intelligence Scale for Children IV) was a better predictor of mathematical ability (measured by the Wechsler Individual Achievement Test II) than domain specific areas of intelligence from the WISC-IV. There is research in ninth graders indicating that fluid intelligence is a stronger predictor of mathematical performance than several measures of visual-spatial intelligence (Kyttala & Lehto, 2008). There is also research showing that verbal intelligence may be a better predictor of mathematical performance than spatial intelligence. Delgado and Prieto (2004) found that lexical access (the ability to quickly and accurately differentiate real words from nonsense words) is a better predictor of mathematical performance than mental rotation ability amongst 13 -14 year olds. These findings are in contrast to research showing that spatial ability is of special importance to mathematics (Rohde & Thompson, 2007; Sherman, 1979; Visser et al., 2006; Wei et al., 2012).

The inconsistency in the literature could be due to any number of mediating factors in the relationship between spatial and mathematical abilities. Visual mental representations of mathematical problems and concepts are a potential mediating factor between spatial and mathematical abilities (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). Hegarty and Kozhevnikov (1999) found that "schematic" visual representations of math problems (visual representations that are sparse and pragmatic) were more conducive to mathematical problem solving than "pictorial" representations of math problems (representations that incorporate superfluous, flowery imagery), in a sample of elementary school aged boys. Hegarty and Kozhevnikov (1999) found that spatial ability and other cognitive abilities predicted mathematical problem solving equally well in terms of simple bivariate correlations, but only spatial ability correlated with the use of schematic visual representations. The superiority of schematic representations for mathematical problem solving in children has been corroborated by subsequent research (Garderen & Montague, 2003; Garderen, 2006). This research brings attention to the possibility that spatial abilities are particularly important for constructing good mental models of arithmetic. It is possible that in some contexts the need for schematic visual representations is of greater importance to mathematics than in other contexts. The important contextual factor could be an individual's stage of mathematical learning. Some of the studies reviewed above suggest that spatial ability is of key importance to acquiring new mathematical knowledge (Sherman, 1979; Wei et al., 2012). Indeed there is a body of research in children showing that visual-spatial working memory is important in early stages of mathematical learning and that verbal working memory becomes important later on (DeSmedt, Jansen, Bouwens, Verschaffel, Boets, & Ghesquiere, 2009; McKenzie, Bull, & Grey, 2003; Rasmussen & Bisanz, 2005). Pre-school children perform better at simple non-verbal arithmetic problems than verbal arithmetic problems, while grade one children perform equally well on both (Rasmussen & Bisanz, 2005). Additionally, visual spatial sketchpad functioning has been shown to be correlated with problem solving performance in preschoolers but not in first graders (Rasmussen & Bisanz, 2005). Conversely, longitudinal research shows that visual-spatial working memory but not verbal working memory predicts mathematical performance in the first grade, but verbal working memory is a better predictor in second grade (DeSmedt et al., 2009). It has also been shown that visual-spatial working memory predicts mathematical performance better than does general intelligence in first graders, but general intelligence is the best predictor of mathematical performance in second graders (DeSmedt et al., 2009). Experimental research shows that disruption of the visual-spatial sketchpad hinders mathematical performance in 6- 7 year olds, while disruption of the phonological loop does not (McKenzie et al., 2003). Conversely, disruption of both the visual-spatial sketchpad and the phonological loop hinders mathematical performance in 8- 9 year olds (McKenzie et al., 2003). The importance of both verbal and visual-spatial working memory in the 8- 9 year olds suggests the importance of general cognitive functioning. Similarly, amongst gifted first grade students, spatial and quantitative abilities are more closely related than verbal abilities are to either (Robinson et al., 1996), but this pattern is even more pronounced in gifted kindergarten students (Robinson., 1996). Thus, visual-spatial ability is shown to be more important to mathematics than verbal ability at earlier stages of development (DeSmedt et al., 2009; McKenzie et al., 2003; Rasmussen & Bisanz, 2005), but this can vary in such a way that visual-spatial working memory ability is predictive of mathematics performance in first grade (DeSmedt et al., 2009; McKenzie et al., 2003) or not predictive for that age group (Rasmussen & Bisanz, 2005). In all of these studies (DeSmedt et al., 2009; McKenzie et al., 2003; Rasmussen & Bisanz, 2005; Robinson et al., 1996), the mathematical measures were controlled so that they were equivalent or comparable across age groups. The key importance of visual-spatial abilities to mathematics does not seem to be localized to an exact age group. Instead, it would appear that visual-spatial abilities are important during the acquisition of mathematical skills, knowledge, and understanding, and that verbal abilities and general intelligence are important after this information has crystallized. Spatial ability has been shown to have increased importance for learning new mathematical concepts in adults (Wei et al., 2012), which implies that visual-spatial cognitive resources play this role across the lifespan.

Rasmussen and Bisanz (2005) theorize that preschoolers who have not formally learned mathematics rely on visual mental models to solve math problems, whereas older children who have learned the symbolic number system rely on verbal cognitive resources to solve math problems. Rasmussen and Bisanz (2005) also theorize that the information from early visual mental models is later recoded into a verbal cognitive format. Rasmussen and Bisanz (2005) largely extrapolate this theoretical framework from Huttenlocher and colleagues (Huttenlocher, Jordan, & Levine, 1994). Here this theoretical framework will be modified to state that when individuals are acquiring any kind of new mathematical information, at any age or stage of cognitive development, they first rely on visual-spatial cognitive resources to develop mental models of the new information; better spatial abilities facilitate better mental models. Eventually, the information represented by visual-spatial mental models is re-encoded into verbal information in semantic memory. At this point general intelligence becomes the most important determinant of how well an individual can use that information to solve problems. Evidence suggests that when verbal abilities take on increased importance later on they do not outstrip the importance of general intelligence (DeSmedt et al., 2009). This is likely because visual-spatial cognitive resources do not lose their significance. This theoretical framework is consistent with findings demonstrating that spatial abilities assist in the construction of schematic visual images which in turn are advantageous to mathematical performance (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). This theoretical framework has important implications for theories about conceptual understanding of mathematics. It is implied that spatial abilities will be important for the acquisition of conceptual knowledge like inversion and associativity by assisting in the development of mental models of those principles.

Given the theoretical framework presented here, it can be inferred that there are two necessary criteria for spatial ability to surpass general intelligence as a predictor of mathematical performance: 1) the individuals in the sample must differ according to the quantity and quality of mathematical concepts or skills that they have grasped, and 2) the measure of mathematical ability must be sensitive to these individual differences in acquired mathematical knowledge. From here an explanation can be offered as to why spatial ability is an important predictor of mathematical ability in some samples but not in others. In three of the studies reviewed above where crystallized or general intelligence primarily predicted mathematical performance, large samples representative of the general population were used (Floyd, Evans, & McGrew, 2003; McGrew & Hessler, 1995; Parkin & Beaujean, 2012). Many of the studies that found a distinct spatial-mathematical relationship were looking at intellectually advanced populations (i.e., gifted students, SAT test takers, university students; Robinson et al., 1996; Rohde & Thompson, 2007; Visser et al., 2006). In the general population, the variability in mathematical performance is due to people with low, average, and high ability levels. In more advanced populations the variability should be restricted to people with average and high ability levels. It is possible that people of low and average ability levels are relatively equal in terms of the fundamental concepts that have been grasped (i.e., counting, computation) and so it is primarily measures of general intelligence that determine how well they can retrieve and manipulate those concepts to solve math problems. It is possible that people of average and high ability levels are somewhat disparate from one another in terms of the mathematical concepts that they have grasped, and if spatial ability is primarily related to the quantity and quality of conceptual mathematical knowledge that people have acquired, then spatial ability will primarily be related to mathematical performance for samples consisting of average to high ability individuals.

RESEARCHING A LINK BETWEEN SPATIAL ABILITY AND CONCEPTUAL KNOWLEDGE OF ARITHMETIC

Purpose and Significance

It is a compelling possibility that spatial ability relates to conceptual knowledge of arithmetic operations more strongly than other intellectual abilities do. There is no research to date directly demonstrating an association between general visual-spatial ability and knowledge of inversion or associativity. In children it has been demonstrated that spatial ability surpasses verbal ability for predicting the use of efficient problem solving strategies (Laski et al., 2013). In Laski and colleagues' (2013) study, first grade girls were given addition and subtraction problems, and the researchers measured how efficiently they counted out the solution to the problem, or if they used a more efficient strategy like retrieval from memory. Assessing the relationship between spatial abilities and the inversion and associativity problem solving strategies in adults will build on Laski and colleagues' (2013) research. If spatial abilities predict use of inversion and associativity above and beyond verbal ability in adults, then it can be inferred that the capacity to build good mental models is of key importance to learning a variety of mathematical problem solving strategies at any age. More generally, this research study tests the assumption that spatial ability is the primary determinant of the mathematical concepts that people have grasped, and that spatial and verbal abilities (i.e., general intelligence) are the primary determinant of a person's ability to apply well learned mathematical skills and knowledge. In this research study self-report data were used to determine when the conceptually less advanced left-to-right strategy was used rather than inversion or associativity. Response time on the left-to-right strategy can be used to assess a person's more basic mathematical skills and knowledge (Dubé & Robinson, 2010a). The theoretical framework presented in the previous section would predict that both verbal and spatial ability (i.e., general intelligence) would be important predictors of response time using the left-to-right strategy. Therefore this research study will test the assumption that spatial ability predicts mathematical performance primarily through its importance to acquiring new mathematical knowledge, such as knowledge of inversion and associativity. If spatial ability is of specific importance to conceptual knowledge of arithmetic and overall arithmetic performance in the sample from this study, but spatial ability and verbal ability are both important to performance using the left-to-right strategy, then the assumptions of the theoretical framework presented above will be supported.

In this study, the measure of verbal reasoning that was used was a measure of verbal deductive reasoning (Ennis & Millman, 1985). This measure requires precise logic to solve verbal problems (Ennis & Millman, 1985). A number of previous studies that found a lack of association between verbal ability and mathematical performance relied on verbal measures that tested vocabulary, understanding of semantics, and similar skills involving knowledge and understanding of words (Laski et al., 2013; Visser et al., 2006; Wei et al., 2012). Verbal measures that test an individual's knowledge and understanding of words are not engaging the same kind of logical deductive reasoning process seen in mathematics. It is therefore possible that some of these studies (Laski et al., 2013; Visser et al., 2006; Wei et al., 2012) failed to find an association between verbal measures and mathematics because the verbal measures did not involve the logical deductive reasoning used in mathematics. The measure of verbal deductive reasoning used in this study surmounts this problem by measuring the logical deductive process as it is used in mathematics. Therefore, a lack of association between this measure and conceptual knowledge would strongly imply that verbal cognitive resources do not contribute to the acquisition of conceptual knowledge.

Hypotheses

It was hypothesized that spatial ability and verbal reasoning ability will both be positively correlated with accuracy on arithmetic problems, and negatively correlated with response time on arithmetic problems, but spatial ability will be more strongly correlated with those measures than verbal reasoning ability. It was hypothesized that spatial ability and verbal reasoning ability will both be positively correlated with the use of associativity and inversion respectively, but spatial ability will be more strongly correlated with those measures than verbal reasoning ability. It was hypothesized that spatial ability and verbal reasoning ability will both be equally correlated with response time using the left-to-right strategy (negative relationship).

METHODS

Participants

A sample of 42 participants (20 males and 22 females, with an average age of 20 years, and age range of 18 to 33) was recruited from the University of Regina psychology department undergraduate participant pool. All participants obtained marks for their class in exchange for their participation.

Materials and procedures

Participants were given arithmetic problems (Appendix A), a verbal deductive reasoning task (Appendix B), and three spatial reasoning tasks (Appendix C). The arithmetic problems and the verbal deductive reasoning task were presented to participants on a computer with the experimental psychology software e-prime. The spatial tasks were presented in a pencil and paper format. The arithmetic problems were presented in two separate orders, one the reverse of the other. The arithmetic problems were always presented first, followed by the verbal reasoning and then spatial problems. Within the spatial problems the SS-3 was presented first, followed by the VZ-2, and then the SS-2. The arithmetic problems were presented first since the solution strategy data may be a fairly sensitive measurement. The order of the verbal and spatial tasks was held constant since they likely produce measurements that are stable regardless of when the testing takes place (National Research Council, 2006; Solon, 2001; 2003).

Arithmetic problems

The arithmetic problems consisted of three-term multiplication and division problems. The arithmetic problems were made up of inversion problems (e.g., 7 x 5 ÷ 5) and standard problems (e.g., 6 x 9 ÷ 2). The size of the arithmetic problems was varied so that they were either small or large problem size. For 19 problems the multiplication operation resulted in a number greater than 30 (large) and for 21 problems the multiplication operation resulted in a number equal to or smaller than 30 (small; Appendix A). Standard problems were either conducive or non-conducive to the use of the associativity shortcut. Eleven standard problems (i.e., conducive problems) were designed so that computing the operation between the last two numbers first will be easier (e.g., 5 x 12 ÷ 6). The primary criterion for a problem to be conducive to associativity was that doing the division operation first will not result in a fraction. The secondary criterion was that doing the division operation first will result in a smaller value than doing the multiplication operation first. Nine standard problems (i.e., non-conducive problems) were designed to offer no advantage of using a conceptually-based shortcut (e.g., 4 x 3 ÷ 2). The unequal numbers in each condition were due to experimenter error during the initial classification of problems. The arithmetic problems appeared on the screen in serial order in such a way as to ensure that the different types and sizes of problems were evenly distributed across all 40 problems (Appendix A). In Appendix A, “order presented” indicates the order at which each problem appeared on the screen. The order given in Appendix A is Order 2. Order 1 presented each problem in the reverse of that order. The arithmetic problems for this study have largely been extrapolated from Dubé and Robinson (2010b).

The arithmetic problems appeared on the computer screen and participants stated their answer out loud and hit the space bar simultaneously so that their reaction time could be recorded. The researcher logged their answer as either accurate or inaccurate. Participants were then asked “how did you solve that problem?” Probing questions such as “could you be more specific” were asked when necessary. When clarification was needed the researcher would reiterate the participant’s description back to them in order to ensure that their solution strategies were recorded accurately. For inversion problems, the solution strategies were coded as inversion, negation, or computation. For standard problems, the solution strategies were coded as associativity or computation. In 0.3% of instances a participant’s answer was recorded as “other.” If a participant was unable to satisfactorily describe what they did then their answer would be classified as “other.” In some instances when participants described using two solution strategies at the same time their descriptions were too ambiguous to classify as either solution strategy and would be classified as “other.” All sessions were video-recorded so that verbal report data could be coded by two raters to allow for computing inter-rater reliability. The problem solving strategies for 13 participants (31% of participants) were reviewed in this way. Inter-rater reliability was 99.03%, indicating good reliability. Disagreements were resolved through discussion. The video recordings were also used to determine reaction time when a participant’s answer was produced before or after the spacebar was pressed. This was necessary for 3.2% of trials.

Verbal deductive reasoning task

The deductive reasoning subscale from the Cornell Critical Thinking Test (CCTT) (Appendix B) was used to measure verbal deductive reasoning ability. The CCTT is shown to have high test re-test reliability (Solon, 2001; 2003). The deductive reasoning subscale is composed of 10 items. Each item is a paragraph where someone argues in favor of an opinion. Each paragraph contains a series of statements and an underlined conclusion. Participants must determine if the conclusion (a) follows from the statements given in the paragraph, (b) contradicts the other statements, or (c) is neutral with regards to them. The 10 items were presented on a computer screen, with the participant automatically advancing to the next item after selecting their answer (a, b, or c) for the current one. Participants were allowed 14 minutes to complete the 10 items which is approximately equivalent to allowing participants around an hour to complete the 52 items on the full test in standard practice (Frisby, 1992; Solon, 2001; 2003).

Spatial reasoning tasks

The VZ-2 Paper Folding Task, SS-2 Choosing a Path Test, and SS-3 Map Planning Test (Appendix C; Ekstrom, French, & Harman, 1976) were used as measures of spatial ability. The VZ-2 requires that participants view a piece of paper which is folded in a unique way and then punctured. The participant must then select what the paper will look like when it is unfolded, and choose from 5 possible answers. The VZ-2 consists of 20 questions and has a time limit of 6 minutes. The SS-2 and SS-3 both involve looking at maps and choosing the most efficient route between two points. The SS-2 consists of 32 questions and has a time limit of 14 minutes. The SS-3 consists of 40 questions and has a time limit of 6 minutes. All three measures are included in the Kit of Factor Referenced Tests (Ekstrom et al., 1976). Participants completed all these tasks in a pencil and paper format. Although all tasks are unique, they are similar in that they measure understanding of concrete spatial relationships. Spatial intelligence is shown to be very stable over a person's lifetime and resistant to change (National Research Council, 2006).

Sex

Females perform better than males in some areas of mathematics and at some stages of development, but this situation is reversed in other areas of mathematics and at other stages of development (Hyde, Fennema, & Lammon, 1990). For this reason sex's effect on arithmetic performance and conceptual knowledge of arithmetic was controlled for in this research study. Although sex is not of interest to this investigation it is an important variable to control for. The effect of sex on the variables of interest was assessed for all analyses. When sex did not affect the relationships between the variables of interest, the statistics for sex were not reported in the text for that analysis, or the analysis was redone with sex excluded, and that analysis was reported in the text.

RESULTS

Response time on arithmetic problems

A mixed design 2 (problem size: small and large) x 3 (problem type: inversion, conducive standard, non-conducive standard) ANOVA with sex and order as between-subjects variables was carried out on mean response time to arithmetic problems. For this ANOVA, and all ANOVAs in this research study, Tukey's Honestly Significant Difference test was used to examine post hoc effects for all main effects involving more than two means, and all interaction effects. An alpha level of .05 (two-tailed) was used for all analyses in this research study. The mean response time (ms) for each problem size and type, for both sexes, is given in Table 1. Response time on large problems (M = 7422 ms) was slower than response time on small problems (M = 3284 ms), $F(1, 38) = 22.57$, $MSE = 48967797$, $p < 0.001$, $\eta_{p}^{2} = 0.37$. Response time on inversion problems (M = 2384 ms) did not differ from response times on standard conducive problems (M = 4763 ms), but both yielded faster response times than standard non-conducive problems (M = 8911 ms), $F(2, 76) = 35.47$, $MSE = 26112286$, $p < 0.001$, $\eta_{p}^{2} = 0.48$, HSD = 2672. Males (M = 3785 ms) were faster than females (M = 6920 ms), $F(1, 38) = 6.85$, $MSE = 97573274$, $p = 0.013$, $\eta_{p}^{2} = 0.15$. There was a problem type by problem size interaction, $F(2, 76) = 20.65$, $MSE = 20602703$, $p > 0.001$, $\eta_{p}^{2} = 0.35$, HSD = 2899. Response times to large standard non-conducive problems were slower than response times to large standard conducive, large inversion problems, and small standard non-conducive problems (see Figure 1). Response times to large standard conducive problems were slower than response times to large inversion problems (see Figure 1). There was a sex by problem size interaction $F(1, 38) = 6.04$, $MSE = 48967797$, $p = 0.019$, $\eta p2 = 0.14$, HSD = 4082. The response time for females on large problems ($M = 10044$ ms) was slower than female response time on small problems ($M = 3796$ ms), and male response time on large problems ($M = 4799$ ms). There were no significant contrasts involving male response time on small problems ($M = 2771$ ms). There was a sex by problem type interaction, $F(2, 76) = 3.67$, $MSE = 26112286$, $p = 0.03$, $\eta p2 = 0.09$, HSD = 3264 (Appendix D). There was a sex by problem size by problem type interaction, $F(2, 76) = 3.60$, $MSE = 20602703$, $p = 0.03$, $\eta p2 = 0.09$, HSD = 3347 (see table 1). Female response times to large non-conducive problems were slower than male response times to large non-conducive problems, as well as female response times to large inversion problems, small non-conducive problems, and large conducive problems (see table 1). Female response times to large conducive problems were slower than female response times to large inversion problems, female response times to small conducive problems, and male response times to large conducive problems (see table 1). Male response times to large non-conducive problems were slower than male response times to large inversion problems, large conducive problems and small non-conducive problems (see table 1). No main effects or interactions were found for order. In summary, participants had difficulty responding quickly to large problems, non-conducive standard problems, and in particular, to large non-conducive standard problems. The effects and interactions for sex are not interpreted here since they are not of interest to this investigation, however the variance due to sex is accounted for in this analysis.

Table 1
Mean response time (milliseconds), on every size and type of problem for males and females.

	Response Time							
	Males				Females			
	Small		Large		Small		Large	
	M	*SE*	*M*	*SE*	*M*	*SE*	*M*	*SE*
Inversion	1681	113.73	1903	172.17	2338	406.11	3609	1407
Standard conducive	2926	219.77	3713	524.13	4165	526.78	8246	1588
Standard non-conducive	3705	576.34	8779	1717	4885	541.0	18274	3535

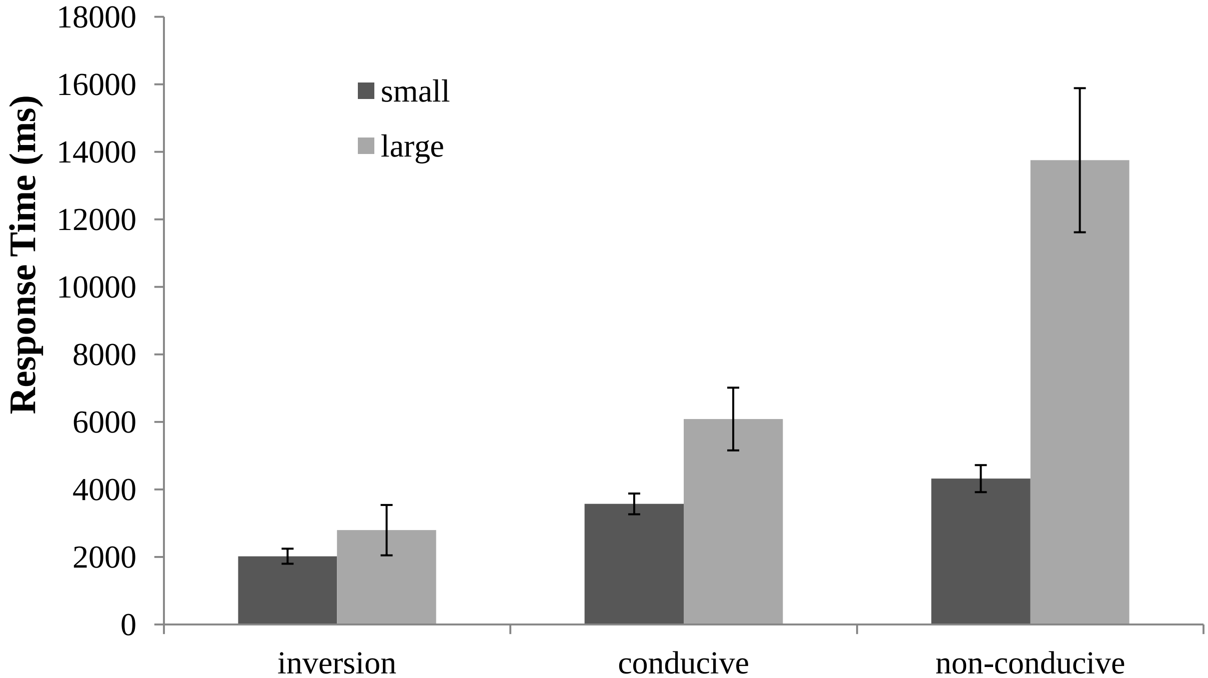

Figure 1. *Mean response time (ms) on each size and type of arithmetic problem. Error bars correspond to one standard error around the mean.*

Accuracy on arithmetic problems

The mean percentage of correct responses was 95%. With a mean accuracy this high overall variability could be reduced by a ceiling effect. A mixed design 2 (problem size: small and large) x 3 (problem type: inversion, conducive standard, non-conducive standard) ANOVA with sex and order as between-subjects variables was carried out on accuracy. The mean accuracy (%) for each problem size and type, in both orders, is given in Table 2. Accuracy on large problems (*M* = 89.5%) was lower than accuracy on small problems (*M* = 97.54%), *F*(1, 38) = 31.18, *MSE* = 133.01, $p < 0.001$, *ηp2* = 0.45. Accuracy on inversion problems (*M* = 98.45%) did not differ from accuracy on conducive standard problems (*M* = 94.60%), but accuracy on non-conducive standard problems (*M* = 87.5%) was lower than on the other two problem types, *F*(2, 76) = 15.27, *MSE* = 170.14, $p < 0.001$, $\eta_p^2 = 0.29$, HSD = 6.82. Participants were more accurate on order 2 (*M* = 96%) than on order 1 (*M* = 94%), *F*(1, 38) = 4.46, *MSE* = 216.41, $p = 0.041$, $\eta_p^2 = 0.10$. There was a problem size by problem type interaction, *F*(2, 76) = 11.92, *MSE* = 132.87, $p < 0.001$, *ηp2* = 0.24, HSD = 7.36 (see Figure 2). Accuracy was lower on large non-conducive problems than on large conducive problems, large inversion problems, and on small non-conducive problems (see Figure 2). There was a problem size by order interaction, *F*(1, 38) = 4.47, *MSE* = 133.01, $p = 0.041$, *ηp2* = 0.10, HSD = 6.73. Accuracy on large order 1 problems (*M* = 85.9%) was lower than accuracy on small order 1 problems (*M* = 97.11%), and large order 2 problems (*M* = 92.9 %). There were no significant contrasts involving accuracy on small order 2 problems (*M* = 97.95%). The reduced accuracy on large order 1 problems could be because in order 1, an inversion problem was presented first, which may have primed participants to not expect to do calculations when necessary. No main effects or interactions were found for sex. In summary, participants had difficulty responding accurately to large problems, non-conducive standard problems, and especially large non-conducive standard problems.

Table 2
Mean accuracy (%), on every type and size of problem for order 1 and order 2.

	Accuracy							
	Order 1				Order 2			
	Small		Large		Small		Large	
	M	*SE*	*M*	*SE*	*M*	*SE*	*M*	*SE*
Inversion	98.5	1.09	98.5	1.09	99.5	0.4	97.3	0.98
Standard conducive	97.0	2.19	89.2	3.3	97.3	1.5	94.7	1.68
Standard non-conducive	95.8	2.06	70.0	6.3	97.0	1.4	86.4	4.7

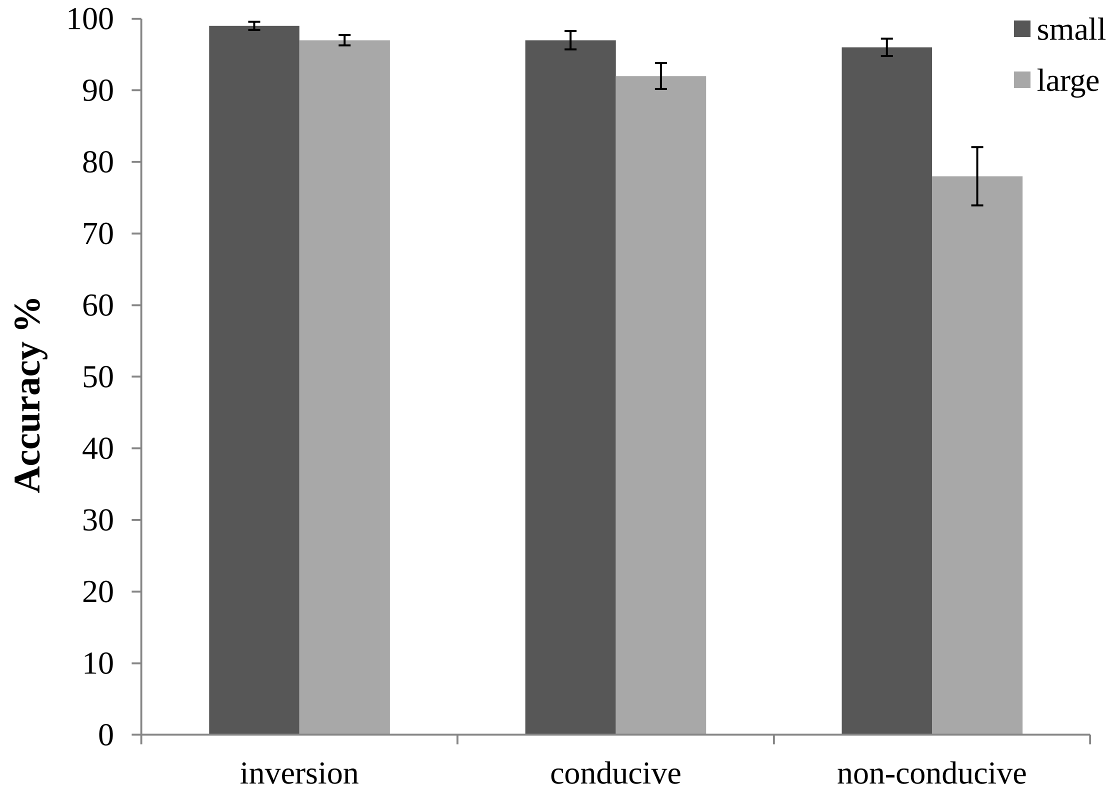

Figure 2. *Mean accuracy (%) on each size and type of arithmetic problem. Error bars correspond to one standard error around the mean.*

Overall strategy use on arithmetic problems

Participants' solution strategies were coded as either left-to-right, negation, inversion, or associativity. Solution strategies for inversion problems were classified as inversion on 72.3% of trials, as negation on 13.2% of trials, and as left-to-right on 13.9% of trials (see Figure 3). Solution strategies for standard problems were classified as left-to-right on 74.8% of trials (40% of non-conducive trials and 34.8% of conducive trials), and as associativity on 24.9% of trials (4.9% of non-conducive trials and 20% of conducive trials; see Figure 4). Participants were unable to describe their solution strategy on 0.1% of problems and their solution descriptions were too ambiguous to classify on 0.2% of problems.

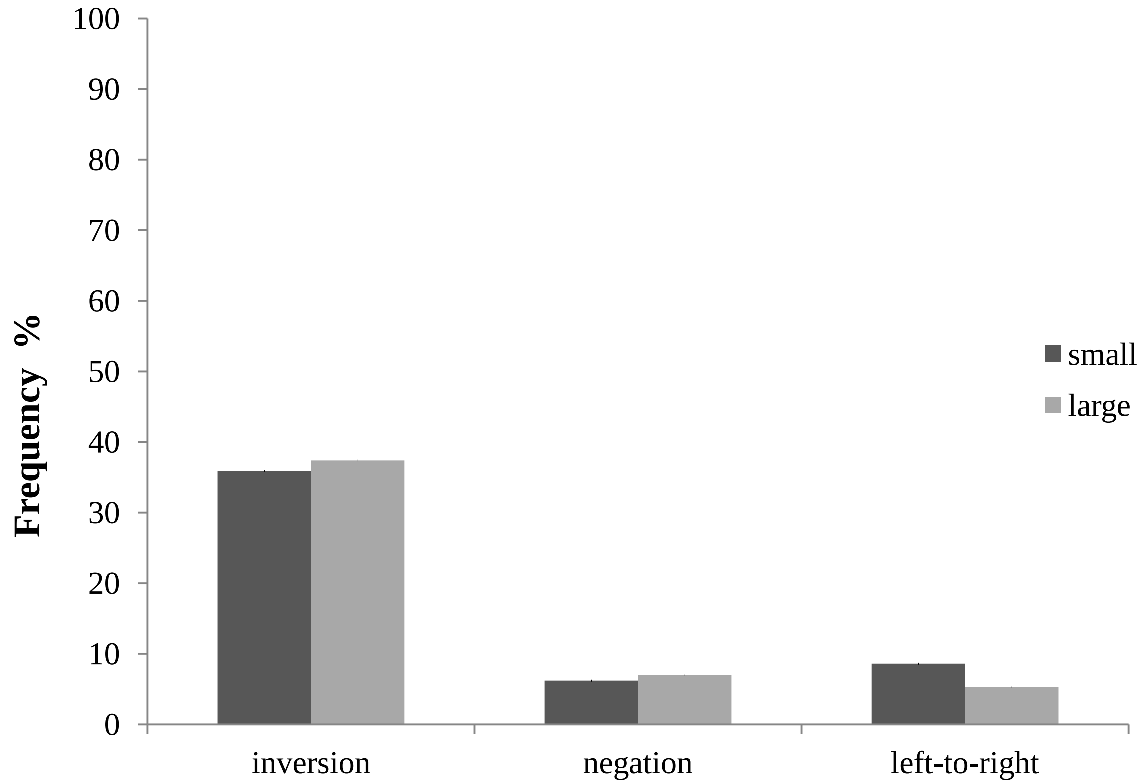

Figure 3. *The frequency (%) of each solution strategy on each size of inversion problem.*

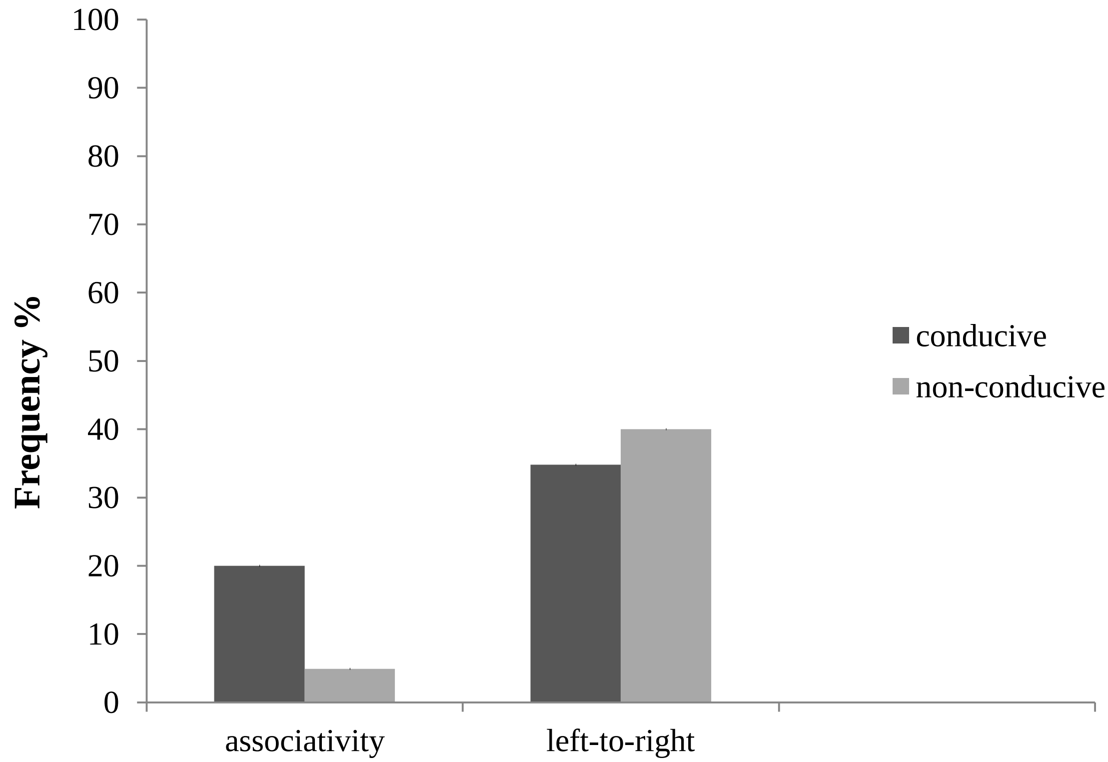

Figure 4. *The frequency (%) of each solution strategy on each type of standard problem.*

Strategy use on inversion problems

A mixed design 2 (problem size: small and large) ANOVA with sex and order as between-subjects variables was carried out on total inversion shortcut use on inversion problems. The inversion shortcut was used almost as often on large problems (74.8%) as on small problems (69.8%), although this effect only approached significance, $F(1, 38) = 3.84$, $MSE = 127.06$, $p = 0.058$, $\eta_p^2 = 0.09$. No main effect or interactions were found for sex or order.

Strategy use on standard problems

A mixed design 2 (problem size: small and large) x 2 (problem type: conducive standard and non-conducive standard) ANOVA with sex and order as between-subjects variables was carried out on associativity use on standard problems. The associativity shortcut was used more often on large problems ($M = 29.6\%$) than on small problems ($M = 19.3\%$), $F(1, 38) = 14.64$, $MSE = 299.96$, $p < 0.001$, $\eta_p^2 = 0.28$. The associativity shortcut was used more often on conducive problems ($M = 36\%$) than on non-conducive problems ($M = 12.9\%$), $F(1, 38) = 27.6$, $MSE = 839.67$, $p < 0.001$, $\eta_p^2 = 0.42$. Problem size and problem type did not interact. No main effects or interactions were found for sex or order. In summary, participants had a tendency to use associativity in an adaptive manner on large problems where harder calculations are required, and on conducive problems.

Assessing the validity of verbal reports

When the inversion shortcut is used on inversion problems it should result in greater speed than the left-to-right strategy, since the first number in the equation is provided as the answer, and no calculations are required (Bisanz & LeFevre, 1990). The inversion shortcut should also result in greater accuracy than the left-to-right strategy since there is less room for error when no calculations are needed (Bisanz & LeFevre, 1990). This implies that the negation strategy will be in an intermediate position for speed and accuracy. Furthermore, problem size should not affect speed and accuracy when the inversion shortcut is being used, but the left-to-right strategy should result in faster solutions on small problems than large problems (Ashcraft, 1992). This is because the size of the numbers in the problem should not matter when no calculations are being carried out with those numbers. In order to test these assumptions a factorial 2 (problem size: small and large) by 2 (sex: male and female) by 3 (solution strategy: inversion, negation, left-to-right) ANOVA was carried out on response time, across all inversion problem trials. The mean response times for each solution strategy and problem size, collapsed across males and females, can be found in Table 3. Sex is not of theoretical interest for validating the verbal reports but it has already been shown to affect response time and interact with problem size to affect response time, so it was included as a factor in this ANOVA. Response time on small inversion problems was shorter than on large inversion problems (see Figure 1), $F(1, 821) = 20.08$, $MSE = 28531019.43$, $p < 0.001$, $\eta p2 = 0.024$. The inversion strategy ($M = 1618$ ms) produced faster responses than the negation strategy ($M = 3364$ ms) which produced faster responses than the left-to-right strategy ($M = 5642$ ms), $F(2, 821) = 35.06$, $MSE = 28531019.43$, $p < 0.001$, $\eta p2 = 0.08$, HSD = 1417. Males ($M = 1778$ ms) were faster than females ($M = 2982$ ms) on inversion problems, $F(2, 821) = 27.73$, $MSE = 28531019.43$, $p < 0.001$, $\eta p2 = 0.03$. There was a solution strategy by problem size interaction, $F(2, 821) = 11.82$, $MSE = 28531019.43$, $p < 0.001$, $\eta p2 = 0.03$, HSD = 2471 (see Figure 5). The left-to-right strategy produced slower response times on large inversion problems than the inversion strategy and negation strategy. The left-to-right strategy also produced slower responses on large inversion problems than it did on small inversion problems (see Figure 5). There was a sex by problem size interaction, $F(1, 821) = 12.51$, $MSE = 28531019.43$, $p > 0.001$, $\eta p2 = 0.01$, HSD = 1345. When females solved large inversion problems (M = 3614 ms) response times were slower than when males solved large inversion problems (M = 1913 ms). No significant contrasts were found involving male response times solving small inversion problems (M = 1641 ms) or female response times solving small inversion problems (M = 2349 ms). There was a solution strategy by sex interaction, $F(2, 821) = 18.94$, $MSE =$ 28531019.43, $p > 0.001$, $\eta p2 = 0.04$, HSD = 2492. Females produced slower response times using the left-to-right strategy (7928 ms) on inversion problems than they did using the negation strategy (3613 ms) or the inversion strategy (1754 ms). Females also produced slower response times using the left-to-right strategy on inversion problems than males did using the left-to-right strategy (2625 ms). No significant contrasts were found involving male response time to inversion problems using the negation strategy (2937 ms), or inversion strategy (1482 ms). There was a sex by problem size by problem type interaction, $F(2, 821) = 11.11$, $MSE = 28531019.43$, $p > 0.001$, $\eta p2 = 0.03$, HSD = 4101 (see table 4). Female response time to large inversion problems using the left-to-right strategy was slower than female response time to small inversion problems using the left-to-right strategy, male response time to large inversion problems using the left-to-right strategy, female response time to large inversion problems using the negation strategy, and female response time to large inversion problems using the inversion strategy (see table 4). The inversion strategy offers an advantage over the negation strategy which offers an advantage over the left-to-right strategy. This is consistent with valid verbal reports. As expected, problem size affected response time only when the left-to-right solution strategy was used, as shown by the problem size by solution strategy interaction effect (see Figure 5). The main effect and interactions for sex are not interpreted here because they are not of theoretical interest for validating verbal reports.

Table 3
Mean response time (milliseconds), for each solution strategy and problem size collapsed across males and females.

Response Time on Inversion Problems

	Small		Large	
	M	*SE*	*M*	*SE*
Inversion	1483	63.33	1743	121.76
Negation	2868	244.20	3800	262.57
Left-to-right	3469	433.89	8945	3378

Table 4
Mean response time (milliseconds), for each solution strategy and problem size for males and females.

	Males				Females			
	Small		Large		Small		Large	
	M	*SE*	*M*	*SE*	*M*	*SE*	*M*	*SE*
Inversion	1402	69.00	1561	90.79	1570	108.42	1915	220.37
Negation	2429	354.73	3334	310.4	3100	314.97	4097	376.89
Left-to-right	2447	176.65	2834	254.37	4266	678.72	15250	6634.83

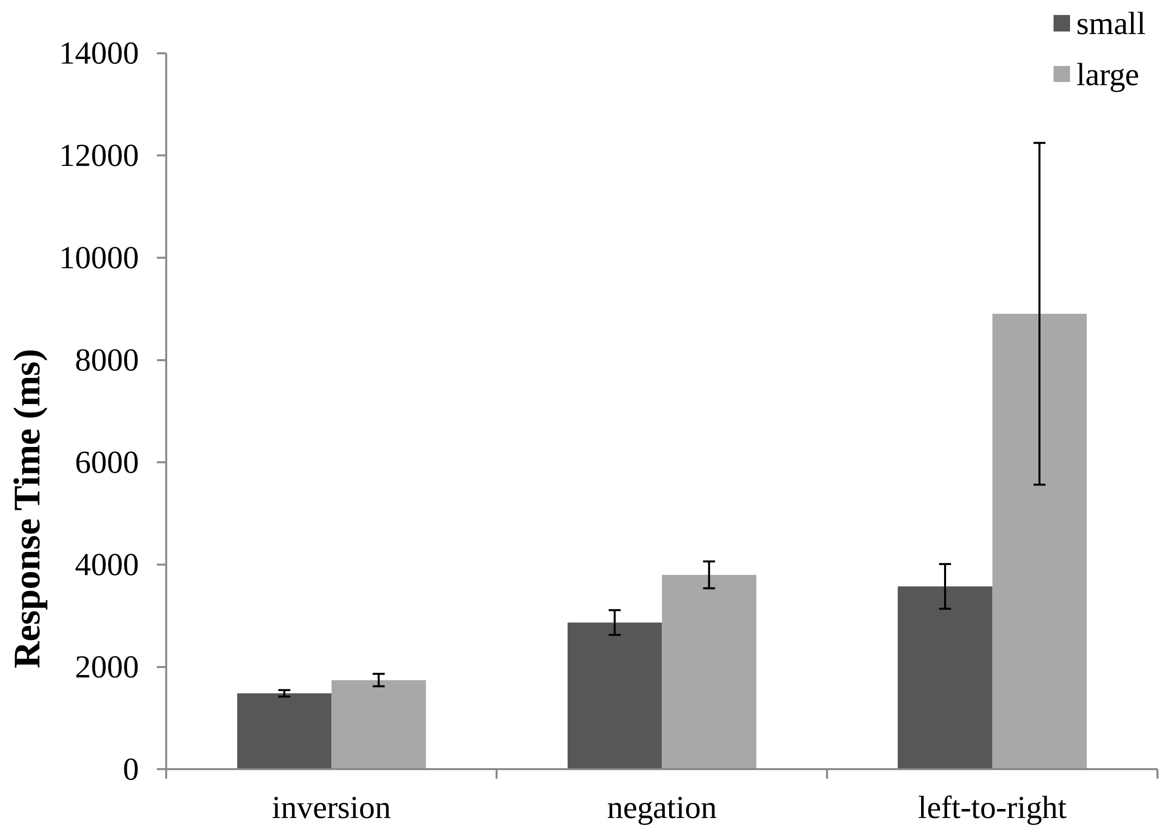

Figure 5. *Mean response time (ms) on each size of inversion problem using each kind of solution strategy. Error bars correspond to one standard error around the mean.*

A 3 (solution strategy: inversion, negation, left-to-right) by 2 (problem size: small and large) factorial ANOVA was carried out on accuracy across all inversion problem trials. The mean accuracies for each solution strategy and problem size can be found in Table 5. Accuracy on small inversion problems (M = 99%) was greater than on large inversion problems (M = 98%), $F(1, 826) = 8.59$, $MSE = 0.01$, $p = 0.003$, $\eta p2 = 0.01$. The inversion shortcut (M = 100%) was not reliably more accurate than the negation strategy (M = 98%) but both were more accurate than the left-to-right strategy (M = 94%), $F(2, 826) = 23.17$, $MSE = 0.01$, $p < 0.001$, $\eta p2 = 0.053$, HSD = 2.66. There was a problem size by solution strategy interaction, $F(2, 826) = 8.74$, $MSE = 0.01$, $p < 0.001$, $\eta p2 = 0.021$, HSD = 4.6 (see Figure 6). Accuracy was lower when the left-to-right strategy was used on large problems than when it was used on small problems. Accuracy was also lower when the left-to-right strategy was used on large problems than when the inversion strategy or negation strategy were used on large problems (see Figure 6). The higher accuracy of the inversion and negation strategies is consistent with valid verbal reports. As expected, problem size affected response time only when the left-to-right strategy was used, as shown by the problem size by solution strategy interaction effect (see Figure 6).

Table 5
Mean accuracy (%), for each solution strategy and problem size.
Accuracy on Inversion Problems

	Small		Large	
	M	*SE*	*M*	*SE*
Inversion	100	0	100	0
Negation	98	1.96	98	1.69
Left-to-right	97	1.94	89	4.84

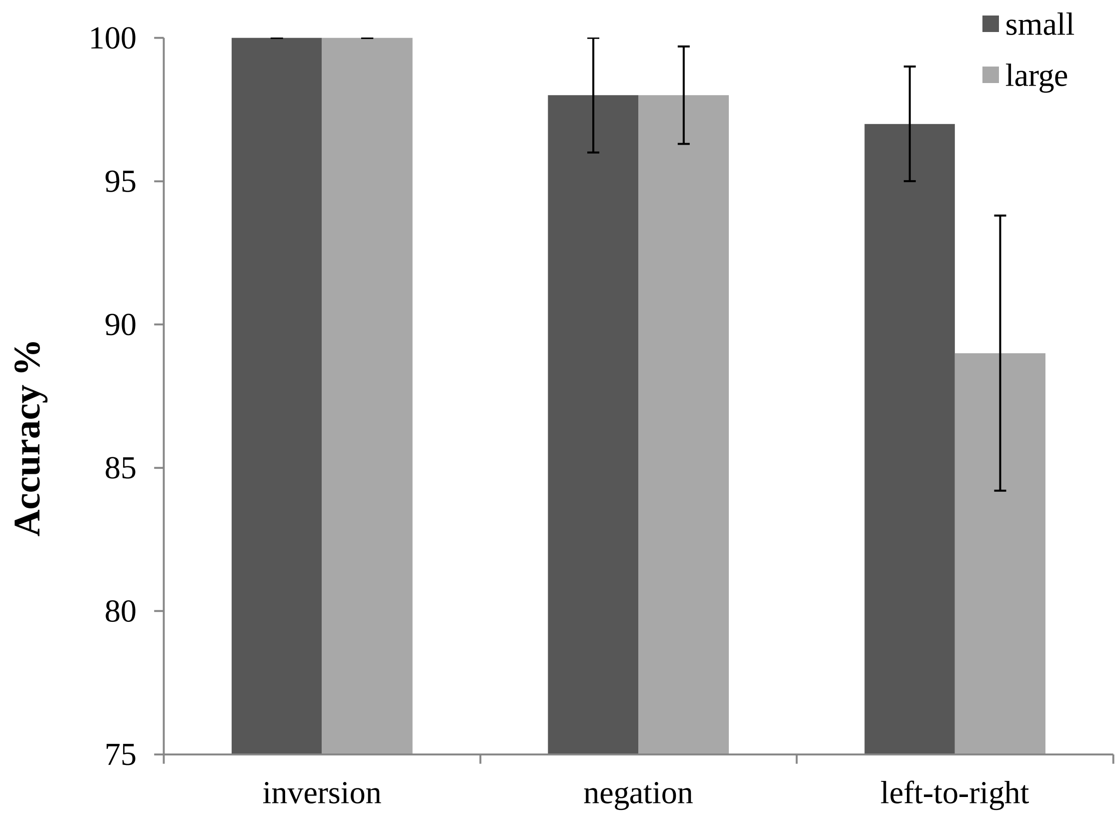

Figure 6. *Mean accuracy (%) on each size of inversion problem using each kind of solution strategy. Error bars correspond to one standard error around the mean.*

The associativity strategy should result in increased speed and accuracy when used on problems that are conducive to associativity (Robinson, Ninowski & Gray, 2006). A 2 (solution strategy: left-to-right and associativity) by 2 (conduciveness: conducive to associativity and not conducive) by 2 (sex: male and female) ANOVA was carried out on response time across all standard problem trials. The mean response times for each solution strategy and problem size can be found in Table 6. Sex was included in this model because it was previously shown to affect response time. Participants were just as fast using the associativity strategy (*M* = 4820 ms) as they were using the left-to-right strategy (*M* = 6447 ms), $F(1, 828) = 0.18$, $MSE = 83988993.56$, $p = 0.675$, $\eta_p^2 < 0.001$. Participants were faster on conducive problems (*M* = 4852 ms) than on non-conducive problems (*M* = 7487 ms), $F(1, 828) = 18.80$, $MSE = 83988993.56$, $p < 0.001$, $\eta_p^2 = 0.022$. Males (*M* = 4275 ms) were faster than females (*M* = 7660 ms) on standard problems, $F(1, 828) = 6.07$, $MSE = 83988993.56$, $p = 0.014$, $\eta_p^2 = 0.007$. There was a solution strategy by conduciveness interaction effect, $F(1, 828) = 6.95$, $MSE = 83988993.56$, $p = 0.009$, $\eta_p^2 = 0.008$, HSD = 3189. The associativity strategy on non-conducive problems produced slower responses than the associativity strategy on conducive problems (see Figure 7). This shows that the associativity strategy offered an advantage on conducive problems and a disadvantage on non-conducive problems, which is consistent with valid verbal reports. An interaction was found between solution strategy and sex, $F(1, 828) = 5.54$, $MSE = 83988993.56$, $p = 0.019$, $\eta_p^2 = 0.007$, HSD = 2658. Response times were equally fast when males used the left-to-right strategy (M = 4161 ms) or the associativity strategy (M = 4601 ms), but when females used the left-to-right strategy (M = 8504 ms) response times were slower than when females used the associativity strategy (M = 5032 ms) or when males used the left-to-right strategy. No other interactions were found involving sex. In summary, associativity offered an advantage on conducive problems compared to non-conducive problems and participants were faster on conducive problems, which is consistent with valid verbal reports. Conversely associativity did not offer an advantage over the left-to-right strategy, but this could be explained by the fact that both strategies require calculations. The main effect and interaction for sex are not interpreted here because they are not of theoretical interest for validating verbal reports.

Table 6
Mean response time (milliseconds), for each solution strategy and type of standard problem.

Response Time on Standard Problems

	Conducive		Non-conducive	
	M	*SE*	*M*	*SE*
Associativity	3593	263.097	9843	1388
Left-to-right	5578	517.79	7199	631.24

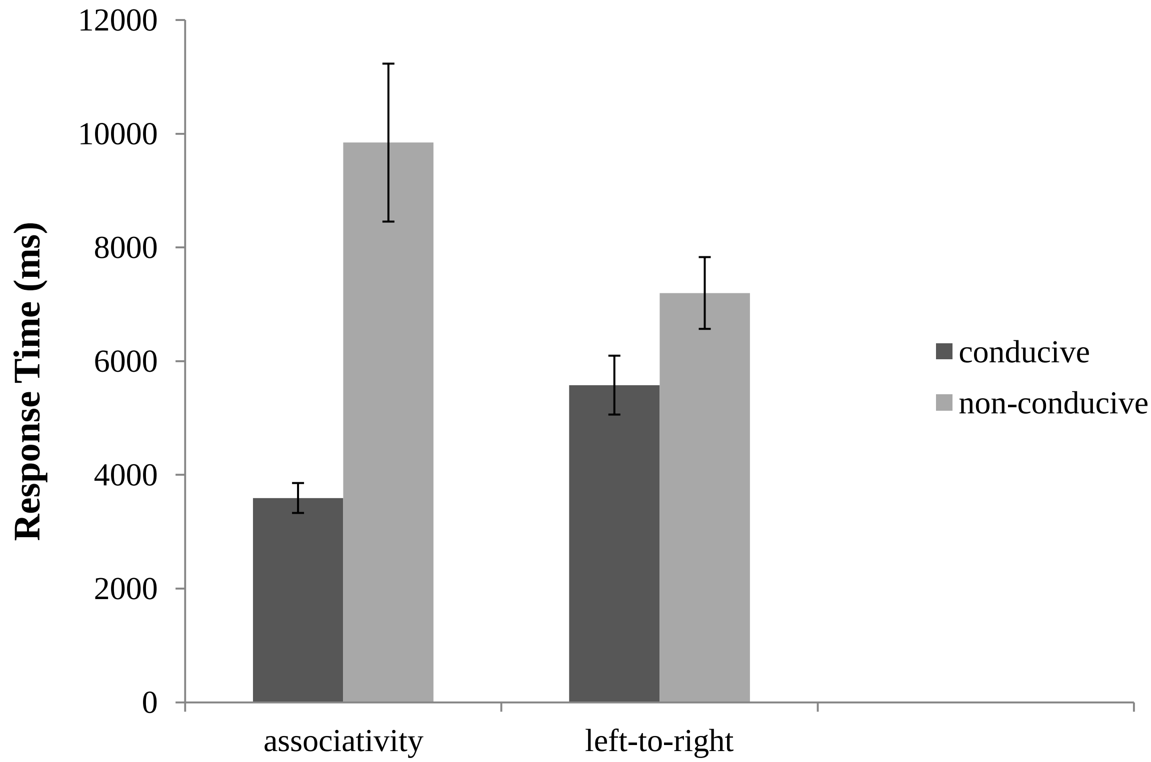

Figure 7. *Mean response time (ms) on each type of standard problem using each kind of solution strategy. Error bars correspond to one standard error around the mean.*

A 2 (solution strategy: left-to-right and associativity) by 2 (conduciveness: conducive to associativity and not conducive) ANOVA was carried out on accuracy across all standard problem trials. The mean accuracies for each solution strategy and problem size can be found in Table 7. The associativity strategy (*M* = 96%) was just as accurate as the left-to-right strategy (*M* = 92%), $F(1, 832) < 0.001$, *MSE* = 0.064, $p = 0.996$, $\eta_p^2 < 0.001$. Accuracy was higher on conducive problems (*M* = 95%) than non-conducive problems (*M* = 91%), $F(1, 832) = 9.41$, *MSE* = 0.064, $p = 0.002$, $\eta_p^2 = 0.011$. There was a solution strategy by conduciveness interaction effect, $F(1, 832) = 6.17$, *MSE* = 0.064, $p = 0.013$, $\eta_p^2 = 0.007$, (HSD = 8.8). The associativity strategy on conducive problems was more accurate than the associativity strategy on non-conducive problems but no differences were found involving the left-to-right strategy on either conducive or non-conducive problems, (see Figure 8). These results are consistent with valid self-report data. In summary, associativity offered an advantage on conducive problems compared to non-conducive problems and participants were more accurate on conducive problems, which is consistent with valid verbal reports. Conversely associativity did not offer an advantage over the left-to-right strategy, but this could be explained by the fact that both strategies require calculations.

Table 7

Mean accuracy (%), for each solution strategy and type of standard problem.

Accuracy on Standard Problems

	Conducive		Non-conducive	
	M	*SE*	*M*	*SE*
Associativity	99%	0.77	85%	5.62
Left-to-right	93%	1.52	91%	1.53

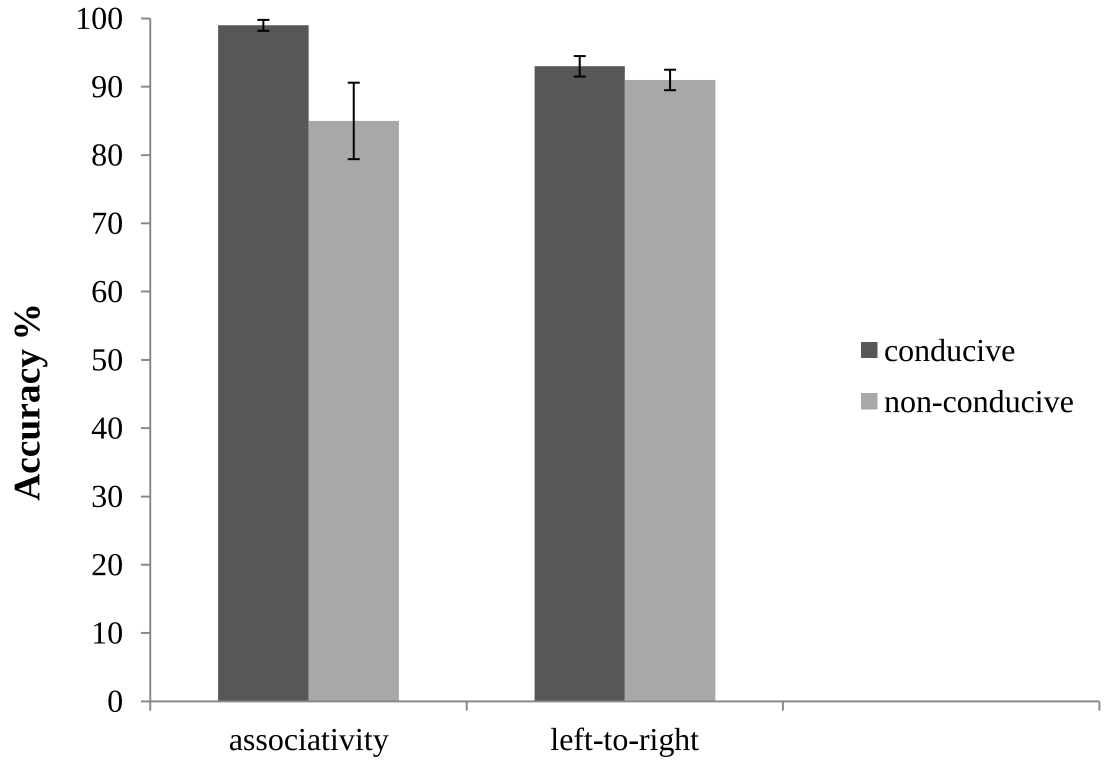

Figure 8. *Mean accuracy (%) on each type of standard problem using each kind of solution strategy. Error bars correspond to one standard error around the mean.*

Descriptive statistics for cognitive components and correlations with conceptual strategy use and arithmetic performance

The mean percentage of correct responses and mean finishing time in minutes for each cognitive task is given in Table 8. The inter-correlation matrix for the cognitive tasks and the outcome measures on the arithmetic problems was obtained (see Table 9). The frequency of inversion use, frequency of associativity use, along with response time and accuracy across all arithmetic problems are included in this correlation matrix. Response time using the left-to-right strategy was included in this correlation matrix. Response time using a default strategy (left-to-right) can serve as a measure of computational ability, and slower left-to-right responses times can be indicative of procedural and factual errors (Bajic & Richard, 2009; Dube & Robinson, 2010a). Since the associativity strategy only offered an advantage on conducive standard problems it may not be appropriate to consider a participant's total use of associativity across all standard problems as indicative of conceptual knowledge. Conversely, a participant's capacity for switching back and forth between left-to-right and associativity as needed could be a better measure of conceptual knowledge than total associativity use. For this reason Simpson's Diversity Index was used to calculate the diversity of problem solving approaches (left-to-right and associativity) used on standard problems for each participant. The diversity statistic, *D*, gives the probability of two randomly selected solution strategies (on two randomly selected standard problems) being the exact same for a given participant (Simpson, 1949). In order to ease interpretation, the diversity statistic was subtracted from 1 for each participant (1 - *D*) so that higher values would correspond to balanced use of both solution strategies. The diversity of solution strategies on standard problems was included in the correlation matrix.

In terms of the simple bivariate correlations it can be seen that the three spatial measures are all positively correlated with one another, which is indicative of general spatial intelligence. The SS-2 (choosing a path) and SS-3 (map planning) predict speed on arithmetic problems collapsed across all solution strategies, and just using the left-to-right strategy. The SS-3 is shown to predict conceptual knowledge along with accuracy on arithmetic problems. The VZ-2 (paper folding) and CCTT (verbal reasoning) predicted response time using the left-to-right strategy but not response time collapsed across all solution strategies. The CCTT negatively correlated with associativity strategy use. In summary, the individual measures of spatial ability suggest an underlying general spatial ability. Spatial ability is correlated with performance on arithmetic problems and some measures of conceptual knowledge. Performance on arithmetic problems is correlated with measures of conceptual knowledge. Verbal reasoning predicts response time using the left-to-right strategy and is negatively correlated with use of the associativity shortcut.

Table 8
The mean percentage of accurate responses and mean finishing time in minutes on each cognitive task.

Performance on Cognitive Measures

	Accuracy		Finishing Time	
	M	*SE*	*M*	*SE*
SS-3 Map Planning	72%	3.09	5.93	0.02
SS-2 Choosing a Path*	37%	2.81	13.96	0.03
VZ-2 Paper folding	50%	3.70	5.64	0.07
CCTT Deductive Reasoning	65%	2.62	4.37	0.02

*One participant was unable to complete the SS-2 and these data were excluded pair-wise for all correlation and regression analyses.

Table 9
Intercorrelation matrix.

	SS-3	SS-2	VZ-2	CCTT	A	I	D	AA	ART	LRRT
SS-3		0.36*	0.48**	0.06	0.32*	0.25	0.38*	0.36*	-0.39**	-0.42**
SS-2			0.68**	0.20	-0.05	0.27	-0.04	0.11	-0.33*	-0.31*
VZ-2				0.17	-0.09	0.24	0.25	0.29	-0.27	-0.37*
CCTT					-0.41**	-0.06	-0.07	0.09	-0.26	-0.32*
Associativity (A)						0.51**	0.65**	0.22	-0.23	-0.13
Inversion (I)							0.45**	0.20	-0.43**	-0.26
Diversity (D)								0.28	-0.35*	-0.25
Arithmetic Accuracy (AA)									-0.68**	-0.71**
Arithmetic Mean Response Time (ART)										0.90**
Left-To-Right Mean Response Time (LRRT)										

**, Correlation is significant at the .01 level two tailed. *, Correlation is significant at the .05 level two tailed. In this table "Associativity" refers to a participant's total use of the associativity shortcut, and "Inversion" refers to their total use of the inversion shortcut. "Diversity" refers to the diversity of problem solving strategies on standard problems. The bivariate inter-correlations between these and other variables are given here.

Predicting conceptual strategy use and arithmetic performance with cognitive components

Five stepwise multiple regression analyses were carried out with the following dependent variables: (1) mean response time on arithmetic problems, (2) total accuracy on arithmetic problems, (3) total use of the inversion shortcut, (4) diversity of solution strategies on standard problems, and (5) mean response time on arithmetic problems using the left-to-right strategy. In each analysis four predictor variables were used; score on the (1) CCTT (verbal reasoning), score on the (2) SS-2 (choosing a path), score on the (3) SS-3 (map planning), and score on the (4) VZ-2 (paper folding). SS-3 (map planning) uniquely predicted response time, and accuracy, on arithmetic problems with none of the other predictor variables accounting for additional variance. SS-3 (map planning) uniquely predicted diversity of strategy use with none of the other predictor variables accounting for additional variance. SS-3 (map planning) and CCTT (verbal reasoning) uniquely predicted response time using the left-to-right strategy, with none of the other predictor variables accounting for additional variance. These regression analyses can be found in Table 10. Since the spatial measures all positively correlated with one another (Table 8), a general spatial measure was derived (mean score across all three spatial tasks with each task given equal weight). The stepwise multiple regression analyses from above were carried out again with general spatial ability entered into the model as a predictor. General spatial ability uniquely predicted use of the inversion shortcut, response time collapsed across all solution strategies, and response time using the left-to-right strategy, with none of the other predictors accounting for additional variance. Map planning continued to uniquely predict arithmetic accuracy, and diversity of solution strategies, with no other predictors accounting for additional variance. The regression analyses with general spatial intelligence included as a predictor can be found in Table 11.

Table 10

Regression analyses without general spatial intelligence.

Variable	Predictor	β	p
Stepwise: Solution Strategy Diversity			
Model 1: $F(1, 39) = 6.68$, $R^2 = .15$, $p = .014$	Map Planning	.38	.014
Stepwise: Response Time			
Model 1: $F(1, 39) = 7.00$, $R^2 = .15$, $p = .012$	Map Planning	-.39	.012
Stepwise: Arithmetic Accuracy			
Model 1: $F(1, 39) = 5.95$, $R^2 = .13$, $p = .019$	Map Planning	.36	.019
Stepwise: Inversion Shortcut Frequency			
(No Variables Entered Into Model)	--	--	--
Stepwise: Left-To-Right Response Time			
Model 1: $F(1, 39) = 8.30$, $R^2 = .17$, $p = .006$	Map Planning	-.42	.006
Model 2: $F(2, 38) = 6.73$, $R^2 = .26$,	Map Planning	-.40	.007
$p = .003$	Verbal Reasoning	-.29	.042

Table 11

Regression analyses with general spatial intelligence.

Variable	Predictor	β	p
Stepwise: Solution Strategy Diversity			
Model 1: $F(1, 39) = 6.68$, $R^2 = .15$, $p = .014$	Map Planning	.38	.014
Stepwise: Response Time			
Model 1: $F(1, 39) = 7.44$, $R^2 = .16$, $p = .010$	Spatial General	-.40	.010
Stepwise: Arithmetic Accuracy			
Model 1: $F(1, 39) = 5.95$, $R^2 = .13$, $p = .019$	Map Planning	.36	.019
Stepwise: Inversion Shortcut Frequency			
Model 1: $F(1, 39) = 5.29$, $R^2 = .12$, $p = .027$	Spatial General	.35	.027
Stepwise: Left-To-Right Response Time			
Model 1: $F(1, 39) = 9.65$, $R^2 = .20$, $p = .004$	Spatial General	-.44	.004

Exploring the relationship between verbal reasoning and associativity strategy use

The negative correlation between CCTT score and use of the associativity strategy was an unexpected result. It has already been demonstrated that the associativity strategy is not advantageous on non-conducive standard problems, so this correlation could be an indicator that participants with low deductive reasoning ability use the associativity strategy when it is not advantageous. It was hypothesized that the negative correlation is primarily due to an association between CCTT score and use of the associativity strategy on non-conducive problems. Controlling for associativity use on conducive problems, CCTT score was negatively correlated with associativity use on non-conducive problems, *partial* $r = -.35$, $p = .02$. Controlling for associativity use on non-conducive problems, CCTT score was not correlated with associativity use on conducive problems, *partial* $r = -.13$, $p = .41$. This result is consistent with the hypothesis that CCTT score is primarily related to excessive associativity strategy use on non-conducive problems.

Two participants spontaneously reported that “BEDMAS had to be followed” (division must be done first) and they were particularly prone to using the associativity strategy across all standard problem trials (both participants were above the 95th percentile for associativity use, and were the highest and second highest associativity users respectively). Both participants were also low scorers on the CCTT (5th and 12th percentiles respectively). In order to further assess the hypothesis that low deductive reasoning ability leads to excessive associativity use, the correlation between associativity use and CCTT score was recalculated with the data from these two participants excluded. With the data from these two participants removed there was no longer a significant bivariate correlation between CCTT score and associativity use, $r = -.23$, $p = .15$. This finding is consistent with the theory that low deductive reasoning ability results in inefficient use of the associativity strategy on non-conducive problems.

DISCUSSION

Overview of main findings

It was hypothesized that spatial ability would be positively related to accuracy on arithmetic problems and negatively related to response time on arithmetic problems. This result was confirmed. However, it was also hypothesized that verbal reasoning ability would be related to speed and accuracy on arithmetic problems, but not as strongly as spatial ability. Verbal reasoning ability was neither related to accuracy on arithmetic problems, nor average response time across all arithmetic problems. It was hypothesized that spatial ability would be positively correlated with use of the inversion shortcut and use of the associativity strategy. Map planning had a significant bivariate correlation with associativity use, but the other two measures of spatial ability did not. The prediction of associativity with cognitive components was not evaluated with regression analysis, because unexpected outcomes in the results made the status of associativity as a measure of conceptual knowledge questionable. No individual measures of spatial ability were positively correlated with use of the inversion shortcut, but general spatial ability was positively correlated with use of the inversion shortcut. It was hypothesized that verbal reasoning would be positively correlated with use of associativity and inversion but not as strongly as spatial ability. To the contrary it was found that verbal reasoning did not predict use of the inversion shortcut, but was negatively correlated with use of the associativity shortcut. It was hypothesized that verbal reasoning and spatial ability would both be equally positively correlated with response time to arithmetic problems using the left-to-right strategy. All three measures of spatial ability were significantly correlated with response time using the left-to-right strategy with correlations ranging from $r = .31$ to $r = .42$. Verbal reasoning was positively correlated with response time using the left-to-right strategy, $r = .32$. These bivariate correlations suggest that verbal reasoning ability and spatial ability are both equally related to response time using the left-to-right strategy. Stepwise multiple regression demonstrated that map planning was able to account for 17% of the variance in response time using the left-to-right strategy and verbal reasoning was able to contribute an additional 9% of the variance. It was demonstrated that general spatial ability was able to account for 20% of the variance in response time using the left-to-right strategy with no other variables accounting for additional variance. These results suggest that although verbal reasoning and spatial ability are both related to response time using the left-to-right strategy, spatial ability is more strongly related than verbal reasoning ability.

There were unexpected results with respect to participant's use of the associativity strategy. Associativity use was not related to shorter response times or increased accuracy on arithmetic problems. A particularly unexpected result was that associativity use was negatively correlated with verbal deductive reasoning. During data collection two participants spontaneously reported that "BEDMAS had to be followed." These two participants displayed a relatively rigid tendency to solve problems from right-to-left, and this appeared to be their default problem solving strategy. Until now this type of problem solving behaviour has not been reported in participants (Dubé & Robinson, 2010b; Robinson & Dubé, 2009b). After obtaining this finding it was theorized that a participant's tendency to shift back and forth between left-to-right and associativity would be a better measure of conceptual knowledge of arithmetic. Simpson's diversity index was used to measure the diversity of problem solving strategies that a participant used on standard problems. Diversity was negatively correlated with response time to arithmetic problems, and positively correlated with use of the inversion shortcut, which would suggest that it is a better measure of conceptual knowledge than associativity use for this sample. It was hypothesized that the negative correlation between associativity use and verbal deductive reasoning was primarily related to the inappropriate use of a default right-to-left strategy on non-conducive problems. It was confirmed that verbal deductive reasoning was negatively correlated with use of the associativity shortcut on non-conducive problems while controlling for the use of associativity on conducive problems. However, verbal deductive reasoning was not correlated with the use of associativity on conducive problems while controlling for its use on non-conducive problems. It was confirmed that when the data for the two participants who used a default right-to-left strategy was removed, associativity was no longer negatively correlated with verbal deductive reasoning. The hypothesis that verbal deductive reasoning was negatively related to inappropriate use of associativity on non-conducive problems was thus confirmed.

Interpretation of main findings

The findings in this study do not completely fit the hypotheses, but they would seem to fit the proposed theory of the relationship between cognitive components and mathematical cognition. The theory proposed in this research study states that spatial ability will be more important than other cognitive components for acquiring new mathematical knowledge, but general cognitive ability will be more important for applying well learned mathematical skills. Spatial ability predicted raw measures of arithmetical performance, as well as measures of conceptual knowledge like inversion and diversity. Previous research has demonstrated that spatial ability is related to university student's performance on arithmetic tasks while verbal ability is not (Visser et al., 2006). However, past research had indicated that verbal ability was related to university students' ability to grasp mathematical concepts, but not as strongly as spatial ability (Wei et al., 2012). Conversely, there is a body of literature suggesting that crystalized intelligence (assessed with measures that are verbal in nature) is more strongly correlated with mathematical ability than spatial ability in the general population (Floyd, Evans, & McGrew, 2003; McGrew & Hessler, 1995). Taken together, previous literature would suggest that verbal and spatial abilities could both be related to arithmetical performance and conceptual knowledge in university students, but spatial ability would be a stronger predictor. The proposed theoretical framework would also lead to the prediction that spatial ability should be a stronger predictor of an individual's ability to acquire knowledge of inversion and associativity, but verbal reasoning and spatial ability should both be related to an individual's ability to apply knowledge of those concepts after they have acquired that knowledge. Contrary to expectations, verbal reasoning made no contribution to performance on arithmetic problems while spatial ability did. This result is consistent with some previous findings (Visser et al., 2006). Similarly it was found that spatial ability was related to measures of conceptual knowledge but verbal reasoning was not related at all. This later finding mirrors the results of some research in children (Laski et al., 2013). These findings support the proposed theory, but they indicate a larger gap between spatial and verbal abilities for acquiring conceptual knowledge of mathematics than was anticipated. Verbal reasoning ability was related to response time using the left-to-right strategy, and the use of associativity on non-conducive standard problems. Since it was theorized that verbal reasoning would be more closely related to the application of well learned mathematical knowledge, it was hypothesized that verbal reasoning and spatial ability would be equally related to response time using the left-to-right strategy. However verbal reasoning was not as strongly related to response time using the left-to-right strategy as spatial ability. This could be due to some participants applying unmeasured mathematical knowledge while using the left-to-right strategy. Participants with lower verbal reasoning ability were also more likely to use the associativity strategy on non-conducive problems in a way that was not adaptive. This latter result was not originally anticipated. Although these results with verbal reasoning are somewhat inconsistent with the original hypotheses, they are compatible with the proposed theoretical framework. Verbal reasoning was not related to measures of conceptual knowledge but it was related to response time using the left-to-right strategy, which is consistent with the idea that verbal reasoning will have more relative importance to the application of well learned mathematical knowledge than it will to the acquisition of new mathematical knowledge. Furthermore the difficulties with associativity use on non-conducive problems were due to participants using a default right-to-left strategy just as rigidly as other participants would use a default left-to-right strategy. This finding is consistent with the proposed theoretical framework if it is assumed that the rigid use of a default right-to-left strategy is indicative of difficulty in applying well learned mathematical knowledge. The two participants who used a default right-to-left strategy cited BEDMAS as a rationale, and their problem solving behaviour is indicative of a misunderstanding of the fact that BEDMAS is arbitrary in some contexts.

Inversion shortcut use

The inversion shortcut was used on 72.3% of inversion problems (see Figure 3) which is comparable to previous studies with adults (Robinson & Dubé, 2009b). The inversion shortcut led to increased speed and accuracy which is consistent with past research (Bisanz & LeFevre, 1990; Robinson & Dubé, 2009a, b, c). It was also found that response time using the left-to-right strategy did not correlate with frequency of inversion shortcut use (see Table 9). This finding is consistent with past research showing that basic calculation abilities are not related to the use of conceptually based shortcuts (Bryant, Christie & Rendu, 1999; Sherman & Bisanz, 2007). Consistent with previous research in adults (Robinson & Dubé, 2009b), the use of the inversion shortcut was correlated with use of the associativity shortcut (see Table 9). Analyses were not carried out to determine if knowledge of inversion is a necessary prerequisite for knowledge of associativity as seen in previous research (Robinson & Dubé, 2009b).

Associativity shortcut use

The associativity shortcut was used on 24.9% of standard problems (see Figure 4). This is somewhat less than in previous research in adults (Robinson & Dubé, 2009b), although this is to be expected since in some previous research (Robinson & Dubé, 2009b) all standard problems were designed to offer an advantage to using the associativity shortcut (Robinson & Dubé, 2009b). The associativity shortcut facilitated faster responses on conducive problems but led to slower responses on non-conducive problems. Furthermore, the associativity strategy was used over 4 times as often on conducive problems as on non-conducive problems (see Figure 4). This finding is consistent with theories that the associativity strategy will be selected based on how adaptive it is to a particular problem (Robinson & LeFevre, 2012). The associativity shortcut did not offer the same speed advantage as the inversion shortcut, and was not as prevalent as the inversion shortcut which is consistent with previous research (Robinson & Dubé, 2009b) and existing theoretical frameworks (Robinson & LeFevre, 2012). Previous discussions have addressed the extent to which individuals are either automatically activating the left-to-right strategy, and then inhibiting it in order to use a shortcut or scanning the problem before activating any solution strategy (Robinson, & LeFevre, 2012). This research study suggests that in a minority of adults, division will be carried out first, by default, making for a default right-to-left strategy. Two participants spontaneously reported that "BEDMAS has to be followed" and then did division first on the majority of problems. Excessively using associativity on non-conducive problems did not enhance speed and accuracy as may be expected. It can be seen that use of the associativity shortcut does not correlate with response time to arithmetic problems (see Table 9). Furthermore, using the associativity strategy on non-conducive problems was related to difficulties with verbal reasoning. It should be noted that there is an evident qualitative distinction between sparingly using the associativity strategy as needed, and using it on the majority of standard problems by default. The problem solving strategy of the two participants who did division first by default could most aptly be described as a right-to-left strategy. It is therefore problematic to classify those participants' problem solving strategies as associativity, and it is equally problematic to try and formulate a rigorous criterion for differentiating associativity from the right-to-left strategy. It was thus determined that the best way to approach this dilemma would be to classify all right-to-left problem solving strategies on standard problems as associativity but not to use associativity as a measure of conceptual knowledge. Instead, a participant's ability to flexibly switch back and forth between associativity and the left-to-right strategy, as needed, was determined to be a better measure of conceptual knowledge.

Simpson's diversity index was used to calculate a diversity statistic for each participant indicating the diversity of problem solving approaches they used on standard problems. Flexibility has long been recognized as a hallmark of conceptual understanding (Boaler, 1998a, b; Robinson & LeFevre, 2012) and for this reason the diversity of problem solving approaches is a theoretically consistent measure of conceptual understanding. The diversity of problem solving approaches on standard problems is positively correlated with use of the inversion shortcut (see Table 9), and unlike associativity but like inversion, was shown to be positively correlated with average response time to arithmetic problems (see Table 9), and like inversion was unrelated to response time using the left-to-right strategy. This pattern of correlations suggests good validity for diversity as a measure of conceptual understanding.

Spatial ability and arithmetic

The findings presented here are consistent with those in Laski et al.'s (2013) article, and support the hypotheses that were given. Spatial ability supersedes verbal ability as a predictor of advantageous solution strategies and by extension conceptual knowledge in arithmetic. These results are also consistent with previous research demonstrating the importance of spatial abilities to the acquisition of mathematical concepts in adults (Wei et al., 2012). Spatial measures were related to accuracy and speed for arithmetic problems, showing a much stronger relationship than verbal reasoning ability. This finding is consistent with a wide body of previous research demonstrating the importance of spatial ability to mathematical performance (Rohde & Thompson, 2007; Visser et al., 2006; Wei et al., 2012). Although no individual spatial measures related to use of the inversion shortcut, general spatial ability accounted for 10% of the variance in inversion shortcut use. Map planning proved to be an important predictor, accounting for 15% of the variance in the participants' tendency to switch between solution strategies on standard problems (diversity), as well as making important contributions to speed and accuracy on arithmetic problems. This is consistent with previous research (Visser et al., 2006) showing that map planning is an important predictor of arithmetical performance. The content of this test may have something to do with its importance to arithmetic performance. The map planning test presents a spatial layout of different locations on a grid and asks participants to find the most efficient route between two points. Arguably the map planning test demands that participants find a spatial shortcut, and perhaps the ability to do this is fundamentally related to the ability to find a conceptual shortcut in arithmetic. In studying the content of this test there are implications for understanding how spatial cognition affects mathematical cognition. A connection may be sought with previous literature on the importance of spatial ability for developing effective mental representations in mathematics (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). It is possible that map planning is specifically related to the cognitive resources used to develop effective visual representations in mathematics. These findings are consistent with past literature showing that spatial ability may be a powerful predictor of mathematical performance in university samples and in other advanced populations (Rohde & Thompson, 2007; Stumpf, 1994, Visser et al., 2006; Wei et al., 2012).

Verbal ability and arithmetic

Verbal reasoning correlated with response time using the left-to-right-strategy but not across all solution strategies. Using stepwise multiple regression, verbal reasoning and map planning both made significant contributions to response time using the left-to-right strategy, accounting for 26% of the variance in response times, when general spatial intelligence was not in the model. With general spatial ability in the model, verbal reasoning did not make a significant contribution to response time using the left-to-right strategy and variance accounted for decreased to 20%. The reason for this is that verbal reasoning contributes unique additional variance to the prediction of response time with the left-to-right strategy but the contribution does not reach significance at step two when general spatial intelligence is entered at step one. With this caveat in mind it would seem reasonable to conclude that both types of cognitive processing are related to procedural knowledge. Response time using the left-to-right strategy is used as a measure of procedural ability (Bajic & Richard, 2009; Dube & Robinson, 2010a). It could also be theoretically sound to think of it as a measure of basic computational ability (Dube & Robinson, 2010a). It would seem that spatial and verbal abilities can both be important when conceptual knowledge is not being used, or when conceptual knowledge is held constant. This finding is consistent with research in children showing that verbal cognitive resources are important to the use of well learned mathematical skills and principles (DeSmedt et al., 2009; McKenzie et al., 2003; Rasmussen & Bisanz, 2005). This finding bolsters the proposed theory that verbal cognitive resources are important to the use of well learned mathematical skills and principals across the lifespan.

Participants with low verbal reasoning ability used a default right-to-left strategy based on BEDMAS. This resulted in those participants' excessively using associativity on non-conducive standard problems. This result is consistent with the theory that verbal reasoning is related to the application of well learned mathematical skills and knowledge. The overuse of a default right-to-left strategy would seem to be based on fundamental difficulties with applying the BEDMAS procedure. It would also seem reasonable to assume that rigidly using BEDMAS could be a way of compensating for difficulties with mathematical knowledge and skills that are more basic than inversion or associativity. The results with respect to verbal reasoning are consistent with the theory that verbal reasoning is primarily related to the application of well learned arithmetical skills and knowledge. Since the deductive reasoning subscale of the CCTT relies on the use of hard logic to solve problems and not a priori knowledge, it can be inferred that the dissociation between verbal ability and higher level problem solving strategies observed in previous research (Laski et al., 2013) is not due to a measurement artifact.

Sex and arithmetic performance

Contrary to previous studies in this area (Robinson et al., 2006; Robinson & Dubé, 2009a, b, c) sex was related to response time on arithmetic problems. Males were faster than females. The body of research on sex differences in mathematics indicates a complex relationship where females outperform males in some areas and at some stages of development but not at others (Hyde et al., 1990). The findings in this study could be due to a cohort effect or some other idiosyncratic factor. A series of complex interactions with sex and other variables affected response time (see Appendixes D, see Table 1). These interactions are not interpreted here because they are outside the scope of this research study. What is important to this investigation is to ensure that sex's relationship with any of the variables of interest has been accounted for. Sex was used as a factor in the ANOVAs that were primarily used to test the relationship between response time and solution strategies. This ensured that the variance due to sex could be accounted for and the relationship between verbally reported solution strategy and response time could be accurately assessed. Similarly, before proceeding with the correlation and regression analyses with arithmetic variables and cognitive components, it was determined that sex was not correlated with any of the cognitive components that were assessed, nor was it correlated with measures of conceptual knowledge.

An overarching theoretical framework

It has been theorized that young children first use visual-spatial cognitive resources to learn and do math (Huttenlocher et al., 1994; Rasmussen & Bisanz, 2005). These theories state that once children learn the linguistically based number system the visually encoded mathematical information is converted to a verbal format and verbal abilities take on increased if not primary importance to mathematics (Huttenlocher et al., 1994; Rasmussen & Bisanz, 2005). Research in children is consistent with this theoretical framework (DeSmedt et al., 2009; McKenzie et al., 2003; Rasmussen & Bisanz, 2005) although studies in young adults (Sherman, 1979) and adults (Wei et al., 2012) indicate that visual-spatial cognitive resources continue to be of heightened importance to acquiring new mathematical knowledge long after childhood. The findings from this investigation are consistent with the theory that spatial ability is of primary importance to the acquisition of novel mathematical knowledge well into adulthood but verbal abilities and spatial abilities are both important to the application of well learned mathematical principles. The present study found that spatial abilities map onto overall arithmetic performance as well as the utilization of relatively sophisticated conceptual shortcuts (i.e., inversion and associativity) and the tendency to use a shortcut flexibly and as needed (i.e., diversity). Conversely, verbal reasoning ability only mapped onto arithmetic performance (response time) when the influence of conceptually based shortcuts was removed, which implies that it must map onto more basic abilities. Participants with lower verbal reasoning ability also had a tendency to use problem solving algorithms (BEDMAS) inefficiently, which implies difficulty with fundamental aspects of mathematics. The results of this study do seem to fit the overarching theory that is proposed here, and this study demonstrates that the importance of spatial ability to mathematical concept acquisition continues into adulthood. It is important to consider the mechanism through which better spatial abilities facilitate the acquisition of new mathematical concepts. Spatial abilities contribute to the construction of effective visual representations for mathematical problem solving while contributions from other cognitive components have not yet been identified (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). When the findings in this research study are considered together with the previous research on visual representations, it can be inferred that visual representations could play an important role in the development of conceptual understanding represented by the inversion shortcut, the associativity shortcut, and the diversity of problem solving strategies. An important clue as to the fundamental cognitive mechanisms that lead to the development of conceptual knowledge is offered by this framework.

Merging the spatial mathematical literature with the conceptual arithmetic strategy literature

There is a rich body of research detailing the relationship between spatial abilities and mathematics (Delgado & Prieto, 2004; Floyd et al., 2003; McGrew & Hessler, 1995; Parkin & Beaujean, 2012; Rohde & Thompson, 2007; Visser et al., 2006; Wei et al., 2012). There is an equally rich body of literature on the use of different strategies to solve arithmetic problems (Bisanz & LeFevre, 1990; Gilmore & Papadatou- Pastou, 2009; Robinson & Dubé, 2009a, b, c; Robinson & LeFevre, 2012). Until now these two fields of study within mathematical cognition have not been combined. Previous research in the latter field has had a strong focus on the role of attentional processes in using conceptually based shortcuts to solve arithmetic problems (Robinson & Dubé, 2013; Robinson & LeFevre, 2012). This line of research has been fruitful, indicating that the ability to inhibit automatic responses contributes to the activation of conceptually based shortcuts (Robinson & Dubé, 2013). Findings indicate that inhibition may fundamentally be more related to a person's ability to apply their acquired conceptual knowledge with a conceptually based shortcut, than to whether or not they have that knowledge at all (Robinson & Dubé, 2013). This study indicates that spatial ability could be a very strong candidate for explaining the process by which individuals "grasp" and acquire conceptual knowledge. Future research should focus on the relationship between spatial abilities, conceptual shortcuts and attentional processes. Research paradigms could test the assumption that spatial abilities are more closely related to the acquisition of conceptual knowledge and that inhibition is more closely related to application. The current study indicates that mental models could be very important to the development of knowledge of the inverse principle and other forms of conceptual mathematical knowledge. It is also possible that attentional processes could mediate the processes by which individuals build mental models for conceptual knowledge of arithmetic.

Other previous research in conceptually based shortcuts is contradictory to the conclusions drawn here (Dubé & Robinson, 2010a). Analogical reasoning, an aspect of verbal intelligence, has been shown to be a predictor of inversion shortcut use in children (Dubé & Robinson, 2010a). This finding casts doubt on the theory presented here and there are several ways that this contradiction can be resolved. It is possible analogical reasoning represents a domain specific ability that facilities conceptual understanding, and is not representative of overall verbal intelligence making a contribution to this. Dubé and Robinson (2010a) do not argue for a general relation between verbal ability and acquisition of conceptual knowledge, but instead they posit that analogical reasoning involves the understanding of relationships, which is crucial to understanding mathematics, particularly the relations between mathematical operators. Future research looking at a wide array of verbal and spatial measures will be necessary to shed further light on this issue. Another explanation for the correlation between analogical reasoning and inversion shortcut use found in Dubé and Robinson's (2010a) study could be the stage of learning about inversion that the children in the sample were at. If these children were at a later stage of learning about the inversion shortcut then the functioning of their verbal cognitive resources would begin to play an important role in how well they apply that conceptual knowledge according to the theory presented in this paper. Furthermore, verbal ability influencing the acquisition of mathematical concepts is not inconsistent with this theoretical framework. It is argued here that spatial abilities are more important than other abilities to the acquisition of mathematical skills but that they are not necessarily of exclusive importance. Previous research demonstrates this principle (Wei et al., 2012).

In order to rigorously understand the relationship between cognitive components and the use of conceptually based shortcuts in arithmetic, research studies with large samples using large batteries of cognitive measures must be utilized. Furthermore the stage of learning that an individual is at along with other key factors must be controlled. A longitudinal design would be ideal for this. It will not only be important to examine the role of a wide array of cognitive components, but a wide array of mathematical measures, and measures of conceptual understanding must be used as well in order to understand the complex relationships among cognitive components, conceptual knowledge of arithmetic, learning stage, and other mathematical measures.

Implications and limitations

There are several limitations to this study. The sample size was small, and so the results must be treated with caution. Furthermore, there were more spatial measures used in this study than verbal reasoning measures. This may not be problematic considering the predictive power of the general spatial factor far exceeded the predictive power of verbal reasoning and in a way that fit the hypothesis. Future studies should utilize much larger batteries of cognitive tests in order to establish or falsify the theory that is proposed here. A longitudinal design will also be essential for rigorously demonstrating the role of different cognitive components during and after the acquisition of mathematical skills and knowledge.

Spatial ability may be more important than a wide array of other narrow cognitive components for learning mathematics, but this doesn't necessarily mean it is more important than the general factor of intelligence. Alternatively, spatial ability could be more important than all other cognitive measures as well as the general factor of intelligence. This is another issue that future research studies can address by using a wide array of cognitive measures.

A further concern is the tendency to overestimate the explanatory power of a nomothetic paradigm. Although at a population level spatial abilities seem to be of heightened importance for the acquisition of mathematical knowledge (DeSmedt et al., 2009; McKenzie et al., 2003; Rasmussen & Bisanz, 2005, & Wei et al, 2012), the effect size has not been shown to exceed 25% of variance accounted for (Wei et al., 2012). With this effect size there can exist people with high spatial ability who do not readily learn mathematics, and the opposite case as well. Some people may be able to use verbal reasoning to acquire mathematical concepts. Researchers must be cognizant of the fact that some individuals will have an intellectual profile that does not fit this pattern.

Mathematics education research draws attention to the value of interactive, dynamic classroom settings that make use of concrete demonstrations (Boaler, 1998a, b; Fuson et al., 2000; Wood & Sellers, 1997). Classroom settings of this type facilitate conceptual knowledge (Boaler, 1998a, b; Fuson et al., 2000; Wood & Sellers, 1997). Similarly, there is research demonstrating that animated visual aids can teach students statistical concepts better than static visual aids (Wender & Muehlboeck, 2003). This latter finding may be related to the finding that individuals who create “schematic” visual representations perform better at mathematics than individuals who create “pictorial” visual representations (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). In the one instance it is an external visual display that enhances someone’s ability to learn mathematics (Wender & Muehlboeck, 2003), and in the other instance it is an internal visual representation that they create (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). Individuals with higher spatial abilities create the more effective “schematic” visual representations when solving math problems (Garderen & Montague, 2003; Garderen, 2006; Hegarty & Kozhevnikov, 1999). It is possible that animated visual aids, good spatial abilities, and interactive classrooms facilitate the acquisition of mathematical knowledge for the same reasons; perhaps they all allow students to build good mental models of mathematical phenomena. An intriguing possibility is that external displays, demonstrations, and techniques can help students to form better mental models of mathematics. This is an especially intriguing possibility for students who are not as endowed with spatial abilities as their peers. Animated visual aids and interactive classrooms could help those students to compensate for lower spatial abilities. The possibility that animated depictions of mathematical concepts and other similar novel techniques may facilitate greater conceptual knowledge is promising.

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APPENDIX A

Arithmetic Problems

Order presented	Problem	Size	Type	Conducive
1	2 x 9 ÷ 3	Small	standard	Yes
2	6 x 11 ÷ 11	Large	inversion	N.A
3	5 x 10 ÷ 2	Large	standard	Yes
4	2 x 12 ÷ 12	Small	inversion	N.A
5	9 x 2 ÷ 3	Small	standard	No
6	7 x 5 ÷ 5	Large	inversion	N.A
7	12 x 10 ÷ 2	Large	standard	Yes
8	4 x 2 ÷ 2	Small	inversion	N.A
9	4 x 6 ÷ 3	Small	standard	Yes
10	7 x 12 ÷ 12	Large	inversion	N.A
11	6 x 20 ÷ 3	Large	standard	No
12	3 x 10 ÷ 10	Small	inversion	N.A
13	3 x 10 ÷ 6	Small	standard	No
14	4 x 9 ÷ 9	Large	inversion	N.A
15	4 x 8 ÷ 2	Large	standard	Yes
16	8 x 3 ÷ 3	Small	inversion	N.A
17	3 x 8 ÷ 2	Small	standard	Yes
18	9 x 7 ÷ 7	Large	inversion	N.A
19	2 x 18 ÷ 4	Large	standard	No
20	2 x 8 ÷ 8	Small	inversion	N.A
21	4 x 3 ÷ 2	Small	standard	No
22	7 x 6 ÷ 6	Large	inversion	N.A
23	7 x 9 ÷ 3	Large	standard	Yes
24	4 x 3 ÷ 3	Small	inversion	N.A
25	2 x 8 ÷ 4	Small	standard	Yes
26	5 x 8 ÷ 8	Large	inversion	N.A
27	6 x 9 ÷ 2	Large	standard	No
28	3 x 5 ÷ 5	Small	inversion	N.A
29	2 x 5 ÷ 10	Small	standard	No
30	6 x 8 ÷ 8	Large	inversion	N.A
31	5 x 12 ÷ 6	Large	standard	Yes
32	7 x 2 ÷ 2	Small	inversion	N.A
33	3 x 4 ÷ 2	Small	standard	Yes
34	5 x 10 ÷ 10	Large	inversion	N.A
35	2 x 12 ÷ 8	Small	standard	No
36	3 x 8 ÷ 8	Small	inversion	N.A
37	2 x 9 ÷ 6	Small	standard	No
38	5 x 9 ÷ 9	Large	inversion	N.A
39	4 x 9 ÷ 3	Large	standard	Yes
40	3 x 4 ÷ 4	Small	inversion	N.A

APPENDIX B

(Ennis & Millman, 1985)

Cornell Critical Thinking Test: Deductive reasoning subscale: Example Question

Section 1A

In the first five items, two men are debating about voting by eighteen-year-olds. Mr. Pinder is the speaker in the first three items. Mr Wilstings in the last two. Each item presents a set of statements and conclusion. In each item, the conclusion is underlined. Do not be concerned with whether or not the conclusions or statements are true.

Mark items 1 through 5 according to the following system:

If the conclusion **follows necessarily** from the statements given, mark **A.**
If the conclusion **contradicts** the statements given, Mark **B** .
If the conclusion neither follows necessarily nor contradicts the statements given, mark **C**

If a conclusion follows necessarily, a person who accepts the statements is unavoidably committed to accepting the conclusion. When two things are contradictory, they cannot both be correct.

<u>CONSIDER EACH ITEM INDEPENDENTLY OF THE OTHERS.</u>

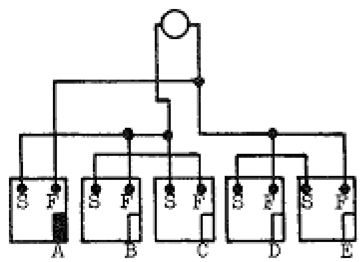
1. "Mr. Wilstings says that eighteen-year-olds haven't faced the problems of the world, and that anyone who hasn't faced these problems should not be able to vote. What he says is correct, but <u>eighteen-year-olds still should be able to vote.</u> They're mature human beings, aren't they?"

Answer:
1. B. The conclusion affirms what is denied by an implication of the statements given.

APPENDIX C

Example question from SS-2

In the problems in this test there will be five such boxes. Only one box will have a line from the S, through the circle, and back to the F in the same box. Dots on the lines show the only places where connections can be made between lines. If lines meet or cross where there is no dot, there is no connection between the lines. Now try this example. Show which box has the line through the circle by blackening the space at the lower right of that box.



The first box is the one which has the line from S, through the circle, and back to F. The space lettered A has therefore been blackened.

Example question from SS-3

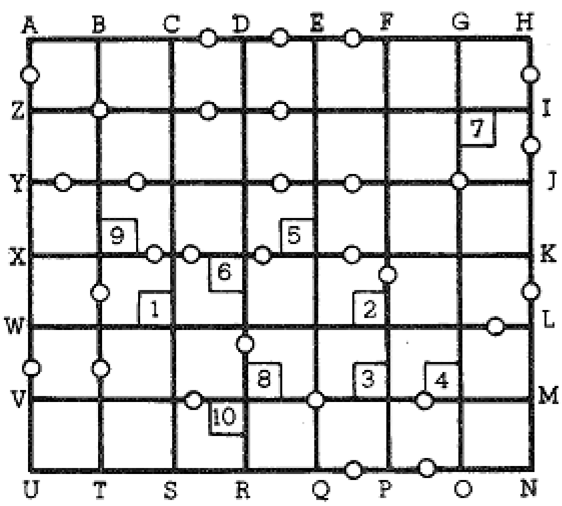
This is a test of your ability to find the shortest route between two places as quickly as possible. The drawing below is a map of a city. The dark lines are streets. The circles are road-blocks, and you cannot pass at the places where there are circles. The numbered squares are buildings. You are to find the shortest route between two lettered points. The number on the building passed is your answer.

Rules: 1. The shortest route will always pass along the side of one and only one of the numbered buildings.

2. A building is not considered as having been passed if a route passes only a corner and not a side.

3. The same numbered building may be used on more than one route.

Look at the sample map below. Practice by finding the shortest route between the various points listed at the right of the map. The first problem has been marked correctly.



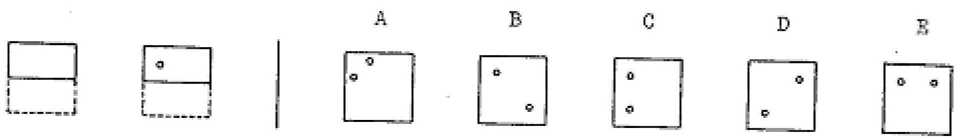
The shortest route from:	Passes building:
1. A to Z	1
2. E to S	
3. P to J	
4. V to K	
5. O to F	
6. G to M	
7. D to Q	
8. F to T	

The answers to the other practice problems are as follows: 2 passes 5; 3 passes 3; 4 passes 2; 5 passes 4; 6 passes 4; 7 passes 6; 8 passes 5.

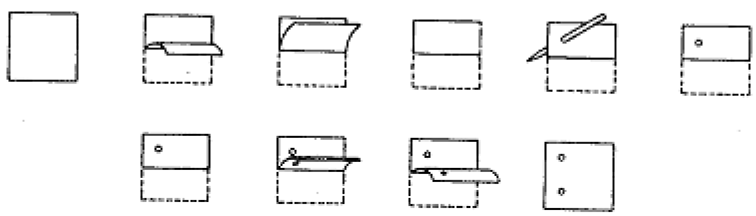
Example question from VZ-2

In this test you are to imagine the folding and unfolding of pieces of paper. In each problem in the test there are some figures drawn at the left of a vertical line and there are others drawn at the right of the line. The figures at the left represent a square piece of paper being folded, and the last of these figures has one or two small circles drawn on it to show where the paper has been punched. Each hole is punched through all the thicknesses of paper at that point. One of the five figures at the right of the vertical line shows where the holes will be when the paper is completely unfolded. You are to decide which one of these figures is correct and draw an X through that figure.

Now try the sample problem below. (In this problem only one hole was punched in the folded paper.)



The correct answer to the sample problem above is C and so it should have been marked with an X. The figures below show how the paper was folded and why C is the correct answer.



APPENDIX D

Response time (milliseconds) pairwise comparisons for the sex by problem type interaction.

	M	MI	MC	MN	FI	FC	FN
Male Inversion (MI)	1792.86		-1527.33	-4449.96*	-1181.46	N.A.	N.A.
Male Conducive (MC)	3320.19			-2922.63	N.A	2885.75	N.A.
Male Non-Conducive (MN)	6242.825				N.A.	N.A.	5337.12*
Female Inversion (FI)	2974.32					3231.62	8605.63*
Female Conducive (FC)	6205.945						5374.0*
Female Non-Conducive (FN)	11579.95						

Note. Significant contrasts are flagged with *.

APPENDIX E

University of Regina

OFFICE OF RESEARCH SERVICES

MEMORANDUM

DATE: August 28, 2012

TO: William Edwards
10 Cartier Court
Peterborough, Ontario K9H 6S6

FROM: Dr. Bruce Plouffe
A/Chair, Research Ethics Board

Re: **Underlying Cognitive Components and Conceptual Knowledge in Arithmetic (File # 06S1213)**

Please be advised that the University of Regina Research Ethics Board has reviewed your proposal and found it to be:

☑ 1. APPROVED AS SUBMITTED. Only applicants with this designation have ethical approval to proceed with their research as described in their applications. For research lasting more than one year (Section 1F). **ETHICAL APPROVAL MUST BE RENEWED BY SUBMITTING A BRIEF STATUS REPORT EVERY TWELVE MONTHS.** Approval will be revoked unless a satisfactory status report is received. Any substantive changes in methodology or instrumentation must also be approved prior to their implementation.

☐ 2. ACCEPTABLE SUBJECT TO MINOR CHANGES AND PRECAUTIONS (SEE ATTACHED). Changes must be submitted to the REB and approved prior to beginning research. Please submit a supplementary memo addressing the concerns to the Chair of the REB.** Do not submit a new application. Once changes are deemed acceptable, ethical approval will be granted.

☐ 3. ACCEPTABLE SUBJECT TO CHANGES AND PRECAUTIONS (SEE ATTACHED). Changes must be submitted to the REB and approved prior to beginning research. Please submit a supplementary memo addressing the concerns to the Chair of the REB.** Do not submit a new application. Once changes are deemed acceptable, ethical approval will be granted.

☐ 4. UNACCEPTABLE AS SUBMITTED. The proposal requires substantial additions or redesign. Please contact the Chair of the REB for advice on how the project proposal might be revised.

O. Bruce Plouffe

Dr. Bruce Plouffe

cc: Dr. Katherine Robinson – Psychology – Campion College

** supplementary memo should be forwarded to the Chair of the Research Ethics Board at the Office of Research Services (Research and Innovation Centre, Room 109) or by e-mail to research.ethics@uregina.ca

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